

Full length article

Korovkin type theorems and approximate Hermite–Hadamard inequalities[☆]

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Received 28 November 2011; received in revised form 18 April 2012; accepted 14 May 2012

Available online 23 May 2012

Communicated by Tamás Erdélyi

Abstract

The main results of this paper offer sufficient conditions in order that an approximate lower Hermite–Hadamard type inequality implies an approximate convexity property. The failure of such an implication with constant error term shows that functional error terms should be considered for the inequalities and convexity properties in question. The key for the proof of the main result is a Korovkin type theorem which enables us to deduce the approximate convexity property from the approximate lower Hermite–Hadamard type inequality via an iteration process.

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Keywords: Convexity; Approximate convexity; Lower and upper Hermite–Hadamard inequalities; Korovkin type theorem

1. Introduction

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively. Let X be a real linear space and $D \subset X$ be a convex set.

[☆] This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402 and by the TÁMOP-4.2.1/B-09/1/KONV-2010-0007, TÁMOP-4.2.2/B-10/1-2010-0024 projects. These projects are co-financed by the European Union and the European Social Fund.

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One can easily see that, for any constant $\varepsilon \geq 0$, the ε -convexity of f (cf. [14]), i.e., the validity of

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \quad (x, y \in D, t \in [0, 1]),$$

implies the following lower and upper ε -Hermite–Hadamard inequalities

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y) dt + \varepsilon \quad (x, y \in D), \quad (1.1)$$

and

$$\int_0^1 f(tx + (1-t)y) dt \leq \frac{f(x) + f(y)}{2} + \varepsilon \quad (x, y \in D). \quad (1.2)$$

The above implication was discovered if $\varepsilon = 0$ by Hadamard [8] in 1893. (See also [21,16,24,7] for a historical account). For $\varepsilon = 0$, the converse is also known to be true (cf. [23,24]), i.e., if a function $f : D \rightarrow \mathbb{R}$ which is continuous over the segments of D satisfies (1.1) or (1.2) with $\varepsilon = 0$, then it is also convex. Concerning the reversed implication for the case $\varepsilon > 0$, Nikodem et al. in [25] have recently shown that the ε -Hermite–Hadamard inequalities (1.1) and (1.2) do not imply the $c\varepsilon$ -convexity of f (with any $c > 0$). Thus, in order to obtain results that establish implications between the approximate Hermite–Hadamard inequalities and the approximate Jensen inequality, one has to consider these inequalities with nonconstant error terms.

In order to describe the old and new results about the connection of an approximate Jensen convexity inequality and the approximate Hermite–Hadamard inequality with variable error terms, we need to introduce the following terminology.

For a function $f : D \rightarrow \mathbb{R}$, we say that f is *hemi- P* , if, for all $x, y \in D$, the mapping

$$t \mapsto f((1-t)x + ty) \quad (t \in [0, 1]) \quad (1.3)$$

has property P . For example f is *hemiintegrable*, if for all $x, y \in D$ the mapping defined by (1.3) is integrable. Analogously, we say that a function $h : (D - D) \rightarrow \mathbb{R}$ is *radially- P* , if for all $u \in D - D$, the mapping

$$t \mapsto h(tu) \quad (t \in [0, 1])$$

has property P on $[0, 1]$.

In [19] the authors established the connections between an upper Hermite–Hadamard type inequality and a Jensen type inequality, which was stated in the following theorem.

Theorem A. Let $\alpha_H : (D - D) \rightarrow \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \geq 0$ and $p > 0$ such that

$$\rho(t) \leq c(-\ln|1-2t|)^{p-1} \quad \left(t \in \left]0, \frac{1}{2}\left[\bigcup\left]\frac{1}{2}, 1\right[\right],\right.$$

and $\lambda \in [0, 1]$. Then every $f : D \rightarrow \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality

$$\int_0^1 f(tx + (1-t)y) \rho(t) dt \leq \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \quad (x, y \in D),$$

fulfills the approximate Jensen inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \alpha_J(x-y) \quad (x, y \in D) \quad (1.4)$$

provided that $\alpha_J : (D - D) \rightarrow \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality

$$\alpha_J(u) \geq \int_0^1 \alpha_J(|1-2t|u) \rho(t) dt + \alpha_H(u) \quad (u \in (D - D))$$

and $\alpha_J(0) \geq \alpha_H(0)$.

In [13], Háyzy and Páles obtained a relationship between a lower Hermite–Hadamard type inequality and a Jensen type inequality by proving the following result. In Remark 3.20, we will deduce this result as a consequence of Corollary 3.19.

Theorem B. Let $\alpha_H : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y) dt + \alpha_H(x-y) \quad (x, y \in D). \quad (1.5)$$

Then f satisfies the approximate Jensen inequality (1.4) where $\alpha_J : 2(D - D) \rightarrow \mathbb{R}_+$ is a nonnegative, radially increasing solution of the functional inequality

$$\alpha_J(u) \geq \int_0^1 \alpha_J(2tu) dt + \alpha_H(u) \quad (u \in D - D). \quad (1.6)$$

The following theorem, which has been stated in [17,20,22,27,26], is also a consequence of our main result. We will also derive it in Remark 3.22 by applying Corollary 3.21.

Theorem C. Let $\alpha_J : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemibounded function satisfying the approximate Jensen inequality (1.4). Then f also satisfies the following convexity type inequality

$$f((1-s)x + sy) \leq (1-s)f(x) + sf(y) + \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha_J(2d_{\mathbb{Z}}(2^n s)(x-y)) \quad (1.7)$$

for all $x, y \in D$, $s \in [0, 1]$, where $d_{\mathbb{Z}}(s) := \text{dist}(s, \mathbb{Z}) := \min\{|s-z| : z \in \mathbb{Z}\}$ for $s \in \mathbb{R}$.

In this paper we examine the implication from a lower Hermite–Hadamard type inequality to a convexity type inequality. In Theorem 3.18 below, we generalize Theorem B replacing the Lebesgue integral by an integral with respect to an arbitrary Borel probability measure. This allows us to view an approximate Jensen inequality as a particular approximate Hermite–Hadamard inequality. The reversed implication is established in the following theorem, which was one of the main results of [18].

Theorem D. Let D be a convex set of a linear space X . Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such

that the support of μ is not a singleton. Denote

$$\mu_1 := \int_{[0,1]} t d\mu(t) \quad \text{and} \\ S(\mu) := \mu([0, \mu_1]) \int_{[\mu_1, 1]} t d\mu(t) - \mu([\mu_1, 1]) \int_{[0, \mu_1]} t d\mu(t).$$

Assume that $f : D \rightarrow \mathbb{R}$ is a hemi- μ -integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \eta_{x,y}(t) \quad ((x, y) \in D^{2*}, t \in [0, 1])$$

where, for all $(x, y) \in D^{2*} := \{(x, y) \mid x, y \in D, x \neq y\}$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a function such that

$$I(x, y) := \int_{[\mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') \eta_{(1-t')x + t'y, (1-t'')x + t''y} \left(\frac{\mu_1 - t'}{t'' - t'} \right) d\mu(t') d\mu(t'')$$

exists in $[-\infty, \infty]$ for all $(x, y) \in D^{2*}$. Then, for all $(x, y) \in D^{2*}$, the function f also satisfies the lower Hermite–Hadamard type inequality

$$f((1 - \mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1-t)x + ty) d\mu(t) + \frac{1}{S(\mu)} I(x, y) \quad ((x, y) \in D^{2*}).$$

The main results of the subsequent sections concern convexity with respect to Chebyshev systems. Thus, we will also need the following terminology. Given a nonempty open real interval I , denote by $\Delta(I)$ and $\Delta^\circ(I)$ the sets

$$\{(x, y) \in I \times I \mid x \leq y\} \quad \text{and} \quad \{(x, y) \in I \times I \mid x < y\},$$

respectively. We say that a pair (ω_0, ω_1) is a *Chebyshev system* over I , if $\omega_0, \omega_1 : I \rightarrow \mathbb{R}$ are continuous functions and

$$\Omega(x, y) := \begin{vmatrix} \omega_0(x) & \omega_0(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix} > 0 \quad ((x, y) \in \Delta^\circ(I)). \quad (1.8)$$

One can easily see, that if ω_0 is a positive function, then (1.8) holds if and only if ω_1/ω_0 is strictly increasing on I . In this latter case, (ω_0, ω_1) will be called a *positive Chebyshev system* over I . On the other hand, we can always assume that ω_0 is a positive function, because for every Chebyshev system (ω_0, ω_1) , there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\omega_0 + \beta\omega_1 > 0$ (cf. [2–4]). In the sequel, for fixed $x, y \in I$, the partial functions $u \mapsto \Omega(u, y)$ and $u \mapsto \Omega(x, u)$ will be denoted by $\Omega(\cdot, y)$ and $\Omega(x, \cdot)$, respectively. Observe that both partial functions are linear combinations of the base functions ω_0 and ω_1 . An important property of Chebyshev systems is that for every two pairs $(x, \xi), (y, \eta) \in I \times \mathbb{R}$ with $x \neq y$ the function ω defined as

$$\omega := \xi \frac{\Omega(\cdot, y)}{\Omega(x, y)} + \eta \frac{\Omega(x, \cdot)}{\Omega(x, y)}$$

is the unique linear combination of ω_0 and ω_1 such that $\omega(x) = \xi$ and $\omega(y) = \eta$ hold.

Given a positive Chebyshev system (ω_0, ω_1) over I and a proper subinterval J of I , a function $f : J \rightarrow \mathbb{R}$ is called (ω_0, ω_1) -convex on J if, for all $x < u < y$ from J ,

$$\begin{vmatrix} f(x) & f(u) & f(y) \\ \omega_0(x) & \omega_0(u) & \omega_0(y) \\ \omega_1(x) & \omega_1(u) & \omega_1(y) \end{vmatrix} \geq 0, \quad (1.9)$$

or equivalently,

$$f(u) \leq \frac{\Omega(u, y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, u)}{\Omega(x, y)} f(y). \quad (1.10)$$

If (1.9) holds with strict inequality sign “>”, then f is said to be *strictly* (ω_0, ω_1) -convex on J .

Lemma 1.1 ([3,4]). *Let (ω_0, ω_1) be a positive Chebyshev system over I , $J \subset I$ be a proper subinterval and $f : J \rightarrow \mathbb{R}$. Then the following statements are equivalent:*

- (i) f is [strictly] (ω_0, ω_1) -convex on J ;
- (ii) $\left(\frac{f}{\omega_0}\right) \circ \left(\frac{\omega_1}{\omega_0}\right)^{-1}$ is [strictly] convex in the standard sense on $\left(\frac{\omega_1}{\omega_0}\right)(J)$;
- (iii) for every $u \in J^\circ$, there exist constants $\alpha, \beta \in \mathbb{R}$ such that $f(u) = \alpha\omega_0(u) + \beta\omega_1(u)$ and, for all $x \in J \setminus \{u\}$, the inequality $f(x) \geq \alpha\omega_0(x) + \beta\omega_1(x)$ [the strict inequality $f(x) > \alpha\omega_0(x) + \beta\omega_1(x)$] holds.

In [5,3], the authors established the following connections between the (ω_0, ω_1) -convexity and the Hermite–Hadamard type inequality.

Theorem E. *Let (ω_0, ω_1) be a positive Chebyshev system on an open interval I and let $\rho : I \rightarrow \mathbb{R}$ be a positive integrable function. Define, for all elements $x < y$ of I , the functions $\xi(x, y)$ and $c(x, y)$ by the formulas*

$$\xi(x, y) = \left(\frac{\omega_1}{\omega_0}\right)^{-1} \left(\frac{\int_x^y \omega_1 \rho}{\int_x^y \omega_0 \rho}\right) \quad \text{and} \quad c(x, y) = \frac{\int_x^y \omega_0 \rho}{\omega_0(\xi(x, y))}. \quad (1.11)$$

If the continuous function $f : I \rightarrow \mathbb{R}$, for all elements $x < y$ of I , satisfies the inequality

$$c(x, y) f(\xi(x, y)) \leq \int_x^y f \rho, \quad (1.12)$$

then it is (ω_0, ω_1) -convex.

In Theorems 2.12 and 3.14 below, these results will also be generalized.

For the reversed implication the following theorem was obtained in [18]. To formulate its result, consider the following basic assumptions.

