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Weak inequalities for maximal functions in Orlicz–Lorentz spaces and applications[☆]

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Abstract

Let $0 < \alpha \leq \infty$ and let $\{B(x, \epsilon)\}_\epsilon$, $\epsilon > 0$, denote a net of intervals of the form $(x - \epsilon, x + \epsilon) \subset [0, \alpha)$. Let $f^\epsilon(x)$ be any best constant approximation of $f \in \Lambda_{w,\phi'}$ on $B(x, \epsilon)$. Weak inequalities for maximal functions associated with $\{f^\epsilon(x)\}_\epsilon$, in Orlicz–Lorentz spaces, are proved. As a consequence of these inequalities we obtain a generalization of Lebesgue's Differentiation Theorem and the pointwise convergence of $f^\epsilon(x)$ to $f(x)$, as $\epsilon \to 0$.

Keywords: Orlicz–Lorentz spaces; Maximal functions; Best constant approximant; a.e. convergence

1. Introduction

Let $\mathcal{M}_0$ be the class of all real extended μ-measurable functions on $[0, \alpha)$, $0 < \alpha \leq \infty$, where μ is the Lebesgue measure. As usual, for $f \in \mathcal{M}_0$ we denote its distribution function by $\mu_f(s) = \mu(\{x \in [0, \alpha) : |f(x)| > s\})$, $s \geq 0$, and its decreasing rearrangement by $f^*(t) = \inf\{s : \mu_f(s) \leq t\}$, $t \geq 0$. For properties of μ_f and f^*, the reader can look at ([2], pp. 36–42).

Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a differentiable and convex function, $\phi(0) = 0$, $\phi(t) > 0$, $t > 0$, and let $w : (0, \alpha) \to (0, \infty)$ be a weight function, non-increasing and locally integrable. If $\alpha = \infty$,

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we also assume $\int_0^\infty w\mathrm{d}\mu = \infty$. We denote by $W : [0, \alpha) \to [0, \infty)$ the function

$$W(r) = \int_0^r w(t)\mathrm{d}t.$$

For $f \in \mathcal{M}_0$, let

$$\Psi_{w,\phi}(f) = \int_0^{\mu_f(0)} \phi(f^*) w\mathrm{d}\mu.$$

In [9,11–13], several authors studied geometric properties of the regular Orlicz–Lorentz space $\{f \in \mathcal{M}_0 : \Psi_{w,\phi}(\lambda f) < \infty$ for some $\lambda > 0\}$. We consider the following subspace:

$$\Lambda_{w,\phi} := \{f \in \mathcal{M}_0 : \Psi_{w,\phi}(\lambda f) < \infty \text{ for all } \lambda > 0\}.$$

Under the Luxemburg norm given by $\|f\|_{w,\phi} = \inf\left\{\epsilon > 0 : \Psi_{w,\phi}\left(\frac{f}{\epsilon}\right) \le 1\right\}$, the Orlicz–Lorentz space is a Banach space (see [11]). If w is constant, it is the Orlicz space L_ϕ (see [20]). On the other hand, setting $\phi(t) = t^p$, $1 \le p < \infty$, we obtain the Lorentz space $L_{w,p}$ and $\Psi_{w,\phi}(f) = \|f\|_{w,p}^p$. These spaces have been studied in [7]. If $w(t) = \frac{p}{q} t^{\frac{q}{p}-1}$, $1 \le q \le p < \infty$, a good reference for a description of these spaces is [10].

A function ϕ satisfies the Δ_2-condition if there exists $K > 0$ such that $\phi(2t) \le K\phi(t)$ for all $t \ge 0$. We denote it briefly by $\phi \in \Delta_2$. We recall that if $\phi \in \Delta_2$, then the subspace $\Lambda_{w,\phi}$ is the Orlicz–Lorentz space.

If ϕ' is the derivative of the function ϕ, the space $\Lambda_{w,\phi'}$ is analogously defined. We write $\phi \in \Phi_0$ if $\phi'(0) = 0$, where $\phi'(0)$ is the right derivative of ϕ at 0.

For $g \in \mathcal{M}_0$, we write $N(g) := \{|g| > 0\}$ and $Z(g) := \{g = 0\}$.

We will denote by $\mathcal{S}$ the class of step functions in $\mathcal{M}_0$ with support in a set of finite measure, i.e., $g \in \mathcal{S}$ if $g = \sum_{k=1}^m a_k \chi_{U_k}$, where a_k are real numbers, U_k are finite measure intervals, and χ_V is the characteristic function of set V.

Observe that the inequalities $\phi(x) \le x\phi'(x) \le \phi(2x)$, $x \ge 0$, hold. Therefore

$$\{f \in \Lambda_{w,\phi} : \mu(N(f)) < \infty\} \subset \Lambda_{w,\phi'}.$$

Let $A \subset [0, \alpha)$ be a finite measure set. For $f \in \Lambda_{w,\phi}$, we write $C(f, A)$ as the set of all constants c minimizing the expression $\Psi_{w,\phi}((f - c)\chi_A)$. It is easy to see that $C(f, A)$ is a nonempty compact interval for every $f \in \Lambda_{w,\phi}$ (see [17]). Each element of $C(f, A)$ is called a best constant approximation of f on A. We put $f_A = \min C(f, A)$ and $f^A = \max C(f, A)$.

We denote by T_A the best constant approximant operator which assigns to each $f \in \Lambda_{w,\phi}$ the set $C(f, A) = [f_A, f^A]$. In [17], T_A is extended from an Orlicz–Lorentz space $\Lambda_{w,\phi}$ to the space $\Lambda_{w,\phi'}$, in the following way: for $f \in \Lambda_{w,\phi'}$, $T_A(f) = [f_A, f^A]$, $f_A = \min\{c : \gamma^+((c - f)\chi_A, \chi_A) \ge 0\}$, and $f^A = \max\{c : \gamma^+((f - c)\chi_A, \chi_A) \ge 0\}$, where $\gamma^+(g, h)$ is defined by (2.11) in ([16], Theorem 2.14) for $g, h \in \Lambda_{w,\phi'}$. Any $c \in T_A(f)$ is said to be a best constant approximation of $f \in \Lambda_{w,\phi'}$ on A. Moreover, the monotonicity property in the sense of Landers and Rogge (see [14]) of its extension is established.

