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Characterization of greedy bases in Banach spaces

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Abstract

We shall present a new characterization of greedy bases and 1-greedy bases in terms of certain functionals defined using distances to one dimensional subspaces generated by the basis. We also introduce a new property that unifies the notions of unconditionality and democracy and allows us to recover a better dependence on the constants.

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1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a semi-normalized Schauder basis of $\mathbb{X}$ with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$, i.e., $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty$ and for each $x \in \mathbb{X}$ there exists a unique expansion $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$. We denote by $c_{00}(\mathbb{N})$ (resp. $c_0(\mathbb{N})$) the space of all sequences of real numbers with a finite number of non-zero terms (resp. converging to zero). As usual $\mathrm{supp}(x) = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}$, $|A|$ stands for the cardinal of A, $P_A(x) = \sum_{n\in A} e_n^*(x)e_n$ and $1_A = \sum_{n\in A} e_n$. Throughout the paper, we write

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$\tilde{x} = (e_n^*(x))_{n\in\mathbb{N}} \in c_0(\mathbb{N})$, $\|\tilde{x}\|_\infty = \sup_n |e_n^*(x)|$ and $xy = 0$ whenever $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$. We use the notation $\mathbb{X}_c$ for the subspace of $\mathbb{X}$ of elements with finite support, i.e. $x \in \mathbb{X}$ and $|\mathrm{supp}(x)| < \infty$ or $\tilde{x} \in c_{00}(\mathbb{N})$. Also for each $m \in \mathbb{N}$, $|A| = m$ and $(\varepsilon_n)_{n\in A} \in \{\pm 1\}$, we denote by $1_{\varepsilon A} = \sum_{n\in A} \varepsilon_n e_n$, by $[1_{\varepsilon A}]$ the one-dimensional subspace generated by $1_{\varepsilon A}$ and by $[e_n, n \in A]$ the m-dimensional subspace generated by $\{e_n, n \in A\}$.

Recall that a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is called *unconditional* if any rearrangement of the series $\sum_{n=1}^\infty e_n^*(x)e_n$ converges in norm to x for any $x \in \mathbb{X}$. This turns out to be equivalent the fact that the projections P_A are uniformly bounded on all sets A, i.e. there exists a constant $C > 0$ such that

$$\|P_A(x)\| \leq C\|x\|, \quad x \in \mathbb{X} \text{ and } A \subset \mathbb{N}. \tag{1}$$

In such a case we say that $\mathcal{B}$ is a C-suppression unconditional basis. The smallest constant that satisfies (1) is the so-called *suppression constant* and it is denoted by K_{su}. Moreover, we have that

$$K_{su} = \sup\{\|P_A\| : A \subseteq \mathbb{N} \text{ is finite}\} = \sup\{\|P_A\| : A \subseteq \mathbb{N} \text{ is cofinite}\}.$$

In particular, for unconditional bases one has that $x = \sum_{n=1}^\infty e_{\pi(n)}^*(x)e_{\pi(n)}$ where $\pi : \mathbb{N} \to \mathbb{N}$ is any permutation so that $|e_{\pi(n)}^*(x)| \geq |e_{\pi(n+1)}^*(x)|$ for all $n \in \mathbb{N}$.

For each $x \in \mathbb{X}$ and $m \in \mathbb{N}$, S.V. Konyagin and V.N. Temlyakov defined in [5] a *greedy sum* of x of order m by

$$G_m(x) = \sum_{n=1}^m e_{\pi(n)}^*(x)e_{\pi(n)},$$

where π is a *greedy ordering*, that is $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ is a permutation such that $\mathrm{supp}(x) = \{j : e_j^*(x) \neq 0\} \subseteq \pi(\mathbb{N})$ and $|e_{\pi(i)}^*(x)| \geq |e_{\pi(j)}^*(x)|$ for $i \leq j$. Any sequence $(G_m(x))_{m=1}^\infty$ is called a *greedy approximation* of x. Of course we can have several greedy sums of the same order whenever the sequence $(e_j^*(x))_{j=1}^\infty$ contains several terms with the same absolute value.

Given $x = \sum_{i=1}^\infty e_i^*(x)e_i \in \mathbb{X}$, we define the *natural greedy ordering* for x as the map $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\mathrm{supp}(x) \subset \rho(\mathbb{N})$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. The mth *greedy sum* of x is

$$\mathcal{G}_m[\mathbb{X}, \mathcal{B}](x) := \mathcal{G}_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x)e_{\rho(j)},$$

and the sequence of maps $\{\mathcal{G}_m\}_{m=1}^\infty$ is known as the *greedy algorithm* associated to $\mathcal{B}$ in $\mathbb{X}$. With this notation out of the way we have that

$$\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\| = 0, \tag{2}$$

for any $x \in \mathbb{X}$ whenever $\mathcal{B}$ is unconditional.

Konyagin and Temlyakov (see [5]) also introduced the term of *quasi-greedy basis* for the basis satisfying the existence of a universal constant $C > 0$ such that

$$\|\mathcal{G}_m(x)\| \leq C\|x\|, \quad x \in \mathbb{X}, \ m \in \mathbb{N}. \tag{3}$$

In such a case the basis is called a C-quasi-greedy basis.

A bit later Wojtaszczyk (see [7]) proved that condition (3) is actually equivalent to (2). Of course, (3) is equivalent to the existence of a universal constant $C' > 0$ such that

$$\|x - \mathcal{G}_m(x)\| \le C'\|x\|, \quad x \in \mathbb{X},\ m \in \mathbb{N}. \tag{4}$$

Since $\mathcal{G}_m(x) = P_\Lambda(x)$ for given Λ with $|\Lambda| = m$, one has that any C-suppression unconditional basis is also a C-quasi-greedy basis.

Recently, Albiac and Ansorena [1, Theorem 2.1] showed that $\mathcal{B}$ is 1-suppression unconditional if and only if $\sup_{m\in\mathbb{N}} \|\mathcal{G}_m(x)\| \le \|x\|$ and if and only if $\sup_{m,k\in\mathbb{N}}\{\|\mathcal{G}_m(x)\|, \|x-\mathcal{G}_k(x)\|\} \le \|x\|$.