- (1) $(T, \mathcal{A}, \mu)$ is a measure space.
- (2) $\Lambda : T \times \Delta^\circ(I) \rightarrow \mathbb{R}_+$ is μ -integrable in its first variable.
- (3) $M : T \times \Delta^\circ(I) \rightarrow \mathbb{R}$ is $\mathcal{A}$ -measurable in its first variable and for all $t \in T$, the map $(x, y) \mapsto M(t, x, y)$ is a two-variable mean on I . $M_0 : \Delta^\circ(I) \rightarrow I$ is a strict mean such that $\mu\{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) \neq M_0(x, y)\} > 0$ if $(x, y) \in \Delta^\circ(I)$.
- (4) There exists an (ω_0, ω_1) -Chebyshev system on I such that ω_0 is positive. Furthermore, for $i \in \{0, 1\}$,

$$\omega_i(M_0(x, y)) = \int_T \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) \quad ((x, y) \in \Delta^\circ(I)).$$

Theorem F. *Assume that (1)–(4) hold. Let $f : I \rightarrow \mathbb{R}$ be a locally upper bounded Borel measurable solution of the approximate (ω_0, ω_1) -convexity type functional inequality*

$$f(u) \leq \frac{\Omega(u, y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, u)}{\Omega(x, y)} f(y) + \varepsilon_{x,y}(u) \quad (u \in [x, y]),$$

where for all $(x, y) \in \Delta^\circ(I)$ and $u \in]x, y[$, the function $(v, w) \mapsto \varepsilon_{v,w}(u)$ is bounded and Borel measurable for $(v, w) \in [x, u] \times [u, y]$. Then f also satisfies the approximate lower Hermite–Hadamard type inequality

$$f(M_0(x, y)) \leq \int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) + E(x, y) \quad ((x, y) \in \Delta(I)),$$

where $E : \Delta^\circ(I) \rightarrow \mathbb{R}$ is defined by the following way

$$E(x, y) = \frac{\int_{T'_{x,y}} \int_{T''_{x,y}} \Phi(x, y, t', t'') \varepsilon_{M(t',x,y), M(t'',x,y)}(M_0(x, y)) d\mu(t'') d\mu(t')}{\int_{T'_{x,y}} \int_{T''_{x,y}} \Phi(x, y, t', t'') d\mu(t'') d\mu(t')},$$

where

$$T'_{x,y} := \{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) < M_0(x, y)\},$$

$$T''_{x,y} := \{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) \geq M_0(x, y)\},$$

$$\Phi(x, y, t', t'') := \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)).$$

In Section 2, certain Korovkin type theorems [15,1] will be proved, which will play an important role in the proof of the main results: Theorems 2.12 and 3.14. In Section 3, the implication from an approximate lower Hermite–Hadamard type inequality to an approximate convexity inequality will be investigated.

Throughout this paper, the notation δ_t stands for the Dirac measure concentrated at the point $t \in [0, 1]$.

2. Korovkin type theorems

The subsequent results are Korovkin type theorems. In the sequel, denote by $C([a, b])$ and $B([a, b])$ the space of continuous and bounded Borel measurable real valued functions defined on the interval $[a, b]$ equipped with the usual supremum norm.

Theorem 2.2. Let $a < b$, $\mathcal{T}_n : B([a, b]) \rightarrow B([a, b])$ ($n \in \mathbb{N}$) be a sequence of positive linear operators and let (ω_0, ω_1) be a positive Chebyshev system over $[a, b]$ such that

$$\lim_{n \rightarrow \infty} (\mathcal{T}_n \omega_0)(u) = \omega_0(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} (\mathcal{T}_n \omega_1)(u) = \omega_1(u) \quad (u \in [a, b]). \quad (2.1)$$

Suppose that there exists a function $g \in C([a, b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a, b[$ such that $\lim_{n \rightarrow \infty} (\mathcal{T}_n g)(u) = 0$ for all $u \in [a, b]$. Then, for all bounded upper semicontinuous functions $f : [a, b] \rightarrow \mathbb{R}$,

$$\limsup_{n \rightarrow \infty} \mathcal{T}_n f(u) \leq f(a) \frac{\Omega(u, b)}{\Omega(a, b)} + f(b) \frac{\Omega(a, u)}{\Omega(a, b)} \quad (u \in [a, b]). \quad (2.2)$$

Remark 2.3. It easily follows from the above theorem that, if f is continuous, then (2.2) holds with equality and the “limsup” can be replaced by “lim”. By a generalization of the classical Korovkin theorem if, for some continuous (ω_0, ω_1) -convex function $g : [a, b] \rightarrow \mathbb{R}$, $\mathcal{T}_n g$ converges to g , then $\mathcal{T}_n f$ converges to f for all $f \in C([a, b])$.

Proof. As a consequence of (2.1), it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathcal{T}_n \Omega(\cdot, b))(u) &= \Omega(u, b) \quad \text{and} \\ \lim_{n \rightarrow \infty} (\mathcal{T}_n \Omega(a, u))(u) &= \Omega(a, u) \quad (u \in [a, b]). \end{aligned} \quad (2.3)$$

To prove (2.2), let $\varepsilon > 0$ be arbitrary and define $\varphi :]a, b[\rightarrow \mathbb{R}$ by

$$\varphi := \frac{\Omega(a, b)f - f(a)\Omega(\cdot, b) - f(b)\Omega(a, \cdot) - \varepsilon\Omega(a, b)\omega_0}{\Omega(a, b)g}.$$

Since f is upper semicontinuous at a and also at b , there exists $0 < \delta < \frac{b-a}{2}$ such that

$$\begin{aligned} \Omega(a, b)f(u) - f(a)\Omega(u, b) - f(b)\Omega(a, u) &< \varepsilon\Omega(a, b)\omega_0(u) \quad \text{if } a \leq u < a + \delta, \\ \Omega(a, b)f(u) - f(a)\Omega(u, b) - f(b)\Omega(a, u) &< \varepsilon\Omega(a, b)\omega_0(u) \quad \text{if } b - \delta < u \leq b. \end{aligned}$$

Hence, the function φ is nonpositive on $[a, a + \delta[\cup]b - \delta, b]$. On the other hand, φ is upper semicontinuous on the compact interval $[a + \delta, b - \delta]$, therefore φ attains its maximum on $[a + \delta, b - \delta]$, which we denote by K . This implies that

$$f \leq f(a)\frac{\Omega(\cdot, b)}{\Omega(a, b)} + f(b)\frac{\Omega(a, \cdot)}{\Omega(a, b)} + Kg + \varepsilon\omega_0 \quad \text{on } [a, b].$$

Applying the linearity and the monotonicity of $\mathcal{T}_n$, for all $n \in \mathbb{N}$ and $u \in [a, b]$, we get

$$\begin{aligned} (\mathcal{T}_n f)(u) &\leq \frac{f(a)}{\Omega(a, b)} (\mathcal{T}_n \Omega(\cdot, b))(u) \\ &\quad + \frac{f(b)}{\Omega(a, b)} (\mathcal{T}_n \Omega(a, \cdot))(u) + K(\mathcal{T}_n g)(u) + \varepsilon(\mathcal{T}_n \omega_0)(u). \end{aligned}$$

Upon taking the limit $n \rightarrow \infty$ and using (2.3), we obtain

$$\limsup_{n \rightarrow \infty} (\mathcal{T}_n f)(u) \leq \frac{f(a)}{\Omega(a, b)} \Omega(u, b) + \frac{f(b)}{\Omega(a, b)} \Omega(a, u) + \varepsilon\omega_0(u) \quad (u \in [a, b]),$$

which results the statement. $\square$

The following result offers a sufficient condition for (2.2) to hold when the sequence $(\mathcal{T}_n)$ is obtained as the sequence of iterates of a positive linear operator.

Corollary 2.4. Let $a < b$, $\mathcal{T} : B([a, b]) \rightarrow B([a, b])$ ($n \in \mathbb{N}$) be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a, b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$, i.e.,

$$\mathcal{T}\omega_0 = \omega_0 \quad \text{and} \quad \mathcal{T}\omega_1 = \omega_1. \quad (2.4)$$

Suppose that there exist a function $g \in C([a, b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a, b[$ and a constant $q \in]0, 1[$ such that

$$(\mathcal{T}g)(u) \leq qg(u) \quad (u \in [a, b]). \quad (2.5)$$

Then, for all upper semicontinuous $f \in B([a, b])$,

$$\limsup_{n \rightarrow \infty} \mathcal{T}^n f(u) \leq f(a)\frac{\Omega(u, b)}{\Omega(a, b)} + f(b)\frac{\Omega(a, u)}{\Omega(a, b)} \quad (u \in [a, b]). \quad (2.6)$$

Remark 2.5. It easily follows from the above corollary that, if f is continuous, then (2.6) holds with equality and the “limsup” can be replaced by “lim”.

Proof. We apply Theorem 2.2 to the sequence of operators $\mathcal{T}_n := \mathcal{T}^n$. Now (2.1) automatically holds by (2.4). On the other hand, by induction on $n \in \mathbb{N}$, we get $0 \leq \mathcal{T}^n g \leq q^n g$. Taking the limit $n \rightarrow \infty$, this inequality yields that $\lim_{n \rightarrow \infty} \mathcal{T}^n g = 0$. Hence, Theorem 2.2 applies and we obtain (2.2), which is now equivalent to (2.6). $\square$

The following two results establish a connection between (ω_0, ω_1) -convexity and the inequality $f \leq \mathcal{T}f$.

Proposition 2.6. Let $a < b$, $\mathcal{T} : B([a, b]) \rightarrow B([a, b])$ ($n \in \mathbb{N}$) be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a, b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$. Then, for all (ω_0, ω_1) -convex functions $f : [a, b] \rightarrow \mathbb{R}$,

$$f(u) \leq (\mathcal{T}f)(u) \quad (u \in]a, b[) \quad (2.7)$$

holds.

Proof. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is an (ω_0, ω_1) -convex function. Then it is continuous on the interior of $[a, b]$ and upper semicontinuous at the endpoints of $[a, b]$, furthermore, it is also bounded. Hence $f \in B([a, b])$. In view of Lemma 1.1, for all $u \in]a, b[$, there exist constants α, β such that

$$\alpha\omega_0(u) + \beta\omega_1(u) = f(u) \quad \text{and} \quad \alpha\omega_0 + \beta\omega_1 \leq f.$$

Applying $\mathcal{T}$ to the last inequality and using that ω_0 and ω_1 are fixed points of $\mathcal{T}$, we get

$$\alpha\omega_0 + \beta\omega_1 \leq \mathcal{T}f.$$

Evaluating both sides at u , it follows that

$$f(u) = \alpha\omega_0(u) + \beta\omega_1(u) \leq (\mathcal{T}f)(u).$$

This proves (2.7). $\square$

Proposition 2.7. Let $a < b$, $\mathcal{T} : B([a, b]) \rightarrow B([a, b])$ ($n \in \mathbb{N}$) be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a, b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$. Assume that there exists a function $g \in C([a, b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a, b[$ such that (2.5) holds for some $q \in [0, 1[$. Then, for all bounded upper semicontinuous functions $f \in B([a, b])$ which satisfy $f \leq \mathcal{T}f$, the inequality

$$f(u) \leq f(a) \frac{\Omega(u, b)}{\Omega(a, b)} + f(b) \frac{\Omega(a, u)}{\Omega(a, b)} \quad (u \in [a, b]) \quad (2.8)$$

holds.

Proof. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is a bounded upper semicontinuous function which satisfies $f \leq \mathcal{T}f$. Then, by induction, $\mathcal{T}^n f \leq \mathcal{T}^{n+1} f$ and hence $f \leq \mathcal{T}^n f$ follows for all $n \in \mathbb{N}$. On the other hand, by Corollary 2.4, (2.6) also holds. Therefore, upon taking the limit $n \rightarrow \infty$ in the inequality $f \leq \mathcal{T}^n f$, and using (2.6), the desired inequality (2.8) follows. $\square$

In what follows, we construct a large family of positive linear operators on $B([a, b])$ which satisfies the assumptions of the previous results and will be instrumental in the investigation of approximate convexity.

For a triplet (μ, λ, m) define a linear operator $\mathcal{T}_{\mu, \lambda, m} : B([a, b]) \rightarrow B([a, b])$ by the following formula:

$$(\mathcal{T}_{\mu, \lambda, m} f)(u) := \int_T \lambda(t, u) f(m(t, u)) d\mu(t), \quad (2.9)$$

where, on the data (μ, λ, m) , we make the following assumptions.

(H1) $(T, \mathcal{A}, \mu)$ is a measure space.

(H2) $\lambda : T \times [a, b] \rightarrow \mathbb{R}_+$ is measurable in its first variable and continuous in its second variable, furthermore, the function $\ell : T \rightarrow \mathbb{R}_+$ defined by

$$\ell(t) := \sup_{u \in [a, b]} \lambda(t, u)$$

is μ -integrable, i.e., $\int_T \ell(t) d\mu(t) < +\infty$.

(H3) $m : T \times [a, b] \rightarrow [a, b]$ is measurable in its first variable, continuous in its second variable on $[a, b]$ differentiable in its second variable at the endpoints of $[a, b]$ and it is uniformly calm at the endpoints of $[a, b]$, i.e., there exist constants $\delta > 0$ and $K \geq 0$ such that

$$|m(t, v) - m(t, u)| \leq K|v - u| \quad \text{if } u \in \{a, b\}, v \in [a, b], |v - u| \leq \delta.$$

Furthermore, for all $t \in T$, $m(t, a) = a$, $m(t, b) = b$, and, for all $u \in [a, b]$,

$$\begin{aligned} \mu\{t \in T \mid \lambda(t, u) \neq 0, m(t, u) \neq u\} &> 0 \quad \text{if } u \notin \{a, b\}, \\ \mu\{t \in T \mid \lambda(t, u) \neq 0, m'(t, u) \neq 1\} &> 0 \quad \text{if } u \in \{a, b\}. \end{aligned} \quad (2.10)$$

(H4) There exists a positive Chebyshev system (ω_0, ω_1) over $[a, b]$ such that ω_0 is positive, $\frac{\omega_1}{\omega_0}$ is differentiable at a and b with $\left(\frac{\omega_1}{\omega_0}\right)'(a) > 0$ and $\left(\frac{\omega_1}{\omega_0}\right)'(b) > 0$. Furthermore, for $i \in \{0, 1\}$,

$$\omega_i(u) = \int_T \lambda(t, u) \omega_i(m(t, u)) d\mu(t) \quad (u \in [a, b]). \quad (2.11)$$

Proposition 2.8. Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4). Then

$$\int_T \lambda(t, a) d\mu(t) = 1, \quad \int_T \lambda(t, b) d\mu(t) = 1, \quad (2.12)$$

$$\int_T \lambda(t, a) m'(t, a) d\mu(t) \leq 1, \quad \int_T \lambda(t, b) m'(t, b) d\mu(t) \leq 1, \quad (2.13)$$

and

$$\int_T \lambda(t, a) (m'(t, a))^p d\mu(t) < 1, \quad \int_T \lambda(t, b) (m'(t, b))^p d\mu(t) < 1 \quad \text{if } p \in]0, 1[. \quad (2.14)$$

Proof. We note that $a \leq m(t, u) \leq b$ for all $(t, u) \in T \times [a, b]$, hence the assumptions $m(\cdot, a) = a$ and $m(\cdot, b) = b$ imply that $m'(t, a) \geq 0$ and $m'(t, b) \geq 0$ for all $t \in T$.

