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Full Length Article

Norm estimates for Chebyshev polynomials, I

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Abstract

In this paper, we extend the sharp upper bound of Christiansen et al. (2017) and the sharp lower bound of Schiefermayr (2008) to the case of weighted Chebyshev polynomials on subsets of $[-1, 1]$ for the weight $w(x) = \sqrt{1 - x^2}$. We then analyse the norm of Chebyshev polynomials on a circular arc, prove monotonicity of the corresponding Widom factors, find exact values of their supremum and infimum, and obtain a new proof for their limit.

MSC: 41A50; 30C10; 30E10

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1. Introduction

Let $K \subset \mathbb{C}$ be a compact set consisting of infinitely many points. We denote by $T_n^{(K)}(z)$ the Chebyshev polynomial of degree $n \in \mathbb{N}$ on K, that is, the minimizer for the supremum norm $\|P_n\|_K := \sup_{z\in K} |P_n(z)|$ among all monic polynomials P_n of degree n. For $K \subseteq [-1, 1]$, we will also consider weighted Chebyshev polynomials on K with respect to the weight function

$$w(x) := \sqrt{1 - x^2}. \tag{1}$$

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The corresponding w-Chebyshev polynomial of degree $n \in \mathbb{N}$, denoted by $T_{n,w}^{(K)}(z)$, is the minimizer of $\|wP_n\|_K$ over all monic polynomials P_n of degree n. We denote the norms of $T_n^{(K)}$ and $wT_{n,w}^{(K)}$ by

$$t_n(K) := \|T_n^{(K)}\|_K \quad \text{and} \quad t_n(K,w) := \|wT_{n,w}^{(K)}\|_K. \tag{2}$$

For basic properties of these polynomials we refer to [8, Chapter 3], [16, Chapter 4], and Theorems 11 and 12.

As usual, let T_n and U_n denote the classical Chebyshev polynomial (of degree n) of the first and second kind, respectively, that is,

$$T_n(x) = \tfrac{1}{2}(z^n + z^{-n}) \quad \text{and} \quad U_n(x) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}, \quad x = \tfrac{1}{2}(z + z^{-1}), \tag{3}$$

or equivalently,

$$T_n(x) = \cos(n\theta) \quad \text{and} \quad U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \quad x = \cos(\theta). \tag{4}$$

In the following, we will assume that K is a proper subset of $\mathbb{R}$ or $\partial\mathbb{D}$ and that it has positive logarithmic capacity, $\operatorname{cap}(K) > 0$. This guarantees that there exists the Green function $g_K(z)$, that is, a unique subharmonic function on $\mathbb{C}$ which is positive and harmonic on $\mathbb{C}\backslash K$,

$$g_K(z) = \log|z| - \log\operatorname{cap}(K) + O(1/z) \quad \text{as} \quad z \to \infty, \tag{5}$$

and $g_K(z) \to 0$ as $z \to z_0$ for quasi-every $z_0 \in K$. For some of our results we will make an additional assumption that K is regular (for potential theory) which means that the Green function g_K is continuous on $\mathbb{C}$ with $g_K(z) = 0$ for all $z \in K$. For basic notions of potential theory, we refer to [15,17,20,21,32] and appendices of [26,28].

Following [11] (see also [4]), we will use the notion of so-called *Widom factors*, defined by

$$\mathcal{W}_n(K) := \frac{t_n(K)}{\operatorname{cap}(K)^n} \quad \text{and} \quad \mathcal{W}_n(K,w) := \frac{t_n(K,w)}{\operatorname{cap}(K)^n}, \quad n \in \mathbb{N}. \tag{6}$$

In this paper, we consider the Widom factors of the circular arc

$$\Gamma_\alpha := \left\{e^{i\theta} : \theta \in [-\alpha, \alpha]\right\}, \quad 0 < \alpha < \pi, \tag{7}$$

and of the "corresponding" set on the real line, given by

$$E_a := [-1, -a] \cup [a, 1], \quad a := \cos(\tfrac{\alpha}{2}). \tag{8}$$

Thiran and Detaille [29] were probably the first who discovered a connection between the Chebyshev polynomials on Γ_α and the Chebyshev polynomials with respect to 1 and w on E_a. In particular, they proved identities for the corresponding norms of these Chebyshev polynomials, see (34) and (35). In light of these identities, finding upper and lower bounds for $\mathcal{W}_n(\Gamma_\alpha)$ is equivalent to finding the corresponding bounds for $\mathcal{W}_{2n+1}(E_a)$ and $\mathcal{W}_{2n-1}(E_a, w)$. If a set K is real then sharp upper and lower bounds for $\mathcal{W}_n(K)$ are known, see Theorem 1. As a first main result, we derive upper and lower bounds in the weighted case for $\mathcal{W}_n(K, w)$, see Theorem 3. Applying these bounds to the case $K = E_a$ then gives sharp upper and lower bounds for $\mathcal{W}_n(\Gamma_\alpha)$, see Theorem 9. Moreover, as a second main result, a monotonicity property for the Widom factors on Γ_α is given, more precisely, we prove that (for Γ_α fixed) the sequence $\left\{\mathcal{W}_n(\Gamma_\alpha)\right\}_{n=1}^{\infty}$ is strictly monotone increasing, see Theorem 8. This monotonicity property generates a second proof for the upper bound in (40) and, in addition, it is essential for proving the lower bound in (40) for odd degrees. The proofs of Theorems 3 and 8, given in Section 4, are based on potential theory and on properties of the Chebyshev polynomials (with respect to 1 and w) on real sets.

This paper is the first part of a series of two papers. In the second part [25], we give estimations for the Widom factors of more general sets on the unit circle (symmetric with respect to the real line).

2. Sharp bounds for real sets

First, let us recall the sharp lower and upper bounds for the Widom factor of real sets, recently given in the literature.

Theorem 1. *Let $K \subset \mathbb{R}$ be an infinite compact set with* $\operatorname{cap}(K) > 0$.

(i) *For all $n \in \mathbb{N}$,*

$$\mathcal{W}_n(K) \geq 2. \tag{9}$$

Equality is attained in (9) if and only if there exists a polynomial P_n of degree n such that $K = P_n^{-1}([-1,1]) = \{z \in \mathbb{C} : P_n(z) \in [-1,1]\}$.