For each $m \in \mathbb{N}$, the *m-term error of approximation* with respect to $\mathcal{B}$ is defined as

$$\sigma_m(x, \mathcal{B}) = \sigma_m(x) := \inf\{d(x, [e_n, n \in A]) :\ A \subset \mathbb{N}, |A| = m\},$$

where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$ for each $Y \subset \mathbb{X}$. Clearly $\sigma_m(x) \le \|x - \mathcal{G}_m(x)\|$. Bases where the greedy algorithm is efficient in the sense that the error we make when approximating x by $\mathcal{G}_m(x)$ is comparable with $\sigma_m(x)$ were first considered in [5] and called greedy bases. Namely a basis $\mathcal{B}$ is said to be *greedy* if there exists an absolute constant $C \ge 1$ such that

$$\|x - \mathcal{G}_m(x)\| \le C\sigma_m(x), \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N}. \tag{5}$$

In this case, we will say that $\mathcal{B}$ is C-greedy. The smallest constant C that satisfies (5) is the *greedy constant* and is denoted by C_g.

In the same paper, a basis $\mathcal{B}$ was said to be *democratic* if there is a constant $D \ge 1$ such that

$$\|1_A\| \le D\,\|1_B\| \tag{6}$$

for all $A, B \subset \mathbb{N}$ finite and the same cardinality. The smallest constant appearing in (6) is called the *democracy constant* and $\mathcal{B}$ is said to be a D-democratic basis.

Theorem KT (*[5,6]*).

(i) *If $\mathcal{B}$ is a C-greedy basis then $\mathcal{B}$ is C-democratic and C-suppression unconditional.*
(ii) *If $\mathcal{B}$ is K_{su}-suppression unconditional and D-democratic then $\mathcal{B}$ is $(K_{su} + K_{su}^3 D)$-greedy.*

Notice from the dependence on the constants that 1-suppression unconditional and 1-democratic only gives 2-greedy. To characterize 1-greedy bases, Albiac and Wojtaszczyk (see [3]) introduced the so-called Property (A). For each $|S| < \infty$ and $x = \sum_{n\in S} e_n^*(x)e_n \in \mathbb{X}$, we write $M(x) := \{n \in S : |e_n^*(x)| = \max_m |e_m^*(x)|\}$. A basis is said *to have Property (A)* whenever

$$\|x\| = \left\| \sum_{n\in M(x)} \theta_n e_n^*(x) e_{\pi(n)} + (x - P_{M(x)}x) \right\|, \tag{7}$$

for all injective maps $\pi : S \to \mathbb{N}$ such that $\pi(j) = j$ if $j \notin M(x)$ and $\theta_n \in \{\pm 1\}$ with $\theta_n = 1$ whenever $\pi(n) = n$ for $n \in M(x)$.

Theorem AW (*[3, Theorem 3.4]*). *Let $\mathbb{X}$ be a Banach space and $\mathcal{B}$ a Schauder basis. Then $\mathcal{B}$ is a 1-greedy basis if and only if $\mathcal{B}$ is 1-suppression unconditional and it has Property* (A).

It has been recently shown by Albiac and Ansorena (see [2, Theorem 3.1]) that the bases with Property (A) coincide with the almost-greedy bases with $C_{ag} = 1$, that is to say

$$\|x - \mathcal{G}_m(x)\| \le \inf_{|A|=m} \|x - P_A(x)\|, \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N}. \tag{8}$$

Later on, Theorems AW and KT were generalized in [4] using the so-called Property (A) with constant C (which has been also called *C-symmetric for largest coefficients* in [2]) where the equality (7) is replaced for an inequality

$$\|x\| \leq C \left\| \sum_{n \in M(x)} \theta_n e_n^*(x) e_{\pi(n)} + (x - P_{M(x)} x) \right\|. \tag{9}$$

Theorem D (*[4, Theorem 2]*)**.**

(i) *If $\mathcal{B}$ is a C-greedy basis then $\mathcal{B}$ is C-suppression unconditional and it has Property* (A) *with constant C.*
(ii) *If $\mathcal{B}$ is K_{su}-suppression unconditional and Property* (A) *with constant C then $\mathcal{B}$ is $K_{su}^2 C$-greedy.*

Let us first reformulate Property (A) in terms useful for our purposes (see [4]).

Lemma 1.1. *Let $\mathcal{B}$ be a Schauder basis of $\mathbb{X}$. The basis $\mathcal{B}$ has the Property* (A) *with constant C if and only if*

$$\|x + 1_{\varepsilon A}\| \leq C \|x + 1_{\varepsilon' B}\|$$

for any $\varepsilon, \varepsilon' \in \{\pm 1\}$, $|A| = |B| < \infty$, $A \cap B = \emptyset$, $x \in \mathbb{X}_c$ with $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$ *and* $\|\tilde{x}\|_\infty \leq 1$.

Proof. Assume $\mathcal{B}$ has Property (A) with constant C. For each $\varepsilon, \varepsilon' \in \{\pm 1\}$, A, B and x such that $|A| = |B|$, $A \cap B = \emptyset$ and $\|\tilde{x}\|_\infty \leq 1$ with $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$, we write $y = 1_{\varepsilon A} + x$. Hence $M(y) = A \cup \{n \in \operatorname{supp}(x) : |e_n^*(x)| = 1\}$. Let $\pi : A \to B$ be a bijection and set $\theta_n = \varepsilon'_{\pi(n)}$ for $n \in A$. Hence $\|y\| \leq C \|1_{\varepsilon' B} + x\|$.