The substitution $u = a$ in (2.11) and $m(\cdot, a) = a$ yield that

$$\omega_0(a) = \int_T \lambda(t, a) \omega_0(m(t, a)) d\mu(t) = \int_T \lambda(t, a) \omega_0(a) d\mu(t) = \omega_0(a) \int_T \lambda(t, a) d\mu(t),$$

which, by $\omega_0(a) > 0$, results the first formula in (2.12). The proof of the second formula is analogous.

To prove (2.13), let $u \in]a, b[$ be arbitrary. Using (2.11), we have that

$$\omega_0(u) \cdot \left(\frac{\omega_1}{\omega_0} \right) (a) = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot \left(\frac{\omega_1}{\omega_0} \right) (m(t, a)) d\mu(t) \quad (2.15)$$

and

$$\omega_0(u) \cdot \left(\frac{\omega_1}{\omega_0} \right) (u) = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot \left(\frac{\omega_1}{\omega_0} \right) (m(t, u)) d\mu(t). \quad (2.16)$$

Subtracting (2.15) from (2.16) and then dividing by $u - a$, we get that

$$\begin{aligned} \omega_0(u) \cdot \frac{\left(\frac{\omega_1}{\omega_0} \right) (u) - \left(\frac{\omega_1}{\omega_0} \right) (a)}{u - a} \\ = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot \frac{\left(\frac{\omega_1}{\omega_0} \right) (m(t, u)) - \left(\frac{\omega_1}{\omega_0} \right) (m(t, a))}{u - a} d\mu(t). \end{aligned} \quad (2.17)$$

Observe that the integrand is a nonnegative measurable function of the variable t for all $u \in [a, b]$. Now, let $u_n > a$ be an arbitrary sequence tending to a and substitute u by u_n in (2.17). Then, taking the limit $n \rightarrow \infty$ and using the Fatou Lemma, we get

$$\begin{aligned} \omega_0(a) \cdot \left(\frac{\omega_1}{\omega_0} \right)' (a) &= \liminf_{n \rightarrow \infty} \omega_0(u_n) \cdot \frac{\left(\frac{\omega_1}{\omega_0} \right) (u_n) - \left(\frac{\omega_1}{\omega_0} \right) (a)}{u_n - a} \\ &= \liminf_{n \rightarrow \infty} \int_T \lambda(t, u_n) \cdot \omega_0(m(t, u_n)) \cdot \frac{\left(\frac{\omega_1}{\omega_0} \right) (m(t, u_n)) - \left(\frac{\omega_1}{\omega_0} \right) (m(t, a))}{u_n - a} d\mu(t) \\ &\geq \int_T \liminf_{n \rightarrow \infty} \lambda(t, u_n) \cdot \omega_0(m(t, u_n)) \cdot \frac{\left(\frac{\omega_1}{\omega_0} \right) (m(t, u_n)) - \left(\frac{\omega_1}{\omega_0} \right) (m(t, a))}{u_n - a} d\mu(t) \\ &= \int_T \lambda(t, a) \cdot \omega_0(a) \cdot \left(\frac{\omega_1}{\omega_0} \right)' (a) \cdot m'(t, a) d\mu(t). \end{aligned}$$

Using $\omega_0(a) \cdot \left(\frac{\omega_1}{\omega_0} \right)' (a) > 0$, the first inequality in (2.13) follows. The proof of the second inequality is analogous.

Let $p \in]0, 1[$ be fixed. The function $s \mapsto s^p$ is strictly concave over $[0, \infty[$, therefore

$$s^p \leq p(s - 1) + 1 \quad (s \in [0, \infty[)$$

and the equality is valid if and only if $s = 1$. Hence,

$$(m'(t, a))^p \leq pm'(t, a) + 1 - p \quad (t \in T), \quad (2.18)$$

and the equality holds if and only if $m'(t, a) = 1$. Thus, by the second inequality in assumption (2.10), on a set of positive μ measures we have strict inequality in (2.18). Multiplying this inequality by $\lambda(t, a)$ and then integrating by t with respect to μ , and using (2.12) and (2.13), it follows that

$$\int_T \lambda(t, a) \cdot (m'(t, a))^p d\mu(t) < p \int_T \lambda(t, a) \cdot m'(t, a) d\mu(t) \\ + (1 - p) \int_T \lambda(t, a) d\mu(t) \leq 1.$$

The proof of the second inequality in (2.14) is completely similar. $\square$

Proposition 2.9. Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu, \lambda, m}$ by (2.9). Then $\mathcal{T}_{\mu, \lambda, m} : B([a, b]) \rightarrow B([a, b])$ is a bounded positive linear operator with

$$\|\mathcal{T}_{\mu, \lambda, m}\| \leq \int_T \ell(t) d\mu(t). \quad (2.19)$$

In addition, $\mathcal{T}_{\mu, \lambda, m}$ has the following properties.

(i) For all $f \in B([a, b])$,

$$(\mathcal{T}_{\mu, \lambda, m} f)(a) = f(a) \quad \text{and} \quad (\mathcal{T}_{\mu, \lambda, m} f)(b) = f(b). \quad (2.20)$$

(ii) If $f : [a, b] \rightarrow \mathbb{R}$ is bounded and lower (upper) semicontinuous then $\mathcal{T}_{\mu, \lambda, m} f$ is also lower (upper) semicontinuous, respectively.

(iii) If $f_n \in B([a, b])$ is a norm-bounded sequence converging pointwise to $f_0 \in B([a, b])$ then

$$\lim_{n \rightarrow \infty} (\mathcal{T}_{\mu, \lambda, m} f_n)(u) = (\mathcal{T}_{\mu, \lambda, m} f_0)(u) \quad (u \in [a, b]).$$

Proof. If $f \in B([a, b])$ then, for all fixed $u \in [a, b]$, the function $t \mapsto \lambda(t, u) f(m(t, u))$ is $\mathcal{A}$ -measurable on T . On the other hand, $|\lambda(t, u) f(m(t, u))| \leq \ell(t) \|f\|$, hence the integral on the right hand side of (2.9) is well-defined and

$$|(\mathcal{T}_{\mu, \lambda, m} f)(u)| \leq \int_T |\lambda(t, u) f(m(t, u))| d\mu(t) \leq \int_T \ell(t) d\mu(t) \|f\|,$$

which proves the boundedness of $\mathcal{T}_{\mu, \lambda, m}$ and (2.19). The linearity and positivity of $\mathcal{T}_{\mu, \lambda, m}$ are obvious. The endpoint properties (2.20) are consequences of hypothesis (H3) and Proposition 2.6.

Now we show that $\mathcal{T}_{\mu, \lambda, m} f$ is lower semicontinuous whenever $f \in B([a, b])$ is also lower semicontinuous. Let (u_n) be an arbitrary sequence in $[a, b]$ converging to $u_0 \in [a, b]$. By the continuity–semicontinuity properties of λ , m and f , and the uniform integrable bound estimate $|\lambda(t, u_n) f(m(t, u_n))| \leq \ell(t) \|f\|$, the Fatou Lemma can be applied. Thus

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\mathcal{T}_{\mu, \lambda, m} f)(u_n) &= \liminf_{n \rightarrow \infty} \int_T \lambda(t, u_n) f(m(t, u_n)) d\mu(t) \\ &\geq \int_T \liminf_{n \rightarrow \infty} \lambda(t, u_n) f(m(t, u_n)) d\mu(t) \\ &\geq \int_T \lambda(t, u_0) f(m(t, u_0)) d\mu(t) = (\mathcal{T}_{\mu, \lambda, m} f)(u_0), \end{aligned}$$

which proves the lower semicontinuity of $\mathcal{T}_{\mu, \lambda, m} f$ at u_0 . Hence $\mathcal{T}_{\mu, \lambda, m} f$ is lower semicontinuous at every point of $[a, b]$ provided that f is also lower semicontinuous on $[a, b]$. The proof for the upper semicontinuity is analogous.

To verify (iii), assume that $f_n \in B([a, b])$ is a norm-bounded sequence converging pointwise to a function $f_0 : [a, b] \rightarrow \mathbb{R}$. Then $f_0 \in B([a, b])$ and, by Lebesgue's Dominated Convergence

Theorem, for all $u \in [a, b]$,

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathcal{T}_{\mu, \lambda, m} f_n)(u) &= \lim_{n \rightarrow \infty} \int_T \lambda(t, u) f_n(m(t, u)) d\mu(t) \\ &= \int_T \lim_{n \rightarrow \infty} (\lambda(t, u) f_n(m(t, u))) d\mu(t) \\ &= \int_T \lambda(t, u) f_0(m(t, u)) d\mu(t) = (\mathcal{T}_{\mu, \lambda, m} f_0)(u), \end{aligned}$$

which proves (iii).

Finally, we prove that $\mathcal{T}_{\mu, \lambda, m} f$ is bounded Borel measurable for all $f \in B([a, b])$. For this purpose, define the class of functions $B_0([a, b])$ by

$$B_0([a, b]) := \{f \in B([a, b]) \mid \mathcal{T}_{\mu, \lambda, m} f \in B([a, b])\}.$$

Obviously, this class is a subspace of $B([a, b])$ which contains $C([a, b])$ because, by (ii), the images of continuous functions are continuous. On the other hand, if f_n is a norm-bounded sequence of functions from $B_0([a, b])$ converging to a function $f_0 \in B([a, b])$, then, by (iii), $\mathcal{T}_{\mu, \lambda, m} f_0$ is the pointwise limit of the sequence of bounded Borel functions $\mathcal{T}_{\mu, \lambda, m} f_n$, and hence, $\mathcal{T}_{\mu, \lambda, m} f_0$ has to be also a Borel function, i.e., $f_0 \in B_0([a, b])$. In other words, the class $B_0([a, b])$ is closed with respect to the pointwise convergence and contains $C([a, b])$. Hence the equality $B_0([a, b]) = B([a, b])$ must hold (cf. [6]), which has to be proved. $\square$

The following proposition is a counterpart of [Proposition 2.6](#)

Proposition 2.10. Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu, \lambda, m}$ by (2.9). Then, for all strictly (ω_0, ω_1) -convex functions $f : [a, b] \rightarrow \mathbb{R}$,

$$f(u) < (\mathcal{T}_{\mu, \lambda, m} f)(u) \quad (u \in]a, b[). \quad (2.21)$$

Proof. Let $u \in]a, b[$ be a fixed element. Then, by [Lemma 1.1](#), there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\omega_0(u) + \beta\omega_1(u) = f(u) \quad \text{and} \quad \alpha\omega_0(v) + \beta\omega_1(v) < f(v) \quad (v \in [a, b] \setminus \{u\}). \quad (2.22)$$

We show that the image of the function $f - (\alpha\omega_0 + \beta\omega_1)$ by $\mathcal{T}_{\mu, \lambda, m}$ is everywhere positive on $[a, b]$. If this were not the case, then there would exist an element $v \in [a, b]$ such that

$$(\mathcal{T}_{\mu, \lambda, m} (f - \alpha\omega_0 - \beta\omega_1))(v) = 0.$$

Definition (2.9) of $\mathcal{T}_{\mu, \lambda, m}$ yields that

$$\int_T \lambda(t, v) \cdot (f - \alpha\omega_0 - \beta\omega_1)(m(t, v)) d\mu(t) = 0.$$

The function $t \mapsto (f - \alpha\omega_0 - \beta\omega_1)(m(t, v))$ vanishes if and only if $m(t, v) = u$. Since the integrand is a nonnegative measurable function of the variable t , it can only vanish if and only if the set

$$S := \{t \in T \mid \lambda(t, v) > 0, m(t, v) \neq u\}$$

has zero μ measure.

