

The Maximal Riesz Operator of Two-Dimensional Fourier Transforms and Fourier Series on $H_p(\mathbf{R} \times \mathbf{R})$ and $H_p(\mathbf{T} \times \mathbf{T})$ ¹

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It is proved that the maximal operator of the two-parameter Riesz means with parameters $\alpha, \beta \leq 1$ is bounded from $L_p(\mathbf{R}^2)$ to $L_p(\mathbf{R}^2)$ ($1 < p < \infty$). The two-dimensional classical Hardy spaces $H_p(\mathbf{R} \times \mathbf{R})$ are introduced and it is shown that the maximal Riesz operator of a tempered distribution is also bounded from $H_p(\mathbf{R} \times \mathbf{R})$ to $L_p(\mathbf{R}^2)$ ($\max\{1/(\alpha+1), 1/(\beta+1)\} < p \leq \infty$) and is of weak type $(H_1^{\sharp}(\mathbf{R} \times \mathbf{R}), L_1(\mathbf{R}^2))$ where the Hardy space $H_1^{\sharp}(\mathbf{R} \times \mathbf{R})$ is defined by the hybrid maximal function. As a consequence we obtain that the Riesz means of a function $f \in H_1^{\sharp}(\mathbf{R} \times \mathbf{R}) \supset L \log L(\mathbf{R}^2)$ converge a.e. to the function in question. Moreover, we prove that the Riesz means are uniformly bounded on the spaces $H_p(\mathbf{R} \times \mathbf{R})$ whenever $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$. Thus, in case $f \in H_p(\mathbf{R} \times \mathbf{R})$, the Riesz means converge to f in $H_p(\mathbf{R} \times \mathbf{R})$ norm. The same results are proved for the conjugate Riesz means and for two-parameter Fourier series, too. © 2000 Academic Press

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1. INTRODUCTION

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{R} \times \mathbf{R})$ of tempered distributions are introduced with the $L_{p,q}(\mathbf{R}^2)$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbf{R} \times \mathbf{R}) = H_{p,p}(\mathbf{R} \times \mathbf{R})$ are the usual Hardy spaces ($0 < p \leq \infty$).

In this paper the Riesz means $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ of two-dimensional tempered distributions are considered where $0 < \alpha, \beta < \infty$ and $1 \leq \gamma, \delta < \infty$. In the one-dimensional case Butzer and Nessel [3] and Stein and Weiss [16] proved for $\gamma = 1, 2$ that the Riesz means $\sigma_T^{\alpha,\gamma} f$ of a function $f \in L_1(\mathbf{R})$ converge a.e. to f as $T \rightarrow \infty$. The author [19] verified the same result for all $\gamma \geq 1$ and,

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moreover, that the one-dimensional maximal Riesz operator $\sigma_*^{\alpha, \gamma} := \sup_{T>0} |\sigma_T^{\alpha, \gamma}|$ is of weak type $(1, 1)$, i.e.,

$$\sup_{\rho>0} \rho \lambda(\sigma_*^{\alpha, \gamma} f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbf{R})).$$

Moreover, we proved in [19] that $\sigma_*^{\alpha, \gamma}$ is bounded from $H_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $1/(\alpha+1) < p < \infty$ and $0 < \alpha \leq 1$.

In Weisz [17] we investigated the Fejér means of two-parameter Fourier series, i.e. if $\alpha = \beta = \gamma = \delta = 1$, and proved that $\sigma_*^{1,1,1,1} := \sup_{n,m \in \mathbf{N}} |\sigma_{n,m}^{1,1,1,1}|$ is bounded from $H_{p,q}(\mathbf{T} \times \mathbf{T})$ to $L_{p,q}(\mathbf{T}^2)$ ($3/4 < p \leq \infty$, $0 < q \leq \infty$) and is of weak type $(H_1^\#(\mathbf{T} \times \mathbf{T}), L_1(\mathbf{T}^2))$, i.e.

$$\sup_{\rho>0} \rho \lambda(\sigma_*^{1,1,1,1} f > \rho) \leq C \|f\|_{H_1^\#(\mathbf{T} \times \mathbf{T})} \quad (f \in H_1^\#(\mathbf{T} \times \mathbf{T})).$$

Moreover, the Fejér means $\sigma_{n,m}^{1,1,1,1} f$ converge a.e. to f as $n, m \rightarrow \infty$ whenever $f \in H_1^\#(\mathbf{T} \times \mathbf{T}) \supset L \log L(\mathbf{T}^2)$ (see Weisz [17] and Zygmund [21] for $L \log L(\mathbf{T}^2)$).

In this paper we use another method and so we can sharpen and generalize these results for the Riesz means of two-dimensional Fourier transforms and Fourier series with $\alpha, \beta > 0$ and $\gamma, \delta \geq 1$.

First we modify the one-dimensional Riesz means by taking the absolute value of the kernel functions and prove that the maximal operator of these modified Riesz means is of weak type $(L_1(\mathbf{R}), L_1(\mathbf{R}))$ and is bounded from $L_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $1 < p < \infty$. From this it follows that the maximal operator $\sigma_*^{\alpha, \gamma}$ of the original one-dimensional Riesz means is also of weak type $(L_1(\mathbf{R}), L_1(\mathbf{R}))$ and is bounded from $L_p(\mathbf{R})$ to $L_p(\mathbf{R})$ ($1 < p < \infty$). Note that this last result was also proved in Weisz [19] with another method, as mentioned above. Using these two results about the one-dimensional Riesz means we verify that the two-dimensional maximal operator $\sigma_*^{\alpha, \beta, \gamma, \delta} := \sup_{T, U>0} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta}|$ ($0 < \alpha, \beta \leq 1$) is bounded from $L_p(\mathbf{R}^2)$ to $L_p(\mathbf{R}^2)$ ($1 < p < \infty$), which is also a new result.

Next we extend this result and investigate the boundedness of $\sigma_*^{\alpha, \beta, \gamma, \delta}$ on Hardy spaces. We will show that $\sigma_*^{\alpha, \beta, \gamma, \delta}$ is bounded from $H_{p,q}(\mathbf{R} \times \mathbf{R})$ to $L_{p,q}(\mathbf{R}^2)$ whenever $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$, $0 < q \leq \infty$ and is of weak type $(H_1^\#(\mathbf{R} \times \mathbf{R}), L_1(\mathbf{R}^2))$. We introduce the conjugate distributions $\tilde{f}^{(i,j)}$, the conjugate Riesz means $\tilde{\sigma}_{T,U}^{(i,j); \alpha, \beta, \gamma, \delta}$ and the conjugate maximal operators $\tilde{\sigma}_*^{(i,j); \alpha, \beta, \gamma, \delta}$ ($i, j = 0, 1$). We obtain that the operator $\tilde{\sigma}_*^{(i,j); \alpha, \beta, \gamma, \delta}$ is also of type $(H_{p,q}(\mathbf{R} \times \mathbf{R}), L_{p,q}(\mathbf{R}^2))$ for $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$, $0 < q \leq \infty$ and of weak type $(H_1^\#(\mathbf{R} \times \mathbf{R}), L_1(\mathbf{R}^2))$.

A usual density argument implies then that the Riesz means $\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} f$ converge a.e. to f and the conjugate Riesz means $\tilde{\sigma}_{T,U}^{(i,j); \alpha, \beta, \gamma, \delta} f$ converge a.e. to $\tilde{f}^{(i,j)}$ ($i, j = 0, 1$) as $T, U \rightarrow \infty$, provided that $f \in H_1^\#(\mathbf{R} \times \mathbf{R})$. Note that $\tilde{f}^{(i,j)}$ is not necessarily in $H_1^\#(\mathbf{R} \times \mathbf{R})$ whenever f is.

We will prove also that the operators $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}$ and $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta}$ ($T, U \in \mathbf{R}_+$) are uniformly bounded from $H_{p,q}(\mathbf{R} \times \mathbf{R})$ to $H_{p,q}(\mathbf{R} \times \mathbf{R})$ if $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty, 0 < q \leq \infty$. From this it follows that $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f \rightarrow f$ and $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f \rightarrow \tilde{f}^{(i,j)}$ ($i, j = 0, 1$) in $H_{p,q}(\mathbf{R} \times \mathbf{R})$ norm as $T, U \rightarrow \infty$, whenever $f \in H_{p,q}(\mathbf{R} \times \mathbf{R})$ and $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty, 0 < q \leq \infty$.