Conversely given $x \in \mathbb{X}_c$ with $\operatorname{supp}(x) = S$ and $\alpha = \max\{|e_n^*(x)| : n \in S\}$ one can consider, for each π and θ in the conditions above, the set $A = \{j \in M(x) : \pi(j) \neq j\}$ and define $\varepsilon_n = \frac{e_n^*(x)}{|e_n^*(x)|}$ for each $n \in A$. Now, selecting $B = \pi(A)$ and $\varepsilon'_n = \theta_n$ for $n \in B$, we have

$$\begin{aligned} \|x\| &= \alpha \left\| 1_{\varepsilon A} + \frac{1}{\alpha}(x - P_A x) \right\| \\ &\leq C\alpha \left\| 1_{\varepsilon' B} + \frac{1}{\alpha}(x - P_A x) \right\| \\ &= C \left\| \sum_{n \in A} \theta_n e_n^*(x) e_{\pi(n)} + (x - P_A x) \right\|. \quad \square \end{aligned}$$

We would like to introduce here two properties which encode the notions of unconditionality and democracy or unconditionality and Property (A) at once.

Definition 1.2. A Schauder basis $\mathcal{B}$ is said to have *Property (Q)* with constant C whenever

$$\|x + 1_A\| \leq C \|x + y + 1_B\| \tag{10}$$

for any $|A| = |B| < \infty$, $A \cap B = \emptyset$ and $x, y \in \mathbb{X}_c$ such that $xy = 0$, $\|\tilde{x}\|_\infty \leq 1$ and $\operatorname{supp}(x + y) \cap (A \cup B) = \emptyset$.

Remark 1.3. Clearly *Property (Q)* with constant C on a basis $\mathcal{B}$ implies that $\mathcal{B}$ is C-democratic and C-suppression unconditional. Conversely if $\mathcal{B}$ is D-democratic and C-suppression unconditional then $\mathcal{B}$ has Property (Q) with constant $C(1+D)$.

Definition 1.4. Let $z \in \mathbb{X}_c$. We write $\Gamma_0 = \mathbb{X}_c$ and for $z \neq 0$ we define

$$\Gamma_z = \{y \in \mathbb{X}_c : zy = 0, |\text{supp}(z)| \leq |\{n : |e_n^*(y)| = 1\}|\}. \tag{11}$$

Definition 1.5. A Schauder basis $\mathcal{B}$ is said to have *Property* (Q^*) with constant C whenever

$$\|x+z\| \leq C\|x+y\| \tag{12}$$

for any $x, z, y \in \mathbb{X}_c$ such that $xz = 0$, $xy = 0$, $\max\{\|\tilde{x}\|_\infty, \|\tilde{z}\|_\infty\} \leq 1$ and $y \in \Gamma_z$.

Remark 1.6. It is clear that Property (Q^*) implies the *Property (Q)* and Property (A) with the same constant.

Conversely if $\mathcal{B}$ is K-suppression unconditional and it has Property (A) with constant C then $\mathcal{B}$ has Property (Q^*) with constant KC.

Indeed, let $x, z, y \in \mathbb{X}_c$ such that $xz = 0$, $xy = 0$, $\max\{\|\tilde{x}\|_\infty, \|\tilde{z}\|_\infty\} \leq 1$ and $y \in \Gamma_z$. If $z = 0$ we have $\|x\| \leq K\|x+y\|$ using that the basis is K-suppression unconditional. Assume now that $z \neq 0$ with $A = \text{supp}(z)$. Select $B \subseteq \{n : |e_n^*(y)| = 1\}$ with $|B| = |A| < \infty$ and $\varepsilon'_n = \frac{e_n^*(y)}{e_n^*(y)}$ for $n \in B$. Therefore

$$\|x + 1_{\varepsilon A}\| \leq C\|x + 1_{\varepsilon' B}\| \leq CK\|x+y\|.$$

Notice that $\|\tilde{z}\|_\infty \leq 1$ implies that $z \in co(\{1_{\varepsilon A} : |\varepsilon_n| = 1\})$. Hence $x + z = \sum_{j=1}^m \lambda_j (x + 1_{\varepsilon^{(j)} A})$ for some $|\varepsilon_n^{(j)}| = 1$ and $0 \leq \lambda_j \leq 1$ with $\sum_{j=1}^m \lambda_j = 1$ and we obtain $\|x+z\| \leq CK\|x+y\|$. $\square$

In this paper we also introduce two functionals depending only on distances to one dimensional subspaces which allow us to characterize the greedy bases and 1-greedy bases.

Definition 1.7. Let $\mathcal{B}$ be a basis in a Banach space $\mathbb{X}$, $x \in \mathbb{X}$ and $m \in \mathbb{N}$. We define

$$\mathcal{D}_m(x, \mathcal{B}) = \mathcal{D}_m(x) := \inf\{d(x, [1_A]) : \ A \subset \mathbb{N}, |A| = m\},$$

and

$$\mathcal{D}_m^*(x, \mathcal{B}) = \mathcal{D}_m^*(x) := \inf\{d(x, [1_{\varepsilon A}]) : \ (\varepsilon_n) \in \{\pm 1\}, A \subset \mathbb{N}, |A| = m\}.$$

In particular

$$\mathcal{D}_m^*(x, \mathcal{B}) = \mathcal{D}_m^*(x) := \inf\{\|x - \alpha(1_{A_1} - 1_{A_2})\| : \ |A_1 \cup A_2| = m, A_1 \cap A_2 = \emptyset, \alpha \in \mathbb{R}\}.$$

Of course $\forall x \in \mathbb{X}$ one has

$$\sigma_m(x) \leq \mathcal{D}_m^*(x) \leq \mathcal{D}_m(x) \leq \|x\|.$$

Our aim is to show that greedy bases can be actually defined using the functionals $\mathcal{D}_m^*$ or $\mathcal{D}_m$ instead of σ_m and the use of the Property (Q^*) allows us to improve the dependence of the constants.

