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Interlacing of zeros of orthogonal polynomials under modification of the measure☆

Dimitar K. Dimitrov[a,*], Mourad E.H. Ismail[b,c], Fernando R. Rafaeli[a]

[a] *Departamento de Matemática Aplicada, Instituto de Biociências, Letras e Ciências Exatas, Universidade Estadual Paulista - UNESP, Brazil*

[b] *Department of Mathematics, University of Central Florida, Orlando, FL, USA*

[c] *Department of Mathematics, College of Science, King Saud University, Saudi Arabia*

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Abstract

We investigate the mutual location of the zeros of two families of orthogonal polynomials. One of the families is orthogonal with respect to the measure $d\mu(x)$, supported on the interval (a, b) and the other with respect to the measure $|x-c|^{\tau}|x-d|^{\gamma}d\mu(x)$, where c and d are outside (a, b). We prove that the zeros of these polynomials, if they are of equal or consecutive degrees, interlace when either $0 < \tau, \gamma \leq 1$ or $\gamma = 0$ and $0 < \tau \leq 2$. This result is inspired by an open question of Richard Askey and it generalizes recent results on some families of orthogonal polynomials. Moreover, we obtain further statements on interlacing of zeros of specific orthogonal polynomials, such as the Askey–Wilson ones.

Keywords: Orthogonal polynomials; Classical orthogonal polynomials; q-orthogonal polynomials; Zeros; Interlacing; Monotonicity

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* Corresponding author.

E-mail addresses: dimitrov@ibilce.unesp.br (D.K. Dimitrov), ismail@math.ucf.edu (M.E.H. Ismail), rafaeli@ibilce.unesp.br (F.R. Rafaeli).

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1. Introduction and statement of results

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials with respect to a positive Borel measure $d\mu(x)$ supported in the finite or infinite interval (a,b). It is well known that the zeros of $p_n(x)$ are real, distinct and lie in (a,b). Moreover, if we denote by $x_{n,k}$, $k=1,\dots,n$, the zeros of $p_n(x)$, then

$$a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1} < b.$$

In other words, the zeros of $p_n(x)$ and $p_{n+1}(x)$ interlace.

If $c,d \notin (a,b)$ and $\tau,\ \gamma \in \mathbb{R}$, let $x_{n,k}(\tau,\gamma,c,d)$, $k=1,\dots,n$, be the zeros of the polynomial $p_n(\tau,\gamma,c,d;x)$, orthogonal with respect to the measure

$$d\mu_{\tau,\gamma}(c,d;x) := |x-c|^{\tau}|x-d|^{\gamma}d\mu(x). \tag{1}$$

Obviously

$$a < x_{n,1}(\tau,\gamma,c,d) < \cdots < x_{n,n}(\tau,\gamma,c,d) < b$$

for every $\tau,\gamma \in \mathbb{R}$ and the zeros of $p_n(x)$ and $p_n(\tau,\gamma,c,d;x)$ coincide when $\tau=\gamma=0$. Then the natural question arises as to whether there is a neighborhood of the origin, such that the zeros of $p_n(\tau,\gamma,c,d;x)$ interlace with those of $p_n(x)$ and $p_{n+1}(x)$ when the parameters τ and γ are in the neighborhood. It is also interesting to know how the location of the points c and d influences this interlacing property.

Richard Askey [1] asked these questions in a particular situation. He conjectured that the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha+2,\beta)}(x)$ interlace. The classical Theorem of Markov (see [19, Theorem 6.12.1] or [12, Theorem 7.1.1]) implies that the zeros of the Jacobi polynomials are decreasing functions of the parameter α. Then the conjecture of Askey is equivalent to the statement that the zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha+\gamma,\beta)}(x)$ interlace if $0<\gamma\le 2$. Observe that in this case $d\mu(x)=(1-x)^{\alpha}(1+x)^{\beta}dx$, $\tau=0$, $a=-1$, $b=d=1$ and $0<\gamma\le 2$. Driver, Jordaan and Mbuyi [10] established the latter generalized version of Askey's conjecture. Moreover, they provided an example which shows that if we stay within the class of Jacobi polynomials then the interval $0<\gamma\le 2$ is the largest possible one with the above property.

