Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 272 (2021) 105642

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

Convergence rate for weighted polynomial approximation on the real line

Anna Kononova

St. Petersburg State University, Department of Mathematics and Mechanics, 198504, Stary Peterhof, Universitetsky pr., 28, St. Petersburg, Russia

Received 10 December 2020; received in revised form 16 June 2021; accepted 6 August 2021
Available online 20 August 2021

Communicated by H. Mhaskar

Abstract

In this note we study a quantitative version of Bernstein's approximation problem when the polynomials are dense in weighted spaces on the real line completing a result of Mergelyan (1960). We estimate in the logarithmic scale the error of the weighted polynomial approximation of the Cauchy kernel.

MSC: 41A10; 41A25

Keywords: Polynomial approximation; Weighted spaces; Rate of convergence

1. Introduction

The polynomial approximation problem in weighted spaces of functions on the real line is a classical subject in analysis since the beginning of the 20th century.

Let $W : \mathbb{R} \to [1, \infty]$ be an upper semicontinuous function on the real line. We denote by $\mathcal{C}_W$ the linear space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim\limits_{|t|\to\infty} \dfrac{f(t)}{W(t)} = 0$ endowed with finite semi-norm

$$\|f\|_{\infty,W} := \sup_{t\in\mathbb{R}} \left|\frac{f(t)}{W(t)}\right| < +\infty.$$

E-mail address: a.kononova@spbu.ru.

https://doi.org/10.1016/j.jat.2021.105642

We can think of $\mathcal{C}_W$ as a normed space (passing, as usual, to the quotient space under the standard equivalence relation $f \underset{W}{\sim} g \quad \Leftrightarrow \quad \|f-g\|_{\infty,W}=0$). Throughout the paper we always assume that W grows to infinity faster than any polynomial:

$$\lim_{|t|\to\infty} \frac{t^n}{W(t)} = 0 \quad \forall n \in \mathbb{N}. \tag{1}$$

This condition ensures that the space $\mathcal{C}_W$ contains all polynomials.

In 1924, S. N. Bernstein [3] posed the following question: *for which functions W satisfying (1) are the polynomials dense in $\mathcal{C}_W$?* The Bernstein weighted approximation problem has been persistently attracting attention of analysts for almost a century.

T. Hall [7] proved in 1938 that if the polynomials are dense in $\mathcal{C}_W$, then necessarily

$$\int_{-\infty}^{\infty} \frac{\log W(t)}{1+t^2} \mathrm{d}t = \infty. \tag{2}$$

This condition fails to be sufficient for the density of polynomials [8, Section VI.H.3].

There are different approaches to Bernstein's problem. We mention here two classical papers by N. I. Akhiezer [2] and S. N. Mergelyan [12], both published in 1956. Let us recall Mergelyan's solution to Bernstein's problem. He introduced the function

$$\Omega_W(z) := \sup\left\{|P(z)| : P \in \mathcal{P}, |P(t)| \le \sqrt{1+t^2}W(t),\ t \in \mathbb{R}\right\}, \quad z \in \mathbb{C}\setminus\mathbb{R},$$

where $\mathcal{P}$ is the space of the polynomials. Mergelyan proved that the density of the polynomials in $\mathcal{C}_W$ is equivalent to each of the following conditions:

- $\Omega_W(i) = \infty$,
- $\displaystyle\int_{-\infty}^{\infty} \frac{\log \Omega_W(t)}{1+t^2} \mathrm{d}t = \infty.$

If the function W is such that the polynomials are dense in the space $\mathcal{C}_W$, it is natural to ask about the approximation rate by polynomials. More precisely, let $\mathcal{P}_n$ denote the space of the polynomials of degree less than or equal to n. For a function $f \in \mathcal{C}_W$ and for positive n, we can define the error of approximation by polynomials of degree n by

$$\mathcal{E}_n(f) = \inf_{P\in\mathcal{P}_n} \|f-P\|_{\infty,W}.$$

The asymptotics of the sequence $\{\mathcal{E}_n(f)\}$ for various functions f were studied by numerous authors, and we refer the reader to the survey papers of D. S. Lubinsky [10] and of H. N. Mhaskar [14] on this subject.

For a particular class of functions f, the values $\mathcal{E}_n(f)$ were estimated by N. I. Akhiezer [2]; G. Wahde [16] found estimates of such kind in the L^2 norm and used them to deal with the uniform weighted approximation problem. M. M. Dzhrbashyan [6] studied the best approximation error of the Cauchy kernel by rational functions.

In this paper, we concentrate on the case when the function f to be approximated is fixed and equals the Cauchy kernel $\mathcal{K}(x) := (x-i)^{-1}$. In 1960, Mergelyan [13] found an upper bound for $\mathcal{E}_n(\mathcal{K})$ for quite a wide class of functions W. The main goal of the present paper is to obtain matching upper and lower bounds for $\mathcal{E}_n(\mathcal{K})$ in the logarithmic scale. An analogous problem in L^p norm is also considered. The proofs use the ideas from the paper [4] by A. Borichev, M. Sodin, and the author.