Our main results establish firstly that Property (Q) and Property (Q^*) are actually equivalent (see Theorem 2.4), secondly that the conditions

$$\|x - \mathcal{G}_m(x)\| \le C\mathcal{D}_m(x), \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N},$$
$$(\text{resp. } \|x - \mathcal{G}_m(x)\| \le C\mathcal{D}^*_m(x),\ \forall x \in \mathbb{X},\ \forall m \in \mathbb{N})$$

imply Property (Q) (resp. Property (Q^*)) with constant C (see Proposition 2.1) and finally that bases having Property (Q^*) with constant C are C^2-greedy bases (see Theorem 2.6). Combining the results above one gets the following chain of equivalent formulations of greedy bases.

Corollary 1.8. *Let $\mathbb{X}$ be a Banach space and $\mathcal{B}$ a Schauder basis of $\mathbb{X}$. The following statements are equivalent:*

(i) *$\mathcal{B}$ is greedy.*
(ii) *There exists an absolute constant $C > 0$ such that*

$$\|x - \mathcal{G}_m(x)\| \le C\mathcal{D}^*_m(x), \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N}.$$

(iii) *There exists an absolute constant $C > 0$ such that*

$$\|x - \mathcal{G}_m(x)\| \le C\mathcal{D}_m(x), \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N}.$$

(iv) *$\mathcal{B}$ satisfies the Q-property.*
(v) *$\mathcal{B}$ satisfies the Q^*-property.*
(vi) *$\mathcal{B}$ is unconditional and democratic.*

Corollary 1.9. *Let $\mathcal{B}$ be a Schauder basis of $\mathbb{X}$. Then $\mathcal{B}$ is 1-greedy if only if $\mathcal{B}$ satisfies the Q^*-property with constant 1 if and only if $\mathcal{B}$ is 1-unconditional and it has Property (A) with constant $C = 1$.*

Our proofs will follow closely the ideas in [2–5].

2. Bases with property (Q) and (Q^*)

Proposition 2.1. *Let $\mathbb{X}$ be a Banach space and $\mathcal{B}$ a Schauder basis of $\mathbb{X}$. The following statements are equivalent:*

(i) *There exists $C > 0$ such that*

$$\|x - \mathcal{G}_m(x)\| \le C\mathcal{D}_m(x), \quad \forall x \in \mathbb{X},\ \forall m \in \mathbb{N}.$$

(ii) *$\mathcal{B}$ has Property (Q).*
(iii) *$\mathcal{B}$ is a greedy basis.*

Proof. Due to Remark 1.3 and Theorem KV only the implication (i) $\Rightarrow$ (ii) requires a proof.

Assume (i). We shall see first that the basis is democratic. Let A, B with $|A| = |B| = n$ and $m = |A \setminus B| = |B \setminus A|$. Define, for each $\varepsilon > 0$, $x = (1+\varepsilon)1_{A\setminus B} + 1_B$ and observe that $\mathcal{G}_m(x) = (1+\varepsilon)1_{A\setminus B}$. Hence,

$$\|1_B\| = \|x - \mathcal{G}_m(x)\| \le C\mathcal{D}_m(x) \le C\|x - 1_{B\setminus A}\| \le C\|1_A\| + C\varepsilon\|1_{A\setminus B}\|.$$

Now take the limit as $\varepsilon \to 0$ to complete the argument.

Let us now prove the unconditionality of $\mathcal{B}$. Let $x \in \mathbb{X}_c$ and $\mathrm{supp}(x) = B$. Let $A \subseteq B$ and write $m = |B \setminus A|$. Select $\alpha > 0$ such that

$$\alpha > \sup_{j\in A} |e^*_j(x)| + \sup_{j\in B\setminus A} |e^*_j(x)|,$$

and define

$$y = x + \alpha 1_{B\setminus A} = \sum_{j\in B\setminus A} (\alpha + e_j^*(x))e_j + \sum_{j\in A} e_j^*(x)e_j.$$

Hence $\mathcal{G}_m(y) = \sum_{j\in B\setminus A}(\alpha + e_j^*(x))e_j$ and $P_A(x) = y - \mathcal{G}_m(y)$. Then,

$$\|P_A(x)\| = \|y - \mathcal{G}_m(y)\| \le C\mathcal{D}_m(y) \le C\|y - \alpha 1_{B\setminus A}\| = C\|x\|. \quad \square$$

Proposition 2.2. *Let $\mathcal{B}$ be a basis of $\mathbb{X}$. The following statements are equivalent:*

(i) *There exists $C > 0$ such that*

$$\|x + 1_{\varepsilon A}\| \le C\|x + 1_{\varepsilon' B} + y\| \tag{13}$$

for any A, B such that $A \cap B = \emptyset$ and $|A| = |B| < \infty$, any $(\varepsilon_n)_{n\in A}, (\varepsilon'_n)_{n\in B} \in \{\pm 1\}$ and any $x, y \in \mathbb{X}_c$ such that $xy = 0$, $\|\tilde{x}\|_\infty \le 1$ and $(A \cup B) \cap (\mathrm{supp}(x + y)) = \emptyset$.

(ii) *$\mathcal{B}$ has Property (Q^*) with constant C.*

(iii) *There exists $C > 0$ such that*

$$\|x\| \le C\|x - P_A(x) + ty\| \tag{14}$$

for any $x \in \mathbb{X}_c$, $t \ge \|\tilde{x}\|_\infty$, finite set A and $y \in \Gamma_{P_A(x)}$ with $xy = 0$.

Proof. (i) $\Rightarrow$ (ii) Let $x, y, z \in \mathbb{X}_c$ with pairwise disjoint supports with $\max\{\|\tilde{x}\|_\infty, \|\tilde{z}\|_\infty\} \le 1$ and $y \in \Gamma_z$.

For $z = 0$ we apply (13) with $A = B = \emptyset$ to obtain $\|x\| \le C\|x + y\|$.