(ii) *If, in addition, K is regular for potential theory, then, for all $n \in \mathbb{N}$,*

$$\mathcal{W}_n(K) \leq 2\exp\big(\mathcal{PW}(K)\big), \tag{10}$$

where

$$\mathcal{PW}(K) := \sum_{z \in \mathcal{C}} g_K(z) \tag{11}$$

and $\mathcal{C} = \{x \in \mathbb{R}\backslash K : \frac{d}{dx} g_K(x) = 0\}$ denotes the set of all critical points of the Green function $g_K(z)$. Equality is attained in (10) if and only if K is an interval.

Proof. (i) Inequality (9) was first proved in [22], whereas the if and only if statement was proved in [30, Theorem 1] and also in [7, Theorem 1.1].

(ii) Inequality (10) was proved in [6, Theorem 1.4], whereas the if and only if statement was proved in [7, Theorem 4.3]. □

Remark 2. The term $\mathcal{PW}$ stands for *Parreau–Widom* who made important contributions to the function theory on infinitely connected Riemann Surfaces with boundaries K satisfying $\mathcal{PW}(K) < \infty$, [18,33], see also [12] for a book presentation.

As our first main result, we give sharp lower and upper bounds for the weighted Chebyshev polynomials.

Theorem 3. *Let $K \subseteq [-1,1]$ be an infinite compact set with* $\operatorname{cap}(K) > 0$ *and $w(x)$ be the weight function (1).*

(i) *For all $n \in \mathbb{N}$,*

$$\mathcal{W}_n(K, w) \geq 2\operatorname{cap}(K). \tag{12}$$

Equality is attained in (12) if and only if there exists a polynomial P_n of degree n such that

$$K = \{z \in \mathbb{C} : (1-z^2)P_n^2(z) \in [0,1]\}. \tag{13}$$

(ii) *If, in addition, K is regular for potential theory, then, for all $n \in \mathbb{N}$,*

$$\mathcal{W}_n(K, w) \le 2\,\mathrm{cap}(K)\exp\big(\mathcal{PW}(K, w)\big), \tag{14}$$

where

$$\mathcal{PW}(K, w) := \tfrac{1}{2}\big(g_K(-1) + g_K(1)\big) + \mathcal{PW}(K). \tag{15}$$

Equality is attained in (14) if and only if $K = [-1, 1]$.

Proof. The proof of Theorem 3 is given in Section 4. □

Remark 4.

(i) Theorem 3 is new even for sets K consisting of finitely many intervals. The upper bound (14) is non trivial for regular Parreau–Widom sets. By a result of Jones and Marshall [13], such sets are known to include homogeneous sets in the sense of Carleson [5] and, in particular, positive measure Cantor sets.

(ii) For $K = [-1, 1]$ the (weighted) extremal polynomials are, $n \in \mathbb{N}$,

$$T_n^{([-1,1])}(z) = 2^{-n+1}T_n(z) \quad\text{and}\quad T_{n,w}^{([-1,1])}(x) = 2^{-n}U_n(x) \tag{16}$$

as can be easily verified via the alternation theorem (Theorems 11(i) and 12(i)) hence, using $\mathrm{cap}([-1, 1]) = \frac{1}{2}$, we have

$$t_n([-1, 1]) = 2\,\mathrm{cap}([-1, 1])^n \quad\text{and}\quad t_n([-1, 1], w) = 2\,\mathrm{cap}([-1, 1])^{n+1}. \tag{17}$$

Unlike the unweighted Chebyshev polynomials $T_n^{([a,b])}$, which can be obtained from $T_n^{([-1,1])}$ by shifting and rescaling, the weighted Chebyshev polynomials $T_{n,w}^{([a,b])}$ are not known for a subinterval $[a, b] \subset [-1, 1]$. We pose it as an open problem to find these polynomials.

(iii) If $\pm 1 \in K$ and $K \subseteq [-1, 1]$ is regular, then $g_K(\pm 1) = 0$ and hence

$$\mathcal{PW}(K, w) = \mathcal{PW}(K). \tag{18}$$

Next, we discuss the set $K = E_a$ given in (8) which is of independent interest due to its application to Chebyshev norms on a circular arc. For this set the lower and upper bounds of Theorems 1 and 3 are the best possible. We start with a technical lemma.

Lemma 5 *([20, Theorem 5.2.5]). Let $P_n(z) = c_n z^n + \cdots$ with $c_n \in \mathbb{C}\backslash\{0\}$ be a polynomial of degree n and $L \subset \mathbb{C}$ be a compact set. Then, for the polynomial pre-image set $K := P_n^{-1}(L) = \{z \in \mathbb{C} : P_n(z) \in L\}$, we have*

$$\mathrm{cap}(K) = \big(\mathrm{cap}(L)/|c_n|\big)^{1/n}. \tag{19}$$

In addition, if $\mathrm{cap}(L) > 0$ then $g_K(z) = \frac{1}{n}g_L(P_n(z))$ and hence K is regular if and only if L is. In particular, if $L = [-1, 1]$ then $\mathrm{cap}(K) = \big(2|c_n|\big)^{-1/n}$ and

$$g_K(z) = \tfrac{1}{n}g_{[-1,1]}(P_n(z)) = \tfrac{1}{n}\log\Big|P_n(z) + \sqrt{P_n(z)^2 - 1}\Big|. \tag{20}$$

Here and in the following, we define the complex square root $\sqrt{z^2 - 1}$ such that it lies in the same quadrant as z (except for $z \in [-1, 1]$, along which the plane must be cut) and $\sqrt{P_n^2(z) - 1}$ is defined analogously.

Now we note that $E_a = [-1, -a] \cup [a, 1]$, $0 \le a < 1$, is a polynomial pre-image of $[-1, 1]$ and hence, by Lemma 5,

$$g_{E_a}(z) = \tfrac{1}{2} \log \left| P(z) + \sqrt{P(z)^2 - 1} \right|, \quad \text{where} \quad P(z) = \frac{2z^2 - 1 - a^2}{1 - a^2}, \tag{21}$$

and

$$\operatorname{cap}(E_a) = \tfrac{1}{2}\sqrt{1 - a^2}. \tag{22}$$

It is easy to see that g_{E_a} has one critical point at $x = 0$ and $g_{E_a}(0) = \frac{1}{2} \log \frac{1+a}{1-a}$ (one has to be careful with the sign of the square root!). Since $\pm 1 \in E_a$ we get, by (18),

$$\exp\big(\mathcal{PW}(E_a, w)\big) = \exp\big(\mathcal{PW}(E_a)\big) = \exp\big(g_{E_a}(0)\big) = \sqrt{\frac{1 + a}{1 - a}}. \tag{23}$$

Let us take the opportunity to point out an error in the paper [23, Eq. (67)], where the $+$ and $-$ signs should be changed.