2. Main result

Definition 1. Let the function W be of the form

$$W(x) = \exp(\varphi(|x|)), \quad x \in \mathbb{R},$$

where φ is a positive continuous function strictly increasing on $\mathbb{R}_+$. We will say that a function W is *a weight* if it satisfies conditions (1) and (2).

Given $n \geq \varphi(0)$, define

$$A_n := \varphi^{-1}(n).$$

Given $\alpha \in \mathbb{R}$ we introduce the perturbed weight

$$W_\alpha(x) := W(x)(x^2 + 1)^{\alpha/2},$$

and the corresponding function $\varphi_\alpha := \log W_\alpha$.

We deal with the space $\mathcal{C}_W$ equipped with the norm $\|f\|_{\infty,W}$ defined in the introduction, and with the weighted L^p spaces defined as follows:

$$L^p_W := \left\{ f : \mathbb{R} \to \mathbb{C} : \|f\|_{p,W} := \left(\int_{\mathbb{R}} \left| \frac{f(x)}{W(x)} \right|^p \mathrm{d}x \right)^{1/p} < \infty \right\},$$

where $p \in [1, \infty)$. For $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, define

$$E_n(p, W) := \inf_{Q \in \mathcal{P}_{n-1}} \left\{ \left\| \frac{1}{x - i} - Q(x) \right\|_{p,W} \right\}.$$

The sequence $(E_n(p, W))_n$ is nonincreasing; if the polynomials are dense in $L^p_W(\mathcal{C}_W)$, then this sequence tends to 0. We are interested in estimating the growth rate of the sequence $|\log E_n(p, W)|$ as $n \to \infty$ in terms of the function φ.

An easy calculation shows that

$$E_n(p, W) = \inf_{P_n(i)=1, P_n \in \mathcal{P}_n} \left\| \frac{P_n(x)}{\sqrt{x^2 + 1}} \right\|_{p,W} = \inf_{P_n(i)=1, P_n \in \mathcal{P}_n} \|P_n(x)\|_{p,W_1}.$$

In particular, for the case $p = 2$ our results relate to the asymptotical properties of the Christoffel function outside the real line (we refer to [9] for more information about the Christoffel function). On the other hand, in terms of Mergelyan's function Ω_W we have

$$\lim_{n \to \infty} E_n(\infty, W) = \frac{1}{\Omega_W(i)}.$$

In our note we deal with the following classes of functions.

Definition 2. Given a continuous increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying condition (2), we say that φ is

- **normally growing** if $\varphi(x)/x^2$ is decreasing and $\varphi(x)$ is a convex function of $\log x$ on $[A, +\infty)$ for some $A > 0$;
- **rapidly growing** if $\varphi(x)/x^{1+\varepsilon}$ is increasing on $[A, +\infty)$ for some $\varepsilon > 0$ and $A > 0$;
- **regularly growing** if it is either normally growing or rapidly growing.

Remark 1. In the definition of normally growing functions by the convexity of $\varphi(x)$ as of function of $\log x$ we mean that the function $t \mapsto \varphi(e^t)$ is convex (for example, it holds in the case when $x\varphi'(x)$ is an increasing function).

Remark 2. There is a nonempty intersection between the classes of rapidly growing and normally growing functions, e.g., the function $\varphi(x) = x^{3/2}$ satisfies both conditions.

Remark 3. Note that the polynomials are dense in $\mathcal{C}_W$ and in L^2_W provided that $\log W$ is a regularly growing function [8, Sections VI.D, VI.G]. For normally growing weights, this can be proved using the convexity of $\varphi(e^t)$, and for rapidly growing weights this follows from the fact that $W(x) \gtrsim e^{|x|}$ for $|x|$ large enough. Therefore, for a regular weight W the sequence $(E_n(p, W))$ tends to 0. In particular, for all but finitely many $n \in \mathbb{N}$ we have $\log E_n < 0$.

We use the following notation: given two positive (or two negative) sequences (α_n) and (β_n) we will write

- $\alpha_n \lesssim \beta_n$ if for some constant $C > 0$ one has $\alpha_n \le C \cdot \beta_n$ for $n \in \mathbb{N}$ and
- $\alpha_n \simeq \beta_n$ if both $\alpha_n \lesssim \beta_n$ and $\beta_n \lesssim \alpha_n$ are true.

Analogous notation is used for functions.

The main result of our paper is

Theorem 1. *Suppose that φ is a regularly growing function. Given $p \in [1, \infty]$, we have*

$$\log E_n(p, W) \simeq -\int_0^1 \min\left(\varphi\left(\frac{1}{x}\right),\ n\right) \mathrm{d}x.$$

The implicit constants may depend on φ but are independent of n.

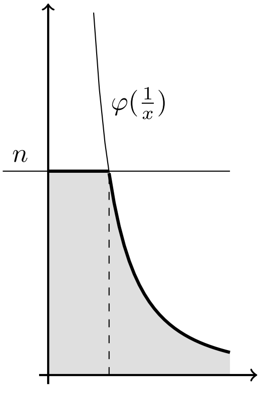

Fig. 1. $\min\left\{\varphi\left(\frac{1}{x}\right), n\right\}$.