Let $\{B(x, \epsilon)\}_\epsilon$, $\epsilon > 0$, denote a net of intervals of the form $(x - \epsilon, x + \epsilon) \subset [0, \alpha)$. For $f \in \Lambda_{w,\phi'}$, we define by $Mf : (0, \alpha) \to \mathbb{R}$ the maximal function

$$Mf(x) = \sup\left\{\frac{\Psi_{w,\phi'}(f\chi_{B(x,\epsilon)})}{\Psi_{w,\phi'}(\chi_{B(x,\epsilon)})} : \epsilon > 0 \text{ and } B(x, \epsilon) \subset (0, \alpha)\right\}.$$

In [1], weak inequalities for Mf have been studied for when $\Lambda_{w,\phi'}$ is the Lorentz space $L_{p,q}$, $1 \le p, q < \infty$.

Let $f^\epsilon(x)$ be any best constant approximation of $f \in \Lambda_{w,\phi'}$ on $B(x,\epsilon)$. For $f \in L_2$, it is easy to check that $f^\epsilon(x)$ is the average $\frac{1}{\mu(B(x,\epsilon))}\int_{B(x,\epsilon)} f$. From [5], we have that if f is differentiable at x, then these averages converge to $f(x)$ a.e., as $\epsilon \to 0$. A more adequate version of this fact is given by Lebesgue's Differentiation Theorem, which says that $f^\epsilon(x) \to f(x)$, as $\epsilon \to 0$, for every locally integrable function f (see [22]). In [15] the authors extend the best approximation operator from L_p to L_{p-1}, when $p > 1$ and the approximation class is a σ lattice of functions. They studied almost everywhere convergence of best approximants. In [19], Lebesgue's Differentiation Theorem was generalized using best approximation by constants over balls in the $L_p(\mathbb{R}^n)$ spaces with $1 \le p < \infty$. They extended the best approximation operator by constants over balls from $L_p(\mathbb{R}^n)$ to $L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, for $1 \le p < \infty$, and they showed the convergence of best constant approximations when the diameters of the balls shrink to 0. Similar results in a subspace of the Orlicz space $L_{\phi'}(\mathbb{R}^n)$ have appeared in [6].

Other generalizations of the classical Lebesgue's Differentiation Theorem can be considered; for example to prove that certain integral averages of a function g from the space converge to g a.e.. The convergence of integral averages of a function from L_p, $1 \le p < \infty$, can be seen in [22,23].

In Section 2, we present a certain type of Dominated Convergence Theorem in $\Lambda_{w,\phi'}$. Moreover, the density of the simple functions and also that of the step functions are established. In Section 3, we show weak inequalities for the maximal function Mf. As a consequence of these inequalities we prove the convergence of integral averages of a function from $\Lambda_{w,\phi'}$, i.e., a generalization of Lebesgue's Differentiation Theorem. In Section 4, weak inequalities are proved for the maximal function associated with the family $\{f^\epsilon(x)\}_\epsilon$, which are used in the study of pointwise convergence of $f^\epsilon(x)$ to $f(x)$, as $\epsilon \to 0$, another extension of Lebesgue's Differentiation Theorem. The results of this paper generalize [19,6] for the case of one-variable functions.

2. Dominated convergence and density in $\Lambda_{w,\phi'}$

We begin this section by proving a type of Dominated Convergence Theorem in $\Lambda_{w,\phi'}$.

Let $h \in \Lambda_{w,\phi'}$ and let $D \subset [0,\alpha)$ be a measurable set such that $N(h) \subset D$. Let $\rho : D \to [0, \mu(D))$ be any measure preserving transformation (m.p.t.). It is easy to see that

$$(w(\rho))^* = w, \quad \text{on } (0, \mu(D))$$

(see [2], pp. 80), and

$$(\phi'(|h|)\chi_{N(h)})^* = \phi'(h^*)\chi_{[0,\mu_h(0))}, \quad \text{on } [0,\alpha).$$

From the Hardy and Littlewood's inequality (see [2], pp. 44) it follows that

$$\int_B w(\rho)\phi'(h)\mathrm{d}\mu \le \int_0^{\mu(B)} \phi'(h^*)w \le \Psi_{w,\phi'}(h), \tag{1}$$

for every measurable set $B \subset N(h)$.

Lemma 2.1. *Let $f, g \in \Lambda_{w,\phi'}$ be nonnegative functions. If* $\min\{f, g\} = 0$*, then* $\Psi_{w,\phi'}(f+g) \le \Psi_{w,\phi'}(f) + \Psi_{w,\phi'}(g)$.

Proof. Since $\lim_{t\to\infty}(f+g)^*(t)=0$, there is a m.p.t. $\rho : N(f+g)\to[0,\mu_{f+g}(0))$ such that $f+g=(f+g)^*\circ\rho$, a.e. on $N(f+g)$ (see [2], pp. 83). By hypothesis, $N(f+g)=N(f)\cup N(g)$ and $N(f)\cap N(g)=\emptyset$. Therefore

$$\begin{aligned}\Psi_{w,\phi'}(f+g) &= \int_{N(f+g)} w(\rho)\phi'(f+g)\mathrm{d}\mu \\ &= \int_{N(f)} w(\rho)\phi'(f)\mathrm{d}\mu + \int_{N(g)} w(\rho)\phi'(g)\mathrm{d}\mu.\end{aligned}$$

Finally, the proof follows from (1). $\square$

Remark 2.2. We observe that $\Psi_{w,\phi'}(f)\le\Psi_{w,\phi'}(g)$ if $|f|\le|g|$, a.e. on $[0,\alpha)$.

