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On the iterates of positive linear operators

Ioan Gavrea, Mircea Ivan*

Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania

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Abstract

We have devised a new method for the study of the asymptotic behavior of the iterates of positive linear operators. This technique enlarges the class of operators for which the limit of the iterates can be computed.

Keywords: Positive linear operators; Iterates

1. Introduction

The remarkable results of Kelisky and Rivlin [12] and Karlin and Ziegler [11] provided a new insight into the study of the asymptotic behavior of the iterates of linear operators on $C[0, 1]$. Several researchers provided useful contributions to this problem, see [12,11,3,5,13,14,17,7,9,8, 19,2,4,1,6] and the references therein.

However, all attempts to calculate the iterate limit for many classical operators failed. For the first time, a solution to the general problem of the asymptotic behavior of the iterates of positive linear operators defined on $C[0, 1]$ was announced by the authors of this paper at the APPCOM08 conference held in Niš, Serbia, on August 2008.

Related results appeared one year later in [15]. See also [16], appeared after the submission of our manuscript.

In the referenced papers, the authors used several techniques from the following areas: spectral theory, probability theory, fixed point theory and the theory of semigroups of operators.

* Corresponding author.
E-mail addresses: Ioan.Gavrea@math.utcluj.ro (I. Gavrea), Mircea.Ivan@math.utcluj.ro (M. Ivan).

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This paper gives a new method for studying the asymptotic behavior of the iterates of positive linear operators defined on the space of bounded real-valued functions on [0, 1] and preserving linear functions.

2. The main result

For the sake of simplicity, we restrict ourselves to the interval [0, 1] and mention that our results are valid for any interval of the form $[a, b]$. Let e_0, e_1, e_2 be the monomial functions, $e_i(x) = x^i$, $x \in [0, 1]$, and $B_\circ[0, 1]$ be the Banach space of bounded real-valued functions on [0, 1] continuous at 0 and 1, endowed with the norm $\|f\| = \max_{x \in [0,1]} |f(x)|$.

Let $L: \mathbb{R}^{[0,1]} \to C[0, 1]$ be the Lagrange interpolation $Lf = f(0)e_0 + (f(1) - f(0))e_1$.

If not specified otherwise, X and Y will denote linear subspaces such that

$$\{e_0, e_1, e_2\} \subset Y \subseteq X \subseteq B_\circ[0, 1].$$

The following theorem is the main result of the paper.

Theorem 1. *Let $U: X \to Y$ be a positive linear operator preserving linear functions. If there exists a function $\varphi \in X$ such that $U\varphi - \varphi$ is continuous and has no roots in* (0, 1)*, then $\lim_{k\to\infty} U^k f = Lf$, for all $f \in X$. If in addition $U^k\varphi$, $k \in \mathbb{N}$, are continuous, then the limiting process is uniform.*

Proof. Let $f \in X$ and let $\varepsilon > 0$. Since the function $Lf - f$ is continuous at 0 and 1, there exists $\delta \in (0, \frac{1}{2})$, such that $|Lf - f| < \varepsilon$, on $[0, \delta) \cup (1 - \delta, 1]$.

Let us consider, for definiteness, that $U\varphi(x) > \varphi(x)$ on $(0, 1)$. Otherwise, we take $-\varphi$ instead of φ.

Let $m_\delta := \inf_{x \in [\delta, 1-\delta]}(U\varphi(x) - \varphi(x)) > 0$. The following inequality is satisfied.

$$|Lf - f| < \varepsilon + \frac{\|Lf - f\|}{m_\delta}(U\varphi - \varphi). \tag{1}$$

Since $UL = L$, by applying the positive linear operator U^k to (1), we get

$$|Lf - U^k f| < \varepsilon + \frac{\|Lf - f\|}{m_\delta}(U^{k+1}\varphi - U^k\varphi). \tag{2}$$