Remark 4. The geometrical meaning of our growth classes can be illustrated by Fig. 1. If the function φ grows rapidly, then the area of the part of the subgraph of the function $\min\left\{\varphi\left(\frac{1}{x}\right), n\right\}$ under the cut-off grows not slower than the area of the remaining part of the subgraph as n tends to infinity, while in the case of the normal growth the part under the cut-off grows not faster than the area of the remaining part of the subgraph.

We also provide a reformulation of Theorem 1 which clarifies a bit our estimates.

Theorem 2. *Under the hypothesis of Theorem 1, for $1 \le p \le \infty$, we have*

(a) $\log E_n(p, W) \simeq -\dfrac{n}{\varphi^{-1}(n)}$ *if φ grows rapidly;*

(b) $\log E_n(p, W) \simeq -\displaystyle\int_0^{\varphi^{-1}(n)} \frac{\varphi(x)\mathrm{d}x}{x^2+1}$ *if φ grows normally.*

Due to Remark 4, Theorem 1 implies Theorem 2. The converse implication will be proven in Section 4.1.

The following corollary can be verified by a simple calculation, which we skip.

Corollary 1. *Let $p \in [1, \infty]$. Denote $E_n := E_n(p, W)$. Then we have the following estimates.*

- *If* $\varphi(x) = \dfrac{x}{\log(2+x)}$, *then*
$$|\log E_n| \simeq \log\log(n+e).$$
- *If* $\varphi(x) = x\log^{\nu}(2+x)$, $\nu > -1$, *then*
$$|\log E_n| \simeq \log^{\nu+1} n.$$
- *If* $\varphi(x) = x^{\nu}$, $\quad \nu > 1$, *then*
$$|\log E_n| \simeq n^{1-1/\nu}.$$
- *If* $\varphi(x) = \exp(x^{\nu})$, $\quad \nu > 0$, *then*
$$|\log E_n| \simeq \frac{n}{(\log n)^{1/\nu}}.$$

In Section 3 we establish some properties of the Tchebyshev polynomials and bring some results by Videnskii and Mergelyan. In Section 4 we prove Theorem 1, first in the uniform case and then in the L^p case.

3. Preliminaries

3.1. Some properties of the Tchebyshev polynomials

In this subsection we have collected some properties of Tchebyshev polynomials that are used in our paper. We start with the classical Tchebyshev inequality.

Let T_n denote the Tchebyshev polynomial of the first kind of degree n:

$$T_n(x) := \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right).$$

Tchebyshev inequality. *For any polynomial P_n of degree n such that $|P_n| \le 1$ on $[-1, 1]$, we have*

$$|P_n| \le |T_n| \text{ on } \mathbb{R} \setminus (-1, 1).$$

We also need two technical results.

Lemma 1. *Let $n = 2k$, $\;k \in \mathbb{N}$, and $a \in \mathbb{R}$. Then*

$$|T_n(i/a)| \ge \frac{1}{2}\left(\frac{1}{a}+1\right)^n.$$

Proof.

$$
\begin{aligned}
|T_n(i/a)| &= \frac{1}{2}\left|\left(\frac{i}{a}+\sqrt{-\frac{1}{a^2}-1}\right)^n + \left(\frac{i}{a}-\sqrt{-\frac{1}{a^2}-1}\right)^n\right| \\
&= \frac{1}{2}\left(\frac{1}{a}+\sqrt{\frac{1}{a^2}+1}\right)^n + \frac{1}{2}\left(\frac{1}{a}-\sqrt{\frac{1}{a^2}+1}\right)^n \\
&= \frac{1}{2}\left(\frac{1}{a}+\sqrt{\frac{1}{a^2}+1}\right)^n + \frac{1}{2}\left(\frac{1}{a}+\sqrt{\frac{1}{a^2}+1}\right)^{-n} \geq \frac{1}{2}\left(\frac{1}{a}+1\right)^n. \quad \square
\end{aligned}
$$

Lemma 2. *Let φ be a rapidly growing function. Then*

$$
\sup_{x \geq A_n}\left\{\frac{|T_n(x/A_n)|}{W(x)}\right\} \lesssim \frac{2^n}{e^n}.
$$

Proof. We have

$$
\begin{aligned}
|T_n(x/A_n)| &= \frac{1}{2}\left|\left(\frac{x}{A_n}+\sqrt{\frac{x^2}{A_n^2}-1}\right)^n + \left(\frac{x}{A_n}-\sqrt{\frac{x^2}{A_n^2}-1}\right)^n\right| \\
&\leq \left(\frac{x}{A_n}+\sqrt{\frac{x^2}{A_n^2}-1}\right)^n \leq \left(2\frac{x}{A_n}\right)^n, \qquad x \geq A_n.
\end{aligned}
$$

Therefore,

$$
\sup_{x \geq A_n}\left\{\frac{|T_n(x/A_n)|}{W(x)}\right\} \leq \frac{2^n}{A_n^n}\sup_{x \geq A_n}\left\{\frac{x^n}{W(x)}\right\}. \tag{3}
$$

Since the function φ is rapidly growing, for some $\varepsilon > 0$ and for large n we have

$$
\log\frac{x^n}{W(x)} = n\log x - \varphi(x) \leq n\log x - \frac{\varphi(A_n)}{A_n^{1+\varepsilon}}x^{1+\varepsilon} =: g_n(x).
$$