With the help of the above observations, we obtain some of the following results.

Theorem 6. *Let $E_a = [-1, -a] \cup [a, 1]$ with $0 \le a < 1$.*

(i) *For even degree, we have, $n \in \mathbb{N}$,*

$$T_{2n}^{(E_a)}(z) = \frac{(1 - a^2)^n}{2^{2n-1}}\, T_n\Big(\frac{2z^2 - 1 - a^2}{1 - a^2}\Big), \tag{24}$$

$$T_{2n,w}^{(E_a)}(z) = \frac{(1 - a^2)^n}{2^{2n}}\, U_{2n}\bigg(\sqrt{\frac{z^2 - a^2}{1 - a^2}}\bigg), \tag{25}$$

therefore,

$$t_{2n}(E_a) = \frac{(1 - a^2)^n}{2^{2n-1}}, \tag{26}$$

$$t_{2n}(E_a, w) = \frac{(1 - a^2)^n}{2^{2n}} \cdot \sqrt{1 - a^2}, \tag{27}$$

and hence

$$\mathcal{W}_{2n}(E_a) = 2, \tag{28}$$

$$\mathcal{W}_{2n}(E_a, w) = \sqrt{1 - a^2}. \tag{29}$$

(ii) *For odd degree, we have, $n \in \mathbb{N}_0$,*

$$2\sqrt{\frac{1 + a}{1 - a}} - \frac{2}{\sqrt{1 - a^2}}\Big(\frac{1 - a}{1 + a}\Big)^n \le \mathcal{W}_{2n+1}(E_a) \le 2\sqrt{\frac{1 + a}{1 - a}} \tag{30}$$

and

$$\mathcal{W}_{2n+1}(E_a, w) \le 1 + a, \tag{31}$$

and therefore,

$$\lim_{n\to\infty} \mathcal{W}_{2n+1}(E_a) = 2\sqrt{\frac{1 + a}{1 - a}}. \tag{32}$$

Proof. Since $T_n(x) = 2^{n-1}x^n + \cdots$ has $n+1$ alternation points on $[-1, 1]$, the function $P_{2n}(z) := \frac{(1-a^2)^n}{2^{2n-1}} T_n\big(\frac{2z^2-1-a^2}{1-a^2}\big)$ is a monic polynomial of degree $2n$ and has $n+1$ alternation points on $[-1, -a]$ and on $[a, 1]$, respectively. Since $P_{2n}(-a) = P_{2n}(a)$, $P_{2n}(z)$ has $2n+1$ alternation points on E_a. Similarly, since $U_{2n}(x)\sqrt{1-x^2} = \sqrt{1-x^2}\cdot(2^{2n}x^{2n}+\cdots)$ is even and has $2n+1$ alternation points on $[-1, 1]$, the function $Q(z)/\sqrt{1-z^2} := \frac{(1-a^2)^n}{2^{2n}} U_{2n}\Big(\sqrt{\frac{z^2-a^2}{1-a^2}}\Big)$ is a monic polynomial of degree $2n$ and $Q(z)$ has $n+1$ alternations points on $[-1, -a]$ and on $[a, 1]$, respectively. Since $Q(-a) = Q(a)$, $Q(z)$ has $2n+1$ alternation points on E_a. Now, (24) and (25) follow from Theorems 11 and 12, respectively. The remaining formulas of (i) follow from (24) and (25). The right inequality of (30) follows immediately from (10) and (23). The left inequality of (30) goes back to the work of Akhiezer [1, p. 320, Eq. (k)], see [23, Section 4]. Inequality (31) follows immediately from (14), (22), and (23). □

Remark 7. A complementary lower bound and the limit for $\mathcal{W}_{2n+1}(E_a, w)$ will be derived in (41) and (42), respectively.

3. Sharp bounds for one circular arc

Let Γ_α and E_a be defined as in (7) and (8), respectively. It is well known [20, Table 5.1] that

$$\operatorname{cap}(\Gamma_\alpha) = \sin(\tfrac{\alpha}{2}) = \sqrt{1-a^2} = 2\operatorname{cap}(E_a). \tag{33}$$

Thiran & Detaille [29], see also Theorem 1 and Theorem 2 of [24], proved that

$$t_{2n}(\Gamma_\alpha) = 2^{2n} t_{2n+1}(E_a), \tag{34}$$

$$t_{2n-1}(\Gamma_\alpha) = 2^{2n-1} t_{2n-1}(E_a, w). \tag{35}$$

Therefore, using (33),

$$\mathcal{W}_{2n}(\Gamma_\alpha) = \operatorname{cap}(E_a)\,\mathcal{W}_{2n+1}(E_a), \tag{36}$$

$$\mathcal{W}_{2n-1}(\Gamma_\alpha) = \mathcal{W}_{2n-1}(E_a, w). \tag{37}$$

Theorem 8. *The sequence of Widom factors $\{\mathcal{W}_n(\Gamma_\alpha)\}_{n=1}^{\infty}$ is strictly monotone increasing, that is,*

$$\mathcal{W}_n(\Gamma_\alpha) < \mathcal{W}_{n+1}(\Gamma_\alpha), \quad n \in \mathbb{N}, \tag{38}$$

and its limit is given by

$$\lim_{n\to\infty} \mathcal{W}_n(\Gamma_\alpha) = 1 + \cos(\tfrac{\alpha}{2}). \tag{39}$$

Proof. The proof of Theorem 8 is given in Section 4. □

Theorem 9. *For $n \in \mathbb{N}$, we have*

$$1 + \cos(\tfrac{\alpha}{2}) - \left(\frac{1-\cos(\frac{\alpha}{2})}{1+\cos(\frac{\alpha}{2})}\right)^{\lfloor \frac{n}{2} \rfloor} \le \mathcal{W}_n(\Gamma_\alpha) \le 1 + \cos(\tfrac{\alpha}{2}). \tag{40}$$

