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Renorming spaces with greedy bases

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Abstract

We study the problem of improving the greedy constant or the democracy constant of a basis of a Banach space by renorming. We prove that every Banach space with a greedy basis can be renormed, for a given $\varepsilon > 0$, so that the basis becomes $(1+\varepsilon)$-democratic, and hence $(2+\varepsilon)$-greedy, with respect to the new norm. If in addition the basis is bidemocratic, then there is a renorming so that in the new norm the basis is $(1+\varepsilon)$-greedy. We also prove that in the latter result the additional assumption of the basis being bidemocratic can be removed for a large class of bases. Applications include the Haar systems in $L_p[0, 1]$, $1 < p < \infty$, and in dyadic Hardy space H_1, as well as the unit vector basis of Tsirelson space.

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0. Introduction

In approximation theory one is often faced with the following problem. We start with a signal, *i.e.,* a vector x in some Banach space X. We then consider the (unique) expansion $\sum_{i=1}^{\infty} x_i e_i$ of x with respect to some (Schauder) basis (e_i) of X. For example, this may be a Fourier expansion of x, or it may be a wavelet expansion in L_p. We then wish to approximate x by considering m-term approximations with respect to the basis. The smallest error is given by

$$\sigma_m(x) = \inf\left\{ \left\| x - \sum_{i\in A} a_i e_i \right\| : A \subset \mathbb{N},\ |A| \le m,\ (a_i)_{i\in A} \subset \mathbb{R} \right\}.$$

We are interested in algorithms that are easy to implement and that produce the best m-term approximation, or at least get close to it. A very natural process is the greedy algorithm which we now describe. For each $x = \sum x_i e_i \in X$ we fix a permutation $\rho = \rho_x$ of $\mathbb{N}$ (not necessarily unique) such that $|x_{\rho(1)}| \ge |x_{\rho(2)}| \ge \cdots$. We then define the *mth greedy approximant to x* by

$$\mathcal{G}_m(x) = \sum_{i=1}^{m} x_{\rho(i)} e_{\rho(i)}.$$

For this to make sense we need $\inf \|e_i\| > 0$, otherwise (x_i) may be unbounded. In fact, since we will be dealing with democratic bases, all our bases will be *seminormalized*, which means that $0 < \inf \|e_i\| \le \sup \|e_i\| < \infty$. It follows that the biorthogonal functionals (e_i^*) are also seminormalized. Note that a space with a seminormalized basis (e_i) can be easily renormed to make (e_i) *normalized, i.e.,* $\|e_i\| = 1$ for all $i \in \mathbb{N}$.

We measure the efficiency of the greedy algorithm by comparing it to the best m-term approximation. We say that (e_i) is a *greedy basis* for X if there exists $C > 0$ (C-*greedy*) such that

$$\|x - \mathcal{G}_m(x)\| \le C\sigma_m(x) \quad \text{for all } x \in X \text{ and for all } m \in \mathbb{N}.$$

The smallest C is the *greedy constant* of the basis. Note that being a greedy basis is a strong property. It implies in particular the strictly weaker property that $\mathcal{G}_m(x)$ converges to x for all $x \in X$. If this weaker property holds, then we say that the basis (e_i) is *quasi-greedy*. This is still a non-trivial property: a Schauder basis need not be quasi-greedy in general.

The simplest examples of greedy bases include the unit vector basis of ℓ_p $(1 \le p < \infty)$ or c_0, or orthonormal bases of a separable Hilbert space. An important and non-trivial example is the Haar basis of $L_p[0, 1]$ $(1 < p < \infty)$ which was shown to be greedy by V.N. Temlyakov [8]. This result was later established by P. Wojtaszczyk [9] using a different method which extended to the Haar system in one-dimensional dyadic Hardy space $H_p(\mathbb{R})$, $0 < p \le 1$. We also mention two recent results. S.J. Dilworth, D. Freeman, E. Odell and Th. Schlumprecht [3] proved that $\left(\oplus_{n=1}^{\infty} \ell_p^n\right)_{\ell_q}$ has a greedy basis whenever $1 \le p \le \infty$ and $1 < q < \infty$. Answering a question raised in [3], G. Schechtman showed that none of the space $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{\ell_q}$, $1 \le p \ne q < \infty$, $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{c_0}$, $1 \le p < \infty$, and $\left(\bigoplus_{n=1}^{\infty} c_0\right)_{\ell_q}$, $1 \le q < \infty$, have greedy bases [7].

Greedy bases are closely related to unconditional bases. We recall that a basis (e_i) of a Banach space X is said to be *unconditional* if there is a constant K (K-*unconditional*) such that

$$\left\| \sum a_i e_i \right\| \le K \cdot \left\| \sum b_i e_i \right\| \quad \text{whenever } |a_i| \le |b_i| \text{ for all } i \in \mathbb{N}.$$

The best constant K is the *unconditional constant* of the basis which we denote by K_U. The property of being unconditional is easily seen to be equivalent to that of being *suppression*

unconditional which means that for some constant K (*suppression K-unconditional*) the natural projection onto any subsequence of the basis has norm at most K:

$$\Big\| \sum_{i\in A} a_i e_i \Big\| \le K \cdot \Big\| \sum_{i=1}^{\infty} a_i e_i \Big\| \quad \text{for all } (a_i) \subset \mathbb{R},\ A \subset \mathbb{N}.$$

The smallest K is the *suppression unconditional constant* of the basis and is denoted by K_S. It is easy to verify that $K_S \le K_U \le 2K_S$. Note that it is trivial to renorm the space X so that in the new norm the basis is suppression 1-unconditional. Indeed, for $A \subset \mathbb{N}$ the map $P_A\colon X \to X$, defined by $P_A\Big(\sum_{i\in\mathbb{N}} x_i e_i\Big) = \sum_{i\in A} x_i e_i$, is bounded in norm by K_S. Hence

$$|||x||| = \sup\{\|P_A(x)\| :\ A \subset \mathbb{N}\}$$

is a K_S-equivalent norm on X in which (e_i) is suppression 1-unconditional. Similarly, using maps $M_\lambda\colon X \to X$ given by $\sum x_i e_i \mapsto \sum \lambda_i x_i e_i$, where $\lambda = (\lambda_i) \in B_{\ell_\infty}$, we can define a K_U-equivalent norm on X in which (e_i) is 1-unconditional.

In [6], S.V. Konyagin and V.N. Temlyakov introduced the notion of greedy and democratic bases and proved the following characterization.

Theorem 1 (*[6, Theorem 1]*)**.** *A basis of a Banach space is greedy if and only if it is unconditional and democratic.*

A basis (e_i) is said to be *democratic* if there is a constant $\Delta \ge 1$ (Δ-*democratic*) such that

$$\Big\| \sum_{i\in A} e_i \Big\| \le \Delta \Big\| \sum_{i\in B} e_i \Big\| \quad \text{whenever } |A| \le |B|.$$

By carefully following the proof of [10, Theorem 1], one obtains the following estimates:

$$K_S \le C, \qquad \Delta \le C \quad \text{and} \quad C \le K_S + K_S K_U^2 \cdot \Delta.$$

One can in fact get slightly better estimates by amalgamating some of the steps in that proof:

$$K_S \le C, \qquad \Delta \le C \quad \text{and} \quad C \le K_S + K_U^2 \cdot \Delta. \tag{1}$$

That is, a C-greedy basis is suppression C-unconditional and C-democratic, and conversely, an unconditional and Δ-democratic basis is C-greedy with $C \le K_S + K_U^2 \cdot \Delta$. In particular a 1-unconditional, 1-democratic basis is 2-greedy. By [5, Theorem 3.1] the constant 2 is best possible. Thus, improving the democracy constant by renorming will not in general improve the greedy constant beyond 2.

In this paper we are concerned with the problem whether a Banach space X with a greedy basis (e_i) can be renormed so that in the new norm the greedy constant of the basis (e_i) is improved ideally to 1 or at least to $1 + \varepsilon$ where $\varepsilon > 0$ can be chosen arbitrarily small. As a byproduct, we also obtain results on renormings that improve the democracy constant. The maps P_A and M_λ that were used above in renormings that improve the unconditional constants are linear. By contrast, the functions $\mathcal{G}_m$ that map vectors to their greedy approximants are not linear, and that is what makes the problem of improving the greedy constant far from trivial.

In the rest of this section we recall what is already known about this problem and state our new results. Definitions will be given in later sections when needed.

In [1] F. Albiac and P. Wojtaszczyk gave a characterization of 1-greedy bases in terms of a weak symmetry property of the basis. They raised several open problems about symmetry properties of 1-greedy bases and about the possibility of improving greedy and democratic constants

by renorming. Most of the problems were answered by four of the authors of this paper in [5]. In Section 1 we recall the Albiac–Wojtaszczyk characterization, and a theorem from [5] which shows that a space with an unconditional, bidemocratic basis can be renormed to make the basis 1-unconditional and 1-bidemocratic. By (1), such a basis is 2-greedy. Here we will obtain the following stronger result.