On the other hand, by assumption (H4), for $i \in \{0, 1\}$, we also have that

$$\begin{aligned}\omega_i(v) &= \int_T \lambda(t, v) \omega_i(m(t, v)) d\mu(t) = \int_{T \setminus S} \lambda(t, v) \omega_i(m(t, v)) d\mu(t) \\ &= \int_{T \setminus S} \lambda(t, v) \omega_i(u) d\mu(t) = \omega_i(u) \int_{T \setminus S} \lambda(t, v) d\mu(t) = \omega_i(u) \int_T \lambda(t, v) d\mu(t).\end{aligned}$$

Dividing the above identities by each other, we get that

$$\frac{\omega_1}{\omega_0}(v) = \frac{\omega_1}{\omega_0}(u).$$

This, by the strict monotonicity of $\frac{\omega_1}{\omega_0}$ contradicts $v \neq u$. Using also (2.11), this means that

$$\alpha\omega_0 + \beta\omega_1 = \mathcal{T}_{\mu, \lambda, m}(\alpha\omega_0 + \beta\omega_1) < \mathcal{T}_{\mu, \lambda, m}f \quad \text{on } [a, b].$$

Substituting the fixed element $u \in]a, b[$ and using also (2.22), we get that

$$f(u) = \alpha\omega_0(u) + \beta\omega_1(u) < (\mathcal{T}_{\mu, \lambda, m}f)(u),$$

which proves (2.21). $\square$

Theorem 2.11. Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu, \lambda, m}$ by (2.9). Then, there exists an (ω_0, ω_1) -concave function $g : [a, b] \rightarrow \mathbb{R}$, with $g(a) = g(b) = 0$, and a constant $q \in [0, 1[$ such that

$$(\mathcal{T}_{\mu, \lambda, m}g)(u) \leq qg(u) \quad (u \in [a, b]). \quad (2.23)$$

Proof. For $p \in]0, 1[$, define $g_p : [a, b] \rightarrow \mathbb{R}$ by the following way:

$$g_p(u) := \omega_0(u) \min \left(\left(\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a) \right)^p, \left(\frac{\omega_1}{\omega_0}(b) - \frac{\omega_1}{\omega_0}(u) \right)^p \right).$$

Observe that $g_p(a) = g_p(b) = 0$, $g_p(u) > 0$ for all $u \in]a, b[$, and

$$g_p(u) = \begin{cases} \omega_0(u) \left(\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a) \right)^p & \text{if } u \in [a, c], \\ \omega_0(u) \left(\frac{\omega_1}{\omega_0}(b) - \frac{\omega_1}{\omega_0}(u) \right)^p & \text{if } u \in [c, b], \end{cases} \quad (2.24)$$

where $c := \left(\frac{\omega_1}{\omega_0} \right)^{-1} \left(\frac{1}{2} \frac{\omega_1}{\omega_0}(a) + \frac{1}{2} \frac{\omega_1}{\omega_0}(b) \right)$, furthermore

$$\begin{aligned}\left(\frac{g_p}{\omega_0} \right) \circ \left(\frac{\omega_1}{\omega_0} \right)^{-1}(v) &= \min \left(\left(v - \frac{\omega_1}{\omega_0}(a) \right)^p, \left(\frac{\omega_1}{\omega_0}(b) - v \right)^p \right) \\ &\quad \left(v \in \left[\frac{\omega_1}{\omega_0}(a), \frac{\omega_1}{\omega_0}(b) \right] \right),\end{aligned}$$

which is the minimum of two strictly concave functions on the interval $\left[\frac{\omega_1}{\omega_0}(a), \frac{\omega_1}{\omega_0}(b) \right]$ in the standard sense. Therefore, by Lemma 1.1, g_p is strictly (ω_0, ω_1) -concave (i.e., $(-g_p)$ is strictly (ω_0, ω_1) -convex). Using Proposition 2.10, we obviously have that

$$\mathcal{T}_{\mu, \lambda, m}g_p < g_p \quad \text{on }]a, b[. \quad (2.25)$$

To prove (2.23), we will first show that

$$\limsup_{u \rightarrow a} \frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u)}{g_p(u)} < 1 \quad \text{and} \quad \limsup_{u \rightarrow b} \frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u)}{g_p(u)} < 1. \quad (2.26)$$

For the calculation of the first limit in (2.26), observe by (2.24) that, for $u \in [a, c]$,

$$\begin{aligned} \frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u)}{g_p(u)} &= \int_T \lambda(t, u) \frac{g_p(m(t, u))}{g_p(u)} d\mu(t) \\ &\leq \int_T \lambda(t, u) \frac{\omega_0(m(t, u))}{\omega_0(u)} \left(\frac{\frac{\omega_1}{\omega_0}(m(t, u)) - \frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a)} \right)^p d\mu(t). \end{aligned} \quad (2.27)$$

In what follows, to ensure the applicability of the Lebesgue Dominated Convergence Theorem, we show that the integrand in the rightmost integral has a u -independent integrable upper bound.

Let $\varepsilon := \frac{1}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a)$. By the differentiability of $\frac{\omega_1}{\omega_0}$, there exists $\eta > 0$ such that

$$\left| \frac{\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a)}{u - a} - \left(\frac{\omega_1}{\omega_0} \right)'(a) \right| \leq \frac{1}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a) \quad (u \in [a, a + \eta]).$$

Therefore,

$$\frac{1}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a) \cdot (u - a) \leq \frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a) \leq \frac{3}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a) \cdot (u - a) \quad (u \in [a, a + \eta]). \quad (2.28)$$

Let $\delta > 0$ and $K \geq 1$ be the constants required in hypothesis (H3) such that $K\delta \leq \eta$. Define the function $\ell : T \rightarrow \mathbb{R}_+$ as in hypothesis (H2). Then ℓ is a μ -integrable function and

$$\lambda(t, u) \leq \ell(t) \quad ((t, u) \in T \times [a, b]).$$

On the other hand, by the uniform calmness assumption on m , for $(t, u) \in T \times]a, a + \delta]$, we have that $|u - a| \leq \eta$ and $|m(t, u) - m(t, a)| \leq \eta$, hence, applying (2.28), we obtain

$$\begin{aligned} \frac{\frac{\omega_1}{\omega_0}(m(t, u)) - \frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a)} &\leq \frac{\frac{3}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a) \cdot (m(t, u) - a)}{\frac{1}{2} \left(\frac{\omega_1}{\omega_0} \right)'(a) \cdot (u - a)} \\ &= 3 \frac{m(t, u) - m(t, a)}{u - a} \leq 3K \quad ((t, u) \in T \times]a, a + \delta]). \end{aligned}$$

Using the above estimates, it follows that, for $(t, u) \in T \times]a, a + \delta]$,

$$\lambda(t, u) \frac{\omega_0(m(t, u))}{\omega_0(u)} \left(\frac{\frac{\omega_1}{\omega_0}(m(t, u)) - \frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a)} \right)^p \leq \ell(t) \cdot \|\omega_0\| \cdot \|\omega_0^{-1}\| \cdot (3K)^p.$$

Therefore, upon taking the limit $u \rightarrow a$ in (2.27), Lebesgue's Theorem can be applied. Let u_n be an arbitrary sequence in $]a, a + \delta]$ converging to a . By taking a subsequence if necessary, we may assume that the sequence $\frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u_n)}{g_p(u_n)}$ is convergent. Then we obtain

$$\lim_{n \rightarrow \infty} \frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u_n)}{g_p(u_n)}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \int_T \lambda(t, u_n) \frac{\omega_0(m(t, u_n))}{\omega_0(u_n)} \left(\frac{\frac{\omega_1}{\omega_0}(m(t, u_n)) - \frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u_n) - \frac{\omega_1}{\omega_0}(a)} \right)^p d\mu(t) \\
&\leq \int_T \lim_{n \rightarrow \infty} \left(\lambda(t, u_n) \frac{\omega_0(m(t, u_n))}{\omega_0(u_n)} \left(\frac{\frac{\omega_1}{\omega_0}(m(t, u_n)) - \frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u_n) - \frac{\omega_1}{\omega_0}(a)} \right)^p \right) d\mu(t) \\
&= \int_T \lambda(t, a) (m'(t, a))^p d\mu(t).
\end{aligned}$$

The choice of the sequence (u_n) being arbitrary, we get, by the first relation in (2.14), that

$$\limsup_{u \rightarrow a} \frac{(\mathcal{T}_{\mu, \lambda, m} g_p)(u)}{g_p(u)} \leq \int_T \lambda(t, a) (m'(t, a))^p d\mu(t) < 1,$$

which results the first inequality of (2.26). The proof of the second inequality is similar.

Let $p \in]0, 1[$. Consider, the function $G_p :]a, b[\rightarrow \mathbb{R}$ defined by

$$G_p(u) := \left(\frac{\mathcal{T}_{\mu, \lambda, m} g_p}{g_p} \right)(u) \quad (u \in]a, b[).$$

Then, by Proposition 2.9, G_p is continuous on $]a, b[$ and, by (2.25), we have that $G_p(u) < 1$ for all $u \in]a, b[$. Using (2.26), we can find constants $0 < \delta < \frac{b-a}{2}$ and $q_0 \in [0, 1[$ such that

$$G_p(u) \leq q_0 \quad \text{if } u \in]a, a + \delta] \cup [b - \delta, b[.$$

Since G_p is continuous on $]a, b[$,

$$q_1 := \sup_{u \in [a + \delta, b - \delta]} G_p(u) < 1.$$

Thus,

$$(\mathcal{T}_{\mu, \lambda, m} g_p)(u) \leq \max(q_0, q_1) \cdot g_p(u) \quad (u \in]a, b[).$$

On the other hand, by (2.20) in Proposition 2.9, we also have that

$$(\mathcal{T}_{\mu, \lambda, m} g_p)(a) = g_p(a) = 0 \quad \text{and} \quad (\mathcal{T}_{\mu, \lambda, m} g_p)(b) = g_p(b) = 0.$$

Then using the previous observations, we get (2.23) with $q := \max(q_0, q_1)$. $\square$

The following theorem is the main result of this section.

Theorem 2.12. Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu, \lambda, m}$ by (2.9). Then $\mathcal{T}_{\mu, \lambda, m} : B([a, b]) \rightarrow B([a, b])$ is a bounded positive linear operator with the properties (i)–(iii) listed in Proposition 2.9. Furthermore, for all bounded upper semicontinuous functions $f : [a, b] \rightarrow \mathbb{R}$,

$$\limsup_{n \rightarrow \infty} (\mathcal{T}_{\mu, \lambda, m}^n f)(u) \leq f(a) \frac{\Omega(u, b)}{\Omega(a, b)} + f(b) \frac{\Omega(a, u)}{\Omega(a, b)} \quad (u \in [a, b]). \quad (2.29)$$

If, in addition, f satisfies the inequality $f \leq \mathcal{T}_{\mu, \lambda, m} f$, then

$$f(u) \leq f(a) \frac{\Omega(u, b)}{\Omega(a, b)} + f(b) \frac{\Omega(a, u)}{\Omega(a, b)} \quad (u \in [a, b]). \quad (2.30)$$

Proof. The first part of this theorem is a consequence of Proposition 2.9. By (2.11) in hypothesis (H4), we obviously have that ω_0 and ω_1 are the fixed points of $\mathcal{T}_{\mu,\lambda,m}$. Using also Theorem 2.11, we get that there exists an (ω_0, ω_1) -concave function $g \in C([a, b])$ such that $g(a) = g(b) = 0$ and (2.23) holds for some $q \in [0, 1[$. Hence Corollary 2.4 can be applied with $\mathcal{T} := \mathcal{T}_{\mu,\lambda,m}$ and we obtain (2.6) which is equivalent to (2.29). If, in addition, f satisfies the inequality $f \leq \mathcal{T}_{\mu,\lambda,m}f$, then Proposition 2.7 can be applied and (2.30) follows. $\square$

Corollary 2.13. Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 = \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Define the operator $\mathcal{T}_\mu : B([a, b]) \rightarrow B([a, b])$ by the following formula:

$$(\mathcal{T}_\mu f)(u) := (\mathcal{T}_\mu f)(u) \\ := \begin{cases} \int_{[0,1]} f\left(\frac{t}{\mu_1}(u-a) + a\right) d\mu(t) & \text{if } a \leq u \leq (1-\mu_1)a + \mu_1 b, \\ \int_{[0,1]} f\left(\frac{1-t}{1-\mu_1}(u-b) + b\right) d\mu(t) & \text{if } (1-\mu_1)a + \mu_1 b \leq u \leq b. \end{cases} \quad (2.31)$$

Then $\mathcal{T}_\mu : B([a, b]) \rightarrow B([a, b])$ is a bounded positive linear operator with $\|\mathcal{T}_\mu\| = 1$, $\mathcal{T}_\mu(C([a, b])) \subseteq C([a, b])$ and

$$(\mathcal{T}_\mu f)(a) = f(a), \quad (\mathcal{T}_\mu f)(b) = f(b), \quad \text{and} \\ \inf_{[a,b]} f \leq (\mathcal{T}_\mu f)(u) \leq \sup_{[a,b]} f \quad (u \in [a, b]) \quad (2.32)$$

for all $f \in B([a, b])$. Furthermore, for all bounded upper semicontinuous functions $f : [a, b] \rightarrow \mathbb{R}$,

$$\limsup_{n \rightarrow \infty} (\mathcal{T}_\mu^n f)(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \quad (t \in [0, 1]). \quad (2.33)$$

If, in addition, f satisfies the inequality $f \leq \mathcal{T}_\mu f$, then

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \quad (t \in [0, 1]). \quad (2.34)$$

Proof. The measure μ is not a Dirac measure, therefore $0 < \mu_1 < 1$ and hence $\mathcal{T}_\mu$ is well-defined. Define $\lambda : [0, 1] \times [a, b] \rightarrow \mathbb{R}_+$ and $m : [0, 1] \times [a, b] \rightarrow [a, b]$ by the following way:

$$\lambda(t, u) = 1 \quad \text{and} \quad m(t, u) := \begin{cases} \frac{t}{\mu_1}(u-a) + a, & \text{if } a \leq u \leq (1-\mu_1)a + \mu_1 b, \\ \frac{1-t}{1-\mu_1}(u-b) + b, & \text{if } (1-\mu_1)a + \mu_1 b \leq u \leq b. \end{cases} \quad (2.35)$$

One can see that $\mathcal{T}_\mu$ defined in (2.31) equals $\mathcal{T}_{\lambda,\mu,m}$ defined in (2.9) with λ and μ from (2.35). In order to make Theorem 2.12 applicable, it is enough to show that the triplet (λ, μ, m) satisfies the conditions (H1)–(H4). It is obvious that (H1) and (H2) hold. Furthermore, the equalities $m(\cdot, a) = a$ and $m(\cdot, b) = b$ are also trivial. Observe that m defined in (2.35) is continuous in its first variable and differentiable in its second variable at every element

$u \in [a, b] \setminus \{(1 - \mu_1)a + \mu_1 b\}$ and

$$m'(t, u) = \begin{cases} \frac{t}{\mu_1} & \text{if } a \leq u < (1 - \mu_1)a + \mu_1 b, \\ \frac{1-t}{1-\mu_1} & \text{if } (1 - \mu_1)a + \mu_1 b < u \leq b. \end{cases}$$

On the other hand, for all $t \in [a, b]$

$$|m(t, u) - a| = \frac{t}{\mu_1}(u - a) \leq \frac{1}{\mu_1}(u - a) \quad \text{if } 0 \leq u - a < \mu_1(b - a),$$

and

$$|m(t, u) - b| = \frac{1-t}{1-\mu_1}(b - a) \leq \frac{1}{1-\mu_1}(b - a) \quad \text{if } 0 \leq b - u < (1 - \mu_1)(b - a).$$

This shows that m is uniformly calm at the endpoints of $[a, b]$. (2.10) also holds, because if $u \notin \{a, b\}$, then $m(t, u) = u$ can hold if and only if $t = \mu_1$ and the support of μ is not the singleton μ_1 . Similarly, if $u \in \{a, b\}$, then $m'(t, u) = 1$ can hold if and only if $t = \mu_1$, which yields the second condition in (2.10). To show (H4), let (ω_0, ω_1) be the standard Chebyshev system, i.e., $\omega_0(u) = 1$ and $\omega_0(u) = u$, if $u \in [a, b]$. Using that μ is a Borel probability measure and the definition of μ_1 , we get that

$$1 = \int_{[0,1]} 1 d\mu(t) \quad \text{and} \quad u = \int_{[0,1]} m(t, u) d\mu(t) \quad (u \in [a, b]),$$

which proves that ω_0 and ω_1 are fixed points of $\mathcal{T}_\mu$ and hence (H4) follows. Thus, all the hypotheses (H1)–(H4) hold and hence Theorem 2.12 can be applied.