There are many results in the literature which deal with interlacing of zeros of classical discrete and continuous orthogonal polynomials, as well as of q-orthogonal polynomials, when some of the parameters differ either by one or by two. To the best of our knowledge, Levit [15] was the first to study these questions in the case of Hahn polynomials. If we adopt the contemporary notation, as in [12], amongst the others, Levit proved that the zeros of $Q_n(x;\alpha,\beta,N)$ interlace with the zeros of $Q_n(x;\alpha,\beta+1,N)$ (see Theorem 5 in [15]) and also that the zeros of $Q_n(x;\alpha,\beta,N)$ and $Q_n(x;\alpha-1,\beta+1,N)$ interlace (Theorem 4 in [15]). Observe that the limit relation between Hahn and Jacobi polynomials [12, formula 6.2.10] immediately implies that the zeros of $P_n^{(\alpha,\beta)}(x)$ interlace with those of $P_n^{(\alpha,\beta+1)}(x)$ and of $P_n^{(\alpha-1,\beta+1)}(x)$. It is worth mentioning that Driver and Jordaan [7] considered the question about interlacing of zeros of hypergeometric polynomials and in one of their applications they rediscovered the interlacing of the zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta+1)}(x)$. Another interesting result of Levit, namely Theorem 6 in [15], is that $Q_n(x;\alpha,\beta,N)$ and $Q_n(x;\alpha,\beta,N+1)$ also have interlacing zeros. The limit relation between Hahn and Kravchuk polynomials (see the formula after Proposition 6.2.2 in [12]), one concludes that the zeros of $K_n(x;p,N)$ and $K_n(x;p,N+1)$ also interlace. The latter statement was rediscovered various times, first by Laura Chihara and Dennis Stanton [3] and recently by Jordaan and Toókos [13].

The recent papers [6,8,9,11,16–18] also deal with problems concerning interlacing of zeros of families of continuous or discrete orthogonal polynomials.

In most cases the authors of the above contributions obtain, exploring specific formulae for the corresponding orthogonal polynomials, relations of the form

$$Cp_n(\tau, \gamma, c, d; x) = Ap_n(x) + Bp_{n-1}(x),$$

where A, B and C are either constants or polynomials independent of n and the latter immediately yields the desired interlacing of the zeros of the polynomials involved.

During the Conference on Special Functions and Applications, in 2007 in Marseille, the second author asked if such techniques could be used to prove similar results for the zeros of q-orthogonal polynomials. This was done by Jordaan and Toókos in [14] where interlacing of zeros of the Al-Salam–Chihara, q-Meixner–Pollaczek, q-ultraspherical and q-Laguerre polynomials with shifted parameters was obtained.

In this paper we approach the problem as suggested above and prove that the zeros of $p_n(\tau, \gamma, c, d; x)$ interlace with those of $p_n(x)$ and $p_{n+1}(x)$ in some important situations. This enables us to obtain most of the above cited results as immediate consequences and to deal with some other families of orthogonal polynomials. At the end of the paper we provide an example which involves the Askey–Wilson polynomials. Furthermore, the method of the proof allows us to understand the role of the points c and d and shed light on some examples furnished in the previous contributions.

Before we state the main results, we prove that $x_{n,k}(\tau, \gamma, c, d)$, $k = 1, \ldots, n$, are monotonic functions with respect to the parameters τ and γ.

Theorem 1. *Let $n \in \mathbb{N}$, $c, d \notin (a, b)$ and $\tau, \gamma \in [0, +\infty)$.*

(i) *If $c \le a$, then each zero $x_{n,k}(\tau, \gamma, c, d)$ is an increasing function of τ.*
(ii) *If $c \ge b$, then each zero $x_{n,k}(\tau, \gamma, c, d)$ is a decreasing function of τ.*
(iii) *If $d \le a$, then each zero $x_{n,k}(\tau, \gamma, c, d)$ is an increasing function of γ.*
(iv) *If $d \ge b$, then each zero $x_{n,k}(\tau, \gamma, c, d)$ is a decreasing function of γ.*

In what follows, when $\gamma = 0$ and

$$d\mu_\tau(c; x) := |x - c|^\tau d\mu(x), \tag{2}$$

the corresponding orthogonal polynomials are denoted by $p_n(\tau, c; x)$ and their zeros by $x_{n,k}(\tau, c)$, $k = 1, \ldots, n$.