Proof. The left inequality of (40) follows from

$$\mathcal{W}_{2n+1}(\Gamma_\alpha) \ge \mathcal{W}_{2n}(\Gamma_\alpha) \qquad \text{by (38)}$$

$$
\begin{aligned}
&= \operatorname{cap}(E_a)\mathcal{W}_{2n+1}(E_a) \qquad \text{by (36)}\\
&\geq 2\operatorname{cap}(E_a)\sqrt{\frac{1+a}{1-a}} - \operatorname{cap}(E_a)\frac{2}{\sqrt{1-a^2}}\Big(\frac{1-a}{1+a}\Big)^n \qquad \text{by (30)}\\
&= 1+a-\Big(\frac{1-a}{1+a}\Big)^n \qquad \text{by (22)}
\end{aligned}
$$

For proving the right inequality of (40), we will distinguish between even and odd degree, respectively. By (36), (33), and the right inequality of (30),

$$\mathcal{W}_{2n}(\Gamma_\alpha) = \operatorname{cap}(E_a)\,\mathcal{W}_{2n+1}(E_a) \leq 1+a.$$

By (37) and (31),

$$\mathcal{W}_{2n-1}(\Gamma_\alpha) = \mathcal{W}_{2n-1}(E_a, w) \leq 1+a. \quad \square$$

Remark 10.

(i) By (37) and (38),

$$\mathcal{W}_{2n+1}(E_a, w) = \mathcal{W}_{2n+1}(\Gamma_\alpha) > \mathcal{W}_{2n}(\Gamma_\alpha) = \tfrac{1}{2}\sqrt{1-a^2}\,\mathcal{W}_{2n+1}(E_a),$$

hence, by (30),

$$\mathcal{W}_{2n+1}(E_a, w) > 1+a-\Big(\frac{1-a}{1+a}\Big)^n. \tag{41}$$

Therefore, together with (31),

$$\lim_{n\to\infty} \mathcal{W}_{2n+1}(E_a, w) = 1+a. \tag{42}$$

(ii) The limit (39) has a long history. It was first conjectured by Thiran & Detaille [29, Section 5], probably based on (32), which is a result of Akhiezer [2, E. 27]. In [9], Eichinger gave a proof of (39) by considering the asymptotics of the corresponding Chebyshev polynomial on Γ_α by means of geometric function theory and, in particular, the machinery developed in [34]. Finally, in [24, Section 3.3], the first author gave another proof using the degenerating behaviour of Jacobi's elliptic and theta functions.

(iii) It is clear that the right inequality of (40) follows from Theorem 8. Nevertheless, above we gave an alternative proof of the right inequality of (40) based on the upper bounds (10) and (14), thus allowing for an alternative derivation of the limit (39) from (40).

(iv) In [24], based on results in [1,14,29], and [19], the first author gave a parametric representation of the Chebyshev polynomials $T_n^{(\Gamma_\alpha)}(z)$ in terms of elliptic and theta functions.

(v) For $n = 1$ and $n = 2$, the Widom factor $\mathcal{W}_n(\Gamma_\alpha)$ can be determined explicitly. The Chebyshev polynomial of degree 1 on E_a with respect to the weight function $w(x)$ is (by symmetry) given by $T_{1,w}^{(E_a)}(x) = x$, hence, by (35),

$$t_1(\Gamma_\alpha) = 2\,t_1(E_a, w) = 2\max_{x\in E_a}|x\sqrt{1-x^2}| = \begin{cases} 2a\sqrt{1-a^2} & \text{for } \frac{1}{\sqrt{2}} \leq a < 1 \\ 1 & \text{for } 0 < a \leq \frac{1}{\sqrt{2}} \end{cases} \tag{43}$$

and therefore

$$\mathcal{W}_1(\Gamma_\alpha) = \frac{t_1(\Gamma_\alpha)}{\operatorname{cap}(\Gamma_\alpha)} = \begin{cases} 2\cos(\frac{\alpha}{2}) & \text{for } 0 < \alpha \leq \frac{\pi}{2} \\ \frac{1}{\sin(\frac{\alpha}{2})} & \text{for } \frac{\pi}{2} \leq \alpha \leq \pi. \end{cases} \tag{44}$$

Due to monotonicity (38), the above value is the sharp lower bound for the Widom factors $\{\mathcal{W}_n(\Gamma_\alpha)\}_{n=1}^{\infty}$.

(vi) The Chebyshev polynomial of degree 3 on E_a can be determined with the help of Theorem 11. Since E_a is symmetric, $T_3^{(E_a)}$ must be odd, hence $T_3^{(E_a)}(x) = x^3 + cx$. Then for $0 < a \leq \frac{1}{2}$, $T_3^{(E_a)}(x) = \frac{1}{4}T_3(x) = x^3 - \frac{3}{4}x$ and for $\frac{1}{2} < a < 1$, the coefficient c can be determined from $T_3^{(E_a)}(a) = -T_3^{(E_a)}(1)$, hence $T_3^{(E_a)}(x) = x^3 - (1 - a + a^2)x$. Thus, by (34),

$$t_2(\Gamma_\alpha) = 4\,t_3(E_a, w) = \begin{cases} 4a(1-a) & \text{for } \frac{1}{2} \leq a < 1 \\ 1 & \text{for } 0 < a \leq \frac{1}{2} \end{cases} \tag{45}$$

and therefore

$$\mathcal{W}_2(\Gamma_\alpha) = \frac{t_2(\Gamma_\alpha)}{\operatorname{cap}(\Gamma_\alpha)^2} = \begin{cases} \frac{4\cos(\frac{\alpha}{2})}{1+\cos(\frac{\alpha}{2})} & \text{for } 0 < \alpha \leq \frac{2\pi}{3} \\ \frac{1}{\sin^2(\frac{\alpha}{2})} & \text{for } \frac{2\pi}{3} \leq \alpha \leq \pi. \end{cases} \tag{46}$$

(vii) Sharp lower bounds for Widom factors associated with orthogonal polynomials (and more generally with L^p-extremal polynomials for $p < \infty$) have been recently investigated in [3,4]. Sharp upper bounds for Widom factors associated with orthogonal polynomials are not yet well understood.