Furthermore,

$$
g_n'(x) = \frac{n}{x} - (1+\varepsilon)x^{\varepsilon}\frac{n}{A_n^{1+\varepsilon}},
$$

and we conclude that the only critical point x^* of the function g_n satisfies the relation

$$
\left(\frac{x^*}{A_n}\right)^{1+\varepsilon} = \frac{1}{1+\varepsilon}
$$

and, hence, the function g_n decreases on $[A_n, \infty)$. Therefore,

$$
\log\frac{x^n}{W(x)} \leq g_n(x) \leq g_n(A_n) = n\log A_n - n
$$

for $x \geq A_n$. Combining this with (3), we get

$$
\sup_{x \geq A_n}\left\{\frac{|T_n(x/A_n)|}{W(x)}\right\} \leq \frac{2^n}{e^n},
$$

for n large enough. $\square$

3.2. Results of Mergelyan and Videnskii

The result of Mergelyan already mentioned in the introduction is based on a lemma by Videnskii ([15], Lemma 3). For the reader's convenience, we provide here the proofs of the versions of both results, which suffice for our purposes.

Theorem 3 (*Mergelyan, [13]*)**.** *Let φ be a function satisfying (2) and such that $\varphi(t)$ is a convex function of $\log t$. Then we have*

$$\log E_n(\infty, W) \lesssim -\int_0^{l_n} \frac{\varphi(x)\mathrm{d}x}{x^2+1},$$

with $l_n = \dfrac{A_{2n}}{2e}$.

Lemma 3 (*Videnskii, [15]*)**.** *Let $\varphi(t)$ be a convex function of $\log t$. Set*

$$M_k = \sup_{x>0} \frac{x^{2k}}{W(x)}$$

and

$$F(x) := \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k M_k}.$$

Then $F(x) \lesssim W(x) \lesssim x^2 F(2x),\ x \geq 1$.

Remark 5. Note that this lemma can also be derived from the results of [1].

3.2.1. Proof of the Videnskii lemma

The sequence $(M_k)_k$ increases for k sufficiently large. Set

$$T(x) := \sup_{k\geq 0} \frac{x^{2k}}{M_k}, \qquad x > 0.$$

Since the function $t \mapsto \varphi(\exp t)$ is convex, the graph of the function

$$\log T(\exp t) = \sup_{k\geq 0} (2kt - \log M_k)$$

is an infinite polygon consisting of the supporting lines of $\varphi(\exp t)$ with even slopes. Therefore, we have $T(x) \leq W(x)$ for large $|x|$. Then

$$F(x) := \sum_{k\geq 0} \frac{x^{2k}}{2^k M_k} \leq \sum_{k\geq 0} \frac{T(x)}{2^k} \lesssim W(x), \qquad x \in \mathbb{R}.$$

Next, let $Kt - B$ be (some) supporting line of the graph of the convex function $\varphi(\exp t)$ at a sufficiently large point t^*:

$$\varphi(\exp(t)) \geq Kt - B, \quad \varphi(\exp(t^*)) = Kt^* - B, \qquad K, B \in \mathbb{R}^+.$$

If the slope K is even, then $B = \log M_{K/2}$ and $T(\exp(t^*)) = W(\exp(t^*))$. Otherwise, let m be the integer part of $K/2$. Then $\log M_m \leq B$ and we have

$$\varphi(\exp(t^*)) = Kt^* - B \leq Kt^* - \log M_m \leq (2m+2)t^* - \log M_m \leq \log\big(T(\exp(t^*))\big) + 2t^*,$$

that is

$$W(x) \le x^2 T(x)$$

for sufficiently large $x = \exp(t^*)$. Thus we obtain

$$F(x) = \sum_{k\ge 0} \frac{x^{2k}}{2^k M_k} \ge \sum_{k\ge 0} \frac{x^{2k}}{2^{2k} M_k} \ge T\left(\frac{x}{2}\right) \gtrsim x^{-2} W(\frac{x}{2}),$$

which proves the lemma. □

3.2.2. Proof of the Mergelyan theorem

We set $M_k = \sup_{x>0} \frac{x^{2k}}{W(x)}$, and $b_k := \frac{1}{2^k M_k}$, so that

$$F(x) = \sum_{k\ge 0} b_k x^{2k}.$$

Fix $n \ge \varphi(0)$. By the definition of M_k it is clear that $b_k \le \frac{W(x)}{2^k x^{2k}}$ for every $x > 0$. Taking $x = A_{2n}$ we see that

$$b_k \le \frac{e^{2n}}{2^k A_{2n}^{2k}}.$$

Consider the polynomial $P_{2n} := \sum_{k=0}^{n} b_k x^{2k}$. For $|x| \le \frac{A_{2n}}{e}$, we have

$$0 \le F(x) - P_{2n}(x) \le \sum_{k=n+1}^{\infty} b_k x^{2k} \le \sum_{k=n+1}^{\infty} \frac{e^{2n}}{2^k A_{2n}^{2k}} x^{2k} \le \sum_{k=n+1}^{\infty} \frac{e^{2n}}{2^k e^{2k}} \le 1/2.$$