For $z \neq 0$, denote $A = \mathrm{supp}(z)$ and $B_1 = \{n \in \mathrm{supp}(y) : |e_n^*(y)| = 1\}$. Since $|B_1| \ge |A|$ we select $B \subseteq B_1$ with $|B| = |A|$ and write $y = P_B(y) + P_{B^c}(y) = 1_{\varepsilon' B} + P_{B^c}(y)$ where $\varepsilon'_n = \frac{e_n^*(y)}{|e_n^*(y)|}$ for $n \in B$. From (13) we have

$$\|x + 1_{\varepsilon A}\| \le C\|x + 1_{\varepsilon' B} + P_{B^c}(y)\| = C\|x + y\|, \quad \forall(\varepsilon_n) \in \pm 1.$$

Notice that $\|\tilde{z}\|_\infty \le 1$ implies that $z \in co(\{1_{\varepsilon A} : |\varepsilon_n| = 1\})$. Hence $x + z = \sum_{j=1}^m \lambda_j (x + 1_{\varepsilon^{(j)} A})$ for some $|\varepsilon_n^{(j)}| = 1$ and $0 \le \lambda_j \le 1$ with $\sum_{j=1}^m \lambda_j = 1$ and we obtain $\|x + z\| \le C\|x + y\|$.

(ii) $\Rightarrow$ (iii) Let $x, y \in \mathbb{X}_c$ with $xy = 0$, $t \ge \|\tilde{x}\|_\infty$ and a finite set A with $y \in \Gamma_{P_A(x)}$.

In the case $A \cap \mathrm{supp}(x) = \emptyset$ we have $P_A x = 0$ and from (12) one gets $\|\frac{x}{t}\| \le C\|\frac{x}{t} + u\|$ for any $u \in \mathbb{X}_c$ with $xu = 0$.

In the case $A \cap \mathrm{supp}(x) \neq \emptyset$, let $x_1 = \frac{x}{t} - P_A(\frac{x}{t})$, $z_1 = P_A(\frac{x}{t})$ and $y_1 = y$. Since $\max\{\|\tilde{x}_1\|_\infty, \|\tilde{z}_1\|_\infty\} \le 1$ and $y \in \Gamma_{z_1}$ we can apply (12) to obtain

$$\|x\| = t\|x_1 + z_1\| \le C\|x - P_A(x) + ty\|.$$

(iii) $\Rightarrow$ (i) Let two finite and disjoint sets A and B with $|A| = |B|$, $(\varepsilon_n)_{n\in A}, (\varepsilon'_n)_{n\in B} \in \{\pm 1\}$, $x, y \in \mathbb{X}_c$ such that $\|\tilde{x}\|_\infty \le 1$ with $xy = 0$ and $(A \cup B) \cap (\mathrm{supp}(x) \cup \mathrm{supp}(y)) = \emptyset$. We apply (14) for $t = 1$, the set A and $u, v \in \mathbb{X}_c$ given by $u = x + 1_{\varepsilon A}$ and $v = 1_{\varepsilon' B} + y$, since $\|\tilde{u}\|_\infty \le 1$, $v \in \Gamma_{1_{\varepsilon A}}$ and $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Therefore

$$\|x + 1_{\varepsilon A}\| = \|u\| \le C\|u - P_A(u) + v\| = C\|x + 1_{\varepsilon' B} + y\|.$$

This finishes the proof. $\square$

Lemma 2.3. *Let $\mathcal{B}$ be a Schauder basis of a Banach space $\mathbb{X}$, $x \in \mathbb{X}$ and a finite set A. Then*

$$\sup\{\|x + 1_{\varepsilon A}\| : |\varepsilon_n| = 1\} = \sup\{\|x + u\| : \operatorname{supp}(u) = A, \|\tilde{u}\|_\infty \leq 1\}$$

and

$$\sup_{B \subset A} \|x + 1_B\| \leq \sup\{\|x + 1_{\varepsilon A}\| : |\varepsilon_n| = 1\} \leq 3 \sup_{B \subset A} \|x + 1_B\|.$$

Proof. Denote

$$I_1 = \sup_{B \subset A} \|x + 1_B\|,$$
$$I_2 = \sup\{\|x + 1_{\varepsilon A}\| : |\varepsilon_n| = 1\},$$
$$I_3 = \sup\{\|x + u\| : \operatorname{supp}(u) = A, \|\tilde{u}\|_\infty \leq 1\}.$$

Of course $I_1 \leq I_2$ since each $B \subseteq A$ can be written as $1_B = \frac{1}{2}\big(1_A + (1_B - 1_{A \setminus B})\big)$.

On the other hand $I_2 \leq I_3$ follows trivially selecting $u = 1_{\varepsilon A}$. The other inequality $I_2 \geq I_3$ follows using the same argument as in Proposition 2.2 since any $u \in \mathbb{X}$ with $\|\tilde{u}\|_\infty \leq 1$ and $\operatorname{supp}(u) = A$ satisfies that $u = \sum_{j \in A} e_j^*(y) e_j \in co(\{1_{\varepsilon A} : |\varepsilon_n| = 1\})$.

For the remaining inequality, denote $A^+ := \{j \in A : \varepsilon_j = 1\}$ and $A^- := \{j \in A : \varepsilon_j = -1\}$. Since $1_{\varepsilon A} = 1_{A^+} - 1_{A^-}$, with $A^+, A^- \subset A$, we can write $x + 1_{\varepsilon A} = 2(x + 1_{A^+}) - (1_A + x)$ and therefore $\|x + 1_{\varepsilon A}\| \leq 3 I_3$ and we obtain $I_2 \leq 3 I_3$. □

Theorem 2.4. *Let $\mathbb{X}$ be a Banach space and $\mathcal{B}$ a Schauder basis of $\mathbb{X}$. $\mathcal{B}$ has Property (Q) if and only if $\mathcal{B}$ has Property (Q^*). Moreover, if we have Property (Q) with constant C_1 and Property (Q^*) with constant C_2, then*

$$C_1 \leq C_2 \leq 6 C_1^3.$$

Proof. Of course Property (Q^*) with constant C_2 implies Property (Q) with the same constant. Assume that $\mathcal{B}$ has the Property (Q) with constant C_1. In particular

$$\|P_M(z)\| \leq C_1 \|z\|, \quad z \in \mathbb{X}_c, \; |M| < \infty. \tag{15}$$