We extend these results also for $\alpha > 1$ and/or $\beta > 1$.

We consider also the Riesz means of two-parameter Fourier series of distributions on \mathbf{T}^2 and prove all the results above in this context.

2. HARDY SPACES ON $\mathbf{R} \times \mathbf{R}$ AND CONJUGATE FUNCTIONS

For a set $\mathbf{X} \neq \emptyset$ let \mathbf{X}^2 be its Cartesian product $\mathbf{X} \times \mathbf{X}$ taken with itself, moreover, let \mathbf{R} denote the real numbers, \mathbf{R}_+ the positive real numbers and let λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I . We briefly write $L_p(\mathbf{R}^2)$ instead of the real $L_p(\mathbf{R}^2, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{R}^2} |f|^p d\lambda)^{1/p}$ ($0 < p \leq \infty$).

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f| > \rho\}) := \lambda(\{x : |f(x)| > \rho\}) \quad (\rho \geq 0).$$

The *weak* $L_p(\mathbf{R}^2)$ space $L_p^*(\mathbf{R}^2)$ ($0 < p < \infty$) consists of all measurable functions f for which

$$\|f\|_{L_p^*(\mathbf{R}^2)} := \sup_{\rho > 0} \rho \lambda(\{|f| > \rho\})^{1/p} < \infty$$

while we set $L_\infty^*(\mathbf{R}^2) = L_\infty(\mathbf{R}^2)$.

The spaces $L_p^*(\mathbf{R}^2)$ are special cases of the more general Lorentz spaces $L_{p,q}(\mathbf{R}^2)$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\tilde{f}(t) := \inf \{ \rho : \lambda(\{|f| > \rho\}) \leq t \}.$$

The Lorentz space $L_{p,q}(\mathbf{R}^2)$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := \left(\int_0^\infty \tilde{f}(t)^q t^{q/p} \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \leq \infty$

$$\|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \tilde{f}(t).$$

Let

$$L_{p,q}(\mathbf{R}^2) := L_{p,q}(\mathbf{R}^2, \lambda) := \{f: \|f\|_{p,q} < \infty\}.$$

One can show the equalities

$$L_{p,p}(\mathbf{R}^2) = L_p(\mathbf{R}^2), \quad L_{p,\infty}(\mathbf{R}^2) = L_p^*(\mathbf{R}^2) \quad (0 < p \leq \infty)$$

(see, e.g., Bennett and Sharpley [1] or Bergh and Löfström [2]).

Let f be a tempered distribution on $C^\infty(\mathbf{R}^2)$ (briefly $f \in \mathcal{S}'(\mathbf{R}^2)$). The Fourier transform of f is denoted by \hat{f} . In special case, if f is an integrable function then

$$\hat{f}(t, u) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x, y) e^{-itx} e^{-iuy} dx dy \quad (t, u \in \mathbf{R}),$$

where $\iota = \sqrt{-1}$.

For $f \in \mathcal{S}'(\mathbf{R}^2)$ and $t, u > 0$ let

$$F(x, y; t, u) := (f * P_t \otimes P_u)(x, y),$$

where $*$ denotes the convolution and

$$P_t(x) := \frac{ct}{t^2 + x^2} \quad (x \in \mathbf{R})$$

is the Poisson kernel. Moreover, let $\Gamma := \{(x, t) : |x| < t\}$ a cone whose vertex is the origin. We denote by $\Gamma(x)$ ($x \in \mathbf{R}$) the translate of Γ so that its vertex is x . The non-tangential maximal function is defined by

$$F^*(x, y) := \sup_{(x', t) \in \Gamma(x), (y', u) \in \Gamma(y)} |F(x', y'; t, u)|.$$

For $0 < p, q \leq \infty$ the Hardy-Lorentz space $H_{p,q}(\mathbf{R} \times \mathbf{R})$ consists of all tempered distributions f for which $F^* \in L_{p,q}(\mathbf{R}^2)$ and set

$$\|f\|_{H_{p,q}(\mathbf{R} \times \mathbf{R})} := \|F^*\|_{p,q}.$$

It is known that if $f \in H_p(\mathbf{R} \times \mathbf{R})$ ($0 < p < \infty$) then $f(x, y) = \lim_{t, u \rightarrow 0} F(x, y; t, u)$ in the sense of distributions (see Gundy and Stein [11], Chang and Fefferman [4]).

Let us introduce the hybrid Hardy spaces. For $f \in L_1(\mathbf{R}^2)$ and $t > 0$ let

$$G(x, y; t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(v, y) P_t(x - v) dv$$

and

$$G^+(x, y) := \sup_{(x', t) \in \Gamma(x)} |G(x', y; t)|.$$

We say that $f \in L_1(\mathbf{R}^2)$ is in the *hybrid Hardy–Lorentz space* $H_{p,q}^\#(\mathbf{R} \times \mathbf{R})$ if

$$\|f\|_{H_{p,q}^\#(\mathbf{R} \times \mathbf{R})} := \|G^+\|_{p,q} < \infty.$$

The equivalences

$$H_{p,q}(\mathbf{R} \times \mathbf{R}) \sim H_{p,q}^\#(\mathbf{R} \times \mathbf{R}) \sim L_{p,q}(\mathbf{R}^2) \quad (1 < p < \infty, 0 < q \leq \infty) \quad (1)$$

were proved in Fefferman and Stein [7], Gundy and Stein [11], and Lin [13]. Note that in case $p=q$ the usual definition of Hardy spaces $H_{p,p}(\mathbf{R} \times \mathbf{R}) = H_p(\mathbf{R} \times \mathbf{R})$ and $H_{p,p}^\#(\mathbf{R} \times \mathbf{R}) = H_p^\#(\mathbf{R} \times \mathbf{R})$ are obtained.

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Lin [13] and also Weisz [18]).

THEOREM A. *If a sublinear (resp. linear) operator V is bounded from $H_{p_0}(\mathbf{R} \times \mathbf{R})$ to $L_{p_0}(\mathbf{R}^2)$ (resp. to $H_{p_0}(\mathbf{R} \times \mathbf{R})$) and from $L_{p_1}(\mathbf{R}^2)$ to $L_{p_1}(\mathbf{R}^2)$ ($p_0 \leq 1 < p_1 < \infty$) then it is also bounded from $H_{p,q}(\mathbf{R} \times \mathbf{R})$ to $L_{p,q}(\mathbf{R}^2)$ (resp. to $H_{p,q}(\mathbf{R} \times \mathbf{R})$) if $p_0 < p < p_1$ and $0 < q \leq \infty$.*

In this paper the constants C are absolute constants and the constants C_p (resp. $C_{p,q}$) are depending only on p (resp. p and q) and may denote different constants in different contexts.

One can prove similarly as in the discrete case (see Weisz [17]) that $L \log L(\mathbf{R}^2) \subset H_1^\#(\mathbf{R} \times \mathbf{R}) \subset H_{1,\infty}(\mathbf{R} \times \mathbf{R})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbf{R} \times \mathbf{R})} = \sup_{\rho > 0} \rho \lambda(F^* > \rho) \leq C \|f\|_{H_1^\#(\mathbf{R} \times \mathbf{R})} \quad (f \in H_1^\#(\mathbf{R} \times \mathbf{R})) \quad (2)$$

and

$$\|f\|_{H_1^\#(\mathbf{R} \times \mathbf{R})} \leq C + C \| |f| \log^+ |f| \|_1 \quad (f \in L \log L(\mathbf{R}^2)),$$

where $\log^+ u = 1_{\{u > 1\}} \log u$.

For a tempered distribution $f \in H_p(\mathbf{R} \times \mathbf{R})$ ($0 < p < \infty$) the *Hilbert transforms* or the *conjugate distributions* $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ are defined by

$$(\tilde{f}^{(1,0)})^\wedge(t, u) := (-i \operatorname{sign} t) \hat{f}(t, u) \quad (t, u \in \mathbf{R})$$

(conjugate with respect to the first variable),

$$(\tilde{f}^{(0,1)})^\wedge(t, u) := (-i \operatorname{sign} u) \hat{f}(t, u) \quad (t, u \in \mathbf{R})$$

(conjugate with respect to the second variable) and

$$(\tilde{f}^{(1,1)})^\wedge(t, u) := (-\operatorname{sign}(tu)) \hat{f}(t, u) \quad (t, u \in \mathbf{R})$$

(conjugate with respect to both variables), respectively. We use the notation $\tilde{f}^{(0,0)} := f$.