Corollary 1. *Let $n \in \mathbb{N}$ and $\tau \in [0, +\infty)$.*

(i) *If $c \le a$, then each zero $x_{n,k}(\tau, c)$ is an increasing function of τ.*
(ii) *If $c \ge b$, then each zero $x_{n,k}(\tau, c)$ is a decreasing function of τ.*

Now we state the interlacing property comparing the zeros of the polynomial $p_n(\tau, c; x)$ and those of $p_n(x)$ and $p_{n+1}(x)$.

Theorem 2. *Let $0 < \tau < 2$.*

(i) *If $c \le a$, then*

$$\begin{aligned} x_{n+1,1} &< x_{n,1} < x_{n,1}(\tau, c) < x_{n,1}(2, c) < \cdots < x_{n+1,n} \\ &< x_{n,n} < x_{n,n}(\tau, c) < x_{n,n}(2, c) < x_{n+1,n+1}. \end{aligned}$$

(ii) *If $c \geq b$, then*

$$x_{n+1,1} < x_{n,1}(2,c) < x_{n,1}(\tau,c) < x_{n,1} < \cdots < x_{n+1,n} < x_{n,n}(2,c) < x_{n,n}(\tau,c) < x_{n,n} < x_{n+1,n+1}.$$

Our next statement concerns the mutual location of the zeros of $p_n(\tau,\gamma,c,d;x)$, $p_n(x)$ and $p_{n+1}(x)$.

Theorem 3. *Let $n \in \mathbb{N}$ and $\tau, \gamma \in (0,1]$.*

(i) *If $c, d \leq a$, then*

$$x_{n+1,1} < x_{n,1} < x_{n,1}(\tau,\gamma,c,d) < \cdots < x_{n+1,n} < x_{n,n} < x_{n,n}(\tau,\gamma,c,d) < x_{n+1,n+1}.$$

(ii) *If $c, d \geq b$, then*

$$x_{n+1,1} < x_{n,1}(\tau,\gamma,c,d) < x_{n,1} < \cdots < x_{n+1,n} < x_{n,n}(\tau,\gamma,c,d) < x_{n,n} < x_{n+1,n+1}.$$

Observe that, if we set $c = d$ in Theorem 3, we immediately obtain the result in Theorem 2. Nevertheless, we prefer to have separate statements because of their applications. Moreover, we provide an independent proof of Theorem 2 first and then use some of the arguments in the proof of Theorem 3.

The principal tools in our proofs are the classical Markov theorem which implies Theorem 1 immediately and an equivalent form of Christoffel's formula [4] (see also [19, Theorem 2.5]) which provides a representation of $p_n(\tau,\gamma,c,d;x)$ in terms of $p_n(x)$ and $p_{n+1}(x)$ for integer values of τ and γ. We emphasize that this form of Christoffel's formula, stated in Lemma 1 below, is very useful in establishing Theorems 2 and 3.

2. Preliminaries

Christoffel's formula [19, Theorem 2.5] implies that the monic polynomial $p_n(\tau,c;x)$, with $\tau = 1$, can be written as

$$(x-c)p_n(1,c;x) = \frac{-1}{p_n(c)} \begin{vmatrix} p_n(x) & p_{n+1}(x) \\ p_n(c) & p_{n+1}(c) \end{vmatrix}, \tag{3}$$

or equivalently,

$$p_n(1,c;x) = \frac{1}{x-c}\left[p_{n+1}(x) - \frac{p_{n+1}(c)}{p_n(c)}p_n(x)\right].$$