Lemma 2.3. *Let $f,g\in\Lambda_{w,\phi'}$. Then $\Psi_{w,\phi'}(f+g)\le\Psi_{w,\phi'}(2f)+\Psi_{w,\phi'}(2g)$. In addition, if $\phi\in\Delta_2$ then there exists $C>0$ such that $\Psi_{w,\phi'}(f+g)\le C\left(\Psi_{w,\phi'}(f)+\Psi_{w,\phi'}(g)\right)$.*

Proof. From Lemma 2.1 it follows that

$$\begin{aligned}\Psi_{w,\phi'}(f+g) &\le \Psi_{w,\phi'}(|f|+|g|)=\Psi_{w,\phi'}\left((|f|+|g|)\chi_{|f|\ge|g|}+(|f|+|g|)\chi_{|f|<|g|}\right)\\ &\le \Psi_{w,\phi'}(2f)+\Psi_{w,\phi'}(2g).\end{aligned}$$

Now, we assume $\phi\in\Delta_2$. Then there exists $K>0$ such that $\phi(2t)\le K\phi(t)$, $t>0$. According to ([6], Lemma 13), we have

$$\phi'(a+b)\le\frac{K^2}{2}(\phi'(a)+\phi'(b)),\quad a,b>0. \tag{2}$$

Therefore, $\Psi_{w,\phi'}(f+g)\le K^2\left(\Psi_{w,\phi'}(f)+\Psi_{w,\phi'}(g)\right)$. $\square$

Theorem 2.4 (*Dominated Convergence*). *Let $g\in\Lambda_{w,\phi'}$. If f_n, $n\in\mathbb{N}$, and f are measurable functions satisfying $|f_n|\le|g|$, and $\lim_{n\to\infty}f_n=f$ a.e., then*

$$\lim_{n\to\infty}\mu_{f_n-f}(s)=0\quad\text{and}\quad\lim_{n\to\infty}\int_0^{\mu_{f_n-f}(s)}\phi'((f_n-f)^*)w=0,\quad s>0. \tag{3}$$

In addition, if $\phi\in\Phi_0$ then $\lim_{n\to\infty}\Psi_{w,\phi'}(f_n-f)=0$.

Proof. Let $s>0$ and set $h_n(x)=\sup_{k\ge n}|f_k(x)-f(x)|$, $n\in\mathbb{N}$. Clearly $|f_n-f|\le|h_n|\le 2|g|$ a.e., which gives $\mu_{f_n-f}\le\mu_{h_n}\le\mu_{2g}$. Since $h_n\downarrow 0$ a.e. and $\mu_{h_1}(s)<\infty$, we see that $\mu_{h_n}(s)\downarrow 0$ and so $\lim_{n\to\infty}\mu_{f_n-f}(s)=0$.
Now, the inequality

$$\int_0^{\mu_{f_n-f}(s)}\phi'((f_n-f)^*)w\le\int_0^{\mu_{f_n-f}(s)}\phi'(2g^*)w$$

implies the second part of (3).
Finally, we assume $\phi\in\Phi_0$. From ([11], Lemma 2.1) we have $h_n^*\downarrow 0$, and consequently $\lim_{n\to\infty}\phi'((f_n-f)^*)=0$. Since

$$\Psi_{w,\phi'}(f_n-f)\le\int_0^{\mu_{2g}(0)}\phi'((f_n-f)^*)w,$$

the Lebesgue Dominated Convergence Theorem implies $\lim_{n\to\infty}\Psi_{w,\phi'}(f_n-f)=0$. $\square$

Next, we prove that the sets of simple functions and step functions are dense.

Lemma 2.5. *Let f be a simple function with finite measure support. Then for each $\epsilon > 0$, there exists $g \in \mathcal{S}$ such that $\mu_{g-f}(s) < \epsilon$, for all $s \geq 0$ and $\Psi_{w,\phi'}(g - f) < \epsilon$.*

Proof. Let $\epsilon > 0$. If $f = 0$, it is obvious. Without loss of generality we can assume that $f = \sum_{k=1}^{m} a_k \chi_{E_k}$, where the sets E_k are pairwise disjoint subsets of $(0, \alpha)$ with finite measure, $a_k \neq 0$, $1 \leq k \leq m$, and $a_i \neq a_j$ if $i \neq j$. Since $\lim_{r\to 0^+} W(r) = 0$, there exists δ, $0 < \delta < \epsilon$, such that

$$W(\delta) \leq \frac{\epsilon}{\phi'(m\|f\|_\infty)}. \tag{4}$$

For each k, $1 \leq k \leq m$, let U_k be a finite union of open intervals such that $\mu(U_k \triangle E_k) < \frac{\delta}{m}$. Set $g = \sum_{k=1}^{m} a_k \chi_{U_k}$. It is clear that $g \in \mathcal{S}$ and $|g - f| \leq m\|f\|_\infty \chi_{\bigcup_{k=1}^{m} U_k \triangle E_k}$. Therefore, we have $\mu_{g-f} \leq \delta \chi_{[0, m\|f\|_\infty)} < \epsilon$ and

$$\Psi_{w,\phi'}(g - f) \leq \phi'(m\|f\|_\infty) W\left(\mu\left(\bigcup_{k=1}^{m} (U_k \triangle E_k)\right)\right) < \phi'(m\|f\|_\infty) W(\delta) < \epsilon. \quad \square$$

Theorem 2.6. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Delta_2$, then there exists a sequence $\{f_n\}_n \subset \mathcal{S}$ such that*

$$\lim_{n\to\infty} \mu_{f_n - f}(s) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_0^{\mu_{f_n-f}(s)} \phi'((f_n - f)^*) w = 0, \quad s > 0.$$

In addition, if $\phi \in \Phi_0$, then $\lim_{n\to\infty} \Psi_{w,\phi'}(f_n - f) = 0$.