Gundy and Stein [10, 11] verified that if $f \in H_p(\mathbf{R} \times \mathbf{R})$ ($0 < p < \infty$) then all conjugate distributions are also in $H_p(\mathbf{R} \times \mathbf{R})$ and

$$\|f\|_{H_p(\mathbf{R} \times \mathbf{R})} = \|\tilde{f}^{(i,j)}\|_{H_p(\mathbf{R} \times \mathbf{R})} \quad (i, j = 0, 1). \quad (3)$$

Furthermore (see also Chang and Fefferman [4], Frazier [9], Duren [5]),

$$\|f\|_{H_p(\mathbf{R} \times \mathbf{R})} \sim \|f\|_p + \|\tilde{f}^{(1,0)}\|_p + \|\tilde{f}^{(0,1)}\|_p + \|\tilde{f}^{(1,1)}\|_p. \quad (4)$$

As is well known, if f is an integrable function then

$$\tilde{f}^{(1,0)}(x, y) = \text{p.v.} \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x-t, y)}{t} dt := \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{\varepsilon < |t|} \frac{f(x-t, y)}{t} dt,$$

$$\tilde{f}^{(0,1)}(x, y) = \text{p.v.} \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x, y-u)}{u} du,$$

and

$$\tilde{f}^{(1,1)}(x, y) = \text{p.v.} \frac{1}{\pi^2} \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{f(x-t, y-u)}{tu} dt du.$$

Moreover, the conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ do exist almost everywhere, but they are not integrable in general. Similarly, if $f \in H_1^\#(\mathbf{R} \times \mathbf{R})$ then $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ are not necessarily in $H_1^\#(\mathbf{R} \times \mathbf{R})$.

3. THE TWO-DIMENSIONAL RIESZ MEANS OF FOURIER TRANSFORMS

Suppose first that $f \in L_p(\mathbf{R}^2)$ for some $1 \leq p \leq 2$. It is known that under certain conditions

$$f(x, y) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} \hat{f}(t, u) e^{ixt} e^{iyu} dt du \quad (x, y \in \mathbf{R}).$$

This motivates the definition of the *Dirichlet integral* $s_{t,u}f$,

$$s_{t,u}f(x, y) := \frac{1}{2\pi} \int_{-t}^t \int_{-u}^u \hat{f}(v, w) e^{ixv} e^{iyw} dv dw \quad (t, u > 0).$$

The *conjugate Dirichlet integrals* are introduced by

$$\tilde{s}_{t,u}^{(1,0)}f(x, y) := \frac{1}{2\pi} \int_{-t}^t \int_{-u}^u (-i \operatorname{sign} v) \hat{f}(v, w) e^{ixv} e^{iyw} dv dw \quad (t, u > 0),$$

$$\tilde{s}_{t,u}^{(0,1)}f(x, y) := \frac{1}{2\pi} \int_{-t}^t \int_{-u}^u (-i \operatorname{sign} w) \hat{f}(v, w) e^{ixv} e^{iyw} dv dw \quad (t, u > 0)$$

and

$$\tilde{s}_{t,u}^{(1,1)}f(x, y) := \frac{1}{2\pi} \int_{-t}^t \int_{-u}^u (-\operatorname{sign}(vw)) \hat{f}(v, w) e^{ixv} e^{iyw} dv dw \quad (t, u > 0),$$

respectively. We write $s_{t,u}f =: \tilde{s}_{t,u}^{(0,0)}f$. It is easy to see that

$$\tilde{s}_{t,u}^{(i,j)}f(x, y) := \int_{\mathbf{R}} \int_{\mathbf{R}} \tilde{f}^{(i,j)}(x-v, y-w) \frac{\sin tv}{\pi v} \frac{\sin uw}{\pi w} dv dw \quad (i, j = 0, 1).$$

For $\alpha, \beta, \gamma, \delta > 0$ the *Riesz* and *conjugate Riesz means* are defined by

$$\begin{aligned} \tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta}f(x, y) := & \frac{\alpha\beta\gamma\delta}{TU} \int_0^T \int_0^U \left(1 - \left(\frac{t}{T}\right)^\gamma\right)^{\alpha-1} \left(\frac{t}{T}\right)^{\gamma-1} \\ & \left(1 - \left(\frac{u}{U}\right)^\delta\right)^{\beta-1} \left(\frac{u}{U}\right)^{\delta-1} \tilde{s}_{t,u}^{(i,j)}f(x, y) dt du, \end{aligned}$$

where $T, U > 0$ and $i, j = 0, 1$. Let $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}f := \tilde{\sigma}_{T,U}^{(0,0);\alpha,\beta,\gamma,\delta}f$. Integrating by parts we get that

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}f(x, y) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x-t, y-u) K_T^{\alpha,\gamma}(t) K_U^{\beta,\delta}(u) dt du,$$

where

$$\begin{aligned} K_T^{\alpha,\gamma}(u) &:= \frac{2}{\sqrt{2\pi}} \int_0^T \left(1 - \left(\frac{t}{T}\right)^\gamma\right)^\alpha \cos tu dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-T}^T \left(1 - \left|\frac{t}{T}\right|^\gamma\right)^\alpha \cos tu dt \end{aligned}$$

is the Riesz kernel. Similarly,

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f(x,y) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} \tilde{f}^{(i,j)}(x-t, y-u) K_T^{\alpha,\gamma}(t) K_U^{\beta,\delta}(u) dt du.$$

We verified in [19] that the Riesz kernel $K_T^{\alpha,\gamma}$ with $0 < \alpha \leq 1 \leq \gamma$ ($T > 0$) satisfies the conditions

$$\int_{\mathbf{R}} |K_T^{\alpha,\gamma}| d\lambda \leq C, \quad (5)$$

$$|K_T^{\alpha,\gamma}(t)| \leq \frac{C}{T^\alpha |t|^{\alpha+1}} \quad (t \in \mathbf{R}, t \neq 0) \quad (6)$$

and

$$|(K_T^{\alpha,\gamma})'(t)| \leq \frac{C}{T^{\alpha-1} |t|^{\alpha+1}} \quad (t \in \mathbf{R}, t \neq 0), \quad (7)$$

where $(K_T^{\alpha,\gamma})'$ denotes the derivative of the Riesz kernel. Note that C may depend on α and γ .

The Riesz means are called typical means if $\gamma = \delta = 1$, Bochner–Riesz means if $\gamma = \delta = 2$ and Fejér means if $\alpha = \beta = \gamma = \delta = 1$. One can prove that (cf. Butzer and Nessel [3]),

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f(x,y) = \frac{1}{2\pi} \int_{-T}^T \int_{-U}^U \left(1 - \left|\frac{t}{T}\right|^\gamma\right)^\alpha \left(1 - \left|\frac{u}{U}\right|^\delta\right)^\beta \hat{f}(t,u) e^{ixt} e^{iyu} dt du,$$

$i, j = 0, 1$.

We extend the definition of the Riesz means to tempered distributions as

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f := f * (K_T^{\alpha,\gamma} \times K_U^{\beta,\delta}) \quad (T, U > 0).$$

One can show that $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ is well defined for all tempered distributions $f \in H_p(\mathbf{R} \times \mathbf{R})$ ($0 < p \leq \infty$) and for all functions $f \in L_p(\mathbf{R}^2)$ ($1 \leq p \leq \infty$) (cf. Fefferman and Stein [7]). The extension of the conjugate Riesz means is

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f := \tilde{f}^{(i,j)} * (K_T^{\alpha,\gamma} \times K_U^{\beta,\delta}) \quad (T, U > 0).$$

The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f := \sup_{T,U>0} |\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f| \quad (i, j = 0, 1).$$

We use again the notation $\sigma_*^{\alpha,\beta,\gamma,\delta} f := \tilde{\sigma}_*^{(0,0);\alpha,\beta,\gamma,\delta} f$.