Chihara [2, p. 37] obtained another representation of $p_n(1,c;x)$ which shows that it is a constant multiple of the monic kernel polynomial $K_n(c,x)$ corresponding to $p_n(x)$,

$$p_n(1,c;x) = \frac{\|p_n\|_\mu^2}{p_n(c)} K_n(c,x),$$

where

$$K_n(c,x) = \sum_{j=0}^{n} \frac{p_j(c)p_j(x)}{\|p_j\|_\mu^2} \quad \text{and} \quad \|p_j\|_\mu^2 = \int_a^b |p_j(x)|^2 d\mu(x). \tag{4}$$

As an immediate consequence of the above formulae, Chihara obtains the following interlacing property involving the zeros of $p_n(1,c;x)$, $p_{n+1}(x)$ and $p_n(x)$ (see [2, Theorem 7.2]):

if $c \le a$, then

$$x_{n+1,1} < x_{n,1} < x_{n,1}(1,c) < x_{n+1,2} < \cdots < x_{n,n} < x_{n,n}(1,c) < x_{n+1,n+1}; \tag{5}$$

if $c \ge b$, then

$$x_{n+1,1} < x_{n,1}(1,c) < x_{n,1} < \cdots < x_{n+1,n} < x_{n,n}(1,c) < x_{n,n} < x_{n+1,n+1}. \tag{6}$$

For the monic polynomial $p_n(2,c;x)$, Christoffel's formula [19, p. 30] reads as

$$(x-c)^2 p_n(2,c;x) = \frac{1}{p_n(c)p'_{n+1}(c) - p_n(c)p_{n+1}(c)} \begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n(c) & p_{n+1}(c) & p_{n+2}(c) \\ p'_n(c) & p'_{n+1}(c) & p'_{n+2}(c) \end{vmatrix}.$$

Using the formulae (3) we obtain the following useful representation for the polynomial $p_n(2,c;x)$.

Lemma 1. *The polynomial $p_n(2,c;x)$ can be represented as*

$$p_n(2,c;x) = \frac{1}{(x-c)^2}\left[p_{n+2}(x) - d_n p_{n+1}(x) + e_n p_n(x)\right], \tag{7}$$

where

$$d_n = \frac{p_{n+2}(c)}{p_{n+1}(c)} + \frac{p_{n+1}(1,c;c)}{p_n(1,c;c)} = \frac{p_{n+2}(c)}{p_{n+1}(c)} + \frac{p_n(c)}{p_{n+1}(c)} e_n$$

and

$$e_n = \frac{p_{n+1}(1,c;c)}{p_n(1,c;c)} \frac{p_{n+1}(c)}{p_n(c)} = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2} \frac{K_{n+1}(c,c)}{K_n(c,c)} > 0.$$

Proof. Having in mind (3), we obtain

$$p_n(2,c;x) = \frac{1}{x-c}\left[p_{n+1}(1,c;x) - \frac{p_{n+1}(1,c;c)}{p_n(1,c;c)} p_n(1,c;x)\right].$$

Now (7) follows from the definition of $p_{n+1}(1,c;x)$ and $p_n(1,c;x)$.

3. Proofs of the theorems

Proof of Theorem 1. Using Markov's theorem (see [19, Thm. 6.12.1] or [12, Thm. 7.1.1]) we shall prove that all the zeros of $p_n(\tau,\gamma,c,d;x)$ are increasing or decreasing functions of the parameters τ and γ, depending on the location of the points c and d.

We recall that the polynomial $p_n(\tau,\gamma,c,d;x)$ is orthogonal with respect to

$$d\mu_{\tau,\gamma}(c,d;x) = |x-c|^\tau |x-d|^\gamma d\mu(x).$$

Since, for every fixed $x \in (a,b)$,

$$\frac{\partial}{\partial \tau} \ln |x-c|^\tau |x-d|^\gamma = \ln|x-c|$$

and

$$\frac{\partial \ln|x-c|}{\partial x} = \frac{1}{x-c}\begin{cases} > 0, & \text{if } c \le a \\ < 0, & \text{if } c \ge b, \end{cases}$$

the statements (i) and (ii) of Theorem 1 follow from Markov's theorem. The other two statements follow in a similar way.