Therefore, $\mathcal{T}_\mu$ is a bounded positive linear operator with the properties listed in the theorem, (2.32) can be checked directly. Using the formula $\Omega(x, y) = y - x$, and putting $u = ta + (1 - t)b$ into (2.29) and (2.30), the inequalities (2.33) and (2.34) follow, respectively. $\square$

Remark. The assumption in Corollary 2.13 that μ is not a Dirac measure is essential, because if $\mu = \delta_\tau$ with $0 < \tau < 1$, then $\mu_1 = \tau$ and $\mathcal{T}_\mu$ is the identity operator on $B([a, b])$. Then, (2.33) and (2.34) cannot hold for all upper semicontinuous functions.

3. From the Hermite–Hadamard inequality to the (ω_0, ω_1) -convexity type inequality

To describe the regularity assumptions for our main results, we introduce the following terminology for real valued functions defined on $\Delta(I)$. Let $\Phi : \Delta(I) \rightarrow \mathbb{R}$. The function Φ is called *separately continuous* (*separately strictly increasing*) if, for all $z \in I$, the mappings $u \mapsto \Phi(u, z)$ and $u \mapsto \Phi(z, u)$ are continuous (strictly increasing) on the intervals $]-\infty, z] \cap I$ and on $[z, \infty[\cap I$, respectively. We say that Φ is *separately partially differentiable at the diagonal* of $I \times I$ if, for all $z \in I$, the mappings $u \mapsto \Phi(u, z)$ and $u \mapsto \Phi(z, u)$ are differentiable at z with derivatives denoted by $\partial_1 \Phi(z, z)$ and $\partial_2 \Phi(z, z)$, respectively. The function Φ is called a *two-variable mean on I* if, for all $(x, y) \in \Delta(I)$, the inequalities $x \leq \Phi(x, y) \leq y$ hold. A mean Φ is called *strict* if both of these inequalities are strict whenever $x < y$.

Consider the following approximate Hermite–Hadamard type functional inequality

$$f(M_0(x, y)) \leq \int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) + E(x, y) \quad ((x, y) \in \Delta(I)), \quad (3.1)$$

for the unknown function $f : I \rightarrow \mathbb{R}$ and error function $E : \Delta(I) \rightarrow \mathbb{R}$, where we make the following assumption.

(A1) $(T, \mathcal{A}, \mu)$ is a measure space.

(A2) $\Lambda : T \times \Delta(I) \rightarrow \mathbb{R}_+$ is measurable in its first variable and separately continuous in its second variable; furthermore, for all $(x, y) \in \Delta(I)$ the function $L_{x,y} : T \rightarrow \mathbb{R}_+$ defined by

$$L_{x,y}(t) := \sup_{u \in [x,y]} \max(\Lambda(t, x, u), \Lambda(t, u, y))$$

is μ -integrable, i.e., $\int_T L_{x,y}(t) d\mu(t) < +\infty$.

(A3) $M : T \times \Delta(I) \rightarrow \mathbb{R}$ is measurable in its first variable and for all $t \in T$, the map $(x, y) \mapsto M(t, x, y)$ is a two-variable mean on I , which is separately continuous and partially differentiable at the diagonal of $I \times I$. $M_0 : \Delta(I) \rightarrow I$ is separately continuous, strictly increasing and partially differentiable at the diagonal of $I \times I$ with $\partial_1 M_0(z, z) > 0$ and $\partial_2 M_0(z, z) > 0$ for all $z \in I$. Furthermore, $M(t, \cdot)$ is separately uniformly calm with respect to M_0 at the diagonal of $I \times I$, i.e., for all $z \in I$, there exist constants $\delta > 0$ and $K \geq 0$ such that, for all $t \in T$,

$$z - M(t, u, z) \leq K(z - M_0(u, z)) \quad (u \in [z - \delta, z]),$$

$$M(t, z, u) - z \leq K(M_0(z, u) - z) \quad (u \in [z, z + \delta]),$$

and

$$\mu\{t \in T \mid \Lambda(t, x, y) \neq 0, M(t, x, y) \neq M_0(x, y)\} > 0 \quad \text{if } (x, y) \in \Delta^\circ(I),$$

$$\mu\{t \in T \mid \Lambda(t, z, z) \neq 0, \partial_i M(t, z, z) \neq \partial_i M_0(z, z)\} > 0 \quad \text{if } z \in I, i \in \{1, 2\}.$$

(3.2)

(A4) There exist functions $\omega_0, \omega_1 : I \rightarrow \mathbb{R}$ such that ω_0 is positive, $\frac{\omega_1}{\omega_0}$ is differentiable on I , with $\left(\frac{\omega_1}{\omega_0}\right)' > 0$. Furthermore, for $i \in \{0, 1\}$,

$$\omega_i(M_0(x, y)) = \int_T \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) \quad ((x, y) \in \Delta(I)). \quad (3.3)$$

Theorem 3.14. Suppose that conditions (A1)–(A4) hold and assume that $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of the functional inequality (3.1), where $E : \Delta(I) \rightarrow \mathbb{R}$. Assume that, for all $(x, y) \in \Delta^\circ(I)$, $\varepsilon_{x,y} : [x, y] \rightarrow \mathbb{R}$ is a lower semicontinuous function with $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$ satisfying the following system of inequalities:

$$\begin{aligned} \varepsilon_{x,y}(M_0(x, u)) &\geq \int_T \Lambda(t, x, u) \varepsilon_{x,y}(M(t, x, u)) d\mu(t) + E(x, u) \quad (u \in [x, y]), \\ \varepsilon_{x,y}(M_0(u, y)) &\geq \int_T \Lambda(t, u, y) \varepsilon_{x,y}(M(t, u, y)) d\mu(t) + E(u, y) \quad (u \in [x, y]). \end{aligned} \quad (3.4)$$

Then, for all fixed $(x, y) \in \Delta^\circ(I)$, the function f also satisfies the approximate (ω_0, ω_1) -convexity inequality:

$$f(u) \leq \frac{\Omega(u, y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, u)}{\Omega(x, y)} f(y) + \varepsilon_{x,y}(u) \quad (u \in [x, y]). \quad (3.5)$$

Proof. To prove (3.5), let $(x, y) \in \Delta^\circ(I)$ be fixed. In what follows, we will apply Theorem 2.12 for functions defined on the interval $[a, b] := [x, y]$. We are going to construct the functions λ and m so that all hypotheses of Theorem 2.12 be satisfied.

By the separate continuity and strict increasingness of the mean M_0 , the functions $u \mapsto M_0(u, y)$ and $u \mapsto M_0(x, u)$ are continuous, strictly increasing and map $[x, y]$ onto $[M_0(x, y), y]$ and $[x, M_0(x, y)]$, respectively. Therefore they are continuously invertible and hence there exist continuous functions $\alpha : [M_0(x, y), y] \rightarrow [x, y]$ and $\beta : [x, M_0(x, y)] \rightarrow [x, y]$ such that

$$M_0(\alpha(u), y) = u \quad (u \in [M_0(x, y), y]) \quad \text{and} \quad M_0(x, \beta(u)) = u \quad (u \in [x, M_0(x, y)]).$$

Define the functions $\lambda, m : T \times [x, y] \rightarrow \mathbb{R}$ and $e : [x, y] \rightarrow \mathbb{R}$ by the following formulas:

$$\begin{aligned} \lambda(t, u) &:= \lambda_{x,y}(t, u) := \begin{cases} \Lambda(t, x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ \Lambda(t, \alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases} \\ m(t, u) &:= m_{x,y}(t, u) := \begin{cases} M(t, x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ M(t, \alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases} \\ e(u) &:= e_{x,y}(u) := \begin{cases} E(x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ E(\alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases} \end{aligned} \quad (3.6)$$

respectively. To see that λ, m and e are correctly defined, observe that $(x, \beta(u)) = (\alpha(u), y)$ for $u = M_0(x, y)$. By $\alpha(y) = y$ and $\beta(x) = x$, we can see that $m(t, x) = x, m(t, y) = y$ for all $t \in T$.

It immediately follows from the definition of λ and m and from the measurability and separate continuity assumptions in (A2) and (A3) that λ and m are measurable in their first variable and continuous in their second variable. Using (A2), we can also see that λ fulfills the integrability condition in hypothesis (H2) with $\ell(t) := L_{x,y}(t)$.

Applying the partial differentiability assumptions of (A3) at $z = x$ and $z = y$ and the chain rule and elementary calculus rule for the differentiation of real inverse functions, it follows that, for all $t \in T$, m is differentiable at the endpoints of $[x, y]$ with

$$m'(t, x) = \frac{\partial_2 M(t, x, x)}{\partial_2 M_0(x, x)} \quad \text{and} \quad m'(t, y) = \frac{\partial_1 M(t, y, y)}{\partial_2 M_0(y, y)}. \quad (3.7)$$

To prove the uniform calmness property of m , we apply the uniform calmness of M with respect to M_0 at $z = x$ and $z = y$ from (A3). Then there exist constants $0 < \delta \leq y - x$ and $K \geq 0$ such that

$$\begin{aligned} y - M(t, v, y) &\leq K(y - M_0(v, y)) \quad (v \in [y - \delta, y]), \\ M(t, x, v) - x &\leq K(M_0(x, v) - x) \quad (v \in [x, x + \delta]). \end{aligned}$$

Substituting $v = \alpha(u)$ and $v = \beta(u)$ into these inequalities, we get

$$\begin{aligned} y - m(t, u) &\leq K(y - u) \quad (u \in [M_0(y - \delta, y), y]), \\ m(t, u) - x &\leq K(u - x) \quad (u \in [x, M_0(x, x + \delta)]). \end{aligned}$$

Choose $\eta := \min(M_0(x, x + \delta) - x, y - M_0(y - \delta, y))$. Then, $\eta > 0$ and

$$[x, x + \eta] \subseteq [x, M_0(x, x + \delta)] \quad \text{and} \quad [y - \eta, y] \subseteq [M_0(y - \delta, y), y].$$

Then, we obtain that

$$\begin{aligned}y - m(t, u) &\leq K(y - u) \quad (u \in [y - \eta, y]), \\m(t, u) - x &\leq K(u - x) \quad (u \in [x, x + \eta]),\end{aligned}$$

which yields the uniform calmness of m at the endpoints of $[a, b]$.

To check condition (2.10), let $u \in]x, y[$. In the case $u \leq M_0(x, y)$ the equalities $\lambda(t, u) = 0$ and $m(t, u) = u$ can hold if and only if $\Lambda(t, v) = 0$ and $M(t, x, v) = M_0(x, v)$ are valid with $v = \beta(u)$, respectively. Therefore, applying the first inequality for $(v, y) \in \Delta^\circ(I)$, we get that the first condition in (2.10) is satisfied. The case $M_0(x, y) \leq u$ is analogous.

Using formulas (3.7), it is completely similar to show that the second part of condition (2.10) in (H3) is a consequence of the second inequality in (3.2).

By condition (A4), for $i \in \{0, 1\}$, we have that

$$\begin{aligned}\omega_i(M_0(x, v)) &= \int_T \Lambda(t, x, v) \omega_i(M(t, x, v)) d\mu(t) \quad (v \in [x, y]), \\ \omega_i(M_0(v, y)) &= \int_T \Lambda(t, v, y) \omega_i(M(t, v, y)) d\mu(t) \quad (v \in [x, y]).\end{aligned}$$

Substituting $v = \beta(u)$ and $v = \alpha(u)$ into the above identities, respectively, we get that

$$\omega_i(u) = \int_T \lambda(t, u) \omega_i(m(t, u)) d\mu(t) \quad (u \in [x, y]).$$

This proves that hypothesis (H4) is also satisfied.