4. THE BOUNDEDNESS OF THE MAXIMAL RIESZ OPERATOR ON $L_p(\mathbf{R}^2)$ SPACES

In order to prove that $\sigma_*^{\alpha, \beta, \gamma, \delta}$ is bounded on the $L_p(\mathbf{R}^2)$ ($1 < p < \infty$) spaces we consider first some one-parameter results. We modify the one-dimensional Riesz means by taking the absolute value of the kernel functions as follows. Let

$$\tau_T^{\alpha, \gamma} f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x-u) |K_T^{\alpha, \gamma}(u)| du$$

and

$$\tau_*^{\alpha, \gamma} f := \sup_{T>0} |\tau_T^{\alpha, \gamma} f|.$$

Obviously,

$$|\sigma_T^{\alpha, \gamma} f| \leq \tau_T^{\alpha, \gamma} |f| \quad (T > 0).$$

With the help of the following theorem that was proved by Schipp *et al.* [15, pp. 262–263], we show that the operator $\tau_*^{\alpha, \gamma}$ is of weak type (1,1).

THEOREM B. *Suppose that the sublinear operator V is bounded from $L_\infty(\mathbf{R})$ to $L_\infty(\mathbf{R})$ and*

$$\int_{\mathbf{R} \setminus 4I} |Vf| d\lambda \leq C \|f\|_1 \quad (8)$$

for all $f \in L_1(\mathbf{R})$ and intervals I which satisfy

$$\text{supp } f \subset I \quad (9)$$

and

$$\int_{\mathbf{R}} f d\lambda = 0, \quad (10)$$

where rI ($r \in \mathbf{N}$) is the interval with the same center as I and with length $r|I|$. Then the operator V is of weak type (1,1), i.e.,

$$\lambda(|Vf| > \rho) \leq \frac{C}{\rho} \|f\|_1$$

for all $f \in L_1(\mathbf{R})$ and $\rho > 0$.

Now we can formulate the main theorem of this section.

THEOREM 1. *Assume that $0 < \alpha \leq 1 \leq \gamma$. Then the operator $\tau_*^{\alpha, \gamma}$ is bounded from $L_\infty(\mathbf{R})$ to $L_\infty(\mathbf{R})$ and*

$$\lambda(\tau_*^{\alpha, \gamma} f > \rho) \leq \frac{C}{\rho} \|f\|_1$$

for all $f \in L_1(\mathbf{R})$ and $\rho > 0$.

Proof. It is easy to see that (5) implies

$$\|\tau_*^{\alpha, \gamma} f\|_\infty \leq C \|f\|_\infty \quad (f \in L_\infty(\mathbf{R})).$$

Let $f \in L_1(\mathbf{R})$ with support I which satisfy the conditions (9) and (10) and suppose that $2^{K-1} < |I| \leq 2^K$ ($K \in \mathbf{Z}$). We can suppose that the center of I is zero. In this case

$$[-2^{K-2}, 2^{K-2}] \subset I \subset [-2^{K-1}, 2^{K-1}].$$

Obviously,

$$\begin{aligned} \int_{\mathbf{R} \setminus 4I} \tau_*^{\alpha, \gamma} f(x) dx &\leq \sum_{|i|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \sup_{T \geq 2^{-K}} |\tau_T^{\alpha, \gamma} f(x)| dx \\ &\quad + \sum_{|i|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \sup_{T < 2^{-K}} |\tau_T^{\alpha, \gamma} f(x)| dx \\ &= (A) + (B). \end{aligned}$$

We can suppose that $i \geq 1$.

Inequality (6) implies

$$|\tau_T^{\alpha, \gamma} f(x)| = \left| \int_I f(t) |K_T^{\alpha, \gamma}(x-t)| dt \right| \leq \int_I \frac{C |f(t)|}{T^\alpha |x-t|^{\alpha+1}} dt.$$

By a simple calculation we get

$$\frac{1}{|x-t|^{\alpha+1}} \leq \frac{C}{(i2^K - 2^{K-1})^{\alpha+1}} \leq \frac{C 2^{-K(\alpha+1)}}{i^{\alpha+1}} \quad (11)$$

if $x \in [i2^K, (i+1)2^K]$ ($i \geq 1$). Hence

$$|\tau_T^{\alpha, \gamma} f(x)| \leq C 2^{-K(\alpha+1)} T^{-\alpha} \frac{1}{i^{\alpha+1}} \int_I |f(t)| dt$$

and so

$$(A) \leq C \sum_{i=1}^{\infty} 2^K 2^{-K(\alpha+1)} 2^{K\alpha} \frac{1}{i^{(\alpha+1)}} \|f\|_1 = C \sum_{i=1}^{\infty} \frac{1}{i^{(\alpha+1)}} \|f\|_1 \leq C \|f\|_1. \quad (12)$$

To estimate (B) observe that by (9) and (10)

$$\tau_T^{\alpha, \gamma} f(x) = \int_I f(t) |K_T^{\alpha, \gamma}(x-t)| dt = \int_I f(t) (|K_T^{\alpha, \gamma}(x-t)| - |K_T^{\alpha, \gamma}(x)|) dt.$$

Thus

$$|\tau_T^{\alpha, \gamma} f(x)| \leq \int_I |f(t)| |K_T^{\alpha, \gamma}(x-t) - K_T^{\alpha, \gamma}(x)| dt.$$

Using Lagrange's mean value theorem, (7) and (11) we conclude

$$\begin{aligned} |K_T^{\alpha, \gamma}(x-t) - K_T^{\alpha, \gamma}(x)| &= |(K_T^{\alpha, \gamma})'(x-\xi)| |t| \\ &\leq \frac{C |I|}{T^{\alpha-1} |x-\xi|^{\alpha+1}} \leq \frac{C 2^K 2^{-K(\alpha+1)} T^{1-\alpha}}{i^{\alpha+1}}, \end{aligned}$$

where $\xi \in I$ and $x \in [i2^K, (i+1)2^K)$. Consequently,

$$|\tau_T^{\alpha, \gamma} f(x)| \leq C 2^{-K\alpha} T^{1-\alpha} \frac{1}{i^{\alpha+1}} \int_I |f(t)| dt$$

and

$$(B) \leq C \sum_{i=1}^{\infty} 2^K 2^{-K\alpha} 2^{-K(1-\alpha)} \frac{1}{i^{(\alpha+1)}} \|f\|_1 = C \sum_{i=1}^{\infty} \frac{1}{i^{(\alpha+1)}} \|f\|_1 \leq C \|f\|_1. \quad (13)$$

Theorem 1 follows now from (12), (13), and Theorem B. \blacksquare

Note that for $\alpha=1$ the proof is simpler because we do not need to estimate (A).

The following result follows by interpolation.

THEOREM 2. *If $0 < \alpha \leq 1 \leq \gamma$ and $1 < p \leq \infty$ then*

$$\|\tau_*^{\alpha, \gamma} f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbf{R})).$$

Since

$$\sigma_*^{\alpha, \gamma} f \leq \tau_*^{\alpha, \gamma} |f|,$$

we get

COROLLARY 1. *Assume that $0 < \alpha \leq 1 \leq \gamma$ and $1 < p \leq \infty$. Then*

$$\lambda(\sigma_*^{\alpha, \gamma} f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (f \in L_1(\mathbf{R}); \rho > 0)$$

and

$$\|\sigma_*^{\alpha, \gamma} f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbf{R})).$$

Note that Corollary 1 was proved also in Weisz [19] with another method.

Now we return to the two-dimensional case and verify the $L_p(\mathbf{R}^2)$ boundedness of $\sigma_*^{\alpha, \beta, \gamma, \delta}$.

THEOREM 3. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $1 < p \leq \infty$. Then*

$$\|\sigma_*^{\alpha, \beta, \gamma, \delta} f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbf{R}^2)).$$

Proof. Applying Theorem 2 and Corollary 1 we have

$$\begin{aligned} & \int_{\mathbf{R}} \int_{\mathbf{R}} \sup_{T, U \in \mathbf{R}_+} \left| \int_{\mathbf{R}} \int_{\mathbf{R}} f(t, u) K_T^{\alpha, \gamma}(x-t) K_U^{\beta, \delta}(y-u) dt du \right|^p dx dy \\ & \leq \int_{\mathbf{R}} \int_{\mathbf{R}} \sup_{U \in \mathbf{R}_+} \left[\int_{\mathbf{R}} \left(\sup_{T \in \mathbf{R}_+} \left| \int_{\mathbf{R}} f(t, u) K_T^{\alpha, \gamma}(x-t) dt \right| \right) |K_U^{\beta, \delta}(y-u)| du \right]^p dy dx \\ & \leq C_p \int_{\mathbf{R}} \int_{\mathbf{R}} \sup_{T \in \mathbf{R}_+} \left| \int_{\mathbf{R}} f(t, y) K_T^{\alpha, \gamma}(x-t) dt \right|^p dx dy \\ & \leq C_p \int_{\mathbf{R}} \int_{\mathbf{R}} |f(x, y)|^p dx dy \end{aligned}$$

which proves the theorem. \blacksquare

Note that using the fact that $\sigma_*^{\alpha, \gamma}$ is bounded from $H_1(\mathbf{R})$ to $L_1(\mathbf{R})$ (see Weisz [19]) we could prove (15) in a similar way.