Theorem A. *Let X be a Banach space with an unconditional, bidemocratic basis (e_i). Then for all $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.*

In particular, the above result applies to the Haar basis of $L_p[0,1]$ for $1 < p < \infty$. In [1] Albiac and Wojtaszczyk raise the problem whether $L_p[0,1]$ can be renormed so that the Haar basis becomes 1-greedy in the new norm. This problem is still open. The result above gets close to giving a positive answer.

Section 3 is concerned with the general case, *i.e.,* when we do not assume bidemocracy. In [1] Albiac and Wojtaszczyk asked whether the democracy constant can be improved to 1. It was already shown in [5] that the answer in general is 'no': the Haar system of dyadic H_1, or an arbitrary unconditional basis of Tsirelson's space T cannot be made 1-democratic by renorming. Here we are able to prove the following positive result.

Theorem B. *Let (e_i) be an unconditional and democratic basis of a Banach space X. For any $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-democratic.*

This answers a question raised by W.B. Johnson. By Eq. (1), it follows from this theorem that if (e_i) is a greedy basis of a Banach space X, then for all $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is 1-unconditional and $(2+\varepsilon)$-greedy. The following problem remains open in its full generality.

Problem C. Let X be a Banach space with a greedy basis (e_i). Given $\varepsilon > 0$, does there exist an equivalent norm on X with respect to which (e_i) is 1-unconditional and $(1+\varepsilon)$-greedy?

In the last section we will give a positive answer for a large class of bases. As an application we obtain, for any $\varepsilon > 0$, a renorming of Hardy space H_1 and of Tsirelson's space T such that the Haar system, respectively, unit vector basis is $(1+\varepsilon)$-greedy.

1. Bidemocratic bases

The aim of this section is to prove Theorem A. We first recall the Albiac–Wojtaszczyk characterization of 1-greedy bases [1]. In fact a trivial modification of their proof gives a characterization of C-greedy bases for an arbitrary $C \geq 1$. For the sake of completeness we shall state and prove their result here in that more general form.

Let (e_i) be a basis of a Banach space X with biorthogonal sequence (e_i^*). For a finite set $A \subset \mathbb{N}$ we denote by $\mathbf{1}_A$ the vector $\sum_{i\in A} e_i$ of X or sometimes the vector $\sum_{i\in A} e_i^*$ in X^*. It will be clear from the context which one is meant. For example the notation $\|\mathbf{1}_A\|$ means the norm of $\sum_{i\in A} e_i$ in X, whereas $\|\mathbf{1}_A\|^*$ indicates the norm in the dual space of $\sum_{i\in A} e_i^*$. The *support* with respect to the basis (e_i) of a vector $x = \sum x_i e_i$ in X is the set $\operatorname{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}$. The subspace of vectors with finite support, *i.e.,* the linear span of (e_i), can be identified in the obvious way with the space c_{00} of real sequences that are eventually zero. The basis (e_i) then

corresponds to the unit vector basis of c_{00}. Given vectors $x = \sum x_i e_i$ and $y = \sum y_i e_i$ in c_{00}, we say *y is a greedy rearrangement of x* if there exist $w, u = (u_i), t = (t_j) \in c_{00}$ of pairwise disjoint support such that $x = w + u$, $y = w + t$, $|\mathrm{supp}(u)| = |\mathrm{supp}(t)|$, and $\|w\|_{\ell_\infty} \le |u_i| = |t_j|$ for all $i \in \mathrm{supp}(u)$, $j \in \mathrm{supp}(t)$. To put it informally, y is obtained from x by moving (and possibly changing the sign of) some of the coefficients of x of maximum modulus to co-ordinates where x is zero. Given $C \ge 1$, we say that (e_i) *has Property (A) with constant C* if for all $x, y \in c_{00}$ we have $\|y\| \le C\|x\|$ whenever y is a greedy rearrangement of x.

Theorem 2 (*cf. [1, Theorem 3.4]*)**.** *Let (e_i) be a basis of a Banach space X. If (e_i) is C-greedy, then it is suppression C-unconditional and has Property* (A) *with constant C. Conversely, if (e_i) is suppression K-unconditional and has Property* (A) *with constant C, then it is greedy with constant at most K^2C.*

In particular, a suppression 1*-unconditional basis of a Banach space is C-greedy if and only if it satisfies Property* (A) *with constant C.*

Proof. First assume that (e_i) is C-greedy. We show that if $x = \sum x_i e_i \in X$ has finite support and $A \subset \mathbb{N}$, then $\big\| \sum_{i\in A} x_i e_i \big\| \le C\|x\|$. Let $B = \mathrm{supp}(x) \setminus A$ and $m = |B|$. Choose a real number $\lambda > \|x\|_{\ell_\infty}$, and set $z = \sum_{i\in A} x_i e_i + \lambda \mathbf{1}_B$. Then $\mathcal{G}_m(z) = \lambda \mathbf{1}_B$ and $w = \sum_{i\in B}(\lambda - x_i)e_i$ is an m-term approximation to z. It follows that

$$\Big\| \sum_{i\in A} x_i e_i \Big\| = \|z - \mathcal{G}_m(z)\| \le C\|z - w\| = C\|x\|,$$

as required. We next show that (e_i) has Property (A) with constant C. Let $y = w + t$ be a greedy rearrangement of $x = w + u$, where $w, u, t \in c_{00}$ are as in the definition above. Fix $\delta > 0$ and set $z = w + (1+\delta)u + t$. Let $m = |\mathrm{supp}(u)| = |\mathrm{supp}(t)|$. Then $\mathcal{G}_m(z) = (1+\delta)u$, whereas t is another m-term approximation to z. It follows that

$$\|y\| = \|z - \mathcal{G}_m(z)\| \le C\|z - t\| = C\|x + \delta u\|.$$

Letting $\delta \to 0$ yields $\|y\| \le C\|x\|$, as required.

To prove the converse, fix $x = \sum x_i e_i \in c_{00}$ and $m \in \mathbb{N}$. Let $\sum_{i\in A} x_i e_i$ be the mth greedy approximant to x, and let $b = \sum_{i\in B} b_i e_i$ be an arbitrary m-term approximation. Let $s = \min\{|x_i| : i \in A\}$, and for each $i \in \mathbb{N}$ let ε_i be the sign of x_i. Note that $|x_i| \ge s \ge |x_j|$ for all $i \in A$ and $j \notin A$. The following is a well known consequence of suppression K-unconditionality. If $0 \le y_i \le z_i$ or $z_i \le y_i \le 0$ for all $i \in \mathbb{N}$, then $\big\| \sum y_i e_i \big\| \le K \big\| \sum z_i e_i \big\|$. We use this in the first and third inequalities below, whereas the second inequality uses Property (A).

$$\begin{aligned}
\|x - b\| &= \Big\| \sum_{i\in A\setminus B} x_i e_i + \sum_{i\in B}(x_i - b_i)e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| \\
&\ge \frac{1}{K}\Big\| \sum_{i\in A\setminus B} s\varepsilon_i e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| \ge \frac{1}{KC}\Big\| \sum_{i\in B\setminus A} s\varepsilon_i e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| \\
&\ge \frac{1}{K^2C}\Big\| \sum_{i\in B\setminus A} x_i e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| = \frac{1}{K^2C}\big\|x - \mathcal{G}_m(x)\big\|.
\end{aligned}$$

This completes the proof. □

Remark. Let (e_i) be a 1-unconditional basis of a Banach space X. For $x \in c_{00}$ define $\|\cdot\|_x$ to be the function

$$\|z\|_x = \big\| z + \|z\|_{\ell_\infty} \cdot x \big\|,$$

which defines a norm on $\overline{\text{span}}\{e_i : i \in \mathbb{N} \setminus \text{supp}(x)\}$. Theorem 2 implies that (e_i) is C-greedy if and only if for every $x \in c_{00}$ with $\|x\|_{\ell_\infty} \le 1$ the norm $\|\cdot\|_x$ is C-democratic. This characterization of greedy bases is slightly different from the one given by Konyagin and Temlyakov [6] where they only assume the democracy of $\|\cdot\|_x$ for $x = 0$. However, for our purposes, the above result has the advantage that the greedy constant is the same as the Property (A) constant.