Thus, we have verified that all the hypotheses (H1)–(H4) of Theorem 2.12 are fulfilled.

Now define the operator $\mathcal{T}_{\mu, \lambda, m} : B([x, y]) \rightarrow B([x, y])$ by (2.9). Using (3.1), we have that

$$\begin{aligned}f(M_0(x, v)) &\leq \int_T \Lambda(t, x, v) f(M(t, x, v)) d\mu(t) + E(x, v) \quad (v \in [x, y]), \\ f(M_0(v, y)) &\leq \int_T \Lambda(t, v, y) f(M(t, v, y)) d\mu(t) + E(v, y) \quad (v \in [x, y]).\end{aligned}\tag{3.8}$$

The function f is upper semicontinuous, therefore, the restriction $f|_{[x, y]}$ is bounded from above (but maybe unbounded from below). Thus the integrals on the right hand side of (3.8) do exist and, by the inequality (3.8), are also finite for all $v \in [x, y]$. With the substitutions $v = \beta(u)$ and $v = \alpha(u)$, respectively, we get that

$$f(u) \leq \int_T \lambda(t, u) f(m(t, u)) d\mu(t) + e(u) \quad (u \in [x, y]).$$

Therefore, we obtain that $f|_{[x, y]}$ satisfies the following inequality

$$f|_{[x, y]} \leq \mathcal{T}_{\mu, \lambda, m}(f|_{[x, y]}) + e.\tag{3.9}$$

Similarly, due to the separate lower semicontinuity of $\varepsilon_{x, y}$, the integrals on the right hand side of (3.4) exist and are also finite. Thus, replacing u by $\beta(u)$ and $\alpha(u)$ in inequality (3.4), respectively, it follows that

$$\varepsilon_{x, y} \geq \mathcal{T}_{\mu, \lambda, m}(\varepsilon_{x, y}) + e.\tag{3.10}$$

Combining (3.9) and (3.10), we get that

$$f|_{[x, y]} - \varepsilon_{x, y} \leq \mathcal{T}_{\mu, \lambda, m}(f|_{[x, y]} - \varepsilon_{x, y}).\tag{3.11}$$

By our semicontinuity assumptions on f and $\varepsilon_{x, y}$, the function $h := f|_{[x, y]} - \varepsilon_{x, y}$ is upper semicontinuous, and (3.11) is equivalent to the inequality $h \leq \mathcal{T}_{\mu, \lambda, m}h$. In order to make

Theorem 2.12 applicable, we approximate h by a sequence of upper semicontinuous and bounded functions. For $n \in \mathbb{N}$, define

$$h_n := \max(h, (-n)\omega_0|_{[x,y]}).$$

Then, one can see that (h_n) is a sequence of upper semicontinuous functions which pointwise converges to h . On the other hand,

$$(-n)\omega_0|_{[x,y]} = \mathcal{T}_{\mu,\lambda,m}(-n\omega_0|_{[x,y]}) \leq \mathcal{T}_{\mu,\lambda,m}h_n \quad \text{and} \quad h \leq \mathcal{T}_{\mu,\lambda,m}h \leq \mathcal{T}_{\mu,\lambda,m}h_n.$$

Therefore, we get that

$$h_n \leq \mathcal{T}_{\mu,\lambda,m}h_n \quad (n \in \mathbb{N}).$$

Now, applying the last statement of **Theorem 2.12**, for $u \in [x, y]$, we obtain that

$$h_n(u) \leq h_n(x) \frac{\Omega(u, y)}{\Omega(x, y)} + h_n(y) \frac{\Omega(x, u)}{\Omega(x, y)} \quad (n \in \mathbb{N}).$$

Upon taking the limit $n \rightarrow \infty$, it follows that

$$\begin{aligned} f(u) - \varepsilon_{x,y}(u) &= h(u) \leq h(x) \frac{\Omega(u, y)}{\Omega(x, y)} + h(y) \frac{\Omega(x, u)}{\Omega(x, y)} \\ &= f(x) \frac{\Omega(u, y)}{\Omega(x, y)} + f(y) \frac{\Omega(x, u)}{\Omega(x, y)}, \end{aligned}$$

which results (3.5). $\square$

The next result provides a necessary and sufficient condition for the existence of the family of lower semicontinuous functions $\varepsilon_{x,y} : [x, y] \rightarrow \mathbb{R}$ satisfying $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$ and condition (3.4) of **Theorem 3.14**.

Proposition 3.15. *Suppose that conditions (A1)–(A4) hold and, in addition, $E : \Delta(I) \rightarrow \mathbb{R}_+$ is a nonnegative separately lower semicontinuous function with $E(z, z) = 0$ for all $z \in I$. Then the following three statements are equivalent.*

- (i) *For all $(x, y) \in \Delta^\circ(I)$, there exists a nonnegative bounded lower semicontinuous function $\varepsilon_{x,y}^* : [x, y] \rightarrow \mathbb{R}$ satisfying $\varepsilon_{x,y}^*(x) = \varepsilon_{x,y}^*(y) = 0$ and condition (3.4) with equality.*
- (ii) *For all $(x, y) \in \Delta^\circ(I)$, there exists a bounded lower semicontinuous function $\varepsilon_{x,y} : [x, y] \rightarrow \mathbb{R}$ satisfying $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$ and inequality (3.4).*
- (iii) *For all $(x, y) \in \Delta^\circ(I)$, the series*

$$\varepsilon_{x,y}^* := \sum_{n=0}^{\infty} \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^n e_{x,y}$$

is pointwise convergent and bounded on $[x, y]$, where the functions $\lambda_{x,y}$, $m_{x,y}$, and $e_{x,y}$ are defined by (3.6).

Furthermore, if the above equivalent conditions are satisfied, then, for all $(x, y) \in \Delta^\circ(I)$, the function $\varepsilon_{x,y}^$ is bounded lower semicontinuous and satisfies $\varepsilon_{x,y}^*(x) = \varepsilon_{x,y}^*(y) = 0$ and condition (3.4) with equality.*

Proof. The implication from (i) to (ii) holds trivially.

Consider the implication from (ii) to (iii). Let $(x, y) \in \Delta^\circ(I)$ and assume that there exists a bounded lower semicontinuous solution $\varepsilon_{x,y} : [x, y] \rightarrow \mathbb{R}$ of (3.4) satisfying $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$. Define the function $e_{x,y} : [x, y] \rightarrow \mathbb{R}$ by (3.6). Since E is nonnegative, separately bounded and separately lower semicontinuous, we have that $e_{x,y}$ is nonnegative, bounded and

lower semicontinuous. As we have seen in the proof of [Theorem 3.14](#), the functions $e_{x,y}$ and $\varepsilon_{x,y}$ satisfy the functional inequality (3.10) and hence

$$e_{x,y} \leq \varepsilon_{x,y} - \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} \varepsilon_{x,y}.$$

Applying the positivity of $\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$, we get by induction on $i \in \mathbb{N}$, that

$$\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^{i-1} e_{x,y} \leq \mathcal{T}_{\mu,\lambda,m}^{i-1} \varepsilon_{x,y} - \mathcal{T}_{\mu,\lambda,m}^i \varepsilon_{x,y}.$$

Summing these equalities from 1 to n , we get

$$\sum_{i=1}^n \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^{i-1} e_{x,y} \leq \varepsilon_{x,y} - \mathcal{T}_{\mu,\lambda,m}^n \varepsilon_{x,y} \leq \varepsilon_{x,y} \quad (n \in \mathbb{N}). \quad (3.12)$$

Hence, by the nonnegativity of $e_{x,y}$, the series $\sum_{i=1}^{\infty} \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^i e_{x,y}$ is pointwise convergent and, by (3.12), is also bounded on $[x, y]$. This proves (iii).

Finally, assume that (iii) holds. By the lower semicontinuity and nonnegativity assumptions, the series $\sum_{i=1}^{\infty} \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^i e_{x,y}$ has nonnegative and lower semicontinuous terms. Therefore, its sum, denoted by $\varepsilon_{x,y}^*$ is also a lower semicontinuous function on $[x, y]$. To complete the proof, we need to show that $\varepsilon_{x,y}^*$ satisfies (3.4) with equality. Indeed,

$$\sum_{i=0}^n \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^i e_{x,y} = e_{x,y} + \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} \left(\sum_{i=0}^{n-1} \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}^i e_{x,y} \right).$$

Taking the limit $n \rightarrow \infty$ and applying the pointwise convergence property of the operator $\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$, we get (i). $\square$

Now we consider the most important particular cases of [Theorem 3.14](#). An immediate consequence of [Corollary 3.16](#) is [Theorem E](#).

Corollary 3.16. *Suppose that conditions (A1)–(A4) hold and let $f : I \rightarrow \mathbb{R}$ be an upper semicontinuous function. Then f is a solution of the functional inequality*

$$f(M_0(x, y)) \leq \int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) \quad ((x, y) \in \Delta(I)) \quad (3.13)$$

if and only if f is (ω_0, ω_1) -convex.

Proof. Assume that $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of the functional inequality (3.13). Then f also fulfills (3.13) with $E \equiv 0$ in (3.1). Then, for all $(x, y) \in \Delta(I)$, $\varepsilon_{x,y} : [x, y] \rightarrow \mathbb{R}$ defined by $\varepsilon_{x,y} \equiv 0$ is a lower semicontinuous solution of (3.4). Hence [Theorem 3.14](#) can be applied, which results (3.5) holds $\varepsilon_{x,y} \equiv 0$. Thus f is (ω_0, ω_1) -convex.

Conversely, assume that $f : I \rightarrow \mathbb{R}$ is (ω_0, ω_1) -convex. Then $f|_{[x,y]}$ is upper semicontinuous and bounded on $[x, y]$. The inequality (3.13) trivially holds if $x = y \in I$. Let $(x, y) \in \Delta^\circ(I)$. Define the triplet $(\mu, \lambda_{x,y}, m_{x,y})$ by (3.18) as in the proof of [Theorem 3.14](#). Then, by [Proposition 2.9](#) we have that $(\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f)(x) = f(x)$ and $(\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f)(y) = f(y)$. On the other hand, $\mathcal{T} := \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$ is a positive linear operator with fixed points $\omega_0|_{[x,y]}$ and $\omega_1|_{[x,y]}$. Hence, by the (ω_0, ω_1) -convexity of f and [Proposition 2.6](#), we get

$$f|_{[x,y]} \leq (\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f|_{[x,y]}),$$

which means that

$$f(u) \leq \int_T \lambda_{x,y}(t, u) f(m_{x,y}(t, u)) d\mu(t) \quad (u \in [x, y]).$$

Substituting $u = M_0(x, y)$ into the above inequality and using the definition of $\lambda_{x,y}$ and $m_{x,y}$, we get (3.13), which completes the proof. $\square$

Remark 3.17. A direct consequence of this corollary is [Theorem E](#). Indeed, suppose that, with the notations introduced in (1.11), the assumptions of [Theorem E](#) hold and ω_0 is positive and (ω_1/ω_0) is differentiable. We show first that the conditions of [Corollary 3.16](#) are also valid. Let μ denote the Lebesgue measure on $[0, 1]$ and define, for all $(x, y) \in \Delta^\circ(I)$, $t \in [0, 1]$,

$$\begin{aligned} M_0(x, y) &:= \xi(x, y), & M(t, x, y) &:= (1-t)x + ty, \\ A(t, x, y) &:= \frac{(y-x)\rho((1-t)x + ty)}{c(x, y)}. \end{aligned}$$

It can be also seen that (A3) holds. We also have

$$\begin{aligned} \int_0^1 A(t, x, y) \omega_1(M(t, x, y)) dt &= \frac{y-x}{c(x, y)} \int_0^1 \rho((1-t)x + ty) \omega_1((1-t)x + ty) dt \\ &= \frac{1}{c(x, y)} \int_x^y \rho \omega_1 = \omega_0(\xi(x, y)) \frac{\int_x^y \omega_1 \rho}{\int_x^y \omega_0 \rho} = \omega_0(\xi(x, y)) \frac{\omega_1}{\omega_0}(\xi(x, y)) = \omega_1(M_0(x, y)) \end{aligned}$$

and, similarly,

$$\int_0^1 A(t, x, y) \omega_0(M(t, x, y)) dt = \frac{1}{c(x, y)} \int_x^y \rho \omega_0 = \omega_0(\xi(x, y)),$$

which proves (3.3). Thus all the assumptions (A1)–(A4) are verified. Therefore, if a function $f : I \rightarrow \mathbb{R}$ satisfies (1.12), then it is (ω_0, ω_1) -convex.