Since $M_0 \le 1$, we have $b_0 \ge 1$, $F(x) \ge 1$, $x \in \mathbb{R}$, and, hence,

$$P_{2n}(x) \ge F(x)/2, \qquad |x| \le \frac{A_{2n}}{e}. \tag{4}$$

Let Q_n be a polynomial of degree n, with no zeros in the upper half plane and such that $|Q_n^2(x)| = P_{2n}(x), \quad x \in \mathbb{R}$. Then

$$F(x) \ge |Q_n^2(x)| = P_{2n}(x) \ge 1.$$

By (4) we have

$$|Q_n^2(x)| = P_{2n}(x) \ge F(x)/2, \qquad |x| \le \frac{A_{2n}}{e}.$$

By the Poisson formula and the Videnskii lemma we obtain that ($l_n := \frac{A_{2n}}{2e}$):

$$\begin{aligned}
\log|Q_n^2(i)| &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\log|Q_n^2(x)|}{x^2+1}\mathrm{d}x \gtrsim \int_0^{2l_n} \frac{\log|Q_n^2(x)|}{x^2+1}\mathrm{d}x \\
&\gtrsim \int_0^{2l_n} \frac{\log F(x)}{x^2+1}\mathrm{d}x + O(1) \gtrsim \int_0^{2l_n} \frac{\log W(\frac{x}{2}) - 2\log x}{x^2+1}\mathrm{d}x + O(1) \\
&\gtrsim \int_0^{2l_n} \frac{\varphi(x/2)}{x^2+1}\mathrm{d}x + O(1) \gtrsim \int_0^{l_n} \frac{\varphi(x)}{x^2+1}\mathrm{d}x, \qquad n \to \infty.
\end{aligned}$$

By the definition of $E_n(\infty, W)$ we see that

$$\log E_n(\infty, W) \le \log \left\| \frac{Q_n}{Q_n(i)} \right\|_{\infty, W_1} \le \log \|Q_n\|_{\infty, W} - \log |Q_n(i)|$$
$$\lesssim \log \|F\|_{\infty, W} - \int_0^{l_n} \frac{\varphi(x)}{x^2+1} \mathrm{d}x \le -\int_0^{l_n} \frac{\varphi(x)}{x^2+1} \mathrm{d}x.$$

which proves the Mergelyan theorem. □

4. Proof of the main theorem

4.1. Equivalence of reformulations of the main theorem

Recall that the main result is presented in the introduction in two equivalent reformulations.

To show how Theorem 2 implies Theorem 1 we use the following lemma (as was noted in the introduction the converse statement holds due to Remark 4).

Lemma 4. *If a function φ is normally growing, then*

$$\int_0^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \gtrsim \frac{\varphi(a)}{a}, \qquad a \ge 1,$$

while for a rapidly growing function φ we have

$$\int_0^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \lesssim \frac{\varphi(a)}{a}, \qquad a \ge 1.$$

Proof. If the function $\varphi(x)/x^2$ decreases for $x \ge A$, then we have

$$\int_0^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \ge \int_A^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \ge \frac{\varphi(a)}{a^2} \int_A^a \frac{x^2\mathrm{d}x}{x^2+1} \gtrsim \frac{\varphi(a)}{a}, \qquad a \ge 2A.$$

If $\varphi(x)/x^{1+\varepsilon}$ increases for $x \ge A$ and for some $\varepsilon > 0$, then

$$\int_0^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \lesssim \int_A^a \frac{\varphi(x)\mathrm{d}x}{x^2+1} \lesssim \frac{\varphi(a)}{a^{1+\varepsilon}} \int_A^a \frac{x^{1+\varepsilon}\mathrm{d}x}{x^2+1} \lesssim \frac{\varphi(a)}{a}, \qquad a \ge A. \quad \square$$

Combining Theorem 3 with Lemma 4 in both cases we get

$$-\log E_n \simeq \int_0^{A_n} \frac{\varphi(x)\mathrm{d}x}{x^2+1} + \frac{\varphi(A_n)}{A_n}.$$

Changing the variables we obtain

$$-\log E_n \simeq \int_0^{A_n} \frac{\varphi(x)\mathrm{d}x}{x^2+1} + \frac{\varphi(A_n)}{A_n} \simeq \int_1^{A_n} \frac{\varphi(x)\mathrm{d}x}{x^2} + \frac{n}{A_n} = \int_0^1 \min\left(\varphi\left(\frac{1}{x}\right), n\right) \mathrm{d}x,$$

which yields the statement of Theorem 1. □

4.2. Estimating $\log E_n$ for the uniform norm

4.2.1. The lower bound

Here we prove the following result.

Lemma 5. *Given a weight W we have*

$$\log E_n \gtrsim -\int_0^{A_n} \frac{\varphi(x)\mathrm{d}x}{x^2+1} - \frac{n}{A_n},$$

where $E_n = E_n(\infty, W)$.

Proof. Let P_n be an extremal polynomial of degree n such that $P_n(i) = 1$ and $E_n = \|P_n\|_{\infty,W_1}$. Then, using the Poisson integral formula for the half-plane and the subharmonicity of $\log|P_n|$, we get

$$0 = \log|P_n(i)| \le \int_{\mathbb{R}} \frac{\log|P_n(x)|\mathrm{d}x}{x^2+1} = \left(\int_{|x|<a} + \int_{|x|\ge a}\right) \frac{\log|P_n(x)|\mathrm{d}x}{x^2+1} =: I_1 + I_2, \qquad (5)$$

where $a > 1$ is some parameter to be chosen later.