Proof. Let $s > 0$ and let $\{h_n\}_n$ be a sequence of simple functions, each with support in a set of finite measure such that $|h_n| \leq |f|$ for all n and $\lim_{n\to\infty} h_n = f$ a.e.. According to Lemma 2.5, there exists a sequence $\{f_n\}_n \subset \mathcal{S}$ such that

$$\mu_{f_n - h_n} \leq \frac{1}{n} \quad \text{and} \quad \Psi_{w,\phi'}(f_n - h_n) < \frac{1}{n}. \tag{5}$$

Since $\mu_{f_n - f}(s) \leq \mu_{f_n - h_n}\left(\frac{s}{2}\right) + \mu_{h_n - f}\left(\frac{s}{2}\right)$, by Theorem 2.4 we get

$$\lim_{n\to\infty} \mu_{f_n - f}(s) = 0. \tag{6}$$

On the other hand, $(f_n - f)^*(t) \leq (f_n - h_n)^*\left(\frac{t}{2}\right) + (h_n - f)^*\left(\frac{t}{2}\right)$, $t > 0$. From (2) it follows that there is a $K > 0$ satisfying

$$\phi'((f_n - f)^*(t)) \leq \frac{K^2}{2}\left(\phi'\left((f_n - h_n)^*\left(\frac{t}{2}\right)\right) + \phi'\left((h_n - f)^*\left(\frac{t}{2}\right)\right)\right), \quad t > 0.$$

As w is a non-increasing function,

$$\phi'((f_n - f)^*(t)) w(t) \leq \frac{K^2}{2}\left(\phi'\left((f_n - h_n)^*\left(\frac{t}{2}\right)\right) + \phi'\left((h_n - f)^*\left(\frac{t}{2}\right)\right)\right) w\left(\frac{t}{2}\right),$$

$t > 0$, and consequently

$$\int_0^{\mu_{f_n-f}(s)} \phi'((f_n - f)^*) w \leq K^2 \int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \left(\phi'((f_n - h_n)^*) + \phi'((h_n - f)^*)\right) w. \tag{7}$$

It is easy to see that

$$\int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'((h_n-f)^*)w \le \int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'(2f^*)w. \tag{8}$$

We observe that if $\mu_{f_n-h_n}(0) < \frac{1}{2}\mu_{f_n-f}(s)$,

$$\int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'((f_n-h_n)^*)w = \Psi_{w,\phi'}(f_n-h_n) + \int_{\mu_{f_n-h_n}(0)}^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'((f_n-h_n)^*)w.$$

Since $(f_n-h_n)^*(t)=0$, for $t \ge \mu_{f_n-h_n}(0)$, we get

$$\int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'((f_n-h_n)^*)w \le \Psi_{w,\phi'}(f_n-h_n) + \phi'(0)W\left(\frac{1}{2}\mu_{f_n-f}(s)\right). \tag{9}$$

Otherwise, (9) is obvious, because

$$\int_0^{\frac{1}{2}\mu_{f_n-f}(s)} \phi'((f_n-h_n)^*)w \le \Psi_{w,\phi'}(f_n-h_n).$$

Thus, (5)–(9) imply $\lim_{n\to\infty}\int_0^{\mu_{f_n-f}(s)} \phi'((f_n-f)^*)w = 0$.
Finally, we assume $\phi \in \Phi_0$. By Theorem 2.4, we get $\lim_{n\to\infty} \Psi_{w,\phi'}(h_n-f)=0$. So, the proof follows from (5) and Lemma 2.3. □

3. Lebesgue's differentiation theorem in $\Lambda_{w,\phi'}$

In this section, we study weak inequalities for the maximal function Mf. As a consequence, we prove the convergence of integral averages of a function from $\Lambda_{w,\phi'}$. More precisely, we extend ([23], Lemma 5). In addition, we also extend ([22], pp. 25) for the case of one-variable functions.
For $\phi \in \Phi_0$, it is easy to see that $(\phi'(|f|))^* = \phi'(f^*)$, $\Psi_{w,\phi'}(f) = \int_0^\infty (\phi'(|f|))^* w$, and

$$\phi'(f^*(t)) \le \frac{1}{W(t)}\Psi_{w,\phi'}(f), \quad t>0. \tag{10}$$

So, from ([4], Theorem 2.1) we obtain

$$\Psi_{w,\phi'}(f) = \int_0^\infty W\left(\mu_{\phi'(|f|)}(s)\right)\mathrm{d}s, \quad f \in \Lambda_{w,\phi'}. \tag{11}$$

Definition 3.1. $\Lambda_{w,\phi'}$ is said to satisfy a *lower W-estimate* if there exists a constant $N<\infty$ such that, for every choice of functions $\{f_k\}_{k=1}^n$ in $\Lambda_{w,\phi'}$ with pairwise disjoint supports, we have

$$N\Psi_{w,\phi'}\left(\sum_{k=1}^n f_k\right) \ge \lambda W\left(\sum_{k=1}^n W^{-1}\left(\frac{\Psi_{w,\phi'}(f_k)}{\lambda}\right)\right), \quad \lambda>0. \tag{12}$$

Remark 3.2. In a special case when $W(t)$=t and $\phi'(t)=t^p$, $1<p<\infty$, we recover the well known notion of a lower p-estimate in L_p (see [18]). If $W(r)=r^{\frac{q}{p}}$ and $\phi'(t)=t^q$, $1\le q\le p<\infty$, then $\Lambda_{w,\phi'}$ is the Lorentz space $L_{p,q}$ and it satisfies a lower W-estimate (see [1]).