Theorem 3.18. Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1-t)x + ty) d\mu(t) + E(x, y) \quad ((x, y) \in D^2), \quad (3.14)$$

where $E : D^2 \rightarrow \mathbb{R}$. Assume that, for all $(x, y) \in D^2$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a lower semicontinuous function with $\eta_{x,y}(0) = \eta_{x,y}(1) = 0$ satisfying the following system of inequalities:

$$\begin{aligned} &\eta_{x,y}(s) \\ &\geq \begin{cases} \int_{[0,1]} \eta_{x,y}\left(\frac{s}{\mu_1}t\right) d\mu(t) + E\left(x, \left(1-\frac{s}{\mu_1}\right)x + \frac{s}{\mu_1}y\right) & \text{if } s \in [0, \mu_1], \\ \int_{[0,1]} \eta_{x,y}\left(1-\frac{1-s}{1-\mu_1}(1-t)\right) d\mu(t) + E\left(\frac{1-s}{1-\mu_1}x + \left(1-\frac{1-s}{1-\mu_1}\right)y, y\right) & \text{if } s \in [\mu_1, 1]. \end{cases} \end{aligned} \quad (3.15)$$

Then, for all $x, y \in D$, the function f also satisfies the approximate convexity inequality:

$$f((1-s)x + sy) \leq (1-s)f(x) + sf(y) + \eta_{x,y}(s) \quad (s \in [0, 1]). \quad (3.16)$$

Proof. Assume that (3.14) holds. Let x, y be arbitrarily fixed elements of D . Substituting x by $ux + (1-u)y$ and y by $vx + (1-v)y$, where $u \leq v$ from $[0, 1]$, we get that

$$\begin{aligned}
& f(((1 - \mu_1)u + \mu_1 v)x + (1 - (1 - \mu_1)u - \mu_1 v)y) \\
& \leq \int_{[0,1]} f(((1 - t)u + tv)x + (1 - (1 - t)u - tv)y) d\mu(t) \\
& \quad + E(ux + (1 - u)y, vx + (1 - v)y).
\end{aligned} \tag{3.17}$$

Define $g := g_{x,y} : [0, 1] \rightarrow \mathbb{R}$ by the following way

$$g(u) := g_{x,y}(u) = f(ux + (1 - u)y) \quad (u \in [0, 1])$$

and

$$F(u, v) := E(ux + (1 - u)y, vx + (1 - v)y) \quad ((u, v) \in \Delta([0, 1])).$$

Then (3.17) reduces to

$$g((1 - \mu_1)u + \mu_1 v) \leq \int_{[0,1]} g((1 - t)u + tv) d\mu(t) + F(u, v) \quad ((u, v) \in \Delta([0, 1])).$$

Define $A : [0, 1] \times \Delta([0, 1]) \rightarrow \mathbb{R}_+$, $M : [0, 1] \times \Delta([0, 1]) \rightarrow [0, 1]$ and $M_0 : \Delta([0, 1]) \rightarrow [0, 1]$ as follows:

$$\begin{aligned}
A(t, u, v) &= 1, \quad M(t, u, v) := (1 - t)u + tv, \quad \text{and} \\
M_0(u, v) &= (1 - \mu_1)u + \mu_1 v.
\end{aligned} \tag{3.18}$$

In order to make Theorem 3.14 applicable for the function g , it is enough to show that A , μ , M , and M_0 satisfy the conditions (A1)–(A4). It is obvious that (A1) and (A2) hold. Observe that M defined in (3.18) is measurable in its first variable and for all $t \in [0, 1]$, the map $(u, v) \mapsto M(t, u, v)$ is a two-variable mean on $[0, 1]$, which is separately continuous and partially differentiable at the diagonal of $[0, 1] \times [0, 1]$, i.e., for all $t \in [0, 1]$

$$\partial_1 M(t, z, z) = 1 - t \quad \text{and} \quad \partial_2 M(t, z, z) = t \quad (z \in [0, 1]).$$

The function $M_0 : \Delta([0, 1]) \rightarrow [0, 1]$ is also separately continuous, strictly increasing and partially differentiable at the diagonal of $[0, 1] \times [0, 1]$, i.e.,

$$\partial_1 M_0(z, z) = 1 - \mu_1 \quad \text{and} \quad \partial_2 M_0(z, z) = \mu_1 \quad (z \in [0, 1]).$$

Observe that these derivatives are positive for all $z \in [0, 1]$. The uniform calmness of M with respect to M_0 also holds with $K := \max\left(\frac{1}{\mu_1}, \frac{1}{1 - \mu_1}\right)$. The equalities

$$\begin{aligned}
M(t, u, v) &= M_0(u, v) \quad ((u, v) \in \Delta^\circ([0, 1])) \quad \text{and} \\
\partial_i M(t, z, z) &= \partial_i M_0(z, z) \quad (z \in [0, 1])
\end{aligned}$$

can hold if and only if $t = \mu_1$, hence $\mu \neq \delta_{\mu_1}$ yields that (2.10) holds. To show (A4), let (ω_0, ω_1) be the standard Chebyshev system, i.e., $\omega_0(u) = 1$ and $\omega_0(u) = u$, if $u \in [0, 1]$. Using that μ is a Borel probability measure and the definition of μ_1 , we get that

$$\begin{aligned}
1 &= \int_{[0,1]} 1 d\mu(t) \quad \text{and} \\
(1 - \mu_1)u + \mu_1 v &= \int_{[0,1]} ((1 - t)u + tv) d\mu(t) \quad ((u, v) \in \Delta([0, 1])),
\end{aligned}$$

which proves that ω_0 and ω_1 fulfill (3.3) and hence (A4) follows.

Let $(u, v) \in \Delta^\circ([0, 1])$ be a fixed element and define $\varepsilon_{u,v} : [u, v] \rightarrow \mathbb{R}$ in the following way:

$$\varepsilon_{u,v}(w) = \eta_{ux+(1-u)y, vx+(1-v)y} \left(\frac{w-u}{v-u} \right) \quad (w \in [u, v]).$$

Then the definition of $\eta_{ux+(1-u)y, vx+(1-v)y}$ is correct and by its semicontinuity, $\varepsilon_{u,v}$ is also lower semicontinuous. We show that $\varepsilon_{u,v}$ is a solution of (3.4). Using (3.15) and the previous notations, for $w \in [u, v]$, we have that

$$\begin{aligned} \varepsilon_{u,v}(M_0(u, w)) &= \varepsilon_{u,v}((1 - \mu_1)u + \mu_1 w) \\ &= \eta_{ux+(1-u)y, vx+(1-v)y} \left(\mu_1 \frac{w-u}{v-u} \right) \\ &\geq \int_{[0,1]} \eta_{ux+(1-u)y, vx+(1-v)y} \left(t \frac{w-u}{v-u} \right) d\mu(t) \\ &\quad + E(ux + (1-u)y, wx + (1-w)y) \\ &= \int_{[0,1]} \varepsilon_{u,v}((1-t)u + tw) d\mu(t) + F(u, w) \\ &= \int_{[0,1]} \varepsilon_{u,v}(M(t, u, w)) d\mu(t) + F(u, w) \end{aligned}$$

and

$$\begin{aligned} \varepsilon_{u,v}(M_0(w, v)) &= \varepsilon_{u,v}((1 - \mu_1)w + \mu_1 v) \\ &= \eta_{ux+(1-u)y, vx+(1-v)y} \left(1 - (1 - \mu_1) \left(1 - \frac{w-u}{v-u} \right) \right) \\ &\geq \int_{[0,1]} \eta_{ux+(1-u)y, vx+(1-v)y} \left(1 - (1-t) \left(1 - \frac{w-u}{v-u} \right) \right) d\mu(t) \\ &\quad + E(wx + (1-w)y, vx + (1-v)y) \\ &= \int_{[0,1]} \varepsilon_{u,v}((1-t)w + tv) d\mu(t) + F(w, v) \\ &= \int_{[0,1]} \varepsilon_{u,v}(M(t, w, v)) d\mu(t) + F(w, v). \end{aligned}$$

Then, $\varepsilon_{u,v}$ satisfies the condition (3.4) in Theorem 3.14. Furthermore, by the lower semicontinuity of $\eta_{x,y}$, the function $\varepsilon_{u,v}$ is also lower semicontinuous. On the other hand $\varepsilon_{u,v}(u) = \eta_{ux+(1-u)y, vx+(1-v)y}(0) = 0$ and $\varepsilon_{u,v}(v) = \eta_{ux+(1-u)y, vx+(1-v)y}(1) = 0$. Thus, we can apply Theorem 3.14 for the function g , which yields that for all $s \in [u, v]$,

$$g(s) \leq \frac{v-s}{v-u} g(u) + \frac{s-u}{v-u} g(v) + \varepsilon_{u,v}(s).$$

Substituting $u = 0$ and $v = 1$, we get that for all $s \in [0, 1]$,

$$\begin{aligned} g(s) &= f(sx + (1-s)y) \leq (1-s)g(0) + sg(1) + \varepsilon_{0,1}(s) \\ &= (1-s)f(y) + sf(x) + \eta_{x,y}(s), \end{aligned}$$

which means that (3.16) holds. $\square$

The following corollary is a generalization of the result of [13] recalled in Theorem B.

Corollary 3.19. Let $\alpha_H : (D - D) \rightarrow \mathbb{R}$ and assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality (1.5). Then f satisfies the approximate convexity inequality

$$f((1-s)x + sy) \leq (1-s)f(x) + sf(y) + \alpha_J(\min(2s, 2(1-s))(x-y)) \quad (3.19)$$

for all $x, y \in D, s \in [0, 1]$, where $\alpha_J : (D - D) \rightarrow \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality

$$\alpha_J(su) \geq \frac{1}{s} \int_0^s \alpha_J(\min(2t, 2(1-t))u) dt + \alpha_H(su) \quad (u \in D - D, s \in]0, 1]) \quad (3.20)$$

with $\alpha_J(0) = 0$.

Proof. We apply Theorem 3.18, when μ is the Lebesgue measure on $[0, 1]$. Then we obviously have that $\mu_1 = \int_0^1 t dt = \frac{1}{2}$. Then, it can be seen that (3.14) is equivalent to the following system of inequalities:

$$\eta_{x,y}(s) \geq \begin{cases} \frac{1}{2s} \int_0^{2s} \eta_{x,y}(t) dt + E(x, (1-2s)x + 2sy) & \text{if } s \in \left]0, \frac{1}{2}\right], \\ \frac{1}{2-2s} \int_{2s-1}^1 \eta_{x,y}(t) d\mu(t) + E((2-2s)x + (2s-1)y, y) & \text{if } s \in \left]\frac{1}{2}, 1\right[. \end{cases} \quad (3.21)$$

Let $\alpha_H : (D - D) \rightarrow \mathbb{R}$ be an even function and $\alpha_J : (D - D) \rightarrow \mathbb{R}_+$ be a lower semicontinuous solution of (3.20). Define $E : D \times D \rightarrow \mathbb{R}$ and for all $x, y \in D, \eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ by the following formulas:

$$\begin{aligned} E(x, y) &:= \alpha_H(x - y) \quad (x, y \in D), \\ \eta_{x,y}(s) &:= \alpha_J(\min(2s, 2(1-s))(x - y)) \quad (s \in [0, 1]). \end{aligned}$$

We show that E and $\eta_{x,y}$ satisfy (3.21). First, let $s \in \left]0, \frac{1}{2}\right]$, then

$$\begin{aligned} \eta_{x,y}(s) &= \alpha_J(2s(x - y)) \geq \frac{1}{2s} \int_0^{2s} \alpha_J(\min(2t, 2(1-t))(x - y)) dt + \alpha_H(2s(x - y)) \\ &= \frac{1}{2s} \int_0^{2s} \eta_{x,y}(t) dt + E(x, (1-2s)x + 2sy). \end{aligned}$$

On the other hand, if $s \in \left]\frac{1}{2}, 1\right[$, we obtain that

$$\begin{aligned} \eta_{x,y}(s) &= \alpha_J(2(1-s)(x - y)) \\ &\geq \frac{1}{2(1-s)} \int_0^{2(1-s)} \alpha_J(\min(2t, 2(1-t))(x - y)) dt + \alpha_H(2(1-s)(x - y)) \\ &= \frac{1}{2(1-s)} \int_{2s-1}^1 \alpha_J(\min(2t, 2(1-t))(x - y)) dt + \alpha_H(2(1-s)(x - y)) \\ &= \frac{1}{2-2s} \int_{2s-1}^1 \eta_{x,y}(t) d\mu(t) + E((2-2s)x + (2s-1)y, y). \end{aligned}$$

Hence (3.21) holds, which means that Theorem 3.18 can be applied. Thus, f satisfies (3.16), which is equivalent to (3.19). $\square$

Remark 3.20. In what follows, we deduce the conclusion of [Theorem B](#) from the above corollary. Let f be a hemicontinuous solution of (1.5) and assume that $\alpha_H : (D - D) \rightarrow \mathbb{R}$ is nonnegative and even. Let $\alpha_J : 2(D - D) \rightarrow \mathbb{R}$ be a nonnegative radially increasing, radially lower semicontinuous solution of (1.6). First, we show that the assumption (3.20) of [Corollary 3.19](#) holds for α_J and α_H . Indeed, for $u \in D - D$ and $s \in]0, 1]$, using (1.6), we get

$$\begin{aligned}\alpha_J(su) &\geq \int_0^1 \alpha_J(2tsu)dt + \alpha_H(su) = \frac{1}{s} \int_0^s \alpha_J(2\tau u)d\tau + \alpha_H(su) \\ &\geq \frac{1}{s} \int_0^s \alpha_J(\min(2\tau, 2(1-\tau))u)d\tau + \alpha_H(su),\end{aligned}$$

which means that (3.20) holds. Thus, by [Corollary 3.19](#), we get that (3.19) holds. This, with the substitution $s = \frac{1}{2}$, implies (1.4).

The following result is related to the main result of the paper [[12](#), Theorem 6] (cf. also [[9,10](#)]).

Corollary 3.21. Let $\tau \in]0, 1[$ and $E : D^2 \rightarrow \mathbb{R}$. Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\tau)x + \tau y) \leq (1-\tau)f(x) + \tau f(y) + E(x, y) \quad ((x, y) \in D^2). \quad (3.22)$$

Assume that, for all $x, y \in D$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a lower semicontinuous function with $\eta_{x,y}(0) = \eta_{x,y}(1) = 0$, satisfying the following system of inequalities:

$$\eta_{x,y}(s) \geq \begin{cases} \tau \eta_{x,y}\left(\frac{s}{\tau}\right) + E\left(x, \left(1 - \frac{s}{\tau}\right)x + \frac{s}{\tau}y\right) & \text{if } s \in [0, \tau], \\ (1-\tau)\eta_{x,y}\left(\frac{s-\tau}{1-\tau}\right) + E\left(\frac{1-s}{1-\tau}x + \left(1 - \frac{1-s}{1-\tau}\right)y, y\right) & \text{if } s \in [\tau, 1]. \end{cases} \quad (3.23)$$

Then, for all $x, y \in D$, the function f also satisfies the approximate convexity inequality (3.16).