Proof of Theorem 2. We shall establish the interlacing of the zeros of $p_n(2, c; x)$ with those of $p_n(x)$ and $p_{n+1}(x)$. For the proof, we apply the recurrence relation

$$p_{n+2}(x) = (x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x)$$

in (7) to obtain

$$p_n(2, c; x) = \frac{1}{(x-c)^2}\left[(x - \beta_{n+1} - d_n)p_{n+1}(x) + (e_n - \gamma_{n+1})p_n(x)\right].$$

It is important to note that

$$e_n - \gamma_{n+1} = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2}\left(\frac{K_{n+1}(c,c)}{K_n(c,c)} - 1\right) > 0. \tag{8}$$

Evaluating $p_n(2, c; x)$ at the zeros $x_{n+1,k}$ we obtain

$$\begin{aligned} sign\left[p_n(2, c; x_{n+1,k})\right] &= sign\left[(e_n - \gamma_{n+1})p_n(x_{n+1,k})\right] \\ &= sign\left[(e_n - \gamma_{n+1})\right]sign\left[p_n(x_{n+1,k})\right] \\ &= sign\left[p_n(x_{n+1,k})\right] = (-1)^{n-k+1}, \end{aligned}$$

for $k = 1, \ldots, n+1$. Since $p_n(x)$ and $p_{n+1}(x)$ have interlacing zeros, we conclude that

$$x_{n+1,1} < x_{n,1}(2, c) < x_{n+1,2} < \cdots < x_{n+1,n} < x_{n,n}(2, c) < x_{n+1,n+1}. \tag{9}$$

On the other hand, evaluating

$$(x-c)^2 p_n(2, c; x) = (x - \beta_{n+1} - d_n)p_{n+1}(x) + (e_n - \gamma_{n+1})p_n(x) \tag{10}$$

at $x = c$ and using (8), we obtain

$$(\beta_{n+1} + d_n - c)\frac{p_{n+1}(c)}{p_n(c)} = e_n - \gamma_{n+1} > 0.$$

Since, when $c \le a$, we have $p_{n+1}(c)/p_n(c) < 0$, and when $c \ge b$, we have $p_{n+1}(c)/p_n(c) > 0$, we deduce that $\beta_{n+1} + d_n < c$ if $c \le a$ and $\beta_{n+1} + d_n > c$ if $c \ge b$. Hence, evaluating (10) at $x = x_{n,k}$, we derive

$$\begin{aligned} sign\left[p_n(2, c; x_{n,k})\right] &= sign\left[(x_{n,k} - \beta_{n+1} - d_n)p_{n+1}(x_{n,k})\right] \\ &= sign[(x_{n,k}) - (\beta_{n+1} + d_n)]sign[p_{n+1}(x_{n,k})], \end{aligned}$$

for $k = 1, \ldots, n$. Since, when $c \le a$, we have $\beta_{n+1} + d_n < c < x_{n,k}$, and when $c \ge b$, we have $\beta_{n+1} + d_n > c > x_{n,k}$ we conclude that

$$sign\left[p_n(2, c; x_{n,k})\right] = sign\left[p_{n+1}(x_{n,k})\right] = (-1)^{n-k+1} \quad \text{if } c \le a$$

and

$$sign\left[p_n(2, c; x_{n,k})\right] = -sign\left[p_{n+1}(x_{n,k})\right] = (-1)^{n-k} \quad \text{if } c \ge b.$$

Thus, if $c \le a$,

$$\beta_{n+1} + d_n < c < x_{n,1} < x_{n,1}(2, c) < \cdots < x_{n,n} < x_{n,n}(2, c), \tag{11}$$

and, if $c \ge b$,

$$x_{n,1}(2, c) < x_{n,1} < \cdots < x_{n,n}(2, c) < x_{n,n} < c < \beta_{n+1} + d_n. \tag{12}$$

Therefore, (9), (11), (12) and Corollary 1 yield the desired result.