Let $|\varepsilon_n| = |\varepsilon'_n| = 1$, $|A| = |B|$, $A \cap B = \emptyset$ and $x, y \in \mathbb{X}_c$ with $xy = 0$, $\|\tilde{x}\|_\infty \leq 1$ and $\operatorname{supp}(x + y) \cap (A \cup B) = \emptyset$. By (15) and Property (Q), for each $A' \subset A$

$$\|x + 1_{A'}\| \leq C_1 \|x + 1_A\| \leq C_1^2 \|x + y + 1_B\|, \quad A' \subset A. \tag{16}$$

Applying Lemma 2.3, together with (15) and (16), we obtain, for $1_{\varepsilon' B} = 1_{B^+} - 1_{B^-}$,

$$\begin{aligned} \|x + 1_{\varepsilon A}\| &\leq 3 \sup_{A' \subset A} \|x + 1'_A\| \leq 3 C_1^2 \|x + y + 1_B\| \\ &\leq 3 C_1^2 (\|x + y + 1_{B^+}\| + \|1_{B^-}\|) \\ &\leq 6 C_1^3 \|x + y + 1_{\varepsilon' B}\|. \end{aligned}$$

This shows (13) and therefore $\mathcal{B}$ has Property (Q^*) invoking Proposition 2.2. □

Let us mention the following result whose proof is borrowed from [3].

Proposition 2.5. *Let $\mathcal{B}$ be a C-suppression unconditional basis of $\mathbb{X}$. Let $x \in \mathbb{X}$, $A \subseteq \operatorname{supp}(x)$ and $\varepsilon_n = \frac{e_n^*(x)}{|e_n^*(x)|}$ for $n \in A$. Then*

$$\left\| \sum_{n \in B} e_n^*(x) e_n + t 1_{\varepsilon A} \right\| \leq C \|x\| \tag{17}$$

for each $B \subset \operatorname{supp}(x) \setminus A$ and $t \leq \min\{|e_n^*(x)| : n \in A\}$.

Proof. Given $B \subset \operatorname{supp}(x) \setminus A$ and $t \leq \min\{|e_n^*(x)| : n \in A\}$ we define

$$f_{t,B}(s) = \sum_{n \in B} e_n^*(x) e_n + \sum_{n \in A} \chi_{\left[0, \frac{t}{|e_n^*(x)|}\right]}(s) e_n^*(x) e_n \in \mathbb{X}_c, \quad 0 \leq s \leq 1.$$

Note that, since $f_{t,B}(s) = P_{A_s} x$, we have that $\|f_{t,B}(s)\| \leq C\|x\|$ and

$$\sum_{n \in B} e_n^*(x) e_n + t 1_{\varepsilon A} = \int_0^1 f_{t,B}(s) ds.$$

Hence, using the vector-valued Minkowski's inequality, (17) is achieved. □

Theorem 2.6. *Let $\mathbb{X}$ be a Banach space and $\mathcal{B}$ a Schauder basis of $\mathbb{X}$.*

(i) *If there exists $C > 0$ such that*

$$\|x - \mathcal{G}_m(x)\| \leq C \mathcal{D}_m^*(x), \quad \forall x \in \mathbb{X}, \ \forall m \in \mathbb{N},$$

then $\mathcal{B}$ has Property (Q^*) with constant C.

(ii) *If $\mathcal{B}$ has Property (Q^*) with constant C then*

$$\|x - \mathcal{G}_m(x)\| \leq C^2 \sigma_m(x), \quad \forall x \in \mathbb{X}, \ \forall m \in \mathbb{N}.$$

Proof. (i) Due to the equivalences in Proposition 2.2 we shall show (13). Let us take $\varepsilon, \varepsilon' \in \{\pm 1\}$, $|A| = |B|$, $A \cap B = \emptyset$ and $x, y \in \mathbb{X}_c$ such that $xy = 0$, $\|\tilde{x}\|_\infty \leq 1$ and $\operatorname{supp}(x + y) \cap (A \cup B) = \emptyset$. Let us write $F = \operatorname{supp}(y)$, $\eta_n = \frac{e_n^*(y)}{|e_n^*(y)|}$ for $n \in F$ and define, for each $\delta > 0$,

$$z = 1_{\varepsilon A} + x + y + 1_{\eta F} + (1 + \delta) 1_{\varepsilon' B}.$$

Using that $|e_n^*(y + 1_{\eta F})| = |\eta_n + e_n^*(y)| = |e_n^*(y)|(1 + \frac{1}{|e_n^*(y)|}) \geq 1$ for each $n \in F$ we have $\mathcal{G}_m(z) = (1 + \delta) 1_{\varepsilon' B} + y + 1_{\eta F}$, where $m = |B| + |F|$. Therefore

$$\begin{aligned} \|1_{\varepsilon A} + x\| &= \|z - \mathcal{G}_m(z)\| \\ &\leq C \mathcal{D}_m^*(z) \leq C \|z - 1_{\varepsilon A} - 1_{\eta F}\| \\ &= C \|x + y + (1 + \delta) 1_{\varepsilon' B}\|. \end{aligned}$$

Now taking the limit as δ goes to 0 one gets (13).

(ii) By density and homogeneity, it suffices to prove the result when x is finitely supported with $\|\tilde{x}\|_\infty \leq 1$. Let $x \in \mathbb{X}_c$, $\|\tilde{x}\|_\infty \leq 1$, $m \in \mathbb{N}$ and let $b \in [e_n : n \in A]$ with $|A| = m$. Select B with $|B| = m$ and $\mathcal{G}_m(x) = P_B(x)$.

Set $t = \min\{|e_n^*(x)| : n \in B \setminus A\}$ and set $\varepsilon_n = \frac{e_n^*(x)}{|e_n^*(x)|}$ for $n \in \operatorname{supp}(x)$.