We next recall the notion of bidemocracy, which was introduced by S.J. Dilworth, N.J. Kalton, D. Kutzarova and V.N. Temlyakov in [4], and the corresponding renorming result [5, Theorem 2.1]. Suppose that (e_i) is a seminormalized basis of a Banach space X with biorthogonal sequence (e_i^*). The *fundamental function* φ of (e_i) is defined by

$$\varphi(n) = \sup_{|A|\le n} \Big\| \sum_{i\in A} e_i \Big\|.$$

The *dual fundamental function* φ^* is given by

$$\varphi^*(n) = \sup_{|A|\le n} \Big\| \sum_{i\in A} e_i^* \Big\|.$$

We recall that $(\varphi(n)/n)$ is a decreasing function of n, since for any $A \subset \mathbb{N}$ with $|A| = n \ge 2$ we have

$$\Big\| \sum_{i\in A} e_i \Big\| = \frac{1}{n-1} \Big\| \sum_{i\in A} \sum_{j\in A\setminus\{i\}} e_j \Big\| \le \frac{n}{n-1}\varphi(n-1).$$

Clearly, $\varphi(n)\varphi^*(n) \ge n$. We say that (e_i) is *bidemocratic* if there is a constant $\Delta \ge 1$ (Δ-*bidemocratic*) such that

$$\varphi(n)\varphi^*(n) \le \Delta n \quad \text{for all } n \in \mathbb{N}.$$

It is known [4, Proposition 4.2] that if (e_i) is bidemocratic with constant Δ, then both (e_i) and (e_i^*) are democratic with constant Δ. In [5] the following result was proved.

Theorem 3. *Suppose that (e_i) is a 1-unconditional and Δ-bidemocratic basis for a Banach space X. Then*

$$|||x||| = \max\Big\{\|x\|, \sup_{|A|<\infty} \frac{\varphi(|A|)}{|A|} \sum_{i\in A} |e_i^*(x)|\Big\} \tag{2}$$

is an equivalent norm on X. Moreover, (e_i) is 1-unconditional and 1-bidemocratic with respect to $|||\cdot|||$. In particular, (e_i) and (e_i^*) are 1-democratic and 2-greedy.

By [5, Theorem 3.1], the conclusion that (e_i) is 2-greedy whenever it is 1-unconditional and 1-democratic cannot be strengthened in general. We now prove a stronger theorem which is the main result of this section. First we introduce two pieces of notation. For a vector $x = \sum x_i e_i$ we write $|x|$ for $\sum |x_i| e_i$, and $x \ge 0$ if $x_i \ge 0$ for all $i \in \mathbb{N}$.

Theorem 4. *Let X be a Banach space with an unconditional, bidemocratic basis (e_i). Then for all $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.*

Proof. After renorming, we may assume that the basis (e_i) is normalized, 1-unconditional and 1-bidemocratic. Let φ and φ^* denote the fundamental and, respectively, dual fundamental function of (e_i). Fix $\varepsilon \in (0, 1)$. Define a new norm $|||\cdot|||$ on X as follows.

$$|||x||| = \sup\left\{\left\langle |x|, x^* + \frac{1}{\varphi^*(n)}\mathbf{1}_A\right\rangle : x^* \in \varepsilon B_{X^*},\ n \in \mathbb{N},\ A \subset \mathbb{N},\ |A| = n\right\}.$$

It is clear that (e_i) is a 1-unconditional basis in $|||\cdot|||$. We next prove that it also satisfies Property (A) with constant $1 + \varepsilon$. Fix $x \in c_{00}$ and $B, \tilde{B} \subset \mathbb{N} \setminus \operatorname{supp}(x)$ such that $\|x\|_{\ell_\infty} \le 1$ and $|B| = |\tilde{B}| < \infty$. It will be sufficient to prove that $|||x + \mathbf{1}_B||| \le (1 + \varepsilon)|||x + \mathbf{1}_{\tilde{B}}|||$. We may of course assume that $x \ge 0$.

Let $n \in \mathbb{N}$, $A \subset \mathbb{N}$ and $x^* \in \varepsilon B_{X^*}$ be such that $|A| = n$ and

$$\begin{aligned}|||x + \mathbf{1}_B||| &= \left\langle x + \mathbf{1}_B, x^* + \frac{1}{\varphi^*(n)}\mathbf{1}_A\right\rangle \\ &= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \frac{1}{\varphi^*(n)}\langle x, \mathbf{1}_A\rangle + \frac{1}{\varphi^*(n)}|B \cap A|. \end{aligned} \tag{3}$$

Without loss of generality we may assume that $\operatorname{supp}(x^*) \cup A \subset \operatorname{supp}(x) \cup B$, and hence $x^* \ge 0$. Note that

$$\begin{aligned}\langle \mathbf{1}_B, x^*\rangle &\le \varepsilon\|\mathbf{1}_B\| \le \varepsilon\varphi(|B|) = \varepsilon\varphi(|\tilde{B}|) \\ &= \varepsilon\left\langle x + \mathbf{1}_{\tilde{B}}, \frac{1}{\varphi^*(|\tilde{B}|)}\mathbf{1}_{\tilde{B}}\right\rangle \le \varepsilon|||x + \mathbf{1}_{\tilde{B}}|||. \end{aligned} \tag{4}$$

Now choose $\tilde{A} \subset \mathbb{N}$ such that $\tilde{A} \cap \operatorname{supp}(x) = A \cap \operatorname{supp}(x)$ and $|\tilde{B} \cap \tilde{A}| = |B \cap A|$. Then $|\tilde{A}| = n$ and

$$\langle x, \mathbf{1}_A\rangle + |B \cap A| = \langle x, \mathbf{1}_{\tilde{A}}\rangle + |\tilde{B} \cap \tilde{A}|. \tag{5}$$

We now obtain

$$\begin{aligned}|||x + \mathbf{1}_B||| &= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \frac{1}{\varphi^*(n)}\langle x, \mathbf{1}_A\rangle + \frac{1}{\varphi^*(n)}|B \cap A| \quad \text{by (3)} \\ &\le \langle x, x^*\rangle + \varepsilon|||x + \mathbf{1}_{\tilde{B}}||| + \frac{1}{\varphi^*(n)}\langle x, \mathbf{1}_{\tilde{A}}\rangle + \frac{1}{\varphi^*(n)}|\tilde{B} \cap \tilde{A}| \quad \text{by (4) and (5)} \\ &\le \left\langle x + \mathbf{1}_{\tilde{B}}, x^* + \frac{1}{\varphi^*(n)}\mathbf{1}_{\tilde{A}}\right\rangle + \varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \quad \text{as } x^* \ge 0 \\ &\le (1 + \varepsilon)|||x + \mathbf{1}_{\tilde{B}}|||, \end{aligned}$$

as required. Thus, so far, we have that (e_i) is 1-unconditional and $(1+\varepsilon)$-greedy in $|||\cdot|||$. It is also clear that $|||e_i||| = 1 + \varepsilon$ for all $i \in \mathbb{N}$. Let ψ and ψ^* denote the fundamental and, respectively, dual fundamental function of (e_i) with respect to $|||\cdot|||$. Let $m, n \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $|A| = m$ and $|B| = n$. By definition of $|||\cdot|||$, in the dual space we have $\|\frac{1}{\varphi^*(m)}\mathbf{1}_A\|^* \le 1$, from which it follows that $\psi^*(m) \le \varphi^*(m)$. Also, for any $x^* \in \varepsilon B_{X^*}$ we have

$$\begin{aligned}\left\langle \mathbf{1}_B, x^* + \frac{1}{\varphi^*(m)}\mathbf{1}_A\right\rangle &\le \varepsilon\|\mathbf{1}_B\| + \frac{\varphi(m)}{m}|B \cap A| \\ &\le \varepsilon\varphi(n) + \frac{\varphi(|B \cap A|)}{|B \cap A|}|B \cap A| \le (1 + \varepsilon)\varphi(n). \end{aligned}$$

Hence $\psi(n) \le (1+\varepsilon)\varphi(n)$, and (e_i) is $(1+\varepsilon)$-bidemocratic in $|||\cdot|||$. So if we replace $|||\cdot|||$ with $\frac{1}{1+\varepsilon}|||\cdot|||$ and apply Theorem 3, then we obtain a new norm $(1+\varepsilon)$-equivalent to $|||\cdot|||$, and with respect to which (e_i) is normalized, 1-unconditional, 1-bidemocratic and $(1+\varepsilon)^2$-greedy. □

Let us now observe that our theorem applies to a large class of Banach spaces and bases. We say that a democratic basis (e_i) (or its fundamental function φ) has the *upper regularity property* (or URP for short) if there exists an integer $r > 2$ such that

$$\varphi(rn) \le \frac{1}{2} r\varphi(n) \quad \text{for all } n \in \mathbb{N}.$$

This is easily seen to be equivalent to the existence of $0 < \beta < 1$ and a constant C such that

$$\varphi(n) \le C\Big(\frac{n}{m}\Big)^{\beta}\varphi(m) \quad \text{for all } m \le n.$$

This property was introduced in [4] where it was shown that a greedy basis of a Banach space with nontrivial type has the URP and that a greedy basis with the URP is bidemocratic. More precisely, they showed that if (e_i) is a greedy basis with fundamental function φ, and there exists a constant C such that

$$\sum_{k=1}^{n} \frac{\varphi(n)}{\varphi(k)} \le Cn \quad \text{for all } n \in \mathbb{N}, \tag{6}$$

then (e_i) is bidemocratic. It is of course clear that the URP implies (6).