Since the set of those functions $f \in L_1(\mathbf{R}^2)$ whose Fourier transform has a compact support is dense in $L_p(\mathbf{R}^2)$ ($1 < p < \infty$) (see Wiener [20]), the usual density argument (see Marcinkiewicz and Zygmund [14]) implies

COROLLARY 2. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $1 < p < \infty$. Then for every $f \in L_p(\mathbf{R}^2)$ we have

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f \rightarrow f \quad \text{a.e. and in } L_p(\mathbf{R}^2) \text{ norm as } T, U \rightarrow \infty.$$

5. THE BOUNDEDNESS OF THE MAXIMAL RIESZ OPERATOR ON HARDY SPACES

In this section we consider the boundedness of $\sigma_*^{\alpha,\beta,\gamma,\delta}$ on the spaces $H_p(\mathbf{R} \times \mathbf{R})$ and extend Theorem 3 and Corollary 2.

A function $a \in L_2$ is called a *rectangle p -atom* if there exists a rectangle $R \subset \mathbf{R}^2$ such that

- (i) $\text{supp } a \subset R$
- (ii) $\|a\|_2 \leq |R|^{1/2-1/p}$
- (iii) for all $x, y \in \mathbf{R}$ and all $N \leq [2/p - 3/2]$,

$$\int_{\mathbf{R}} a(x, y) x^N dx = \int_{\mathbf{R}} a(x, y) y^N dy = 0.$$

For a rectangle $R = I \times J$ let $rR = rI \times rJ$ ($r \in \mathbf{N}$). An operator V which maps the set of tempered distributions into the collection of measurable functions, will be called *p -quasi-local* if there exist a constant $C_p > 0$ and $\eta > 0$ such that for every rectangle p -atom a supported on the rectangle R and for every $r \geq 2$ one has

$$\int_{\mathbf{R}^2 \setminus 2^r R} |Ta|^p d\lambda \leq C_p 2^{-\eta r}.$$

Although $H_p(\mathbf{R} \times \mathbf{R})$ cannot be decomposed into rectangle p -atoms, in the next theorem it is enough to take these atoms (see Weisz [17], Fefferman [8]).

THEOREM C. Suppose that the operator V is sublinear and p -quasi-local for some $0 < p \leq 1$. If V is bounded from $L_2(\mathbf{R}^2)$ to $L_2(\mathbf{R}^2)$ then

$$\|Vf\|_p \leq C_p \|f\|_{H_p(\mathbf{R} \times \mathbf{R})} \quad (f \in H_p(\mathbf{R} \times \mathbf{R})).$$

Now we are in a position to state our main result.

THEOREM 4. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$. Then

$$\|\sigma_*^{\alpha, \beta, \gamma, \delta} f\|_{p, q} \leq C_{p, q} \|f\|_{H_{p, q}(\mathbf{R} \times \mathbf{R})} \quad (f \in H_{p, q}(\mathbf{R} \times \mathbf{R})) \quad (14)$$

for every $\max\{1/(\alpha + 1), 1/(\beta + 1)\} < p < \infty$ and $0 < q \leq \infty$. Especially, if $f \in H_1^\#(\mathbf{R} \times \mathbf{R})$ then

$$\lambda(\sigma_*^{\alpha, \beta, \gamma, \delta} f > \rho) \leq \frac{C}{\rho} \|f\|_{H_1^\#(\mathbf{R} \times \mathbf{R})} \quad (\rho > 0). \quad (15)$$

Proof. First we will show that the operator $\sigma_*^{\alpha, \beta, \gamma, \delta}$ is p -quasi-local for each $\max\{1/(\alpha + 1), 1/(\beta + 1)\} < p \leq 1$. To this end let a be an arbitrary rectangle p -atom with support $R = I \times J$ and

$$2^{K-1} < |I| \leq 2^K, \quad 2^{L-1} < |J| \leq 2^L \quad (K, L \in \mathbf{Z}).$$

We can suppose again that

$$[-2^{K-2}, 2^{K-2}] \subset I \subset [-2^{K-1}, 2^{K-1}]$$

and

$$[-2^{L-2}, 2^{L-2}] \subset J \subset [-2^{L-1}, 2^{L-1}].$$

To prove the p -quasi-locality of the operator $\sigma_*^{\alpha, \beta, \gamma, \delta}$ we have to integrate $|\sigma_*^{\alpha, \beta, \gamma, \delta} a|^p$ over

$$\mathbf{R}^2 \setminus 2^r R = (\mathbf{R} \setminus 2^r I) \times J \cup (\mathbf{R} \setminus 2^r I) \times (\mathbf{R} \setminus J)$$

$$\cup I \times (\mathbf{R} \setminus 2^r J) \cup (\mathbf{R} \setminus I) \times (\mathbf{R} \setminus 2^r J),$$

where $r \geq 2$ is an arbitrary integer.

First we integrate over $(\mathbf{R} \setminus 2^r I) \times J$. We have

$$\begin{aligned} & \int_{\mathbf{R} \setminus 2^r I} \int_J |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \leq \sum_{|i|=2^{r-2}}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_J \sup_{T < 2^{-K}, U \in \mathbf{R}_+} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \quad + \sum_{|i|=2^{r-2}}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_J \sup_{T \geq 2^{-K}, U \in \mathbf{R}_+} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & = (A) + (B). \end{aligned} \quad (16)$$

For $x, y \in \mathbf{R}$ let

$$A_{1,0}(x, y) := \int_{-\infty}^x a(t, y) dt, \quad A_{0,1}(x, y) := \int_{-\infty}^y a(x, u) du$$

and

$$A_{1,1}(x, y) := \int_{-\infty}^x \int_{-\infty}^y a(t, y) dt du.$$

By (iii) of the definition of the rectangle atom we can show that $\text{supp } A_{k,l} \subset R$ and $A_{k,l}$ is zero at the limit points of R ($k, l = 0, 1$). Moreover, using (ii) we can compute that

$$\|A_{k,l}\|_2 \leq |I|^k |J|^l (|I| |J|)^{1/2-1/p} \quad (k, l = 0, 1). \quad (17)$$

Integrating by parts we can see that

$$\begin{aligned} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)| &= \left| \int_I \int_J A_{1,0}(t, u) (K_T^{\alpha,\gamma})' (x-t) K_U^{\beta,\delta} (y-u) dt du \right| \\ &\leq \int_I \left| \int_J A_{1,0}(t, u) K_U^{\beta,\delta} (y-u) du \right| |(K_T^{\alpha,\gamma})' (x-t)| dt. \end{aligned}$$

Apply (7) and (11) to obtain

$$\begin{aligned} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)| &\leq \int_I \left| \int_J A_{1,0}(t, u) K_U^{\beta,\delta} (y-u) du \right| \frac{C}{T^{\alpha-1} |x-t|^{\alpha+1}} dt \\ &\leq \frac{C 2^{-K(\alpha+1)} T^{1-\alpha}}{i^{\alpha+1}} \int_I \left| \int_J A_{1,0}(t, u) K_U^{\beta,\delta} (y-u) du \right| dt \end{aligned}$$

provided that $x \in [i2^K, (i+1)2^K)$ ($i \geq 1$). Hölder's inequality implies

$$\begin{aligned} &\int_J \sup_{T < 2^{-K}, U \in \mathbf{R}_+} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)|^p dy \\ &\leq \frac{C_p 2^{-K(\alpha+1)p} 2^{-K(1-\alpha)p}}{i^{(\alpha+1)p}} |J|^{1-p} \\ &\quad \left(\int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t, u) K_U^{\beta,\delta} (y-u) du \right| dy dt \right)^p. \quad (18) \end{aligned}$$