Proposition 3.3. *If $W(r) = c r^{\frac{1}{a}}$, $a \geq 1$, $c > 0$, and $\phi \in \Phi_0$, then $\Lambda_{w,\phi'}$ satisfies a lower W-estimate.*

Proof. If $a = 1$, it is obvious. Now assume $a > 1$. Let $\lambda > 0$ and let $\{f_k\}_{k=1}^n$ be functions in $\Lambda_{w,\phi'}$ with pairwise disjoint supports. From (11) and Minkowski's vector-valued inequality ([8], pp. 148), we have

$$\lambda W\left(\sum_{k=1}^{n} W^{-1}\left(\frac{\Psi_{w,\phi'}(f_k)}{\lambda}\right)\right) = c\left\|\left\{\int_0^\infty \left(\mu_{\phi'(|f_k|)}(s)\right)^{\frac{1}{a}} \mathrm{d}s\right\}_{k=1}^{n}\right\|_{l_a(\mathbb{R}^n)}$$
$$\leq c\int_0^\infty \left\|\left\{\left(\mu_{\phi'(|f_k|)}(s)\right)^{\frac{1}{a}}\right\}_{k=1}^{n}\right\|_{l_a(\mathbb{R}^n)} \mathrm{d}s$$
$$= c\int_0^\infty \left(\sum_{k=1}^{n} \mu_{\phi'(|f_k|)}(s)\right)^{\frac{1}{a}} \mathrm{d}s. \tag{13}$$

Since $\phi \in \Phi_0$ and $\{f_k\}_{k=1}^n$ have pairwise disjoint supports, it is clear that

$$\sum_{k=1}^{n} \mu_{\phi'(|f_k|)}(s) = \mu_{\sum_{k=1}^{n} \phi'(|f_k|)}(s) = \mu_{\phi'\left(\sum_{k=1}^{n} |f_k|\right)}(s), \quad s > 0.$$

So, (11) and (13) imply (12). □

Let $f \in \Lambda_{w,\phi'}$ and $\epsilon > 0$. We denote by $f_\epsilon : (0,\alpha) \to \mathbb{R}$ the function

$$f_\epsilon(x) = \frac{\Psi_{w,\phi'}\left(f\chi_{B(x,\epsilon)}\right)}{\Psi_{w,\phi'}\left(\chi_{B(x,\epsilon)}\right)}.$$

Lemma 3.4. *Let $f \in \Lambda_{w,\phi'}$ and $\epsilon > 0$. If $\phi \in \Phi_0$, then f_ϵ is a measurable function on $(0,\alpha)$.*

Proof. Let $h = \sum_{k=1}^n a_k \chi_{E_k}$ be a nonnegative simple function where the sets E_k are pairwise disjoint subsets of $(0,\alpha)$ with $a_1 > a_2 > \cdots > a_n > 0$. Then, $\left(h\chi_{B(x,\epsilon)}\right)^* = \sum_{k=1}^n a_k \chi_{[m_{k-1}(x), m_k(x))}$, where $m_0 = 0$ and

$$m_k(x) = \sum_{i=1}^{k} \mu(E_i \cap B(x,\epsilon)), \quad 1 \leq k \leq n.$$

Thus, $h_\epsilon(x) = \sum_{k=1}^n \frac{\phi'(a_k)}{\phi'(1)W(2\epsilon)} (W(m_k(x)) - W(m_{k-1}(x)))$. Since $\{m_k\}_{k=0}^n$ are measurable functions, it follows that h_ϵ is a measurable function. Now, let $\{f_n\}_{n=1}^\infty$ be a sequence of nonnegative simple functions such that $f_n \uparrow |f|$. Then

$$\left(f_n\chi_{B(x,\epsilon)}\right)^* \uparrow \left(|f|\chi_{B(x,\epsilon)}\right)^*, \quad x \in (0,\alpha).$$

Therefore, the Monotone Convergence Theorem implies $\lim_{n\to\infty}(f_n)_\epsilon = f_\epsilon$, on $(0,\alpha)$. So, f_ϵ is a measurable function. □

Lemma 3.5. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Phi_0$, then Mf is a measurable function.*

Proof. Given $\epsilon > 0$, it is easy to see that for each $x \in (0,\alpha)$, $\lim_{r\to\epsilon^-} f_r(x) = f_\epsilon(x)$. Therefore, $Mf(x) = \sup\{f_\epsilon(x) : \epsilon > 0, \epsilon \in \mathbb{Q} \text{ and } B(x,\epsilon) \subset (0,\alpha)\}$. Since the family is countable, from Lemma 3.4, Mf is a measurable function. □

Theorem 3.6. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Phi_0$ and $\Lambda_{w,\phi'}$ satisfies a lower W-estimate, then there exists a constant $C > 0$ such that*

$$W\left(\mu_{Mf}(s)\right) \leq \frac{C}{s}\Psi_{w,\phi'}(f), \quad s > 0. \tag{14}$$

Proof. Let $s > 0$. For each $x \in \Omega_s := \{Mf > s\}$, there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset (0, \alpha)$ and

$$f_{\epsilon_x}(x) > s. \tag{15}$$

Let $c < \mu(\Omega_s)$ and let $B := \bigcup_{x\in\Omega_s} B(x, \epsilon_x)$. Then $c < \mu(B)$. As μ is a regular measure, there exists a compact set $K \subset B$ such that $c < \mu(K)$. Since $\mathcal{C} = \{B(x, \epsilon_x)\}_{x\in\Omega_s}$ is an open covering of K, we can extract a finite subcovering $\mathcal{D} \subset \mathcal{C}$. Therefore, by Lemma 7.3 in [21], there is a pairwise disjoint finite collection $\{B(x_k, \epsilon_{x_k})\}_{k=1}^n \subset \mathcal{D}$ such that

$$c < 3\sum_{k=1}^{n} \mu(B(x_k, \epsilon_{x_k})). \tag{16}$$

As $W(3r) \leq 3W(r)$, $r > 0$, from (15) and (16) we obtain

$$W(c) < 3W\left(\sum_{k=1}^{n}\mu(B(x_k, \epsilon_{x_k}))\right) \leq 3W\left(\sum_{k=1}^{n} W^{-1}\left(\frac{\Psi_{w,\phi'}(f\chi_{B(x_k,\epsilon_{x_k})})}{s\phi'(1)}\right)\right).$$