It is well known that $L_p[0,1]$ for $1 < p < \infty$ has nontrivial type. Thus we obtain the following corollary.

Corollary 5. *Let $1 < p < \infty$. For all $\varepsilon > 0$ there is an equivalent norm on $L_p[0,1]$ in which the Haar basis is normalized, 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.*

Remark. It follows from the Albiac–Wojtaszczyk characterization that a 1-greedy basis is suppression 1-unconditional, and hence 2-unconditional. As shown in [5, Theorem 4.1], the unconditional constant 2 is in general the best one can say about a 1-greedy basis. This is why the 1-unconditionality was included in the above results.

2. The class of quasi-concave functions

We denote by $\mathbb{R}^+$ the set of (strictly) positive real numbers. Recall that the fundamental function $\varphi: \mathbb{N} \to \mathbb{R}^+$ of a basis of a Banach space is increasing and $n \mapsto \frac{\varphi(n)}{n}$ is decreasing. Let us now call a function $\varphi: [1,\infty) \to \mathbb{R}^+$ defined on the *real* interval $[1,\infty)$ a *fundamental function* if it is increasing and $x \mapsto \frac{\varphi(x)}{x}$ is decreasing. Observe that every fundamental function φ is subadditive. Indeed, for $x, y \in [1,\infty)$ we have

$$\varphi(x+y) = \frac{\varphi(x+y)}{x+y}\cdot x + \frac{\varphi(x+y)}{x+y}\cdot y \le \frac{\varphi(x)}{x}\cdot x + \frac{\varphi(y)}{y}\cdot y = \varphi(x)+\varphi(y).$$

The fundamental function of a basis of a Banach space *is* the restriction to $\mathbb{N}$ of a fundamental function in the above sense. Indeed, if $\varphi: \mathbb{N} \to \mathbb{R}^+$ is the fundamental function of a basis, then we can extend it to a function on $[1,\infty)$ by linear interpolation. A straightforward calculation shows that this extended function is a fundamental function in the above sense. The converse is also true, *i.e.,* if $\varphi: [1,\infty) \to \mathbb{R}^+$ is a fundamental function, then its restriction to $\mathbb{N}$ is the

fundamental function of a basis. This will be shown in Proposition 8 at the end of this section. Given a fundamental function $\varphi\colon [1,\infty) \to \mathbb{R}^+$ and a basis (e_i) of a Banach space, we say that φ is *a fundamental function for* (e_i) if the restriction of φ to $\mathbb{N}$ is the fundamental function of (e_i).

Remark. In the literature fundamental functions in the above sense are also known as *quasi-concave functions*. See for example [2, Definition 5.6 on page 69], where quasi-concave functions are defined on the interval $[0,\infty)$ and are naturally associated with rearrangement-invariant spaces. Since we work with discrete lattices corresponding to unconditional bases which in general are not symmetric, for us it will be more convenient to work with the definition above instead.

We will now introduce a parameter δ which provides information on the growth of fundamental functions. After that we will show that the concave envelope of a fundamental function is also a fundamental function.

Let $\varphi\colon [1,\infty) \to \mathbb{R}^+$ be a fundamental function. It will sometimes be more convenient to work with the function $\lambda\colon [1,\infty) \to \mathbb{R}^+$ defined by $\lambda(x) = \frac{\varphi(x)}{x}$. Note that λ is decreasing and $x\lambda(x)$ is increasing. For $y \in [1,\infty)$ define

$$\delta_\varphi(y) = \liminf_{x\to\infty} \frac{\varphi(yx)}{y\varphi(x)} = \liminf_{x\to\infty} \frac{\lambda(yx)}{\lambda(x)}.$$

It follows from properties of λ that δ_φ is decreasing and bounded above by 1. Hence

$$\delta(\varphi) = \inf_{y\geq 1} \delta_\varphi(y) = \lim_{y\to\infty} \delta_\varphi(y) \in [0,1].$$

Let us now observe that the function δ_φ and the parameter $\delta(\varphi)$ depend only on the values of φ on $\mathbb{N}$. Fix $m \in \mathbb{N}$. For any real $x \in [1,\infty)$, putting $n = \lfloor x\rfloor + 1$, we have

$$\frac{\lambda(mx)}{\lambda(x)} = x\cdot\frac{\lambda(mx)}{x\lambda(x)} \geq x\cdot\frac{\lambda(mn)}{n\lambda(n)} = \frac{x}{\lfloor x\rfloor+1}\cdot\frac{\lambda(mn)}{\lambda(n)}.$$

It follows that for each $y \in [1,\infty)$ we have

$$\inf_{n\in\mathbb{N},\ n\geq\lfloor y\rfloor+1} \frac{\lambda(mn)}{\lambda(n)} \geq \inf_{x\in\mathbb{R},\ x\geq y} \frac{\lambda(mx)}{\lambda(x)} \geq \frac{\lfloor y\rfloor}{\lfloor y\rfloor+1}\cdot \inf_{n\in\mathbb{N},\ n\geq\lfloor y\rfloor+1} \frac{\lambda(mn)}{\lambda(n)},$$

and hence we obtain

$$\delta_\varphi(m) = \liminf_{n\to\infty} \frac{\lambda(mn)}{\lambda(n)}.$$

Thus we have

$$\delta(\varphi) = \lim_{m\to\infty} \liminf_{n\to\infty} \frac{\lambda(mn)}{\lambda(n)}.$$

One consequence of all this is that if (e_i) is a basis of a Banach space, then the parameter $\delta(\varphi)$ is the same for *any* fundamental function φ for (e_i).

Two fundamental functions φ and ψ are said to be *equivalent* if there exist positive real numbers a and b such that $a\varphi(x) \leq \psi(x) \leq b\varphi(x)$ for all $x \in [1,\infty)$. In this case we write $\varphi \sim \psi$. Note that equivalence also only depends on the restrictions to $\mathbb{N}$ of φ and ψ. Indeed, if for some

$b > 0$ we have $\psi(n) \le b\varphi(n)$ for all $n \in \mathbb{N}$, then

$$\psi(x) = x \cdot \frac{\psi(x)}{x} \le x \cdot \frac{\psi(\lfloor x \rfloor)}{\lfloor x \rfloor} \le 2b\varphi(\lfloor x \rfloor) \le 2b\varphi(x)$$

for all $x \in [1, \infty)$. So for a basis (e_i) of a Banach space with a fundamental function φ, the property of having $\delta(\varphi) > 0$ is invariant under renormings. We now prove a result about fundamental functions with positive δ-parameter. This will be used in Theorem 11 in the next section.

Lemma 6. *Let φ be a fundamental function with $\delta(\varphi) > 0$. Then for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$ there exists a fundamental function $\psi \sim \varphi$ such that $\delta_\psi(m) > \frac{1}{1+\varepsilon}$.*

Proof. It is enough to show that if $\delta_\varphi(m^2) > \delta$, then there exists a fundamental function $\psi \sim \varphi$ such that $\delta_\psi(m) > \sqrt{\delta}$. Indeed, assuming this result, we fix $0 < \delta < \delta(\varphi)$, choose $k \in \mathbb{N}$ with $\delta^{\frac{1}{2^k}} > \frac{1}{1+\varepsilon}$, and obtain fundamental functions $\varphi = \varphi_0 \sim \varphi_1 \sim \cdots \sim \varphi_k$ such that $\delta_{\varphi_j}\left(m^{2^{k-j}}\right) > \delta^{\frac{1}{2^j}}$ for $j = 0, 1, 2, \ldots, k$. Putting $\psi = \varphi_k$ completes the proof.

Set $\lambda(x) = \frac{\varphi(x)}{x}$, $x \in [1, \infty)$. To prove our initial claim, choose $n_0 \in \mathbb{N}$ such that $\lambda(m^2 x) > \delta\lambda(x)$ for all real $x \ge n_0$. We now define a new function $\mu\colon [1, \infty) \to \mathbb{R}^+$ as follows. We set $\mu(x) = \lambda(x)$ for all real $x \in [1, n_0]$ and for all integers x of the form $x = m^{2k} n_0$, $k = 0, 1, 2, \ldots$. We then extend the definition of μ by interpolation as follows. Given a real number $x \in [n_0, \infty)$, we fix an integer $k \ge 0$ such that $x \in [n, m^2 n]$, where $n = m^{2k} n_0$. (Note that k is unique unless $x \in \{m^{2j} n_0 : j \in \mathbb{N}\}$.) Then there is a unique $\theta \in [0, 1]$ such that $x = n^{1-\theta}(m^2 n)^\theta$. We define

$$\mu(x) = \lambda(n)^{1-\theta}\lambda(m^2 n)^\theta.$$

Note that for $x = n$ and $x = m^2 n$ this agrees with the previous definition of $\mu(x) = \lambda(x)$. It follows that $\mu(x)$ is well-defined, and in particular it does not depend on the choice of k when $x \in \{m^{2j} n_0 : j \in \mathbb{N}\}$. We now prove the following properties for each integer $k \ge 0$ with $n = m^{2k} n_0$.