We start by estimating the first integral:

$$\begin{aligned} I_1 &= \int_{-a}^{a} \frac{\log|P_n(x)|}{x^2+1}\mathrm{d}x \\ &\le \int_{-a}^{a} \log\left|\frac{P_n(x)}{\sqrt{x^2+1}W(x)}\right| \frac{\mathrm{d}x}{x^2+1} + \int_0^a \frac{\log(x^2+1)}{x^2+1}\mathrm{d}x + 2\int_0^a \frac{\varphi(x)}{x^2+1}\mathrm{d}x \\ &\lesssim \int_{-a}^{a} \log\|P_n(x)\|_{\infty,W_1} \frac{\mathrm{d}x}{x^2+1} + \int_0^a \frac{\varphi(x)}{x^2+1}\mathrm{d}x \lesssim \log E_n + \int_0^a \frac{\varphi(x)}{x^2+1}\mathrm{d}x, \end{aligned} \qquad (6)$$

for a large enough.

The next step is to estimate the second integral in (5) with the help of the Tchebyshev inequality (Section 3.1):

$$\begin{aligned} I_2 &\le \int_a^\infty \frac{\log|T_n(x/a)|}{x^2+1}\mathrm{d}x + \int_a^\infty \frac{\log\left(\max_{[-a,a]}|P_n|\right)}{x^2+1}\mathrm{d}x \\ &\lesssim \int_a^\infty \frac{n\cdot\log(x/a)}{x^2+1}\mathrm{d}x + \int_a^\infty \frac{\log\left(\max_{[-a,a]}|P_n|\right)}{x^2+1}\mathrm{d}x \\ &\lesssim \frac{n}{a} + \int_a^\infty \log\|P_n\|_{\infty,W_1}\frac{\mathrm{d}x}{x^2+1} + \int_a^\infty \frac{\log\left(\sup_{[-a,a]} W_1\right)}{x^2+1}\mathrm{d}x \\ &\lesssim \frac{n}{a} + \log E_n + \frac{1}{a}\log(W_1(a)) \lesssim \frac{n}{a} + \log E_n + \frac{\varphi(a)}{a}. \end{aligned}$$

By (5) and (6) we conclude that

$$-\log E_n \lesssim \int_0^a \frac{\varphi(x)}{x^2+1}\mathrm{d}x + \frac{n+\varphi(a)}{a}.$$

Finally, let $a = A_n = \varphi^{-1}(n)$. Then

$$-\log E_n \lesssim \int_0^{A_n} \frac{\varphi(x)}{x^2+1}\mathrm{d}x + \frac{n}{A_n},$$

which proves the lemma. $\square$

4.2.2. The upper bound

For normally growing φ we just use the Mergelyan Theorem. Indeed, since the function $\varphi(x)/x^2$ decreases for x large enough, we have $\varphi(2ex) \lesssim \varphi(x)$. Therefore,

$$\int_0^{A_n} \frac{\varphi(x)\mathrm{d}x}{x^2+1} \le \int_0^{A_{2n}} \frac{\varphi(x)\mathrm{d}x}{x^2+1} = \int_0^{A_{2n}/(2e)} \frac{\varphi(2ex)}{4e^2x^2+1}2e\mathrm{d}x \lesssim \int_0^{A_{2n}/(2e)} \frac{\varphi(x)\mathrm{d}x}{x^2+1}. \qquad (7)$$

To obtain an upper bound in the case of rapidly growing functions φ we use the Tchebyshev polynomials. Taking into account Lemma 1, Lemma 2, and the fact that

$$\sup_{0<x\le A_n}\left\{\frac{|T_n(x/A_n)|}{W(x)}\right\}\le 1,$$

we get by the definition of E_n:

$$\begin{aligned}E_n &\le \frac{\|T_n(x/A_n)\|_{\infty,W_1}}{|T_n(i/A_n)|}\le\frac{\|T_n(x/A_n)\|_{\infty,W}}{|T_n(i/A_n)|}\\ &\le \frac{1}{|T_n(i/A_n)|}\max\left(1,\ \sup_{x>A_n}\left\{\frac{|T_n(x/A_n)|}{W(x)}\right\}\right)\\ &\lesssim \max\left(1,\ \frac{2^n}{e^n}\right)\left(\frac{1}{A_n}+1\right)^{-n}\le\left(\frac{1}{A_n}+1\right)^{-n},\end{aligned}$$

and finally

$$\log E_n\lesssim -\frac{n}{A_n}. \tag{8}$$

4.2.3. *Conclusion*

Estimates (7), (8) together with Lemma 4 and Lemma 5 give Theorem 2 in the uniform case.