(viii) In the case of orthogonal polynomials with respect to the equilibrium measure on a circular arc, monotone increasing behaviour of the associated Widom factors has been recently shown in [3, Theorem 5.1].

4. Proofs

We start this section by briefly reviewing properties of the unweighted and weighted Chebyshev polynomials on a real set and basic elements of potential theory needed for the proofs.

Theorem 11. *Let $K \subset \mathbb{R}$ be an infinite compact set and define $x_{\min} := \inf K$ and $x_{\max} := \sup K$.*

(i) *A monic polynomial P_n of degree n is the Chebyshev polynomial on K, that is, $P_n = T_n^{(K)}$, if and only if there exist points $x_{\min} = x_0 < x_1 < \cdots < x_n = x_{\max}$ on K such that*

$$P_n(x_j) = (-1)^{n-j}\|P_n\|_K, \quad j = 0, 1, \ldots, n. \tag{47}$$

The extremal points $x_1, \ldots, x_{n-1}$ are not necessarily unique.

(ii) *$T_n^{(K)}$ exists, is unique and real.*

(iii) *Define*

$$K_{\max} := \left(T_n^{(K)}\right)^{-1}([-t_n(K), t_n(K)]) = \left\{z \in \mathbb{C} : T_n^{(K)}(z) \in [-t_n(K), t_n(K)]\right\} \tag{48}$$

then

$$K \subseteq K_{\max} \subseteq [x_{\min}, x_{\max}] \subset \mathbb{R}, \tag{49}$$

$K_{\max}$ is the union of m disjoint intervals, $m \leq n$, and $T_n^{(K_{\max})} = T_n^{(K)}$.

(iv) *If, in addition, K is symmetric (that is, $z \in K \Leftrightarrow -z \in K$) then $T_{2n}^{(K)}$ and $T_{2n-1}^{(K)}$ are even and odd polynomials, respectively.*

Proof. (i) and (ii) are well-known, see for example [6, Theorem 1.1] or in a more general setting [8, Chapter 3], (iii) is a consequence of (i). (iv) For any K, it is clear that $T_{2n}^{(-K)}(z) = T_{2n}^{(K)}(-z)$ and $T_{2n-1}^{(-K)}(z) = -T_{2n-1}^{(K)}(-z)$. Since by assumption $K = -K$, the assertion follows from uniqueness of the Chebyshev polynomials. $\square$

Theorem 12. *Let $K \subseteq [-1, 1]$ be an infinite compact set.*

(i) *A monic polynomial P_n of degree n is the Chebyshev polynomial on K with respect to the weight w, that is, $P_n = T_{n,w}^{(K)}$, if and only if there exist points $-1 < x_0 < x_1 < \cdots < x_{n-1} < x_n < 1$ on K such that*

$$w(x_j)\, P_n(x_j) = (-1)^{n-j} \|w P_n\|_K, \quad j = 0, 1, \ldots, n. \tag{50}$$

The extremal points $x_0, \ldots, x_n$ are not necessarily unique.

(ii) *$T_{n,w}^{(K)}$ exists, is unique and real.*

(iii) *Define*

$$K_{\max} := \left\{ z \in \mathbb{C} : \left(w(z)\, T_{n,w}^{(K)}(z) \right)^2 \in [0, t_n(K, w)^2] \right\} \tag{51}$$

then

$$K \subseteq K_{\max} \subseteq [-1, 1], \tag{52}$$

$K_{\max}$ is the union of m disjoint intervals, $1 \le m \le n + 2$, and $\pm 1 \in K_{\max}$. Moreover, $T_{n,w}^{(K_{\max})} = T_{n,w}^{(K)}$, and

$$\tfrac{1}{2} t_n(K, w)^2 - \left(w T_{n,w}^{(K)} \right)^2 = T_{2n+2}^{(K_{\max})}. \tag{53}$$

(iv) *If, in addition, K is symmetric (that is, $z \in K \Leftrightarrow -z \in K$) then $T_{2n,w}^{(K)}$ and $T_{2n-1,w}^{(K)}$ are even and odd polynomials, respectively.*

Proof. (i) and (ii) are well known, see for example [16, Section 4.2] (the argument given in [6] also readily extends to this weighted setting). (iii) By (i), (ii), and the intermediate value theorem the zeros $y_0, y_1, \ldots, y_{n+1}$ of $w T_{n,w}^{(K)}$ are interlaced with the extremal points,

$$-1 =: y_0 < x_0 < y_1 < x_1 < y_2 < \cdots < x_{n-1} < y_n < x_n < y_{n+1} := 1. \tag{54}$$

Now consider the real polynomial $P_{2n+2} := \left(w T_{n,w}^{(K)} \right)^2$ of degree $2n + 2$ which attains the values 0 and $t_n(K, w)^2$ at the above interlaced points. Then again by the intermediate value theorem, each point in the interval $\left(0, t_n(K, w)^2\right)$ has $2n+2$ distinct pre-images in the intervals (y_k, x_k), (x_k, y_{k+1}), $k = 0, \ldots, n$, which accounts for all the pre-images since P_{2n+2} has degree $2n + 2$ and hence (52) holds. Moreover, $Q_{2n+2} := \frac{1}{2} t_n(K, w)^2 - P_{2n+2}$ is a monic polynomial of degree $2n+2$ and has $2n+3$ alternation points on $K_{\max}$, hence $Q_{2n+2} = T_{2n+2}^{(K_{\max})}$ by Theorem 11(i). The proof of (iv) is the same as in the unweighted case. $\square$

Next, we recall some elements of potential theory. Again we refer to [15,17,20,21,32] for an in depth exposition. First, recall that the equilibrium measure μ_K of a compact set K with $\mathrm{cap}(K) > 0$ is the unique Borel probability measure on K that minimizes the logarithmic energy

$$I(\mu) = \iint \log \frac{1}{|z - \zeta|}\, \mathrm{d}\mu(\zeta)\, d\mu(z) \tag{55}$$

among all Borel probability measures μ on K. The logarithmic capacity of K then can be defined by $\operatorname{cap}(K) = \exp(-I(\mu_K))$ ($\operatorname{cap}(K) = 0$ if $I(\mu) = \infty$ for all μ) and the Green function can be expressed in terms of $\operatorname{cap}(K)$ and μ_K as

$$g_K(z) = -\log \operatorname{cap}(K) + \int \log|z-\zeta|\,\mathrm{d}\mu_K(\zeta), \quad z \in \mathbb{C}. \tag{56}$$