(i) $\mu(x)$ is decreasing and $x\mu(x)$ is increasing on $[n, m^2 n]$.
(ii) $\delta\lambda(x) \le \mu(x) \le m^2\lambda(x)$ for all $x \in [n, m^2 n]$,
(iii) $\mu(mx) \ge \sqrt{\delta}\mu(x)$ for all $x \in [n, m^2 n]$.

We will then set $\psi(x) = x\mu(x)$ for each $x \in [1, \infty)$. Since $\mu = \lambda$, and hence $\psi = \varphi$, on the set $[1, n_0] \cup \{m^{2k} n_0 : k \ge 0\}$, property (i) implies that ψ is a fundamental function, which is equivalent to φ by (ii), and satisfies $\delta_\psi(m) \ge \sqrt{\delta}$ by (iii). This proves the initial claim, and hence the lemma.

To see (i) simply differentiate the functions

$$\lambda(n)^{1-\theta}\lambda(m^2 n)^\theta \quad \text{and} \quad n^{1-\theta}(m^2 n)^\theta \lambda(n)^{1-\theta}\lambda(m^2 n)^\theta$$

with respect to θ.

Next, fix $\theta \in [0, 1]$ and set $x = n^{1-\theta}(m^2 n)^\theta$. By the properties of λ, we have

$$\begin{aligned}\mu(x) &= \lambda(n)^{1-\theta}\lambda(m^2 n)^\theta \le \lambda(n) = n\lambda(n) \cdot \frac{1}{n} \le x\lambda(x) \cdot \frac{1}{n} \\ &= n^{1-\theta}(m^2 n)^\theta \cdot \lambda(x) \cdot \frac{1}{n} = m^{2\theta}\lambda(x) \le m^2\lambda(x),\end{aligned}$$

and, since $\lambda(m^2 n) \ge \delta\lambda(n)$, we have

$$\mu(x) = \lambda(n)^{1-\theta}\lambda(m^2 n)^\theta \ge \delta^\theta\lambda(n) \ge \delta\lambda(x).$$

Hence (ii) follows. Finally, fix $0 \le \theta \le 1$. In order to verify (iii) we need to show that

$$\frac{\mu\big(m \cdot n^{1-\theta}(m^2n)^\theta\big)}{\mu\big(n^{1-\theta}(m^2n)^\theta\big)} \ge \sqrt{\delta}. \tag{7}$$

We consider two cases. When $0 \le \theta \le \frac{1}{2}$, we can write $m \cdot n^{1-\theta}(m^2n)^\theta = n^{1-\theta'}(m^2n)^{\theta'}$, where $\theta + \frac{1}{2} = \theta'$. Then, since $\lambda(m^2n) \ge \delta\lambda(n)$ and $\theta' - \theta > 0$, the left-hand side of (7) becomes

$$\frac{\lambda(n)^{1-\theta'}\lambda(m^2n)^{\theta'}}{\lambda(n)^{1-\theta}\lambda(m^2n)^{\theta}} \ge \lambda(n)^{\theta-\theta'} \cdot \big(\delta\lambda(n)\big)^{\theta'-\theta} = \sqrt{\delta}.$$

In the second case, we have $\frac{1}{2} \le \theta \le 1$ and $m \cdot n^{1-\theta}(m^2n)^\theta = (m^2n)^{1-\theta'}(m^4n)^{\theta'}$, where $\theta + \frac{1}{2} = 1 + \theta'$. Then the left-hand side of (7) becomes

$$\begin{aligned}\frac{\lambda(m^2n)^{1-\theta'}\lambda(m^4n)^{\theta'}}{\lambda(n)^{1-\theta}\lambda(m^2n)^{\theta}} &\ge \frac{\lambda(m^2n)^{1-\theta'}\big(\delta\lambda(m^2n)\big)^{\theta'}}{\lambda(n)^{1-\theta}\lambda(m^2n)^{\theta}} \\ &= \delta^{\theta'} \cdot \frac{\lambda(m^2n)^{1-\theta}}{\lambda(n)^{1-\theta}} \ge \delta^{\theta'+1-\theta} = \sqrt{\delta},\end{aligned}$$

as required. □

We next prove that every fundamental function is equivalent to a concave one. This is standard (see for example [2, Proposition 5.10]), but we repeat the simple proof here as we need a further property concerning the δ parameter.

Lemma 7. *Let φ be a fundamental function. Then there exists a concave fundamental function ψ such that $\varphi(x) \le \psi(x) \le 2\varphi(x)$ for all $x \in [1, \infty)$. Moreover, we have $\delta_\psi(y) \ge \delta_\varphi(y)$ for all $y \in [1, \infty)$.*

Proof. We let $\psi\colon [1, \infty) \to \mathbb{R}^+$ be the concave envelope of φ. Recall that this is the (pointwise) smallest concave function dominating φ, and is given by

$$\psi(x) = \sup \sum_{i=1}^n t_i \varphi(x_i),$$

where the supremum is taken over all convex combinations $x = \sum_{i=1}^n t_i x_i$ of numbers $x_1, \dots, x_n \in [1, \infty)$. Of course this concave envelope exists if and only if the above supremum is finite for every x. To verify that this is true in our case, note that either $x_i < x$ and $\varphi(x_i) \le \varphi(x)$, or $x_i \ge x$ and we have $\varphi(x_i) = \frac{\varphi(x_i)}{x_i} x_i \le \frac{\varphi(x)}{x} x_i$. It follows that

$$\sum_{i=1}^n t_i \varphi(x_i) \le \varphi(x) \sum_{x_i < x} t_i + \frac{\varphi(x)}{x} \sum_{x_i \ge x} t_i x_i \le 2\varphi(x).$$

It follows that ψ exists, and $\varphi(x) \le \psi(x) \le 2\varphi(x)$ for all x. We next show that ψ is a fundamental function. Let $1 \le x \le y$, and let $x = \sum_{i=1}^n t_i x_i$ be a convex combination of elements of $[1, \infty)$. Then $\sum_{i=1}^n t_i (x_i + y - x) = y$, and hence

$$\psi(y) \ge \sum_{i=1}^n t_i \varphi(x_i + y - x) \ge \sum_{i=1}^n t_i \varphi(x_i).$$

Taking supremum yields $\psi(y) \geq \psi(x)$, and so ψ is increasing. Next, consider a convex combination $y = \sum_{i=1}^n t_i y_i$. Let $z = \frac{x}{y}$. Then $x = \sum_{i=1}^n t_i z y_i$, and so

$$\begin{aligned}\frac{\psi(x)}{x} &\geq \frac{1}{x}\sum_{i=1}^n t_i \varphi(z y_i) = \sum_{i=1}^n t_i \frac{\varphi(z y_i)}{z y_i}\cdot\frac{y_i}{y} \\ &\geq \sum_{i=1}^n t_i \frac{\varphi(y_i)}{y_i}\cdot\frac{y_i}{y} = \frac{1}{y}\sum_{i=1}^n t_i \varphi(y_i).\end{aligned}$$

After taking supremum, this implies $\frac{\psi(x)}{x} \geq \frac{\psi(y)}{y}$, which completes the proof that ψ is a fundamental function.

To show the "moreover" part of the lemma, first observe that if φ is bounded, then $\lim_{x\to\infty} \varphi(x)$ exists, and hence $\delta_\varphi(y) = \frac{1}{y}$ for all $y \in [1,\infty)$. Since in this case ψ is also bounded, we have $\delta_\varphi(y) = \delta_\psi(y) = \frac{1}{y}$ for all y. Assume now that φ is unbounded. Fix $y \geq 1$. It will be enough to show that if $0 < \delta < \delta_\varphi(y)$ and $\varepsilon > 0$, then $\delta \leq (1+\varepsilon)\delta_\psi(y)$. Choose $w \in [1,\infty)$ such that $\varphi(yx) \geq \delta y \varphi(x)$ for all $x \geq w$. Since $\varphi(x)$ tends to infinity, we can then choose $z > w$ such that $\frac{\delta y}{\varepsilon}\varphi(w) < \varphi(z)$.