Since, by the hypotheses, there exists $N > 0$ satisfying (12), we have

$$W(c) \leq \frac{3N}{s\phi'(1)}\Phi_{w,\phi'}\left(\sum_{k=1}^{n} f\chi_{B(x_k,\epsilon_{x_k})}\right) \leq \frac{3N}{s\phi'(1)}\Psi_{w,\phi'}(f).$$

Finally, if $c \uparrow \mu(\Omega_s)$, the proof is complete. □

Corollary 3.7. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Phi_0$ and $\Lambda_{w,\phi'}$ satisfies a lower W-estimate, then there exists a constant $C > 0$ such that*

$$(Mf)^*(t) \leq \frac{C}{W(t)}\Psi_{w,\phi'}(f), \quad t > 0. \tag{17}$$

Proof. Since

$$\sup_{s>0} sW\left(\mu_h(s)\right) = \sup_{t>0} W(t)h^*(t), \quad h \in \mathcal{M}_0 \tag{18}$$

(see [3]), the corollary is an immediate consequence of Theorem 3.6. □

Theorem 3.8. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Phi_0 \cap \Delta_2$ and $\Lambda_{w,\phi'}$ satisfies a lower W-estimate, then*

$$\lim_{\epsilon\to 0} \frac{\Psi_{w,\phi'}\left((f - f(x))\chi_{B(x,\epsilon)}\right)}{\Psi_{w,\phi'}\left(\chi_{B(x,\epsilon)}\right)} = 0 \quad \text{a.e. } x \in (0, \alpha).$$

Proof. For $h \in \Lambda_{w,\phi'}$, we denote by $Lh : (0, \alpha) \to \mathbb{R}$ the function $Lh(x) = \limsup_{\epsilon\to 0} h_\epsilon(x)$. Let $c \in \mathbb{R}$ and $g \in \mathcal{S}$. For a.e. $x \in (0, \alpha)$, there exists $\epsilon(x) > 0$ such that

$$(g - c)\chi_{B(x,\epsilon)} = (g(x) - c)\chi_{B(x,\epsilon)}, \quad 0 < \epsilon < \epsilon(x). \tag{19}$$

Let $x \in (0, \alpha)$, and let $\epsilon(x) > 0$ satisfy (19). Assume $0 < \epsilon < \epsilon(x)$.

If $g(x) = c$, we get

$$\Psi_{w,\phi'}\left((g-c)\chi_{B(x,\epsilon)}\right) = 0,$$

because $\mu_{(g-c)\chi_{B(x,\epsilon)}}(0) = 0$.
If $g(x) \neq c$, then $\mu_{(g-c)\chi_{B(x,\epsilon)}}(0) = \mu(B(x,\epsilon))$ and $\left((g-c)\chi_{B(x,\epsilon)}\right)^* = |g(x)-c|\chi_{[0,\mu(B(x,\epsilon))]}$.
In consequence,

$$\Psi_{w,\phi'}\left((g-c)\chi_{B(x,\epsilon)}\right) = \phi'(|g(x)-c|)W(\mu(B(x,\epsilon))).$$

Since $\phi \in \Phi_0$ and $\Psi_{w,\phi'}\left(\chi_{B(x,\epsilon)}\right) = \phi'(1)W(\mu(B(x,\epsilon)))$ we have

$$h_\epsilon(x) = \frac{\Psi_{w,\phi'}\left((g-c)\chi_{B(x,\epsilon)}\right)}{\Psi_{w,\phi'}\left(\chi_{B(x,\epsilon)}\right)} = \frac{1}{\phi'(1)}\phi'(|g(x)-c|), \quad 0 < \epsilon < \epsilon(x).$$

Then,

$$L(g-c)(x) = \frac{1}{\phi'(1)}\phi'(|g(x)-c|) \quad \text{a.e. } x \in (0,\alpha).$$

From Lemma 2.3, there exists $C > 0$ such that

$$\begin{aligned} L(f-c)(x) &\le C\left(L(f-g)(x) + \phi'(|g(x)-c|)\right) \\ &\le C\left(M(f-g)(x) + \phi'(|g(x)-c|)\right), \quad \text{a.e. } x \in (0,\alpha). \end{aligned}$$

For $f(x)$ in place of c, it follows that

$$L(f-f(x))(x) \le C\left(M(f-g)(x) + \phi'(|(f-g)(x)|)\right), \quad \text{a.e. } x \in (0,\alpha). \tag{20}$$

Set $E_s = \{x \in (0,\alpha) : L(f-f(x))(x) > sC\}$, $s > 0$. Then, (20) implies

$$\mu(E_s) \le \mu_{M(f-g)}\left(\frac{s}{2}\right) + \mu_{\phi'(|f-g|)}\left(\frac{s}{2}\right), \quad s > 0, \tag{21}$$

Since

$$W(a+b) \le 2(W(a) + W(b)), \quad a, b > 0, \tag{22}$$

from (21) we have

$$W(\mu(E_s)) \le 2\left(W\left(\mu_{M(f-g)}\left(\frac{s}{2}\right)\right) + W\left(\mu_{\phi'(|f-g|)}\left(\frac{s}{2}\right)\right)\right), \quad s > 0. \tag{23}$$

As $(\phi'(|f-g|))^* = \phi'((f-g)^*)$, according to (10) and (18), we get

$$W\left(\mu_{\phi'(|f-g|)}\left(\frac{s}{2}\right)\right) \le \frac{2}{s}\Psi_{w,\phi'}(f-g), \quad s > 0. \tag{24}$$

By Theorem 3.6, there is $C' > 0$ satisfying (14). Thus, (23) and (24) show that

$$W(\mu(E_s)) \le \frac{4(C'+1)}{s}\Psi_{w,\phi'}(f-g), \quad s > 0.$$

In consequence, from Theorem 2.6, $\mu(E_s) = 0$, $s > 0$. The proof is complete. □

In [6], a family $\{B(x,\epsilon)\}_\epsilon$ is said to differentiate $L_{\phi'}$ if for every $f \in L_{\phi'}$ integrable locally,

$$\lim_{\epsilon \to 0} \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} \phi'(|f-f(x)|) = 0 \quad \text{a.e. } x \in (0,\alpha).$$

As an immediate consequence of Proposition 3.3 and Theorem 3.8 we have the following corollary.