Proof of Theorem 3. Recall that the polynomial $p_n(\tau, c; x)$ is orthogonal with respect to

$$d\mu_\tau(c; x) = |x - c|^\tau d\mu(x)$$

and, for $\tau = 1$, $p_n(1, c; x)$ reduces to the Christoffel polynomial which, by (3), can be represented in the form

$$p_n(1, c; x) = \frac{1}{x - c}\left[p_{n+1}(x) - \frac{p_{n+1}(c)}{p_n(c)} p_n(x)\right] = \frac{\|p_n\|_\mu^2}{p_n(c)} K_n(c, x).$$

Set $\tau = \gamma = 1$ in (1). We have

$$d\mu_{1,1}(c, d; x) = |x - c||x - d| d\mu(x) = |x - d| d\mu_1(c; x). \tag{13}$$

Similar reasoning to that used in the proof of Theorem 2 yields

$$p_n(1, 1, c, d; x) = \frac{1}{(x - c)(x - d)}\left[(x - \beta_{n+1} - \widetilde{d}_n) p_{n+1}(x) + (\widetilde{e}_n - \gamma_{n+1}) p_n(x)\right], \tag{14}$$

where

$$\widetilde{d}_n = \frac{p_{n+2}(c)}{p_{n+1}(c)} + \frac{p_n(c)}{p_{n+1}(c)} \widetilde{e}_n,$$

$$\widetilde{e}_n = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2} \frac{K_{n+1}(c, d)}{K_n(c, d)} \ne 0,$$

$$\widetilde{e}_n - \gamma_{n+1} = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2}\left(\frac{K_{n+1}(c, d)}{K_n(c, d)} - 1\right) \ne 0,$$

and then, as a consequence of (14), we obtain:

- if $c, d \le a$, then

$$\begin{aligned} x_{n+1,1} &< x_{n,1} < x_{n,1}(1, 1, c, d) < \cdots < x_{n+1,n} \\ &< x_{n,n} < x_{n,n}(1, 1, c, d) < x_{n+1,n+1}; \end{aligned} \tag{15}$$

- if $c, d \ge b$, then

$$\begin{aligned} x_{n+1,1} &< x_{n,1}(1, 1, c, d) < x_{n,1} < \cdots < x_{n+1,n} \\ &< x_{n,n}(1, 1, c, d) < x_{n,n} < x_{n+1,n+1}. \end{aligned} \tag{16}$$

On the other hand, by (13), (5) and (6), we conclude that, for every $\tau \ge 0$,

- if $c, d \le a$, then

$$\begin{aligned} x_{n+1,1}(\tau, c) &< x_{n,1}(\tau, c) < x_{n,1}(\tau, 1, c, d) < \cdots < x_{n+1,n}(\tau, c) \\ &< x_{n,n}(\tau, c) < x_{n,n}(\tau, 1, c, d) < x_{n+1,n+1}(\tau, c); \end{aligned} \tag{17}$$

- if $c, d \geq b$, then

$$\begin{aligned} x_{n+1,1}(\tau, c) &< x_{n,1}(\tau, 1, c, d) < x_{n,1}(\tau, c) < \cdots < x_{n+1,n}(\tau, c) \\ &< x_{n,n}(\tau, 1, c, d) < x_{n,n}(\tau, c) < x_{n+1,n+1}(\tau, c). \end{aligned} \tag{18}$$

Having in mind the monotonicity of $x_{n,k}(\tau, \gamma, c, d)$ with respect to τ, $0 < \tau \leq 1$, as well as (15)–(18), we conclude that, for every $0 < \tau \leq 1$,

- if $c, d \leq a$, then

$$\begin{aligned} x_{n+1,1} &< x_{n,1} < x_{n,1}(\tau, 1, c, d) < \cdots < x_{n+1,n} \\ &< x_{n,n} < x_{n,n}(\tau, 1, c, d) < x_{n+1,n+1}; \end{aligned}$$

- if $c, d \geq b$, then

$$\begin{aligned} x_{n+1,1} &< x_{n,1}(\tau, 1, c, d) < x_{n,1} < \cdots < x_{n+1,n} \\ &< x_{n,n}(\tau, 1, c, d) < x_{n,n} < x_{n+1,n+1}. \end{aligned}$$

Now fix τ and vary γ. Since $x_{n,k}(\tau, \gamma, c, d)$ is monotonic with respect to γ and $x_{n,k}(\tau, c) = x_{n,k}(\tau, 0, c, d)$, from the above results and Theorem 2, we finally obtain the desired inequalities for the zeros of $p_n(\tau, \gamma, c, d; x)$, $p_n(x)$ and $p_{n+1}(x)$.