We will show that if $x \geq z$, then $(1+\varepsilon)\psi(yx) \geq \delta y \psi(x)$, which will complete the proof. Fix a convex combination $x = \sum_{i=1}^n t_i x_i$. By the definition of ψ, we have

$$\psi(yx) \geq \sum_{i=1}^n t_i \varphi(y x_i).$$

Let $I = \big\{i \in \{1,\dots,n\} : x_i \geq w\big\}$. By the choice of z, we have

$$\delta y \sum_{i\notin I} t_i \varphi(x_i) \leq \delta y \varphi(w) < \varepsilon \varphi(z) \leq \varepsilon \psi(yx).$$

It follows from this and from the choice of w that

$$\psi(yx) \geq \sum_{i\in I} t_i \varphi(y x_i) \geq \delta y \sum_{i\in I} t_i \varphi(x_i) \geq \delta y \sum_{i=1}^n t_i \varphi(x_i) - \varepsilon\psi(yx).$$

Since this holds for all convex combinations $\sum_{i=1}^k t_i x_i$ for x, we obtain that $(1+\varepsilon)\psi(yx) \geq \delta y \psi(x)$, as required. □

Our next result shows that every fundamental function arises from a basis in a Banach space.

Proposition 8. *Let $\varphi\colon [1,\infty) \to \mathbb{R}^+$ be a fundamental function. Then there is a Banach space with a 1-unconditional basis (e_i) whose fundamental function is the restriction of φ to $\mathbb{N}$.*

Proof. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. The only condition we impose on $\mathcal{F}$ that it should contain for every $n \in \mathbb{N}$ a set of size n. By scaling we may assume that $\varphi(1) = 1$. Define a norm $\|\cdot\|$ on the space c_{00} of finite sequences as follows:

$$\|x\| = \|x\|_{\ell_\infty} \vee \sup\Big\{\frac{\varphi(|A|)}{|A|}\sum_{i\in A}|x_i| : A \in \mathcal{F}\Big\}, \quad x = (x_i) \in c_{00}.$$

It is clear that (e_i) is a normalized, 1-unconditional basis of the completion X of $\big(c_{00}, \|\cdot\|\big)$. Now let $m, n \in \mathbb{N}$, let $A \in \mathcal{F}$ with $m = |A|$, and let $B \subset \mathbb{N}$ with $|B| = n$. Then

$$\frac{\varphi(|A|)}{|A|}|B\cap A| \leq \frac{\varphi(|B\cap A|)}{|B\cap A|}|B\cap A| = \varphi(|B\cap A|) \leq \varphi(n).$$

It follows that $\left\| \sum_{i\in B} e_i \right\| \le \varphi(n)$. On the other hand, since $A \in \mathcal{F}$, we have $\left\| \sum_{i\in A} e_i \right\| \ge \varphi(m)$. Thus the fundamental function of (e_i) is indeed φ. □

Remark. As we mentioned at the start of this section, fundamental functions with a slightly different definition were introduced in [2]. There, a continuous version of Proposition 8 shows that every quasi-concave function is the fundamental function of a rearrangement-invariant space (see [2, Proposition 5.8]). However, in our result the basis constructed clearly need not be symmetric, or indeed even democratic. If for some $\delta > 0$ every finite $E \subset \mathbb{N}$ has a subset $A \in \mathcal{F}$ with $|A| \ge \delta|E|$, then (e_i) is $\frac{1}{\delta}$-democratic. If $\mathcal{F}$ is the set of *all* subsets of $\mathbb{N}$, then (e_i) is bidemocratic.

It is also possible for (e_i) to be democratic but not bidemocratic. For this to happen φ cannot be arbitrary. For example, if φ has the URP and (e_i) is democratic, then (e_i) is automatically bidemocratic [4, Proposition 4.4]. However, if $\varphi(n) = n$, say, and $\mathcal{F}$ is the family $\mathcal{S}$ of Schreier sets, *i.e.,* sets $A \subset \mathbb{N}$ with $|A| \le \min A$, then the dual fundamental function cannot be bounded otherwise X^* would be isomorphic to c_0, and hence X would be isomorphic to ℓ_1.

We will later prove a renorming result for bases with fundamental function φ satisfying $\delta(\varphi) > 0$. Note that a basis with the URP clearly has $\delta(\varphi) = 0$. We conclude this section by observing that there are bases without the URP and with $\delta(\varphi) = 0$. Fix integers $1 = n_1 < n_2 < \cdots$. Define $\varphi\colon \mathbb{N} \to \mathbb{R}^+$ by setting $\varphi(1) = 1$ and keeping $\frac{\varphi(n)}{n}$ constant on intervals $[n_k, n_{k+1}]$ when k is odd, and keeping φ constant on intervals $[n_k, n_{k+1}]$ when k is even. Extend φ to a fundamental function defined on $[1, \infty)$. If the n_k are sufficiently rapidly increasing, then $\delta_\varphi(m) = 0$ for all $m \in \mathbb{N}$. By Proposition 8 this φ is the fundamental function for some Schauder basis which fails the URP provided the n_k are sufficiently rapidly increasing.

3. The general case

In this section we will prove Theorem B and give a positive answer to Problem C in the case the fundamental function φ has $\delta(\varphi) > 0$. We will require the following crucial lemma.

Lemma 9. *Let (e_i) be a normalized, 1-unconditional, Δ-democratic basis of a Banach space X with fundamental function φ. Given $0 < q < 1$, fix $C > \frac{\Delta}{q(1-q)}$ and set*

$$\mathcal{A} = \left\{ A \subset \mathbb{N} : \ A \text{ finite and } \left\| \frac{\varphi(|A|)}{|A|} \mathbf{1}_A \right\|^* \le C \right\}.$$

Then for every finite $E \subset \mathbb{N}$ there exists $A \in \mathcal{A}$ such that $A \subset E$ and $|A| \ge q|E|$.

Proof. Choose $\delta > 0$ such that $C > \frac{(1+\delta)\Delta}{\delta q(1-q)}$. Let $n \in \mathbb{N}$ and $E \subset \mathbb{N}$ with $|E| = n$. We inductively construct $z^{(1)}, z^{(2)}, \ldots$ in B_{X^*} and pairwise disjoint subsets $E_1, E_2, \ldots$ of E as follows. Assume that for some $k \in \mathbb{N}$ we have already defined $z^{(1)}, \ldots, z^{(k-1)}$ and $E_1, \ldots, E_{k-1}$. Set $F_k = E \setminus \cup_{i=1}^{k-1} E_i$ (so, in particular, $F_1 = E$). If $|F_k| < (1-q)n$, then we stop. Otherwise we choose $z^{(k)} \ge 0$ in B_{X^*} satisfying

$$\operatorname{supp} z^{(k)} \subset F_k \quad \text{and} \quad \langle \mathbf{1}_{F_k}, z^{(k)} \rangle = \|\mathbf{1}_{F_k}\|,$$

and define

$$E_k = \{ i \in F_k : z_i^{(1)} + \cdots + z_i^{(k)} \ge \delta \}.$$

This completes the induction step. We will see in a moment that this process terminates after a finite number of steps. Assume that $E_1, \ldots, E_m$ and $F_1, \ldots, F_{m+1}$ have been defined for some

$m \geq 1$ (note that $|F_1| = |E| > (1-q)n$, so at least one set E_1 is defined). For each $k = 1, \ldots, m$ and for each $i \in E_k$ we have

$$z_i^{(1)} + \cdots + z_i^{(k-1)} < \delta,$$

and hence

$$z_i^{(1)} + \cdots + z_i^{(m)} = z_i^{(1)} + \cdots + z_i^{(k)} < 1 + \delta.$$

We also have

$$z_i^{(1)} + \cdots + z_i^{(m)} < \delta \quad \text{for all } i \in F_{m+1}.$$

It follows that

$$\begin{aligned}\left\langle \mathbf{1}_E, z^{(1)} + \cdots + z^{(m)} \right\rangle &= \sum_{k=1}^{m} \left\langle \mathbf{1}_{E_k}, z^{(1)} + \cdots + z^{(m)} \right\rangle + \left\langle \mathbf{1}_{F_{m+1}}, z^{(1)} + \cdots + z^{(m)} \right\rangle \\ &< (1+\delta)n.\end{aligned}$$

On the other hand, since $|F_k| \geq (1-q)n$ for each $k = 1, \ldots, m$, and since $\varphi(x)/x$ is decreasing, we have

$$\left\langle \mathbf{1}_E, z^{(1)} + \cdots + z^{(m)} \right\rangle = \sum_{k=1}^{m} \left\langle \mathbf{1}_{F_k}, z^{(k)} \right\rangle \geq m \frac{\varphi\big((1-q)n\big)}{\Delta} \geq m(1-q)\frac{\varphi(n)}{\Delta}.$$

Thus, we can deduce that

$$m \leq \frac{(1+\delta)\Delta}{(1-q)} \cdot \frac{n}{\varphi(n)}, \tag{8}$$

which in particular shows that the process does indeed terminate. Let m denote the time when this happens, *i.e.,* when $|F_{m+1}| < (1-q)n$. Let us now set $A = \bigcup_{k=1}^{m} E_k$. It is clear that $|A| \geq qn$. It remains to show that $A \in \mathcal{A}$. Since

$$z_i^{(1)} + \cdots + z_i^{(m)} \geq \delta \quad \text{for all } i \in A,$$

it follows that $\|\delta \mathbf{1}_A\|^* \leq \|z^{(1)} + \cdots + z^{(m)}\|^* \leq m$. Combining this observation with (8), we obtain

$$\Big\| \frac{\varphi(|A|)}{|A|} \mathbf{1}_A \Big\|^* \leq \frac{m\varphi(|A|)}{\delta|A|} \leq \frac{(1+\delta)\Delta}{(1-q)\delta} \cdot \frac{n}{|A|} \cdot \frac{\varphi(|A|)}{\varphi(n)} \leq \frac{(1+\delta)\Delta}{(1-q)\delta} \cdot \frac{1}{q} \cdot \frac{\varphi(|A|)}{\varphi(n)} \leq C,$$

which completes the proof. □

We are now ready to prove Theorem B on improving the democracy constant.