4.3. *Estimating* $\log E_n$ *for the weighted* L^p *space,* $1\le p<\infty$

4.3.1. *The upper bound*

Recall that for $\alpha\in\mathbb{R}$ we use the notation

$$W_\alpha(x):=W(x)(x^2+1)^{\alpha/2}.$$

Given $1\le p<\infty$, $n\ge 0$, we choose a polynomial P_n of degree n such that

$$E_n(\infty,W_{-2})=\left\|\frac{P_n}{\sqrt{x^2+1}}\right\|_{\infty,W_{-2}}=\|P_n\|_{\infty,W_{-1}}$$

and $|P_n(i)|=1$. Then

$$\begin{aligned}E_n^p(p,W)&\le\left\|\frac{P_n}{\sqrt{x^2+1}}\right\|_{p,W}^p=\int_{\mathbb{R}}\left|\frac{P_n(x)}{W(x)\sqrt{x^2+1}}\right|^p\mathrm{d}x\\ &=\int_{\mathbb{R}}\left|\frac{P_n(x)}{W_{-1}(x)}\right|^p\frac{\mathrm{d}x}{(x^2+1)^p}\lesssim\sup_{x\in\mathbb{R}}\left|\frac{P_n(x)}{W_{-1}(x)}\right|^p=E_n^p(\infty,W_{-2}).\end{aligned}$$

Note, that for a rapidly growing function φ we have $W(x)\gtrsim e^{|x|}$, while for a normally growing function $W(x)\gtrsim(x^2+1)^2$ (see Lemma 3). In both cases it follows that

$$W_{-2}(x)=\frac{W(x)}{x^2+1}\gtrsim\sqrt{W(x)},$$

and we get

$$\log E_n(p,W)\lesssim\log E_n(\infty,W^{1/2}).$$

Since the function $\varphi/2$ grows regularly, we can use Theorem 1 for the case $p=\infty$ (which is already proved) to obtain that

$$\log E_n(p,W) \lesssim -\int_0^1 \min\left(\frac{1}{2}\varphi\left(\frac{1}{x}\right), n\right)\mathrm{d}x \lesssim -\int_0^1 \frac{1}{2}\min\left(\varphi\left(\frac{1}{x}\right), n\right)\mathrm{d}x,$$

which gives the upper estimate in Theorem 1 for $1\le p<\infty$.

4.3.2. The lower bound

Now, let P_n be the extremal polynomial of degree n for the L^p norm, such that $P_n(i)=1$ and

$$E_n(p,W) = \|P_n\|_{p,W_1}.$$

Then

$$0 \le \int_{\mathbb{R}} \frac{\log|P_n(x)|\mathrm{d}x}{x^2+1} = \left(\int_{|x|<A_n} + \int_{|x|\ge A_n}\right)\frac{\log|P_n(x)|\mathrm{d}x}{x^2+1} =: I_1+I_2, \tag{9}$$

where $A_n=\varphi^{-1}(n)$ as before.

To estimate the first integral we use the Jensen inequality:

$$\begin{aligned} I_1 &= \int_{-A_n}^{A_n} \log\left|\frac{P_n(x)}{W(x)\sqrt{x^2+1}}\right|^p \frac{\mathrm{d}x}{x^2+1} + 2p\int_0^{A_n}\left(\varphi(x)+\log\sqrt{x^2+1}\right)\frac{\mathrm{d}x}{x^2+1} \\ &\lesssim \log\left(\int_{-A_n}^{A_n}\left|\frac{P_n(x)}{W(x)\sqrt{x^2+1}}\right|^p\frac{\mathrm{d}x}{x^2+1}\right) + \int_0^{A_n}\frac{\varphi(x)\mathrm{d}x}{x^2+1} \\ &\lesssim \log E_n(p,W) + \int_0^{A_n}\frac{\varphi(x)\mathrm{d}x}{x^2+1}. \end{aligned} \tag{10}$$

To estimate I_2 we use the following lemma.

Lemma 6. *Let Q be a polynomial of degree n, $a>1$ and $p\ge 1$. Then*

$$\max_{[-a,a]}|Q|^p \lesssim \frac{n^2}{a}\int_{-a}^{a}|Q|^p\mathrm{d}x$$

Proof. Let $x_0\in J := [-a,a]$ be such that

$$\max_J|Q| = |Q(x_0)|.$$

Then for every $x\in J_1 := \left[x_0-\frac{a}{2n^2}, x_0+\frac{a}{2n^2}\right]\cap J$ there exists ξ on the interval $J\cap J_1$ such that

$$|Q(x)-Q(x_0)| = |x-x_0|\cdot|Q'(\xi)| \le \frac{a}{2n^2}|Q'(\xi)|$$

Therefore, applying the classical Markov inequality [11]; [5, Theorem 5.1.8] on the interval $[-a,a]$, we obtain

$$|Q(x)-Q(x_0)| \le \frac{a}{2n^2}|Q'(\xi)| \le \frac{a}{2n^2}\frac{n^2}{a}|Q(x_0)| \le \frac{1}{2}|Q(x_0)|,$$

and, hence, we have

$$|Q(x)| \ge \frac{1}{2}|Q(x_0)|, \quad |x-x_0| \le \frac{a}{2n^2}.$$

Thus,

$$\int_{-a}^{a} |Q|^p \mathrm{d}x \geq \int_{J\cap J_1} |Q|^p \mathrm{d}x \geq \frac{a}{4n^2}\frac{1}{2^p}|Q(x_0)|^p,$$

proving the lemma. □

Finally, we turn to estimating I_2. By the Tchebyshev inequality,

$$\begin{aligned} I_2 &\lesssim \int_{A_n}^{\infty} \frac{\log|T_n(x/A_n)|\mathrm{d}x}{x^2+1} + \int_{A_n}^{\infty} \log\left(\max_{[-A_n,A_n]}|P_n|\right)\frac{\mathrm{d}x}{x^2+1} \\ &\lesssim \int_{A_n}^{\infty} \frac{n\cdot\log(x/A_n)\mathrm{d}x}{x^2+1} + \log\left(\max_{[-A_n,A_n]}|P_n|\right)\int_{A_n}^{\infty}\frac{\mathrm{d}x}{x^2+1}. \end{aligned}$$