Using again Hölder's inequality, Corollary 1, and (17) we can conclude that

$$\begin{aligned}
 & \int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t, u) K_U^{\beta, \delta}(y-u) du \right| dy dt \\
 & \leq |J|^{1/2} \int_I \left(\int_{\mathbf{R}} \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t, u) K_U^{\beta, \gamma}(y-u) du \right|^2 dy \right)^{1/2} dt \\
 & \leq C |J|^{1/2} \int_I \left(\int_J |A_{1,0}(t, y)|^2 dy \right)^{1/2} dt \\
 & \leq C |I|^{1/2} |J|^{1/2} \left(\int_I \int_J |A_{1,0}(t, y)|^2 dy dt \right)^{1/2} \\
 & \leq C |I|^{2-1/p} |J|^{1-1/p}.
 \end{aligned} \tag{19}$$

Consequently,

$$\begin{aligned}
 (A) & \leq C_p \sum_{i=2^{r-2}}^{\infty} 2^K \frac{2^{-2Kp}}{i^{(\alpha+1)p}} 2^{L-Lp} 2^{2Kp-K} 2^{Lp-L} \\
 & \leq C_p \sum_{i=2^{r-2}}^{\infty} \frac{1}{i^{(\alpha+1)p}} \leq C_p 2^{-r((\alpha+1)p-1)}.
 \end{aligned} \tag{20}$$

Similarly, we get by (6) that

$$\begin{aligned}
 |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)| & \leq \int_I \left| \int_J a(t, u) K_U^{\beta, \delta}(y-u) du \right| |K_T^{\alpha, \gamma}(x-t)| dt \\
 & \leq \frac{C 2^{-K(\alpha+1)} T^{-\alpha}}{i^{\alpha+1}} \int_I \left| \int_J a(t, u) K_U^{\beta, \delta}(y-u) du \right| dt
 \end{aligned}$$

whenever $x \in [i 2^K, (i+1) 2^K)$. Then

$$\begin{aligned}
 & \int_J \sup_{T \geq 2^{-K}, U \in \mathbf{R}_+} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dy \\
 & \leq \frac{C_p 2^{-K(\alpha+1)p} 2^{K\alpha p}}{i^{(\alpha+1)p}} |J|^{1-p} \left(\int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J a(t, u) K_U^{\beta, \gamma}(y-u) du \right| dy dt \right)^p.
 \end{aligned}$$

Applying an analogous inequality to (19) we can establish that

$$\begin{aligned}
 (B) & \leq C_p \sum_{i=2^{r-2}}^{\infty} 2^K \frac{2^{-Kp}}{i^{(\alpha+1)p}} 2^{L-Lp} 2^{2Kp-K} 2^{Lp-L} \\
 & \leq C_p \sum_{i=2^{r-2}}^{\infty} \frac{1}{i^{(\alpha+1)p}} \leq C_p 2^{-r((\alpha+1)p-1)}.
 \end{aligned} \tag{21}$$

Inequalities (20) and (21) imply

$$\int_{\mathbf{R} \setminus 2^r I} \int_J |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \leq C_p 2^{-\eta r}. \quad (22)$$

Next we integrate over $(\mathbf{R} \setminus 2^r I) \times (\mathbf{R} \setminus J)$. Similarly to (16),

$$\begin{aligned} & \int_{\mathbf{R} \setminus 2^r I} \int_{\mathbf{R} \setminus J} |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \leq \sum_{|i|=2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_{j2^L}^{(j+1)2^L} \sup_{T < 2^{-K}, U < 2^{-L}} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \quad + \sum_{|i|=2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_{j2^L}^{(j+1)2^L} \sup_{T < 2^{-K}, U \geq 2^{-L}} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \quad + \sum_{|i|=2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_{j2^L}^{(j+1)2^L} \sup_{T \geq 2^{-K}, U < 2^{-L}} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & \quad + \sum_{|i|=2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_{j2^L}^{(j+1)2^L} \sup_{T \geq 2^{-K}, U \geq 2^{-L}} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p dx dy \\ & = (C) + (D) + (E) + (F). \end{aligned}$$

Integrating by parts and using (7), (11), and (17) we conclude that

$$\begin{aligned} |\sigma_{T,U}^{\alpha, \beta, \gamma, \delta} a(x, y)| &= \left| \int_I \int_J A_{1,1}(t, u) (K_T^{\alpha, \gamma})' (x-t) (K_U^{\beta, \delta})' (y-u) dt du \right| \quad (23) \\ &\leq \int_I \int_J |A_{1,1}(t, u)| \frac{C}{T^{\alpha-1} |x-t|^{\alpha+1}} \frac{C}{U^{\beta-1} |y-u|^{\beta+1}} dt du \\ &\leq \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{1-\alpha} U^{1-\beta}}{i^{\alpha+1} j^{\beta+1}} \int_I \int_J |A_{1,1}(t, u)| dt du \\ &\leq \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{1-\alpha} U^{1-\beta}}{i^{\alpha+1} j^{\beta+1}} |I|^{1/2} |J|^{1/2} \|A_{1,1}\|_2 \\ &\leq \frac{C 2^{-K\alpha + K - K/p} 2^{-L\beta + L - L/p} T^{1-\alpha} U^{1-\beta}}{i^{\alpha+1} j^{\beta+1}}, \end{aligned}$$

where $x \in [i2^K, (i+1)2^K)$, $y \in [j2^L, (j+1)2^L)$. Henceforth

$$(C) \leq C_p \sum_{i=2^{r-2}}^{\infty} \sum_{j=1}^{\infty} 2^{K+L} \frac{2^{-K} 2^{-L}}{i^{(\alpha+1)p} j^{(\beta+1)p}} \leq C_p 2^{-r((\alpha+1)p-1)}. \quad (24)$$

We get in the same way as in (23) that

$$\begin{aligned}
 |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)| &= \left| \int_I \int_J A_{1,0}(t,u) (K_T^{\alpha,\gamma})'(x-t) K_U^{\beta,\delta}(y-u) dt du \right| \\
 &\leq \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{1-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}} \int_I \int_J |A_{1,0}(t,u)| dt du \\
 &\leq \frac{C 2^{-K\alpha+K-K/p} 2^{-L\beta-L/p} T^{1-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}}
 \end{aligned}$$

in case $x \in [i2^K, (i+1)2^K)$, $y \in [j2^L, (j+1)2^L)$. Hence the inequality (24) is also true for (D). The estimation of (E) is similar.

To estimate (F) let us observe that

$$\begin{aligned}
 |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)| &= \left| \int_I \int_J a(t,u) K_T^{\alpha,\gamma}(x-t) K_U^{\beta,\delta}(y-u) dt du \right| \\
 &\leq \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}} \int_I \int_J |a(t,u)| dt du \\
 &\leq \frac{C 2^{-K\alpha-K/p} 2^{-L\beta-L/p} T^{-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}}
 \end{aligned}$$

provided that $x \in [i2^K, (i+1)2^K)$, $y \in [j2^L, (j+1)2^L)$. This implies that (F) satisfies also (24).

Consequently,

$$\int_{\mathbf{R} \setminus 2^r I} \int_{\mathbf{R} \setminus J} |\sigma_*^{\alpha,\beta,\gamma,\delta} a(x,y)|^p dx dy \leq C_p 2^{-nr}. \quad (25)$$

The integrations over $I \times (\mathbf{R} \setminus 2^r J)$ and over $(\mathbf{R} \setminus I) \times (\mathbf{R} \setminus 2^r J)$ are similar. Hence (22) and (25) imply the p -quasi-locality of $\sigma_*^{\alpha,\beta,\gamma,\delta}$.

Inequality (14) for $\max\{1/(\alpha+1), 1/(\beta+1)\} < p = q \leq 1$ follows now from Theorems 3 and C. Applying Theorems A and 3 we obtain (14).

Let us point out this result for $p = 1$ and $q = \infty$. If $f \in H_1^\#(\mathbf{R} \times \mathbf{R})$ then (2) implies

$$\|\sigma_*^{\alpha,\beta,\gamma,\delta} f\|_{1,\infty} = \sup_{\rho > 0} \rho \lambda(\sigma_*^{\alpha,\beta,\gamma,\delta} f > \rho) \leq C \|f\|_{H_{1,\infty}(\mathbf{R} \times \mathbf{R})} \leq C \|f\|_{H_1^\#(\mathbf{R} \times \mathbf{R})}$$

which shows (15). The proof of the theorem is complete. \blacksquare

Remark. The proof of Theorem 4 is simpler if $\alpha = \beta = 1$ because we do not need the estimations of (B), (D), (E), and (F).