The equilibrium measure μ_K can be written explicitly for $K \subset \mathbb{R}$ which is a polynomial pre-image of an interval. For example, if $P_n(z)$ is a polynomial of degree n such that

$$K_n := P_n^{-1}\big([-1,1]\big) = \big\{z \in \mathbb{C} : P_n(z) \in [-1,1]\big\} \subset \mathbb{R},$$

then (see for example, [10,31], [6, Theorem 2.3], [27, Theorem 5.4.5]),

$$\mathrm{d}\mu_{K_n}(x) = \frac{1}{n\pi}\frac{|P_n'(x)|}{\sqrt{1-P_n^2(x)}}\mathbf{1}_{K_n}(x)\mathrm{d}x. \tag{57}$$

Moreover, P_n^{-1} has n branches that map $[-1,1]$ monotonically onto n intervals $[a_k,b_k]$, $k=1,\dots,n$, and it follows from (57) and $\int_{-1}^{1}\frac{\mathrm{d}t}{\pi\sqrt{1-t^2}} = 1$ that

$$\mu_{K_n}\big([a_k,b_k]\big) = \tfrac{1}{n}, \quad k=1,\dots,n. \tag{58}$$

Lemma 13. *Let $K \subset L$ be two compact subsets of $\mathbb{R}$ of positive capacity and let μ_L be the equilibrium measure of L and $g_K(z)$ the Green function of K. Then*

$$\log\left(\frac{\operatorname{cap}(L)}{\operatorname{cap}(K)}\right) = \int_{L\setminus K} g_K(z)\mathrm{d}\mu_L(z). \tag{59}$$

In addition, if L is regular for potential theory and $\operatorname{cap}(K) = \operatorname{cap}(L)$ *then $K = L$.*

Proof. Let $g_L(z)$ denote the Green function for L and define $h(z) := g_K(z) - g_L(z)$. Then it follows from the properties of the Green function that $h(z)$ is harmonic on $\overline{\mathbb{C}}\setminus L$ with $h(\infty) = \log\big(\frac{\operatorname{cap}(L)}{\operatorname{cap}(K)}\big)$. By [20, Theorem 4.3.14] the equilibrium measure μ_L is the harmonic measure at infinity for the domain $\overline{\mathbb{C}}\setminus L$ and hence

$$h(\infty) = \int_L h(x)\mathrm{d}\mu_L(x). \tag{60}$$

By (56) and [20, Theorem 3.3.4(b)] the Green functions $g_K(z)$ and $g_L(z)$ vanish on K resp. L except possibly on a Borel set E of capacity zero, hence $h(z) = g_K(z)$ for $z \in L\setminus E$ and $h(z) = 0$ for $z \in K\setminus E$. By [20, Theorem 4.3.6] the harmonic measure μ_L is zero on sets of capacity zero, in particular, $\mu_L(E) = 0$. Combining these observations with (60) yields (59).

If $\operatorname{cap}(L) = \operatorname{cap}(K)$ then the integral in (59) is zero and since $g_K(z) > 0$ for $z \in \mathbb{C}\setminus K$ we have $\mu_L(L\setminus K) = 0$ and hence $\operatorname{supp}(\mu_L) \subset K$. Next, we will show that $L \subset \operatorname{supp}(\mu_L)$ which then implies $L = K$ since by assumption $K \subset L$. By [20, Theorem 3.1.2] it follows from (56) applied to the Green function $g_L(z)$ that $g_L(z)$ is harmonic on $\mathbb{C}\setminus\operatorname{supp}(\mu_L)$ with $g_L(z) = \log|z| + O(1)$ as $z \to \infty$ and hence, by the minimum principle, $g_L(z) \neq 0$ for $z \in \mathbb{C}\setminus\operatorname{supp}(\mu_L)$ since otherwise g_L would be identically zero. Thus, $\{z \in \mathbb{C} : g_L(z) = 0\} \subset \operatorname{supp}(\mu_L)$. Since L is regular by assumption we have $L \subset \{z \in \mathbb{C} : g_L(z) = 0\} \subset \operatorname{supp}(\mu_L)$. $\square$

Proof of Theorem 3. As in the proof of Theorem 12(iii), let $P_{2n+2} := \big(wT_{n,w}^{(K)}\big)^2$, denote the alternation points of $T_{n,w}^{(K)}$ by $\{x_k\}_{k=0}^{n} \subset K$ and the interlaced zeros of P_{2n+2} by $\{y_k\}_{k=0}^{n+1}$ (cf.

(54)), and define

$$K_n := P_{2n+2}^{-1}\big([0, t_n(K, w)^2]\big) = \big\{z \in \mathbb{C} : 0 \le P_{2n+2}(z) \le t_n(K, w)^2\big\}. \tag{61}$$

Since P_{2n+2} has degree $2n+2$ and its leading coefficient has absolute value 1 it follows from Lemma 5 that

$$\operatorname{cap}(K_n)^{2n+2} = \operatorname{cap}\big([0, t_n(K, w)^2]\big) = \tfrac{1}{4} t_n(K, w)^2. \tag{62}$$

By Theorem 12(iii), $K \subseteq K_n$ and hence, by (62),

$$t_n(K, w) = 2\operatorname{cap}(K_n)^{n+1} \ge 2\operatorname{cap}(K)^{n+1} \tag{63}$$

which yields the lower bound (12).

If $\mathcal{W}_n(K, w) = 2\operatorname{cap}(K)$ then, by (62), $\operatorname{cap}(K_n) = \operatorname{cap}(K)$ and since K_n is regular by Lemma 5 we have $K = K_n$ by Lemma 13. Then (13) with $P_n = t_n(K, w)^{-1} T_{n,w}^{(K)}$ follows from (61). Conversely, suppose (13) holds and let c_n denote the leading coefficient of P_n. Then $\operatorname{cap}(K)^{2n+2} = (4|c_n|^2)^{-1}$ by Lemma 5 and the monic polynomial $Q_n = c_n^{-1} P_n$ satisfies $\|w Q_n\|_K = |c_n|^{-1}$. Thus, $t_n(K, w) \le \|w Q_n\|_K = 2\operatorname{cap}(K)^{n+1}$ and hence equality holds in (12).