Taking into account that $t \geq \|\tilde{x} - \widetilde{P_B(x)}\|_\infty$, we take in the formula (14) the vector x as $x - P_B(x)$, the set A as $A \setminus B$ and $y = 1_{\varepsilon(B\setminus A)}$. Now from Proposition 2.2 we obtain

$$\begin{aligned}\|x - \mathcal{G}_m(x)\| &\leq C\|x - P_B(x) - P_{A\setminus B}(x) + t1_{\varepsilon(B\setminus A)}\| \\ &= C\|P_{(A\cup B)^c}(x - b) + t1_{\varepsilon(B\setminus A)}\|.\end{aligned}$$

Finally, since $t \leq |e_n^*(x - b)|$ for $n \in B \setminus A$, applying Proposition 2.5 one gets

$$\|x - \mathcal{G}_m(x)\| \leq C^2\|x - b\|.$$

This gives that $\|x - \mathcal{G}_m(x)\| \leq C^2\sigma_m(x)$ and the proof is complete. $\square$

Remark 2.7. If we assume that $\mathcal{B}$ is a K-suppression unconditional basis satisfying Property (A) with constant C we would have Property (Q^*) with constant KC (see Remark 1.6), since the argument in Theorem 2.6 part (ii) only makes use of Proposition 2.2 with constant KC and Proposition 2.5 with constant K, we would obtain that $\mathcal{B}$ is a K^2C-greedy basis. Hence we recover the result in part (ii) in Theorem D.

3. Some properties of the new functionals $\mathcal{D}_m$ and $\mathcal{D}_m^*$

Of course $\mathcal{D}_1(x) = \mathcal{D}_1^*(x) = \|x - \mathcal{G}_1(x)\| = \|x - e_{\rho(1)}^*(x)e_{\rho(1)}\|$. However calculating the functionals $\mathcal{D}_m(\cdot)$ and $\mathcal{D}_m^*(\cdot)$ for $m \geq 2$ is not easy in general. Let us study the situation for Hilbert spaces and $\mathbb{X} = \ell^p$ with $1 \leq p < \infty$.

For Hilbert spaces and for orthonormal bases one can compute the functionals explicitly using the inner product.

Proposition 3.1. *Let $\mathbb{H}$ be a Hilbert space and $\mathcal{B} = (e_n)_n$ be an orthonormal basis of $\mathbb{H}$. Then, for $x \in \mathbb{H}$,*

$$\mathcal{D}_m(x) = \sqrt{\|x\|^2 - \frac{1}{m}\sup\left\{\langle x, 1_A\rangle^2 : |A| = m\right\}},$$

$$\mathcal{D}_m^*(x) = \sqrt{\|x\|^2 - \frac{1}{m}\sup\left\{\langle x, 1_{\varepsilon A}\rangle^2 : |A| = m, (\varepsilon_n) \in \{\pm 1\}\right\}}.$$

Proof. Let $\alpha \in \mathbb{R}$, $(\varepsilon_n) \in \{\pm 1\}$ and $|A| = m$. Then

$$\|x - \alpha 1_{\varepsilon A}\|^2 = \|x\|^2 - 2\langle x, \alpha 1_{\varepsilon A}\rangle + \alpha^2|A|.$$

Therefore the minimum of $\|x - \alpha 1_{\varepsilon A}\|^2$ is achieved at $\alpha_0 = \frac{\sum_{k\in A}\varepsilon_k e_k^*(x)}{m}$ and its value is $\|x\|^2 - \frac{(\langle x, \alpha 1_{\varepsilon A}\rangle)^2}{m}$. Taking infimum over the corresponding families we obtain the result. $\square$

Theorem 3.2. *If $\mathbb{H}$ is a Hilbert space and $\mathcal{B} = (e_n)_n$ is an orthonormal basis of $\mathbb{H}$, then*

$$\lim_{m\longrightarrow\infty}\mathcal{D}_m(x) = \lim_{m\longrightarrow\infty}\mathcal{D}_m^*(x) = \|x\|, \quad \forall x \in \mathbb{H}.$$

Proof. Since $\mathcal{D}_m^*(x) \leq \mathcal{D}_m(x) \leq \|x\|$, it suffices to see that $\lim_{m\to\infty}\mathcal{D}_m^*(x) = \|x\|$. Assume first that $x \in \mathbb{X}_c$ and $\mathrm{supp}(x) = B$ with $N = |B|$. For each $(\varepsilon_n) \in \{\pm 1\}$ and A such that $|A| = m$, we have

$$\frac{1}{|A|}\langle x, 1_{\varepsilon A}\rangle^2 = \frac{1}{|A|}\left(\sum_{k\in A\cap B}\varepsilon_k e_k^*(x)\right)^2 \leq \|x\|^2\frac{|A\cap B|}{|A|} \leq \frac{N\|x\|^2}{m}.$$

From Proposition 3.4 we conclude that

$$\|x\|\sqrt{1-N/m} \le \mathcal{D}^*_m(x) \le \|x\|,$$

which gives the result for $x \in \mathbb{X}_c$.

For general $x \in \mathbb{X}$, given $\varepsilon > 0$, take first $y \in \mathbb{X}_c$ with $\|x-y\| < \varepsilon/2$ and observe that

$$\mathcal{D}^*_m(x) \ge \mathcal{D}^*_m(y) - \|x-y\|,$$

to conclude that

$$\liminf_m \mathcal{D}^*_m(x) \ge \|y\| - \varepsilon/2 \ge \|x\| - \varepsilon. \tag{18}$$

As we know that $\mathcal{D}^*_m(x) \le \|x\|$ for all $x \in \mathbb{X}$ and $m \in \mathbb{N}$, then $\lim_m \sup \mathcal{D}^*_m(x) \le \|x\|$. Hence, using this fact and taking the limit as ε goes to 0 in (18), we obtain the result. $\square$

Now, we are going to show that for any finite set B and the canonical basis $\mathcal{B}$ in $\mathbb{X} = \ell^p$, we have the same property as in Theorem 3.2 for concrete elements. To prove this, we shall use the following elementary lemma.