4. Some applications

4.1. Interlacing of zeros of Jacobi polynomials

Consider the sequence of Jacobi polynomials $p_n(x) = P_n^{(\alpha,\beta)}(x)$ which are orthogonal with respect to $d\mu(\alpha, \beta; x) = \omega(\alpha, \beta; x)dx$ in the interval $(-1, 1)$, where $\omega(\alpha, \beta; x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$ (see [12, p. 80]). If we take, for $-1 < x < 1$,

$$d\mu(\tau, \alpha, \beta; x) := |x+1|^\tau \omega(\alpha, \beta; x)dx = (1-x)^\alpha(1+x)^{\beta+\tau}dx \tag{19}$$

and

$$d\mu(\gamma, \alpha, \beta; x) := |x-1|^\gamma \omega(\alpha, \beta; x)dx = (1-x)^{\alpha+\gamma}(1+x)^\beta dx, \tag{20}$$

then the polynomials which are orthogonal with respect to (19) and (20) are $P_n^{(\alpha,\beta+\tau)}(x)$ and $P_n^{(\alpha+\gamma,\beta)}(x)$, respectively. Thus, by Theorem 2, we obtain the following.

Corollary 2. *If $x_{n,1}(\alpha, \beta) < \cdots < x_{n,n}(\alpha, \beta)$ denote the zeros of $P_n^{(\alpha,\beta)}(x)$, then the inequalities*

$$\begin{aligned} x_{n+1,1}(\alpha, \beta) &< x_{n,1}(\alpha, \beta) < x_{n,1}(\alpha, \beta+\tau) < \cdots < x_{n+1,n}(\alpha, \beta) \\ &< x_{n,n}(\alpha, \beta) < x_{n,n}(\alpha, \beta+\tau) < x_{n+1,n+1}(\alpha, \beta) \end{aligned}$$

and

$$\begin{aligned} x_{n+1,1}(\alpha, \beta) &< x_{n,1}(\alpha+\gamma, \beta) < x_{n,1}(\alpha, \beta) < \cdots < x_{n+1,n}(\alpha, \beta) \\ &< x_{n,n}(\alpha+\gamma, \beta) < x_{n,n}(\alpha, \beta) < x_{n+1,n+1}(\alpha, \beta) \end{aligned}$$

hold for $0 < \tau \leq 2$ and $0 < \gamma \leq 2$.

These are exactly the results in Theorems 2.1 and 2.2 in the paper of Driver, Jordaan and Mbuyi [10].

4.2. Interlacing of zeros of Laguerre polynomials

Let $p_n(x) = L_n^{(\alpha)}(x)$ be the Laguerre polynomials which are orthogonal in $(0, +\infty)$ with respect to the measure $d\mu(\alpha; x) = \omega(\alpha; x)dx$, where $\omega(\alpha; x) = x^\alpha e^{-x}$, with $\alpha > -1$ (see [12, p. 98]). If we take, for $x > 0$,

$$d\mu(\tau, \alpha; x) := |x|^\tau \omega(\alpha; x)dx = x^{\alpha+\tau} e^{-x} dx$$

then, by Theorem 2, we have the following.

Corollary 3. *If $x_{n,1}(\alpha) < \cdots < x_{n,n}(\alpha)$ denote the zeros of $L_n^{(\alpha)}(x)$ then the inequalities*

$$\begin{aligned} x_{n+1,1}(\alpha) &< x_{n,1}(\alpha) < x_{n,1}(\alpha+\tau) < \cdots < x_{n+1,n}(\alpha) \\ &< x_{n,n}(\alpha) < x_{n,n}(\alpha+\tau) < x_{n+1,n+1}(\alpha) \end{aligned}$$

hold for every $0 < \tau \le 2$.