Next, we prove the upper bound (14). By (62) it suffices to prove an appropriate upper bound on $\operatorname{cap}(K_n)/\operatorname{cap}(K)$. By Lemma 13 we have

$$\log\left(\frac{\operatorname{cap}(K_n)}{\operatorname{cap}(K)}\right) = \int_{K_n\setminus K} g_K(x)\mathrm{d}\mu_{K_n}(x) \le \int_{[-1,1]\setminus K} g_K(x)\mathrm{d}\mu_{K_n}(x). \tag{64}$$

Let the open intervals $\{I_j\}_{j\ge1}$ denote the gaps of K, that is, the bounded components of $\mathbb{R}\setminus K$. Then

$$[-1, 1]\setminus K = [-1, \inf K) \cup (\sup K, 1] \cup \bigcup_{j\ge1} I_j.$$

Since $K \subseteq [-1, 1]$ is regular for potential theory, the Green function g_K is continuous on $\mathbb{C}$ with $g_K = 0$ on K, hence in each gap I_j the maximum value of g_K is attained at a critical point $c_j \in I_j$. In addition, it follows from (56) that g_K is decreasing on $[-1, \inf K)$ and increasing on $(\sup K, 1]$ hence the maximum values of g_K on these intervals are attained at -1 and 1, respectively. Observe also that since $\{x_k\}_{k=0}^n \subset K$ and $x_{k-1} < y_k < x_k$ by (54), at most 2 of the intervals $[y_k, x_k]$, $[x_k, y_{k+1}]$, $k = 0, \ldots, n$, may overlap with each I_j and at most 1 with each $[-1, \inf K)$ and $(\sup K, 1]$. By (58) the equilibrium measure μ_{K_n} satisfies

$$\mu_{K_n}\big([y_k, x_k]\big) = \mu_{K_n}\big([x_k, y_{k+1}]\big) = \frac{1}{2n+2}, \quad k = 0, \ldots, n, \tag{65}$$

and hence $\mu_{K_n}(I_j) \le \frac{1}{n+1}$ and $\mu_{K_n}\big([-1, \inf K)\big) \le \frac{1}{2n+2}$, $\mu_{K_n}\big([-1, \inf K)\big) \le \frac{1}{2n+2}$. Substituting these estimates into (64) then gives

$$\log\left(\frac{\operatorname{cap}(K_n)}{\operatorname{cap}(K)}\right) \le \frac{1}{2n+2}\big(g_K(-1) + g_K(1)\big) + \frac{1}{n+1}\sum_{j\ge1} g_K(c_j) \tag{66}$$

which, by (15), is equivalent to

$$\operatorname{cap}(K_n)^{n+1} \le \operatorname{cap}(K)^{n+1} \exp\big(\mathcal{PW}(K, w)\big).$$

Combining this inequality with (62) then yields the upper bound (14). Since g_K is not identically zero, harmonic on $\mathbb{C}\setminus K$, and symmetric with respect to the real line all the local maxima of g_K on $[-1, 1]\setminus K$ are strict and hence the inequality in (66) is strict unless

$K = [-1, 1]$. Thus, the equality in (14) holds only for $K = [-1, 1]$. The converse follows from (17). □

Proof of Theorem 8. Let E_a be as in (8) and define the sets

$$K_{2n} := \left\{z \in \mathbb{C} : T_{2n+1}^{(E_a)}(z) \in \left[-t_{2n+1}(E_a), t_{2n+1}(E_a)\right]\right\}, \tag{67}$$

$$K_{2n-1} := \left\{z \in \mathbb{C} : \left(w(z)T_{2n-1,w}^{(E_a)}(z)\right)^2 \in \left[0, t_{2n-1}(E_a, w)^2\right]\right\}, \quad n \in \mathbb{N}. \tag{68}$$

Since K_{2n} and K_{2n-1} are polynomial pre-images of intervals we have by Lemma 5,

$$t_{2n+1}(E_a) = 2\,\mathrm{cap}\big([-t_{2n+1}(E_a), t_{2n+1}(E_a)]\big) = 2\,\mathrm{cap}(K_{2n})^{2n+1},$$

$$t_{2n-1}(E_a, w) = 2\sqrt{\mathrm{cap}\big(\left[0, t_{2n-1}(E_a, w)^2\right]\big)} = 2\,\mathrm{cap}(K_{2n-1})^{2n}.$$

Then, by (34) and (35), we get for all $n \in \mathbb{N}$,

$$t_n(\Gamma_\alpha) = 2^{n+1}\,\mathrm{cap}(K_n)^{n+1}, \tag{69}$$

which combined with (33) yields

$$\mathcal{W}_n(\Gamma_\alpha) = \frac{t_n(\Gamma_\alpha)}{\mathrm{cap}(\Gamma_\alpha)^n} = \mathrm{cap}(\Gamma_\alpha)\left(\frac{\mathrm{cap}(K_n)}{\mathrm{cap}(E_a)}\right)^{n+1}. \tag{70}$$

Using Theorems 11 and 12 we see that both sets K_{2n} and K_{2n-1} are subsets of $\mathbb{R}$. Moreover, since E_a is symmetric, the Chebyshev polynomial $T_{2n+1}^{(E_a)}$ resp. $T_{2n-1,w}^{(E_a)}$ is odd so the function $T_{2n+1}^{(E_a)}$ resp. $wT_{2n-1,w}^{(E_a)}$ maps some interval $[-a_{2n}, a_{2n}]$, resp., $[-a_{2n-1}, a_{2n-1}]$ monotonically onto $[-t_{2n+1}(E_a), t_{2n+1}(E_a)]$ resp. $[-t_{2n-1}(E_a, w), t_{2n-1}(E_a, w)]$ and thus we have

$$K_n = E_a \cup [-a_n, a_n], \quad 0 < a_n \le 1, \quad n \in \mathbb{N}. \tag{71}$$