Lemma 3.3. *Let $1 < p < \infty$ and $m, N \in \mathbb{N}$ such that $m \ge N$. Define, for $\alpha \in \mathbb{R}$ and $1 \le k \le N$,*

$$H(\alpha, k) = |1-\alpha|^p k + |\alpha|^p (m-k) + (N-k)$$

and, for $\alpha \in \mathbb{R}$, $k_1, k_2 \in \mathbb{N}$ and $1 \le k_1 + k_2 \le N$,

$$L(\alpha, k_1, k_2) = |1-\alpha|^p k_1 + |1+\alpha|^p k_2 + |\alpha|^p (m-(k_1+k_2)) + (N-(k_1+k_2)).$$

Then

$$\min_{\alpha\in\mathbb{R}, 1\le k\le N} H(\alpha, k) = \min_{\alpha\in\mathbb{R}, 1\le k_1+k_2\le N} L(\alpha, k_1, k_2) = N\left(1 + \left(\frac{m-N}{N}\right)^{-1/(p-1)}\right)^{-(p-1)}.$$

Proof. Using that $H(\alpha, k) \ge H(|\alpha|, k)$ and $L(\alpha, k_1, k_2) = L(-\alpha, k_2, k_1)$ we can restrict α to $\alpha \in \mathbb{R}^+$. Also since $(\alpha-1)^p k + \alpha^p(m-k)$ and $(\alpha-1)^p k_1 + (1+\alpha)^p k_2 + \alpha^p k_3$ are increasing for $\alpha \ge 1$, the minima are achieved over $0 \le \alpha \le 1$.

Let $0 \le \alpha \le 1$ and $0 \le k, k_1, k_2 \le N$ and $k_1 + k_2 \le N$. We write $H(\alpha, k) = H_\alpha(k) = J_k(\alpha)$, that is

$$H_\alpha(k) = \Big((1-\alpha)^p - \alpha^p - 1\Big)k + N + \alpha^p m.$$

Similarly we write $L(\alpha, k_1, k_2) = L_\alpha(k_1, k_2)$, that is

$$L_\alpha(k_1, k_2) = \Big((1-\alpha)^p - \alpha^p - 1\Big)k_1 + \Big((1+\alpha)^p - \alpha^p - 1\Big)k_2 + N + \alpha^p m.$$

Since $(1-\alpha)^p \le \alpha^p + 1$ and $(1+\alpha)^p \ge \alpha^p + 1$ we obtain that

$$\begin{aligned}\min\{L_\alpha(k_1, k_2) : 0 \le k_1 + k_2 \le N\} &= \min\{H_\alpha(k) : 0 \le k \le N\} \\ &= (1-\alpha)^p N + \alpha^p (m-N).\end{aligned}$$

Now the minimum of $J_N(\alpha)$, $0 \le \alpha \le 1$, is achieved at $\alpha_{\min} = (1 + (\frac{m-N}{N})^{\frac{1}{p-1}})^{-1}$ and

$$J_N(\alpha_{\min}) = N\left(1 + \left(\frac{m-N}{N}\right)^{-\frac{1}{p-1}}\right)^{-(p-1)}. \quad \square$$

Proposition 3.4. *Let $\mathbb{X} = \ell^p$ for some $1 < p < \infty$ and $\mathcal{B}$ the canonical basis. If $B \subset \mathbb{N}$ and $|B| = N$ then*

$$\mathcal{D}_m(1_B) = \mathcal{D}_m^*(1_B) = (N-m)^{1/p}, \quad m \le N, \tag{19}$$

$$\mathcal{D}_m(1_B) = \mathcal{D}_m^*(1_B) = N^{1/p}\left(1 + \left(\frac{m}{N} - 1\right)^{-1/(p-1)}\right)^{-1/p'}, \quad m \ge N, \tag{20}$$

where $p' = \frac{p}{p-1}$.

Proof. Assume first that $m \le N$. Let $\alpha \in \mathbb{R}$, $|\varepsilon_n| = 1$ and $A \subset \mathbb{N}$ with $|A| = m$. Set $1_{\varepsilon A} = 1_{A_1} - 1_{A_2}$. Observe that

$$\|1_B - \alpha 1_{\varepsilon A}\|^p = |1-\alpha|^p \|1_{A_1 \cap B}\|^p + |1+\alpha|^p \|1_{A_2 \cap B}\|^p + |\alpha|^p \|1_{A \setminus B}\|^p + \|1_{B \setminus A}\|^p. \tag{21}$$

In particular

$$\|1_B - \alpha 1_A\|^p = |1-\alpha|^p \|1_{A \cap B}\|^p + |\alpha|^p \|1_{A \setminus B}\|^p + \|1_{B \setminus A}\|^p. \tag{22}$$

Therefore $\|1_B - \alpha 1_{\varepsilon A}\| \ge \|1_{B \setminus A}\| \ge (N-m)^{1/p}$. This gives $\mathcal{D}_m^*(1_B) \ge (N-m)^{1/p}$.

On the other hand, choosing $A \subseteq B$ and $\alpha = 1$ one concludes that $(N-m)^{1/p} = \|1_B - 1_A\| \ge \mathcal{D}_m(1_B)$. Therefore we obtain (19).

Assume now that $m \ge N$. Denoting $k = |A \cap B| = \|1_{A \cap B}\|^p$, $k_1 = |A_1 \cap B| = \|1_{A_1 \cap B}\|^p$ and $k_2 = |A_1 \cap B| = \|1_{A_2 \cap B}\|^p$, we can apply (21) and (22) together with Lemma 3.3 to obtain (20). $\square$

Remark 3.5. Similar arguments show that for $\mathbb{X} = \ell^1$ and $\mathcal{B}$ the canonical basis and $B \subset \mathbb{N}$ with $|B| = N$ one has $\mathcal{D}_m(1_B) = \begin{cases} N-m, & m \le N; \\ m-N, & N \le m \le 2N; \\ N, & m \ge 2N. \end{cases}$

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