We can state the same for the maximal conjugate Riesz operators.

THEOREM 5. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. Then

$$\|\tilde{\sigma}_{*}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{R} \times \mathbf{R})} \quad (f \in H_{p,q}(\mathbf{R} \times \mathbf{R}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ and $0 < q \leq \infty$. Especially, if $f \in H_1^{\#}(\mathbf{R} \times \mathbf{R})$ then

$$\lambda(\tilde{\sigma}_{*}^{(i,j);\alpha,\beta,\gamma,\delta} f > \rho) \leq \frac{C}{\rho} \|f\|_{H_1^{\#}(\mathbf{R} \times \mathbf{R})} \quad (\rho > 0).$$

Proof. By Theorem 4 for $p = q$ and (3) we obtain

$$\|\tilde{\sigma}_{*}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_p = \|\sigma_{*}^{\alpha,\beta,\gamma,\delta} \tilde{f}^{(i,j)}\|_p \leq C_p \|\tilde{f}^{(i,j)}\|_{H_p(\mathbf{R} \times \mathbf{R})} = C_p \|f\|_{H_p(\mathbf{R} \times \mathbf{R})}$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$. Now Theorem 5 follows from Theorem A and from (2). ■

Since the set of those functions $f \in L_1(\mathbf{R}^2)$ whose Fourier transform has a compact support is dense in $H_1^{\#}(\mathbf{R} \times \mathbf{R})$, the weak type inequalities of Theorems 4 and 5 imply

COROLLARY 3. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. If $f \in H_1^{\#}(\mathbf{R} \times \mathbf{R})$ ($\sup L \log L(\mathbf{R}^2)$) then

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f \rightarrow \tilde{f}^{(i,j)} \quad \text{a.e. as } T, U \rightarrow \infty.$$

Note that $\tilde{f}^{(i,j)}$ is not necessarily in $H_1^{\#}(\mathbf{R} \times \mathbf{R})$ whenever f is.

Now we consider the norm convergence of $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ and extend Corollary 2.

THEOREM 6. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$, $i, j = 0, 1$ and $T, U \in \mathbf{R}_+$. Then

$$\|\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{H_{p,q}(\mathbf{R} \times \mathbf{R})} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{R} \times \mathbf{R})} \quad (f \in H_{p,q}(\mathbf{R} \times \mathbf{R}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ and $0 < q \leq \infty$.

Proof. Since $(\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f)^{\sim(i,j)} = \tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f$, we have by Theorem 5 that

$$\|(\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f)^{\sim(i,j)}\|_p \leq C_p \|f\|_{H_p(\mathbf{R} \times \mathbf{R})} \quad (f \in H_p(\mathbf{R} \times \mathbf{R}))$$

for all $T, U \in \mathbf{R}_+$ and $i, j = 0, 1$. The inequality

$$\|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f\|_{H_p(\mathbf{R} \times \mathbf{R})} \leq C_p \|f\|_{H_p(\mathbf{R} \times \mathbf{R})} \quad (f \in H_p(\mathbf{R} \times \mathbf{R}); T, U \in \mathbf{R}_+)$$

follows from (4). Hence, for $i, j = 0, 1$,

$$\|\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{H_p(\mathbf{R} \times \mathbf{R})} \leq C_p \|f\|_{H_p(\mathbf{R} \times \mathbf{R})} \quad (f \in H_p(\mathbf{R} \times \mathbf{R}); T, U \in \mathbf{N}).$$

Now Theorem 6 follows from Theorem A. \blacksquare

COROLLARY 4. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. If $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$, $0 < q \leq \infty$ and $f \in H_{p,q}(\mathbf{R} \times \mathbf{R})$ then*

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f \rightarrow \tilde{f}^{(i,j)} \quad \text{in } H_{p,q}(\mathbf{R} \times \mathbf{R}) \text{ norm as } T, U \rightarrow \infty.$$

We suspect that Theorems 4, 5, and 6 for $p \leq \max\{1/(\alpha+1), 1/(\beta+1)\}$ are not true though we could not find any counterexample.

We will extend the results to $\alpha > 1$ and $\beta > 1$. By integrating by parts we proved in [19] that

$$\sigma_{T,U}^{1+h,\beta,\gamma,\delta} f(x, y) = \frac{h(h+1)\gamma}{T} \int_0^T \left(1 - \left(\frac{s}{T}\right)^\gamma\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \sigma_{s,U}^{1,\beta,\gamma,\delta} f(x, y) ds,$$

where $h > 0$. In other words

$$\sigma_*^{\alpha,\beta,\gamma,\delta} f \leq C \sigma_*^{\alpha \wedge 1, \beta \wedge 1, \gamma, \delta} f$$

which shows that Theorems 3–5 hold also for $\alpha > 1$ and/or $\beta > 1$. The extension of Theorem 6 can be proved in the same way.

COROLLARY 5. *If $0 < \alpha, \beta < \infty$ and $1 \leq \gamma, \delta$ then all inequalities of Theorems 3–6 and all convergence results of Corollaries 2–4 hold for every $\max\{1/(1+\alpha \wedge 1), 1/(1+\beta \wedge 1)\} < p < \infty$ and $0 < q \leq \infty$.*

In the next sections we verify the results above in the periodic case, i.e., for the Riesz summability of two-parameter Fourier series.

6. HARDY SPACES ON THE BIDISC AND CONJUGATE FUNCTIONS

The Lorentz spaces on the measure space $(\mathbf{T}^2 := [-\pi, \pi]^2, \lambda)$ are denoted by $L_{p,q}(\mathbf{T}^2)$. Let f be a distribution on $C^\infty(\mathbf{T}^2)$ (briefly $f \in \mathcal{D}'(\mathbf{T}^2)$). The (n, m) th Fourier coefficient is defined by $\hat{f}(n, m) := f(e^{-inx} e^{-imy})$. In special case, if f is an integrable function then

$$\hat{f}(n, m) = \frac{1}{(2\pi)^2} \int_{\mathbf{T}} \int_{\mathbf{T}} f(x, y) e^{-inx} e^{-imy} dx dy.$$

For simplicity, we assume that, for a distribution $f \in \mathcal{D}'(\mathbf{T}^2)$, we have $\hat{f}(n, 0) = \hat{f}(0, n) = 0$ ($n \in \mathbf{N}$).

For $f \in \mathcal{D}'(\mathbf{T}^2)$ and $z_1 := re^{ix}$, $z_2 := se^{iy}$ ($0 < r, s < 1$) let

$$u(z_1, z_2) = u(re^{ix}, se^{iy}) := (f * P_r \otimes P_s)(x, y),$$

where

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{ikx} = \frac{1-r^2}{1+r^2-2r \cos x} \quad (x \in \mathbf{T})$$

is the periodic Poisson kernel.

The non-tangential maximal function is defined by

$$u^*(x, y) := \sup_{z_1 \in \Omega(x)} \sup_{z_2 \in \Omega(y)} |u(z_1, z_2)|,$$

where $\Omega(x)$ is the usual Stolz domain (see, e.g., Kashin and Saakyan [12] or Weisz [17]).