While $wT_{2n-1,w}^{(E_a)}$ is not a polynomial, its square is a polynomial hence both $[-a_{2n-1}, 0]$ and $[0, a_{2n-1}]$ are monotone polynomial pre-images of $[0, t_{2n-1}(E_a, w)^2]$. Then, by (58), the equilibrium measure μ_{K_n} of K_n satisfies

$$\mu_{K_n}\big([-a_n, a_n]\big) = \frac{1}{n+1}, \quad n \in \mathbb{N}. \tag{72}$$

Next, we will show that the sequence a_n is strictly decreasing to 0 which, in particular, implies $K_n \subseteq K_{n-1}$ by (71). First, note that by (71) and (72),

$$\mu_{K_n}(E_a \backslash [-a_n, a_n]) = \mu_{K_n}(K_n \backslash [-a_n, a_n]) = 1 - \frac{1}{n+1}, \quad n \in \mathbb{N}. \tag{73}$$

Suppose by contradiction that $a_{n-1} \le a_n$ for some $n \ge 2$. Then $K_{n-1} \subseteq K_n$ by (71) and hence, by the subordination principle [20, Corollary 4.3.9] and monotonicity of measure,

$$\mu_{K_{n-1}}(E_a \backslash [-a_{n-1}, a_{n-1}]) \ge \mu_{K_n}(E_a \backslash [-a_{n-1}, a_{n-1}]) \ge \mu_{K_n}(E_a \backslash [-a_n, a_n]), \tag{74}$$

contradicting (73). Thus, the sequence a_n must be strictly decreasing and hence it has a limit $a_\infty \ge 0$. Suppose $a_\infty > 0$, then the above argument with a_{n-1}, K_{n-1} replaced by a_∞, $K_\infty = E_a \cup [-a_\infty, a_\infty]$ show that for all n,

$$\mu_{K_\infty}(E_a \backslash [-a_\infty, a_\infty]) \ge \mu_{K_n}(E_a \backslash [-a_\infty, a_\infty]) \ge \mu_{K_n}(E_a \backslash [-a_n, a_n]) = 1 - \frac{1}{n+1}. \tag{75}$$

Taking $n \to \infty$ yields $\mu_{K_\infty}([-a_\infty, a_\infty]) = 0$, a contradiction. Thus, $a_n \to 0$.

For small n, $[-a_n, a_n]$ might have a non trivial overlap with E_a. For convenience, in the following we will write

$$K_n = E_a \cup [-b_n, b_n], \quad b_n = \min\{a_n, a\}, \quad n \in \mathbb{N}, \tag{76}$$

where the union is disjoint except possibly at the end points $\pm b_n$. We note that $b_n \le b_{n-1}$ for all $n \ge 2$ since the sequence a_n is monotone decreasing. Next, we show that

$$(n+1)\mu_{K_n}\big([-b_n, b_n]\big) \ge n\mu_{K_{n-1}}\big([-b_{n-1}, b_{n-1}]\big), \quad n \ge 2. \tag{77}$$

Indeed, if $b_n = a$ then $b_{n-1} = a$ so $K_{n-1} = K_n = [-1, 1]$ hence (77) trivially holds in this case. If $b_n < a$ then $b_n = a_n$ while $b_{n-1} \le a_{n-1}$ hence, by (72), $(n+1)\mu_{K_n}\big([-b_n, b_n]\big) = 1$ while $n\mu_{K_{n-1}}\big([-b_{n-1}, b_{n-1}]\big) \le n\mu_{K_{n-1}}\big([-a_{n-1}, a_{n-1}]\big) = 1$. Thus, (77) holds in all cases.

By (70) and (59) we have

$$\log \mathcal{W}_n(\Gamma_\alpha) = \log \operatorname{cap}(\Gamma_\alpha) + \int_{K_n \setminus E_a} g_{E_a}(x)\,(n+1)\, d\mu_{K_n}(x). \tag{78}$$

Since $a_n \to 0$, for large n we have $b_n = a_n$ by (76) and so $(n+1)\mu_{K_n}(K_n \setminus E_a) = 1$ by (72). Then since g_{E_a} is continuous at $x = 0$, it follows from (78), (33), and (23) that

$$\log \mathcal{W}_n(\Gamma_\alpha) \to \log \operatorname{cap}(\Gamma_\alpha) + g_{E_a}(0) = \log(1+a) \tag{79}$$

and exponentiation yields (39).

It remains to show that the integral in (78) is strictly monotone increasing with respect to n. First, note that, by (21), $g_{E_a}(x)$ is even, strictly monotone increasing on $[-a, 0]$, strictly monotone decreasing on $[0, a]$ and hence, by (76), $g_{E_a}(x) \ge g_{E_a}(b_n)$ for $x \in K_n \setminus E_a$ and, since $b_n \le b_{n-1}$, $g_{E_a}(x) \le g_{E_a}(b_n)$ for $x \in K_{n-1} \setminus K_n$ with both inequalities strict except at the points $x = \pm b_n$. Next, introduce the measure $\nu_n = (n+1)\mu_{K_n} - n\mu_{K_{n-1}}$ and note that it is positive on K_n by the subordination principle [20, Corollary 4.3.9] since $K_n \subseteq K_{n-1}$. Then

$$\begin{aligned}
&\int_{K_n \setminus E_a} g_{E_a}(x)\,(n+1)\, d\mu_{K_n}(x) - \int_{K_{n-1} \setminus E_a} g_{E_a}(x)\, n\, d\mu_{K_{n-1}}(x) \\
&\quad = \int_{K_n \setminus E_a} g_{E_a}(x)\, d\nu_n(x) - \int_{K_{n-1} \setminus K_n} g_{E_a}(x)\, n\, d\mu_{K_{n-1}}(x) \\
&\quad > g_{E_a}(b_n)\, \nu_n(K_n \setminus E_a) - g_{E_a}(b_n)\, n\, \mu_{K_{n-1}}(K_{n-1} \setminus K_n) \\
&\quad = g_{E_a}(b_n) \big[(n+1)\, \mu_{K_n}(K_n \setminus E_a) - n\, \mu_{K_{n-1}}(K_{n-1} \setminus E_a)\big] \ge 0, \quad n \ge 2,
\end{aligned}$$

where the last inequality follows from (77). □

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