Theorem 10. *Let (e_i) be an unconditional and democratic basis of a Banach space X. For any $\varepsilon > 0$ there is an equivalent norm $|||\cdot|||$ on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-democratic.*

Proof. We can assume that (e_i) is a normalized, 1-unconditional basis. Let Δ be the democracy constant and φ be a fundamental function for (e_i). Given $\varepsilon > 0$, set $q = \frac{1}{1+\varepsilon}$, fix $C > \frac{\Delta}{q(1-q)}$, and let $\mathcal{A}$ be the family given by Lemma 9. Then the following defines a C-equivalent norm on X:

$$|||x||| = \|x\| \vee \sup\Big\{ \Big\langle |x|, \ \frac{\varphi(|A|)}{|A|} \mathbf{1}_A \Big\rangle : \ A \in \mathcal{A} \Big\}.$$

Clearly, (e_i) is still normalized and 1-unconditional in the new norm. We need to verify that it is $(1+\varepsilon)$-democratic. Fix $E \subset \mathbb{N}$ and let $n = |E|$. Taking $A \in \mathcal{A}$ with $A \subset E$ and $|A| \geq q|E|$, we obtain

$$|||\mathbf{1}_E||| \geq \left\langle \mathbf{1}_E, \frac{\varphi(|A|)}{|A|}\mathbf{1}_A \right\rangle = \frac{\varphi(|A|)}{|A|} \cdot |A| \geq \frac{\varphi(|E|)}{|E|} \cdot |A| \geq q\varphi(n).$$

It remains to verify that $|||\mathbf{1}_E||| \leq \varphi(n)$. On the one hand, by definition, we have $\|\mathbf{1}_E\| \leq \varphi(n)$. On the other hand, for an arbitrary $A \in \mathcal{A}$ we have

$$\left\langle \mathbf{1}_E, \frac{\varphi(|A|)}{|A|}\mathbf{1}_A \right\rangle = \frac{\varphi(|A|)}{|A|} \cdot |A \cap E| \leq \frac{\varphi(|A \cap E|)}{|A \cap E|} \cdot |A \cap E| \leq \varphi(n). \quad \square$$

Remark. The upper bound on the equivalence constant on $|||\cdot|||$ given by the proof of Theorem 10 (which in turn comes from the proof of Lemma 9) is of order $\frac{1}{\varepsilon}$. In special cases this can be improved. For example, it is not hard to see that in Tsirelson's space we get a constant of order $\log \frac{1}{\varepsilon}$. However, in general, the best constant must converge to infinity as ε goes to zero. Indeed, assume that (e_i) is a greedy basis of X for which there exists a constant C such that for all $\varepsilon > 0$ there is a C-equivalent norm $\|\cdot\|_\varepsilon$ on X with respect to which (e_i) is normalized and $(1+\varepsilon)$-democratic. Fix a non-trivial ultrafilter $\mathcal{U}$ and define $|||x||| = \lim_{\mathcal{U}} \|x\|_{\frac{1}{n}}$ for $x \in X$. Then $|||\cdot|||$ is a C-equivalent norm on X with respect to which (e_i) is 1-democratic. As mentioned in the Introduction, there are greedy bases for which such renorming is not possible.

Theorem 11. *Let (e_i) be a greedy basis of a Banach space X with fundamental function φ. Assume that $\delta(\varphi) > 0$. Then for all $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-greedy.*

Proof. We can assume that (e_i) is normalized and 1-unconditional. Let Δ be the democracy constant of (e_i). Given $\varepsilon > 0$, set $q = \frac{1}{1+\varepsilon}$, fix $C > \frac{\Delta}{q(1-q)}$, and let $\mathcal{A}$ be the family given by Lemma 9. Next, fix $m \geq 2$ in $\mathbb{N}$ such that $\frac{m}{m-1} \leq 1+\varepsilon$. With the given ε and m we apply Lemma 6 and then Lemma 7 to obtain a concave fundamental function ψ and positive constants a and b such that $\delta_\psi(m) > q$ and $a\varphi(x) \leq \psi(x) \leq b\varphi(x)$ for all $x \in [1, \infty)$. By the definition of δ_ψ, we can choose an integer $n_0 > \frac{1}{\varepsilon}$ such that $\psi(x) > qm\psi\left(\frac{x}{m}\right)$ for all $x \geq n_0$. Set $s = \frac{\varepsilon a}{1+\varepsilon}$, $L = \frac{m\psi(1)}{\varepsilon}$ and

$$\mathcal{F}_m = \left\{ \sum_{i=1}^m \frac{\psi(|A_i|)}{|A_i|}\mathbf{1}_{B_i} : B_i \subset A_i \in \mathcal{A},\ A_1, \ldots, A_m \text{ pairwise disjoint} \right\}.$$

We are now ready to define a new norm $|||\cdot|||$ as follows.

$$|||x||| = \sup\left\{ \langle |x|, x^* + f + L\mathbf{1}_A \rangle : x^* \in sB_{X^*},\ f \in \mathcal{F}_m,\ |A| \leq n_0 \right\}.$$

It is easy to verify that $s\|x\| \leq |||x||| \leq (s + mbC + Ln_0)\|x\|$ for all $x \in X$, and it is clear that (e_i) is a 1-unconditional basis in $|||\cdot|||$. We next prove that it also satisfies Property (A) with constant $1+4\varepsilon$. Fix $x \in c_{00}$ with $x \geq 0$ and $B, \tilde{B} \subset \mathbb{N} \setminus \operatorname{supp}(x)$ such that $\|x\|_{\ell_\infty} \leq 1$ and $|B| = |\tilde{B}| < \infty$. It is sufficient to prove that $|||x + \mathbf{1}_B||| \leq (1+4\varepsilon)|||x + \mathbf{1}_{\tilde{B}}|||$.

For some $x^* \in sB_{X^*}$, $f = \sum_{i=1}^m \frac{\psi(|A_i|)}{|A_i|}\mathbf{1}_{B_i} \in \mathcal{F}_m$ and $A \subset \mathbb{N}$ with $|A| \leq n_0$ we have

$$\begin{aligned} |||x + \mathbf{1}_B||| &= \langle x + \mathbf{1}_B, x^* + f + L\mathbf{1}_A \rangle \\ &= \langle x, x^* \rangle + \langle \mathbf{1}_B, x^* \rangle + \langle x, f \rangle + \langle \mathbf{1}_B, f \rangle + L\langle x, \mathbf{1}_A \rangle + L|B \cap A|. \end{aligned} \tag{9}$$

Without loss of generality we may assume that $x^* \geq 0$. We now estimate some of the terms above. First, we have

$$\langle \mathbf{1}_B, x^* \rangle \leq s\|\mathbf{1}_B\| \leq s\varphi(|B|) \leq \frac{s}{a}\psi(|\tilde{B}|).$$

On the other hand, we can choose $C \subset \tilde{B}$ such that $C \in \mathcal{A}$ and $|C| \geq q|\tilde{B}|$. Then $g = \frac{\psi(|C|)}{|C|}\mathbf{1}_C \in \mathcal{F}_m$, and so

$$|||x + \mathbf{1}_{\tilde{B}}||| \geq \langle x + \mathbf{1}_{\tilde{B}}, g\rangle = \frac{\psi(|C|)}{|C|}|C| \geq \frac{\psi(|\tilde{B}|)}{|\tilde{B}|}|C| \geq q\psi(|\tilde{B}|).$$

Hence, by the choice of s, we have

$$\langle \mathbf{1}_B, x^* \rangle \leq \frac{s}{a}\psi(|\tilde{B}|) \leq \frac{s(1+\varepsilon)}{a}|||x + \mathbf{1}_{\tilde{B}}||| = \varepsilon|||x + \mathbf{1}_{\tilde{B}}|||. \tag{10}$$