Since $U\varphi \geq \varphi$, we obtain that $\varphi \leq U^k\varphi \leq U^{k+1}\varphi \leq \|\varphi\|$, $k = 1, 2, \ldots$. The sequence $(U^k\varphi)_{k\geq 1}$ is monotone and bounded. It follows that it is pointwise convergent. Since ε was arbitrarily taken, it follows, based on (2), that $U^k f \overset{pointwise}{\longrightarrow} Lf$. In particular, $U^k\varphi \overset{pointwise}{\longrightarrow} L\varphi \in C[0, 1]$. If, in addition, $U^k\varphi$ are continuous, by Dini's Theorem, we obtain that $U^k\varphi \overset{uniformly}{\longrightarrow} L\varphi$. We have, based on the inequality

$$\|Lf - U^k f\| \leq \varepsilon + \frac{\|Lf - f\|}{m_\delta}\|U^{k+1}\varphi - U^k\varphi\|,$$

that $U^k f \overset{uniformly}{\longrightarrow} Lf$. $\square$

Remark 1. It is worth mentioning that condition $X \subseteq B_\circ[0, 1]$ is essential, in the sense that, if we try to enlarge the class $B_\circ[0, 1]$, then the statement of Theorem 1 might not be true. To prove this, let $B_\star[0, 1]$ be the subspace of all bounded functions $f: [0, 1] \to \mathbb{R}$ possessing the one-sided limits $f(0_+)$ and $f(1_-)$ and $L_\star: B_\star[0,1] \to C[0,1]$ be the Lagrange type operator $L_\star f = f(0_+)e_0 + (f(1_-) - f(0_+))e_1$.

The operator $L_\star$ is positive, linear and preserves linear functions.

Moreover $L_\star e_2 - e_2 = e_1 - e_2 > 0$ on $(0,1)$. However $L_\star^k = L_\star \not\to L$.

In what follows, let us suppose that $\varphi \in X$ is such that e_0, e_1 and φ form a Chebyshev system. In particular, $\varphi = e_2$.

Corollary 2. *Let $U: X \to C[0,1]$ be a positive linear operator preserving linear functions. Then the following statements are pairwise equivalent.*

(i) *$U^k f \to Lf$, uniformly for all $f \in X$;*
(ii) *$U\varphi(x) \neq \varphi(x)$, for all $x \in (0,1)$;*
(iii) *there is no $\alpha \in (0,1)$ such that $Uf(\alpha) = f(\alpha)$ for any continuous $f \in X$ (i.e., the operator U has no interior interpolation points).*

Proof. The implication (ii)⇒(i) follows from Theorem 1.

To prove (iii)⇒(ii), let us suppose that there exists $\alpha \in (0,1)$ such that $U\varphi(\alpha) = \varphi(\alpha)$ and let $f \in X$ be continuous.

The functional $F(f) = Uf(\alpha)$ satisfies the conditions: $F(e_0) = e_0(\alpha)$, $F(e_1) = e_1(\alpha)$ and $F(\varphi) = \varphi(\alpha)$. By Šaškin's Theorem [18], we deduce that $F(f) = f(\alpha)$ for all continuous $f \in X$.

The implication (i)⇒(iii) is trivial. □

Remark 2. The preservation of the linear functions is also an essential condition. Indeed, the operator $U: C[0,1] \to C[0,1]$ defined by $Uf(x) = Lf(1-x)$, satisfies condition (iii) from Corollary 2, but it does not preserve the linear functions. The sequence $(U^k e_1)_{k\geq 1}$ has no limit.

Let us now impose supplementary conditions upon the spaces X and Y.

Suppose that Y contains differentiable functions only, and that $X \subseteq C[0,1]$ contains all functions $|\cdot - \alpha|$, $\alpha \in (0,1)$. In particular, $Y = C^1[0,1]$ and $X = C[0,1]$, or X be the space of linear spline functions.

Under these conditions, we obtain the following result.

Corollary 3. *If $U : X \to Y$ is a positive linear operator preserving linear functions then $\lim_{k\to\infty} U^k f = Lf$ uniformly for all $f \in X$.*

Proof. Let us suppose that there exists $\alpha \in (0,1)$ such that $Uf(\alpha) = f(\alpha)$, for all $f \in C[0,1]$. The function $t \mapsto |t-\alpha|$ is convex on $[0,1]$. By Jessen's inequality for functionals [10], we get that $U(|t-\alpha|)(x) \geq |x-\alpha|$, for all $x \in [0,1]$. Moreover, we have $U(|t-\alpha|)(\alpha) = 0$. It follows that $\limsup_{x\to\alpha_-} \frac{U(|t-\alpha|)(x)}{x-\alpha} \leq -1$ and $\liminf_{x\to\alpha_+} \frac{U(|t-\alpha|)(x)}{x-\alpha} \geq 1$, and hence $U(|t-\alpha|)$ is not differentiable at α, which contradicts the hypothesis. By using condition (iii) of Corollary 2, the proof is concluded. □

We conclude by mentioning that Corollary 2 is applicable to all classical positive operators preserving linear functions.

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