Corollary 3.9. *If $\phi \in \Phi_0$, then the family $\{B(x,\epsilon)\}_\epsilon$ differentiates $L_{\phi'}$.*

4. Convergence of best constant approximants

In this section, we prove weak inequalities for the maximal function associated with the family $\{f^\epsilon(x)\}_\epsilon$ of best constant approximants of $f \in \Lambda_{w,\phi'}$ on $B(x,\epsilon)$, which are used in the study of pointwise convergence of $f^\epsilon(x)$ to $f(x)$, another extension of Lebesgue's Differentiation Theorem.

Lemma 4.1. *Let $f \in \Lambda_{w,\phi'}$ be a nonnegative function and let $A \subset [0,\alpha)$ be a finite measure set. If $\phi \in \Phi_0 \cap \Delta_2$, then there exists $C > 0$ such that*

$$\phi'(f^A)W(\mu(A)) \le C\,\Psi_{w,\phi'}(f\chi_A). \tag{25}$$

Proof. From ([17], Theorem 2.9), $f^A = \max\{c : \gamma^+((f-c)\chi_A, \chi_A) \ge 0\}$. As $\gamma^+(f\chi_A, \chi_A) \ge 0$, then $f^A \ge 0$.
By assumption, there exists $K > 0$, satisfying (2). Therefore

$$\phi'(f^A) \le \frac{K^2}{2}\Big(\phi'(f\chi_A) + \phi'((f^A - f)\chi_A)\Big), \quad \text{on } \{f < f^A\} \cap A. \tag{26}$$

It follows easily that

$$\phi'(f^A)W(\mu(A)) = \int_A w\left(\rho_{(f-f^A)\chi_A,\chi_A}\right)\phi'(f^A)\mathrm{d}\mu,$$

where $\rho_{(f-f^A)\chi_A,\chi_A} : A \to [0,\mu(A))$ is the m.p.t. defined in [16]. For simplicity of notation, we write ρ instead of $\rho_{(f-f^A)\chi_A,\chi_A}$. Thus, (26) implies

$$\begin{aligned}\phi'(f^A)W(\mu(A)) \le{}& \int_{\{f\ge f^A\}\cap A} w(\rho)\phi'(f^A)\mathrm{d}\mu + \frac{K^2}{2}\int_{\{f<f^A\}\cap A} w(\rho)\phi'(f\chi_A)\mathrm{d}\mu \\ &+ \frac{K^2}{2}\int_{\{f<f^A\}\cap A} w(\rho)\phi'((f^A - f)\chi_A)\mathrm{d}\mu.\end{aligned} \tag{27}$$

From ([17], Theorem 2.9), we have

$$\int_{\{f<f^A\}\cap A} w(\rho)\phi'((f^A - f)\chi_A)\mathrm{d}\mu \le \int_{\{f\ge f^A\}\cap A} w(\rho)\phi'((f - f^A)\chi_A)\mathrm{d}\mu. \tag{28}$$

But

$$\phi'((f - f^A)\chi_A) \le 2\phi'(f\chi_A), \quad \text{on } \{f \ge f^A\} \cap A \tag{29}$$

since

$$\phi'(a) + \phi'(b) \le 2\phi'(a+b), \quad a, b \ge 0. \tag{30}$$

According to (27)–(29), and $\phi \in \Phi_0$, we get

$$\phi'(f^A)W(\mu(A)) \le C\int_A w(\rho)\phi'(f\chi_A)\mathrm{d}\mu = C\int_{N(f)\cap A} w(\rho)\phi'(f\chi_A)\mathrm{d}\mu,$$

where $C = K^2 + 1$. Finally, (1) implies $\phi'(f^A)W(\mu(A)) \le C\,\Psi_{w,\phi'}(f\chi_A)$. $\square$

Remark 4.2. Let $f \in \Lambda_{w,\phi'}$ and let $A \subset [0,\alpha)$ be a finite measure set. If $\phi \in \Phi_0 \cap \Delta_2$, then there exists $C > 0$ such that

$$\phi'(|m|)W(\mu(A)) \le C\,\Psi_{w,\phi'}(f\chi_A), \quad m \in T_A(f). \tag{31}$$

In fact, from ([17], Theorems 2.9 and 3.9) we have $\max\{|f_A|, |f^A|\} \le |f|^A$. Therefore, (31) is an immediate consequence of Lemma 4.1.

Definition 4.3. Let $f \in \Lambda_{w,\phi'}$. Let $\Gamma f : (0,\alpha) \to \mathbb{R}$ be the maximal function defined by

$$\Gamma f(x) = \sup\left\{|m| \,:\, m \in T_{B(x,\epsilon)}(f), \epsilon > 0 \text{ and } B(x,\epsilon) \subset (0,\alpha)\right\}.$$

Theorem 4.4. *Let $f \in \Lambda_{w,\phi'}$. If $\phi \in \Delta_2$, then there exists a constant $C > 0$ such that:*

- $W(\mu_*(\{\Gamma f > s\})) \le \frac{C}{\phi'(s)}\Psi_{w,\phi'}(f)$, $s > 0$, *if* $\phi \in \Phi_0$;
- $W(\mu_*(\{\Gamma f > s\})) \le \frac{C}{\phi'(0)}\int_0^{\mu_f(s)} \phi'(f^*)w$, $s > 0$, *if* $\phi'(0) > 0$,

where μ_* is the Lebesgue outer measure.