Next, without loss of generality, we may assume that

$$\frac{\psi(|A_m|)}{|A_m|}\langle x, \mathbf{1}_{B_m}\rangle = \min_{1\leq i\leq m} \frac{\psi(|A_i|)}{|A_i|}\langle x, \mathbf{1}_{B_i}\rangle,$$

and hence we obtain

$$\langle x, f\rangle = \sum_{i=1}^{m} \frac{\psi(|A_i|)}{|A_i|}\langle x, \mathbf{1}_{B_i}\rangle \leq \frac{m}{m-1}\sum_{i=1}^{m-1} \frac{\psi(|A_i|)}{|A_i|}\langle x, \mathbf{1}_{B_i}\rangle. \tag{11}$$

Using the fact that $\frac{\psi(x)}{x}$ is decreasing, we then obtain

$$\langle \mathbf{1}_B, f\rangle = \sum_{i=1}^{m} \frac{\psi(|A_i|)}{|A_i|}|B_i \cap B| \leq \sum_{i=1}^{m} \psi(|B_i \cap B|). \tag{12}$$

We now consider two cases. In the first case we assume that $|B| = |\tilde{B}| \leq n_0$. Then by the choice of L we have

$$\sum_{i=1}^{m} \psi(|B_i \cap B|) \leq m\psi(|B|) \leq \varepsilon L|\tilde{B}| = \varepsilon\langle x + \mathbf{1}_{\tilde{B}}, L\mathbf{1}_{\tilde{B}}\rangle \leq \varepsilon|||x + \mathbf{1}_{\tilde{B}}|||. \tag{13}$$

Choose $\tilde{A} \subset \mathbb{N}$ such that $\tilde{A} \cap \mathrm{supp}(x) = A \cap \mathrm{supp}(x)$, $|\tilde{A} \cap \tilde{B}| = |A \cap B|$ and $|\tilde{A}| = |A|$. We then deduce that

$$\begin{aligned}
|||x + \mathbf{1}_{\tilde{B}}||| &= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \langle x, f\rangle + \langle \mathbf{1}_B, f\rangle + L\langle x, \mathbf{1}_A\rangle + L|B \cap A| \quad \text{by (9)}\\
&= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \langle x, f\rangle + \langle \mathbf{1}_B, f\rangle\\
&\quad + L\langle x, \mathbf{1}_{\tilde{A}}\rangle + L|\tilde{B} \cap \tilde{A}| \quad \text{by choice of } \tilde{A}\\
&\leq \langle x, x^*\rangle + \langle x, f\rangle + L\langle x, \mathbf{1}_{\tilde{A}}\rangle + L|\tilde{B} \cap \tilde{A}| + 2\varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \quad \text{by (10), (12), (13)}\\
&\leq \langle x + \mathbf{1}_{\tilde{B}}, x^* + f + L\mathbf{1}_{\tilde{A}}\rangle + 2\varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \quad \text{as } x^* \geq 0\\
&\leq (1 + 2\varepsilon)|||x + \mathbf{1}_{\tilde{B}}|||.
\end{aligned}$$

We now turn to the second case when $|B| = |\tilde{B}| > n_0$. Then by concavity of ψ and by the choice of n_0 we obtain the estimate

$$\sum_{i=1}^{m} \psi(|B_i \cap B|) \leq m\psi\Big(\frac{|B|}{m}\Big) \leq (1+\varepsilon)\psi(|B|). \tag{14}$$

Now choose $\tilde{A}$ as in the previous case, set $\tilde{A}_i = A_i$ and $\tilde{B}_i = B_i \cap \mathrm{supp}(x)$ for $1 \le i < m$, and choose $\tilde{B}_m = \tilde{A}_m \in \mathcal{A}$ such that $\tilde{A}_m \subset \tilde{B}$ and $|\tilde{A}_m| \ge q|\tilde{B}|$. Then $g = \sum_{i=1}^{m} \frac{\psi(|\tilde{A}_i|)}{|\tilde{A}_i|}\mathbf{1}_{\tilde{B}_i} \in \mathcal{F}_m$, and by (11) and the choice of m, we have

$$\begin{aligned}\langle x + \mathbf{1}_{\tilde{B}}, g\rangle &= \sum_{i=1}^{m-1} \frac{\psi(|A_i|)}{|A_i|}\langle x, \mathbf{1}_{B_i}\rangle + \frac{\psi(|\tilde{A}_m|)}{|\tilde{A}_m|}|\tilde{A}_m| \\ &\ge \frac{m-1}{m}\langle x, f\rangle + \frac{\psi(|\tilde{B}|)}{|\tilde{B}|}|\tilde{A}_m| \ge q\langle x, f\rangle + q\psi(|\tilde{B}|). \end{aligned} \tag{15}$$

It follows that

$$\begin{aligned}|||x + \mathbf{1}_{\tilde{B}}||| &= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \langle x, f\rangle + \langle \mathbf{1}_B, f\rangle + L\langle x, \mathbf{1}_A\rangle + L|B \cap A| \quad \text{by (9)} \\ &= \langle x, x^*\rangle + \langle \mathbf{1}_B, x^*\rangle + \langle x, f\rangle + \langle \mathbf{1}_B, f\rangle \\ &\quad + L\langle x, \mathbf{1}_{\tilde{A}}\rangle + L|\tilde{B} \cap \tilde{A}| \quad \text{by choice of } \tilde{A} \\ &\le \langle x, x^*\rangle + \langle x, f\rangle + \langle \mathbf{1}_B, f\rangle + L\langle x, \mathbf{1}_{\tilde{A}}\rangle + L|\tilde{B} \cap \tilde{A}| + \varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \quad \text{by (10)} \\ &\le \langle x, x^*\rangle + (1+\varepsilon)^2\langle x + \mathbf{1}_{\tilde{B}}, g\rangle + L\langle x, \mathbf{1}_{\tilde{A}}\rangle + L|\tilde{B} \cap \tilde{A}| + \varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \\ &\qquad \text{by (12), (14), (15)} \\ &\le (1+\varepsilon)^2\langle x + \mathbf{1}_{\tilde{B}}, x^* + g + L\mathbf{1}_{\tilde{A}}\rangle + \varepsilon|||x + \mathbf{1}_{\tilde{B}}||| \\ &\le (1+4\varepsilon)|||x + \mathbf{1}_{\tilde{B}}|||, \end{aligned}$$

as required. Finally, it is easy to see that $|||e_i||| = s + \psi(1) + L$ for all $i \in \mathbb{N}$. So by scaling the new norm, we make (e_i) normalized, 1-unconditional and $(1+4\varepsilon)$-greedy. □

The condition $\delta(\varphi) > 0$ says that the growth of φ on intervals of any given fixed size is eventually linear. For example, when $\varphi(x) \sim x$ or $\varphi(x) \sim \frac{x}{\log x}$, then $\delta(\varphi) > 0$, so Theorem 11 applies. Note also that when φ has the URP, then $\delta(\varphi) = 0$. However, in that case the basis is bidemocratic and Theorem 4 can be used. We next give an application of Theorem 11 in two special cases. Note that neither of these bases is bidemocratic, so Theorem 4 cannot be applied.

Corollary 12. *For all $\varepsilon > 0$ there is an equivalent norm on dyadic Hardy space H_1 and on Tsirelson's space T such that the Haar system, respectively, the unit vector basis is normalized, 1-unconditional and $(1+\varepsilon)$-greedy.*

4. Open problems

For bidemocratic bases we were able to achieve the best possible renorming for the democracy constant (Theorem 3). For the greedy constant Theorem 4 gets arbitrarily close, but the following remains open.

Problem 13. Let (e_i) be a bidemocratic basis of a Banach space X. Does there exist an equivalent norm on X with respect to which (e_i) is 1-greedy?

The following special case of interest was raised by Albiac and Wojtaszczyk.

Problem 14 (*[1, Problem 6.2]*)**.** Let $1 < p < \infty$. Does there exist an equivalent norm on $L_p[0,1]$ with respect to which the Haar basis is 1-greedy?

The other main problem that remains open concerns the greedy constant in the general, not necessarily bidemocratic, case.

Problem 15. Let (e_i) be a greedy basis of a Banach space X. Does there exist for any $\varepsilon > 0$ an equivalent norm on X with respect to which the basis is $(1+\varepsilon)$-greedy?

This paper gives a positive answer for a large family of bases. In terms of the behaviour of the fundamental function φ, if φ has the URP, or if, on the other extreme, $\delta(\varphi) > 0$, then the answer is 'yes'. If the basis is bidemocratic, or, more generally, if there is a constant C such that the family $\mathcal{A}$ defined in Lemma 9 consists of *all* finite subsets of $\mathbb{N}$, then the proof of Theorem 4 furnishes a positive answer. However, as pointed out at the end of Section 2, there are fundamental functions φ with $\delta(\varphi) = 0$ and with

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{\varphi(mn)}{m\varphi(n)} = 1.$$

Note that this latter condition rules out properties like the URP. So there is still a gap.

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