Proof. We apply [Theorem 3.18](#), when μ is a measure on $[0, 1]$ defined by $\mu := (1-\tau)\delta_0 + \tau\delta_1$. Then we obviously have that $\mu_1 = \int_{[0,1]} t d\mu(t) = \tau$. Now we can see that (3.22) and (3.23) are equivalent to (3.14) and (3.15), respectively. $\square$

Remark 3.22. In what follows, we deduce the conclusion of [Theorem C](#) from the above corollary under stronger regularity assumption on α_J and f . Let $\alpha_J : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative, even and radially lower semicontinuous function with $\alpha_J(0) = 0$ and assume that $f : D \rightarrow \mathbb{R}$ is upper hemicontinuous and approximately Jensen convex in a sense of (1.4). For all $x, y \in D$, define $E : D^2 \rightarrow \mathbb{R}$ and $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$, by

$$E(x, y) := \alpha_J(x - y) \quad \text{and} \quad \eta_{x,y}(s) := \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha_J(2d_{\mathbb{Z}}(2^n s)(x - y)) \quad (s \in [0, 1]).$$

Then, f is also a solution of (3.22) with $\tau = \frac{1}{2}$. By the nonnegativity and radial lower semicontinuity of α_J , for all $x, y \in D$, we get that $\eta_{x,y}$ is lower semicontinuous on $[0, 1]$. Obviously, $\eta_{x,y}(0) = \eta_{x,y}(1) = 0$. We show that $\eta_{x,y}$ satisfies the functional equation (3.23)

with equality, which can equivalently be written as

$$\eta_{x,y}(s) = \begin{cases} \frac{1}{2}\eta_{x,y}(2s) + \alpha_J(2s(x-y)) & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \frac{1}{2}\eta_{x,y}(2s-1) + \alpha_J((2-2s)(x-y)) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases} \quad (3.24)$$

Let $s \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} \eta_{x,y}(s) &= \alpha_J(2s(x-y)) + \sum_{n=1}^{\infty} \frac{1}{2^n} \alpha_J(2d_{\mathbb{Z}}(2^n s)(x-y)) \\ &= \alpha_J(2s(x-y)) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha_J(2d_{\mathbb{Z}}(2^n 2s)(x-y)) \\ &= \alpha_J(2s(x-y)) + \frac{1}{2} \eta_{x,y}(2s). \end{aligned}$$

The case $s \in \left[\frac{1}{2}, 1\right]$ is similar. Thus, [Corollary 3.21](#) yields that f is approximately convex in the sense of [\(3.16\)](#) which is equivalent to [\(1.7\)](#).

We note that, in the case $\alpha_J(u) = \|u\|$, [Theorem C](#) reduces to a result of Háyzy and Páles [\[11\]](#) which was the first to express approximate convexity properties in terms of the Takagi function. In [\[17\]](#) the sharpness of the error term in [\(1.7\)](#) has been obtained for a large class of functions α_J , in particular when $\alpha_J(u) = \|u\|^p$ holds for some $p \in]0, 1]$.

In what follows, we examine the case, when X is a normed space and E is a linear combination of the powers of the norm with positive exponents, i.e., if E is of the form

$$E(x, y) := \int_{]0,1[} \|x - y\|^q d\nu(q) \quad (x, y \in D), \quad (3.25)$$

where ν is a Borel measure on the open interval $]0, 1[$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i \delta_{q_i}$, where $c_1, \dots, c_k > 0$, $q_1, \dots, q_k \in]0, 1[$.

Theorem 3.23. *Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Let ν be a Borel measure on $]0, 1[$ such that*

$$\int_{]0,1[} \gamma(q) d\nu(q) < \infty, \quad (3.26)$$

where

$$\gamma(q) := \max \left(\frac{\mu_1^q}{\mu_1^q - \int_{[0,1]} \tau^q d\mu(\tau)}, \frac{(1 - \mu_1)^q}{(1 - \mu_1)^q - \int_{[0,1]} (1 - \tau)^q d\mu(\tau)} \right) \quad (q \in]0, 1[).$$

Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$\begin{aligned} f((1 - \mu_1)x + \mu_1 y) &\leq \int_{[0,1]} f((1 - t)x + ty) d\mu(t) \\ &\quad + \int_{]0,1[} \|x - y\|^q d\nu(q) \quad ((x, y) \in D^2). \end{aligned} \quad (3.27)$$

Then f also fulfills the convexity type inequality (3.16) for all $x, y \in D$, where for all $x, y \in D$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\eta_{x,y}(s) := \int_{]0,1[} \gamma(q) \min \left(\left(\frac{s}{\mu_1} \right)^q, \left(\frac{1-s}{1-\mu_1} \right)^q \right) \|x - y\|^q d\nu(q) \quad (s \in [0, 1]). \quad (3.28)$$

Proof. Let E be defined by (3.25). Then (3.27) is equivalent to (3.14). To deduce (3.16) using Theorem 3.18, it suffices to show that for all $x, y \in D$ the function $\eta_{x,y}$ defined by (3.28) satisfies (3.15). Observe that, for all $x, y \in D$, $\eta_{x,y}$ is finite-valued and continuous since (3.26) holds. It is also obvious that $\eta_{x,y}(0) = \eta_{x,y}(1) = 0$. For all $q \in]0, 1[$, let $\varphi_q : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$\varphi_q(s) := \gamma(q) \min \left(\left(\frac{s}{\mu_1} \right)^q, \left(\frac{1-s}{1-\mu_1} \right)^q \right).$$

Observe that $\eta_{x,y}(s) = \int_{]0,1[} \varphi_q(s) \|x - y\|^q d\nu(q)$. First we show that φ_q satisfies the following functional inequality,

$$\varphi_q(s) \geq \begin{cases} \int_{]0,1[} \varphi_q \left(\frac{s}{\mu_1} t \right) d\mu(t) + \left(\frac{s}{\mu_1} \right)^q & \text{if } 0 \leq s \leq \mu_1, \\ \int_{]0,1[} \varphi_q \left(1 - \frac{1-s}{1-\mu_1} (1-t) \right) d\mu(t) + \left(\frac{1-s}{1-\mu_1} \right)^q & \text{if } \mu_1 < s \leq 1. \end{cases} \quad (3.29)$$

To see that (3.29) holds, let $0 \leq s \leq \mu_1$ be arbitrarily fixed. Using the definition of γ , we have that for all $q \in]0, 1[$,

$$\gamma(q) \geq \frac{\mu_1^q}{\mu_1^q - \int_{]0,1[} t^q d\mu(t)}.$$

Rearranging the previous inequality, we get that

$$\gamma(q) \geq \gamma(q) \frac{\int_{]0,1[} t^q d\mu(t)}{\mu_1^q} + 1 \quad (q \in]0, 1[).$$

Multiplying this inequality by $\left(\frac{s}{\mu_1} \right)^q$, it follows that

$$\gamma(q) \left(\frac{s}{\mu_1} \right)^q \geq \int_{]0,1[} \gamma(q) \left(\frac{s}{\mu_1^2} t \right)^q d\mu(t) + \left(\frac{s}{\mu_1} \right)^q \quad (q \in]0, 1[)$$

which implies that (3.29) holds for $s \in [0, \mu_1]$. The proof in the case $\mu_1 \leq s \leq 1$ is similar.

Multiplying (3.29) by $\|x - y\|^q$, then integrating on $]0, 1[$ for q with respect to ν , and finally using Fubini's theorem, we get that

$$\begin{aligned} \eta_{x,y}(s) &= \int_{]0,1[} \varphi_q(s) \|x - y\|^q d\nu(q) \\ &\geq \int_{]0,1[} \int_{]0,1[} \varphi_q \left(\frac{s}{\mu_1} t \right) \|x - y\|^q d\mu(t) d\nu(q) + \int_{]0,1[} \left(\frac{s}{\mu_1} \right)^q \|x - y\|^q d\nu(q) \\ &= \int_{]0,1[} \int_{]0,1[} \varphi_q \left(\frac{s}{\mu_1} t \right) \|x - y\|^q d\nu(q) d\mu(t) + \int_{]0,1[} \left(\frac{s}{\mu_1} \right)^q \|x - y\|^q d\nu(q) \\ &= \int_{]0,1[} \eta_{x,y} \left(\frac{s}{\mu_1} t \right) d\mu(t) + \int_{]0,1[} \left(\frac{s}{\mu_1} \right)^q \|x - y\|^q d\nu(q), \quad \text{if } 0 \leq s \leq \mu_1. \end{aligned}$$

In the case $\mu_1 < s \leq 1$, we similarly obtain that

$$\eta_{x,y}(s) \geq \int_{[0,1]} \eta_{x,y} \left(1 - \frac{1-s}{1-\mu_1} (1-t) \right) d\mu(t) + \int_{]0,1[} \left(\frac{1-s}{1-\mu_1} \right)^q \|x-y\|^q dv(q).$$

Thus, $\eta_{x,y}$ satisfies (3.15). Therefore, by Theorem 3.18, the statement follows. $\square$

Corollary 3.24. *Let ν be a Borel measure on $]0, 1[$ such that*

$$\int_{]0,1[} \frac{q+1}{q+1-2^q} dv(q) < \infty. \quad (3.30)$$

Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_{[0,1]} f((1-t)x + ty) dt + \int_{]0,1[} \|x-y\|^q dv(q) \quad ((x, y) \in D^2).$$

Then, for all $x, y \in D$ and $s \in [0, 1]$, f also fulfills the convexity type inequality

$$\begin{aligned} f((1-s)x + sy) &\leq (1-s)f(x) + sf(y) \\ &\quad + \int_{]0,1[} \frac{(q+1)2^q}{q+1-2^q} \min(s^q, (1-s)^q) \|x-y\|^q dv(q). \end{aligned}$$

Proof. We want to apply Theorem 3.23. Let μ be the Lebesgue measure, then $\mu_1 = \frac{1}{2}$ and

$$\begin{aligned} \gamma(q) &= \max \left(\frac{\left(\frac{1}{2}\right)^q}{\left(\frac{1}{2}\right)^q - \int_0^1 \tau^q d\tau}, \frac{\left(\frac{1}{2}\right)^q}{\left(\frac{1}{2}\right)^q - \int_0^1 (1-\tau)^q d\tau} \right) \\ &= \frac{1}{1-2^q \int_0^1 \tau^q d\tau} = \frac{q+1}{q+1-2^q}. \end{aligned}$$

By (3.30), we get that (3.26) holds in Theorem 3.23. For all $x, y \in D$ and $s \in [0, 1]$, we obtain that

$$\begin{aligned} \eta_{x,y}(s) &= \int_{]0,1[} \frac{q+1}{q+1-2^q} \min \left(\left(\frac{s}{1/2} \right)^q, \left(\frac{1-s}{1/2} \right)^q \right) \|x-y\|^q dv(q) \\ &= \int_{]0,1[} \frac{(q+1)2^q}{q+1-2^q} \min(s^q, (1-s)^q) \|x-y\|^q dv(q), \end{aligned}$$

which proves the statement. $\square$

Corollary 3.25. *Let $\tau \in]0, 1[$ and let ν be a Borel measure on $]0, 1[$ such that*

$$\int_{]0,1[} \frac{1}{1 - (\max(\tau, 1-\tau))^{1-q}} dv(q) < \infty.$$

Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\tau)x + \tau y) \leq \tau f(x) + (1-\tau)f(y) + \int_{]0,1[} \|x-y\|^q dv(q) \quad ((x, y) \in D^2).$$

Then, for all $x, y \in D$ and $s \in [0, 1]$, f also fulfills the convexity type inequality

$$f((1-s)x + sy) \leq (1-s)f(x) + sf(y) + \int_{[0,1]} \frac{1}{1 - (\max(\tau, 1-\tau))^{1-q}} \min\left(\left(\frac{s}{\tau}\right)^q, \left(\frac{1-s}{1-\tau}\right)^q\right) \|x - y\|^q dv(q).$$

Proof. We apply Theorem 3.23, when μ is a measure on $[0, 1]$ defined by $\mu := (1-\tau)\delta_0 + \tau\delta_1$. Then, we obviously have that $\mu_1 = \tau$ and

$$\begin{aligned} \gamma(q) &= \max\left(\frac{\tau^q}{\tau^q - \int_{[0,1]} \sigma^q d\mu(\sigma)}, \frac{(1-\tau)^q}{(1-\tau)^q - \int_{[0,1]} (1-\sigma)^q d\mu(\sigma)}\right) \\ &= \max\left(\frac{\tau^q}{\tau^q - \tau}, \frac{(1-\tau)^q}{(1-\tau)^q - (1-\tau)}\right) = \frac{1}{1 - (\max(\tau, 1-\tau))^{1-q}}. \end{aligned}$$

Thus, for all $x, y \in D$ and $s \in [0, 1]$,

$$\eta_{x,y}(s) = \int_{[0,1]} \frac{1}{1 - \max(\tau, 1-\tau)^{1-q}} \min\left(\left(\frac{s}{\tau}\right)^q, \left(\frac{1-s}{1-\tau}\right)^q\right) \|x - y\|^q dv(q),$$

which proves the statement. $\square$

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