By Lemma 6, we obtain that

$$\begin{aligned} I_2 &\lesssim \frac{n}{A_n} + \frac{1}{A_n}\cdot\log\left(\frac{n^2}{A_n}\int_{-A_n}^{A_n}|P_n|^p\mathrm{d}x\right) \\ &\lesssim \frac{n}{A_n} + \frac{\log n^2}{A_n} + \frac{1}{A_n}\log\left(\frac{W^p(A_n)(A_n^2+1)^{p/2}}{A_n}\int_{-A_n}^{A_n}\left|\frac{P_n(x)}{W(x)\sqrt{x^2+1}}\right|^p\mathrm{d}x\right) \\ &\lesssim \frac{n}{A_n} + \frac{p\log(A_n^2+1)-2\log A_n}{A_n} + \frac{1}{A_n}\log\left(W^p(A_n)\int_{-A_n}^{A_n}\left|\frac{P_n(x)}{W(x)\sqrt{x^2+1}}\right|^p\mathrm{d}x\right) \\ &\lesssim \frac{n}{A_n} + \frac{\varphi(A_n)}{A_n} + \frac{1}{A_n}\log E_n(p,W) \\ &\lesssim \frac{n}{A_n} + \log E_n(p,W), \end{aligned}$$

because $n = \varphi(A_n)$.

Combining this estimate with (9) and (10) we get

$$-\log E_n(p,W) \lesssim \int_0^{A_n}\frac{\varphi(x)\mathrm{d}x}{x^2+1} + \frac{\varphi(A_n)}{A_n}$$

which completes the proof of Theorem 2 and, hence, of Theorem 1, in the case $1 \leq p < \infty$. □

Acknowledgments

The author is grateful to A. Borichev for attracting attention to this problem and for constructive suggestions, to E. Abakumov and M. Sodin for their helpful remarks and recommendations.

This work is supported by Russian Science Foundation Grant No. 20-61-46016.

References

[1] E. Abakumov, E. Doubtsov, Approximation by proper holomorphic maps and tropical power series, Constr. Approx. 47 (2) (2018) 321–338.

[2] N.I. Akhiezer, On the weighted approximation of continuous functions by polynomials on the entire real axis, Uspekhi Mat. Nauk (N.S.) 11 (4) (1956) 33–43, (70); Amer. Math. Sot. Trans. Ser. 2 (22) (1962) 95–137.

[3] S.N. Bernstein, Le problème de l'approximation des fonctions continues sur tout l'axe réel et l'une de ses applications, Bull. Math. Soc. France 52 (1924) 399–410.

[4] A. Borichev, A. Kononova, M. Sodin, Notes on the Szegő minimum problem. I. Measures with deep zeroes, Isr. J. Math. 240 (2020) 725–743.

[5] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer-Verlag, New York, 1995.
[6] M.M. Dzhrbashyan, Biorthogonal systems of rational functions and best approximation of the Cauchy kernel on the real axis, Mathemat. USSR-Sbornik 24 (3) (1974) 409–433.
[7] T. Hall, Sur l'approximation polynômiale des fonctions continues d'une variable réelle, Neuvième Congrès des Mathématiciens (1938), Helsingfors (1939) 367–369.
[8] P. Koosis, The Logarithmic Integral I, Cambridge University Press, Cambridge, 1988.
[9] A.L. Levin, D.S. Lubinsky, Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, Constr. Approx. 8 (1992) 463–535.
[10] D.S. Lubinsky, Survey of weighted polynomial approximation with exponential weights, Surv. Approx. Theory 3 (2007) 1–105.
[11] A.A. Markov, On a problem of D.I. Mendeleev, Zap. Imp. Akad. Nauk. 62 (1889) 1–24, (In Russian).
[12] S.N. Mergelyan, Weighted approximation by polynomials, Uspekhi Mat. Nauk 11 (1956) 107–152; Amer. Math. Soc. Transl. 10 (1958) 59–106.
[13] S.N. Mergelyan, Best approximations with the weight on a straight line, Dokl. Akad. Nauk SSSR 132 (2) (1960) 287–290; Sov. Math. 1 (1960) 552–556.
[14] H.N. Mhaskar, Weighted polynomial approximation, J. Approx. Theory 46 (1986) 100–110.
[15] V.S. Videnskii, On normally growing functions, Meetings of the Moscow Mathematical Society, Uspekhi Mat. Nauk, 9 2 (60) (1954) 207–213, (In Russian).
[16] G. Wahde, An extremal problem related to Bernstein's approximation problem, Math. Scand. 15 (1964) 131–141.