Proof. Let $Hf : (0,\alpha) \to \mathbb{R}$ be the maximal function defined by

$$Hf(x) = \sup\left\{|f|^{B(x,\epsilon)} \,:\, \epsilon > 0 \text{ and } B(x,\epsilon) \subset (0,\alpha)\right\}.$$

From ([17], Theorems 2.9 and 3.9), we have $\max\{|f_{B(x,\epsilon)}|, |f^{B(x,\epsilon)}|\} \le |f|^{B(x,\epsilon)}$. Then, $\Gamma f \le Hf$ on $(0,\alpha)$. The proof is completed showing that the results hold for Hf.
Let $s > 0$. For each $x \in \Omega_s := \{Hf > s\}$, there exists $\epsilon_x > 0$ such that $B(x,\epsilon_x) \subset (0,\alpha)$ and

$$|f|^{B(x,\epsilon_x)} > s. \tag{32}$$

Let $c < \mu_*(\Omega_s)$ and let $B := \bigcup_{x\in\Omega_s} B(x,\epsilon_x)$. Clearly $c < \mu(B)$. Analogously to the case for the proof of Theorem 3.6, there is a pairwise disjoint finite collection $\{B(x_k,\epsilon_{x_k})\}_{k=1}^n$ such that $c < 3\sum_{k=1}^n \mu(B(x_k,\epsilon_{x_k}))$. As $W(3r) \le 3W(r)$, $r > 0$, we obtain

$$W(c) < 3W\left(\sum_{k=1}^n \mu(B(x_k,\epsilon_{x_k}))\right) = 3W(\mu(B_*)), \tag{33}$$

where $B_* := \bigcup_{k=1}^n B(x_k,\epsilon_{x_k})$.
Suppose $\phi \in \Phi_0$. From Lemma 4.1, there exists $K > 0$ such that

$$\phi'(|f|^{B_*})W(\mu(B_*)) \le K\,\Psi_{w,\phi'}\left(f\chi_{B_*}\right). \tag{34}$$

As $|f|\chi_{B(x_k,\epsilon_{x_k})} \le |f|\chi_{B_*}$, $1 \le k \le n$, by ([17], Theorem 3.9) we have $|f|^{B(x_k,\epsilon_k)} \le |f|^{B_*}$, $1 \le k \le n$. Then, (32)–(34) imply $\phi'(s)W(c) \le 3K\,\Psi_{w,\phi'}(f)$. Thus, if $c \uparrow \mu_*(\Omega_s)$, the proof in this case is complete.
Now suppose $\phi'(0) > 0$. Since

$$\phi'(0)W(\mu(B_*)) \le \int_{B_*} w\left(\rho_{(|f|-|f|^{B_*})\chi_{B_*},\chi_{B_*}}\right)\phi'(|(|f| - |f|^{B_*})\chi_{B_*}|)\mathrm{d}\mu,$$

from ([17], Theorem 2.9), (30) and (32) we have

$$\begin{aligned}\phi'(0)W(\mu(B_*)) &\le 2\int_{\{|f|\ge|f|^{B_*}\}\cap B_*} w\left(\rho_{(|f|-|f|^{B_*})\chi_{B_*},\chi_{B_*}}\right)\phi'((|f| - |f|^{B_*})\chi_{B_*})\mathrm{d}\mu \\ &\le 4\int_{\{|f|>s\}\cap B_*} w\left(\rho_{(|f|-|f|^{B_*})\chi_{B_*},\chi_{B_*}}\right)\phi'(|f|\chi_{B_*})\mathrm{d}\mu.\end{aligned}$$

So, (1) and (33) imply $\phi'(0)W(c) \leq 12\int_0^{\mu_f(s)} \phi'(f^*)w$. Finally, if $c \uparrow \mu_*(\Omega_s)$, the proof is complete. $\square$

As we have mentioned in Section 1, we extend ([19], Corollary 3.2) and ([6], Theorems 4 and 9) for the case of one-variable functions. In fact we have:

Theorem 4.5. *Let $f \in \Lambda_{w,\phi'}$, $x \in (0,\alpha)$, and let $f^\epsilon(x) \in T_{B(x,\epsilon)}(f)$ be any best constant approximation of f on $B(x,\epsilon)$. If $\phi \in \Delta_2$, then* $\lim_{\epsilon\to 0} f^\epsilon(x) = f(x)$, *a.e.* $x \in (0,\alpha)$.

Proof. Let $Lf(x) = \limsup_{\epsilon\to 0} |f^\epsilon(x) - f(x)|$ and let $g \in \mathcal{S}$. For a.e. $x \in (0,\alpha)$, there exists a net $\{(f-g)^\epsilon(x)\}_\epsilon \subset T_{B(x,\epsilon)}(f-g)$ such that

$$Lf(x) = \limsup_{\epsilon\to 0} |(f-g)^\epsilon(x) - (f(x) - g(x))|.$$

Then $Lf(x) \leq \Gamma(f-g)(x) + |f(x) - g(x)|$, a.e. $x \in (0,\alpha)$, and consequently $\mu_*(\{Lf > 2s\}) \leq \mu_*(\{\Gamma(f-g) > s\}) + \mu_{f-g}(s)$, $s > 0$. From (22), it follows that

$$W(\mu_*(\{Lf > 2s\})) \leq 2\left(W(\mu_*(\{\Gamma(f-g) > s\})) + W\left(\mu_{f-g}(s)\right)\right), \quad s > 0.$$

Therefore, Theorems 4.4 and 2.6 show that $Lf(x) = 0$, a.e. $x \in (0,\alpha)$. This completes the proof. $\square$

Remark 4.6. In [6], the authors assume that the family $\{B(x,\epsilon)\}_\epsilon$ differentiates $L_{\phi'}$ in order to prove Theorem 4, in the case $\phi'(0) = 0$. However, by Corollary 3.9, we prove that this property is always satisfied.

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