For $0 < p, q \leq \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{T} \times \mathbf{T})$ consists of all distributions f for which $u^* \in L_{p,q}(\mathbf{T}^2)$ and set

$$\|f\|_{H_{p,q}(\mathbf{T} \times \mathbf{T})} := \|u^*\|_{p,q}.$$

For $f \in L_1(\mathbf{T}^2)$ and $z := re^{ix}$ ($0 < r < 1$) let

$$v(z, y) = v(re^{ix}, y) := \frac{1}{2\pi} \int_{\mathbf{T}} f(t, y) P_r(x-t) dt$$

and

$$v^+(x, y) := \sup_{z \in \Omega(x)} |v(z, y)|.$$

We say that $f \in L_1(\mathbf{T}^2)$ is in the *hybrid Hardy–Lorentz space* $H_{p,q}^\#(\mathbf{T} \times \mathbf{T})$ if

$$\|f\|_{H_{p,q}^\#(\mathbf{T} \times \mathbf{T})} := \|v^+\|_{p,q} < \infty.$$

For a distribution

$$f \sim \sum_{k,l \in \mathbf{Z}} \hat{f}(k, l) e^{ikx + il y}$$

the *conjugate distributions* are defined by

$$\begin{aligned}\tilde{f}^{(1,0)} &\sim \sum_{k,l \in \mathbf{Z}} (-\imath \operatorname{sign} k) \hat{f}(k, l) e^{ikx + \imath ly}, \\ \tilde{f}^{(0,1)} &\sim \sum_{k,l \in \mathbf{Z}} (-\imath \operatorname{sign} l) \hat{f}(k, l) e^{ikx + \imath ly}\end{aligned}$$

and

$$\tilde{f}^{(1,1)} \sim \sum_{k,l \in \mathbf{Z}} (-\operatorname{sign}(kl)) \hat{f}(k, l) e^{ikx + \imath ly},$$

respectively. We use again the notation $\tilde{f}^{(0,0)} := f$. If f is an integrable function then

$$\begin{aligned}\tilde{f}^{(1,0)}(x, y) &= \text{p.v.} \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x-t, y)}{2 \tan(t/2)} dt, \\ \tilde{f}^{(0,1)}(x, y) &= \text{p.v.} \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x, y-u)}{2 \tan(u/2)} du\end{aligned}$$

and

$$\tilde{f}^{(1,1)}(x, y) = \text{p.v.} \frac{1}{\pi^2} \int_{\mathbf{T}} \int_{\mathbf{T}} \frac{f(x-t, y-u)}{4 \tan(t/2) \tan(u/2)} dt du.$$

We remark that the analogues of (1)–(4) and the analogues of Theorem A, B and C are true in this case (cf. Weisz [17] and the references there).

7. RIESZ SUMMABILITY OF TWO-PARAMETER FOURIER SERIES

The *Riesz means* of a distribution f are defined by

$$\begin{aligned}\sigma_{n,m}^{\alpha,\beta,\gamma,\delta} f(x, y) &:= \sum_{k=-n}^n \sum_{l=-m}^m \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1 - \left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} \hat{f}(k, l) e^{ikx} e^{\imath ly} \\ &=: f * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x, y)\end{aligned}$$

where

$$\kappa_n^{\alpha,\gamma}(x) := \sum_{k=-n}^n \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} e^{ikx} \quad (x \in \mathbf{T})$$

is the periodic Riesz kernel. Similarly, we introduce the *conjugate Riesz means* of a distribution f by

$$\begin{aligned}\tilde{\sigma}_{n,m}^{(1,0);\alpha,\beta,\gamma,\delta}f(x,y) &= \sum_{k=-n}^n \sum_{l=-m}^m \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1 - \left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} \\ &\quad (-\iota \operatorname{sign} k) \hat{f}(k,l) e^{ikx} e^{\iota ly} \\ &= \tilde{f}^{(1,0)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y), \\ \tilde{\sigma}_{n,m}^{(0,1);\alpha,\beta,\gamma,\delta}f(x,y) &= \sum_{k=-n}^n \sum_{l=-m}^m \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1 - \left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} \\ &\quad (-\iota \operatorname{sign} l) \hat{f}(k,l) e^{ikx} e^{\iota ly} \\ &= \tilde{f}^{(0,1)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y)\end{aligned}$$

and

$$\begin{aligned}\tilde{\sigma}_{n,m}^{(1,1);\alpha,\beta,\gamma,\delta}f(x,y) &= \sum_{k=-n}^n \sum_{l=-m}^m \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1 - \left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} \\ &\quad (-\operatorname{sign}(kl)) \hat{f}(k,l) e^{ikx} e^{\iota ly} \\ &= \tilde{f}^{(1,1)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y),\end{aligned}$$

respectively. The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta}f := \sup_{n,m \in \mathbb{N}} |\tilde{\sigma}_{n,m}^{(i,j);\alpha,\beta,\gamma,\delta}f|,$$

where $\tilde{\sigma}_{n,m}^{(0,0);\alpha,\beta,\gamma,\delta}f := \sigma_{n,m}^{\alpha,\beta,\gamma,\delta}f$ and we define again $\sigma_*^{\alpha,\beta,\gamma,\delta}f := \tilde{\sigma}_*^{(0,0);\alpha,\beta,\gamma,\delta}f$.

We proved in [19] that

$$\kappa_n^{\alpha,\gamma}(x) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{m+1}^{\alpha,\gamma}(x + 2k\pi),$$

where $n \in \mathbb{N}$, $\alpha, \gamma > 0$ and $x \in \mathbb{T}$ (cf. also Butzer and Nessel [3]). From this it follows that the analogues to (5)–(7) hold, namely, for $n \in \mathbb{N}$ and $0 < \alpha \leq 1 \leq \gamma$,

$$\int_{\mathbb{T}} |\kappa_n^{\alpha,\gamma}| d\lambda \leq C, \quad |\kappa_n^{\alpha,\gamma}(x)| \leq \frac{C}{n^{\alpha} |x|^{\alpha+1}} \quad (x \in \mathbb{T}, x \neq 0)$$

and

$$|(\kappa_n^{\alpha, \gamma})'(x)| \leq \frac{C}{n^{\alpha-1} |x|^{\alpha+1}} \quad (x \in \mathbf{T}, x \neq 0).$$

Using these estimates we can prove the following results in the same way as in Sections 4 and 5, so we omit the proofs.

THEOREM 7. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $1 < p \leq \infty$. Then*

$$\|\sigma_*^{\alpha, \beta, \gamma, \delta} f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbf{T}^2)).$$

COROLLARY 6. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $1 < p < \infty$. Then for every $f \in L_p(\mathbf{T}^2)$ we have*

$$\sigma_{n,m}^{\alpha, \beta, \gamma, \delta} f \rightarrow f \quad \text{a.e. and in } L_p(\mathbf{T}^2) \text{ norm as } n, m \rightarrow \infty.$$

THEOREM 8. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. Then*

$$\|\tilde{\sigma}_*^{(i,j); \alpha, \beta, \gamma, \delta} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{T} \times \mathbf{T})} \quad (f \in H_{p,q}(\mathbf{T} \times \mathbf{T}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ and $0 < q \leq \infty$. Especially, if $f \in H_1^\#(\mathbf{T} \times \mathbf{T})$ then

$$\lambda(\tilde{\sigma}_*^{(i,j); \alpha, \beta, \gamma, \delta} f > \rho) \leq \frac{C}{\rho} \|f\|_{H_1^\#(\mathbf{T} \times \mathbf{T})} \quad (\rho > 0).$$

Note that a very special case of Theorem 8, i.e., if $i = j = 0$, $\alpha = \beta = \gamma = \delta = 1$ was proved in Weisz [17] only for $3/4 < p < \infty$.

COROLLARY 7. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. If $f \in H_1^\#(\mathbf{T} \times \mathbf{T})$ ($\supset L \log L(\mathbf{T}^2)$) then*

$$\tilde{\sigma}_{n,m}^{(i,j); \alpha, \beta, \gamma, \delta} f \rightarrow \tilde{f}^{(i,j)} \quad \text{a.e. as } n, m \rightarrow \infty.$$

THEOREM 9. *Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$, $i, j = 0, 1$ and $n, m \in \mathbf{N}$. Then*

$$\|\tilde{\sigma}_{n,m}^{(i,j); \alpha, \beta, \gamma, \delta} f\|_{H_{p,q}(\mathbf{T} \times \mathbf{T})} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{T} \times \mathbf{T})} \quad (f \in H_{p,q}(\mathbf{T} \times \mathbf{T}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ and $0 < q \leq \infty$.

COROLLARY 8. Assume that $0 < \alpha, \beta \leq 1 \leq \gamma, \delta$ and $i, j = 0, 1$. If $\max\{1/(\alpha + 1), 1/(\beta + 1)\} < p < \infty$, $0 < q \leq \infty$ and $f \in H_{p,q}(\mathbf{T} \times \mathbf{T})$ then

$$\tilde{\sigma}_{n,m}^{(i,j);\alpha,\beta,\gamma,\delta} f \rightarrow \tilde{f}^{(i,j)} \quad \text{in } H_{p,q}(\mathbf{T} \times \mathbf{T}) \text{ norm as } n, m \rightarrow \infty.$$

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