Thus we have obtained the result in Theorems 2.1 and 2.2 in the paper of Driver, Jordaan and Mbuyi [11].

4.3. Interlacing of zeros of Meixner polynomials

The Meixner polynomials (see [12, p. 174])

$$M_n(x; \beta, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \,\middle|\, 1 - \frac{1}{c} \right),$$

for $\beta > 0$ and $c \in (0, 1)$, are orthogonal with respect to a discrete measure whose distribution has jumps $(\beta)_x c^x / x!$ at $x = 0, 1, \ldots,$ where $(\beta)_x = \beta(\beta+1)\cdots(\beta+x-1)$ is the Pochhammer symbol. In other words,

$$d\mu(\beta, c; x) := \frac{(\beta)_x}{x!} c^x, \quad x = 0, 1, \ldots.$$

Then

$$d\mu(\beta+1, c; x) = \left(1 + \frac{x}{\beta}\right) d\mu(\beta, c; x) = \frac{(\beta+1)_x}{x!} c^x, \quad x = 0, 1, \ldots,$$

and

$$d\mu(\beta+2, c; x) = \left(1 + \frac{x}{\beta+1}\right)\left(1 + \frac{x}{\beta}\right) d\mu(\beta, c; x) = \frac{(\beta+2)_x}{x!} c^x, \\ x = 0, 1, \ldots.$$

Thus, denoting by $m_{n,k}(\beta, c)$ the zeros of $M_n(x; \beta, c)$, by Theorem 2, we obtain the following.

Corollary 4. *For $\beta > 0$, $0 < c < 1$ and $0 < \tau \le 2$, the inequalities*

$$\begin{aligned} m_{n+1,1}(\beta, c) &< m_{n,1}(\beta, c) < m_{n,1}(\beta+\tau, c) < \cdots < m_{n+1,n}(\beta, c) \\ &< m_{n,n}(\beta, c) < m_{n,n}(\beta+\tau, c) < m_{n+1,n+1}(\beta, c) \end{aligned}$$

hold.

The latter results coincide with those in Theorem 2.1 and Corollary 2.2 in Jordaan and Toókos' paper [13].

4.4. Interlacing of zeros of Hahn polynomials

The Hahn polynomials (see [12, p. 177])

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{array}{c} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{array}\middle| 1\right),$$

for $\alpha, \beta > -1$ and $n \le N$, are orthogonal with respect to a discrete measure

$$d\mu(\alpha,\beta,N;x) := \frac{(\alpha+1)_x}{x!}\frac{(\beta+1)_{N-x}}{(N-x)!}, \quad x = 0, 1, \ldots, N.$$

Note that

$$d\mu(\alpha+1,\beta,N;x) = \left(1+\frac{x}{\alpha+1}\right) d\mu(\alpha,\beta,N;x) = \frac{(\alpha+2)_x}{x!}\frac{(\beta+1)_{N-x}}{(N-x)!},$$

and

$$\begin{aligned} d\mu(\alpha+2,\beta,N;x) &= \left(1+\frac{x}{\alpha+2}\right)\left(1+\frac{x}{\alpha+1}\right) d\mu(\alpha,\beta,N;x) \\ &= \frac{(\alpha+3)_x}{x!}\frac{(\beta+1)_{N-x}}{(N-x)!}, \end{aligned}$$

for $x = 0, 1, \ldots, N$.

We denote by $q_{n,k}(\alpha,\beta,N)$, $1 \le k \le n$, the zeros of $Q_n(x;\alpha,\beta,N)$. Then, by Theorem 2, we obtain the following.

Corollary 5. *For $\alpha, \beta > -1$, $n \le N$ and $0 < \tau \le 2$, the inequalities*

$$\begin{aligned} q_{n+1,1}(\alpha,\beta,N) &< q_{n,1}(\alpha,\beta,N) < q_{n,1}(\alpha+\tau,\beta,N) < \cdots < q_{n+1,n}(\alpha,\beta,N) \\ &< q_{n,n}(\alpha,\beta,N) < q_{n,n}(\alpha+\tau,\beta,N) < q_{n+1,n+1}(\alpha,\beta,N) \end{aligned}$$

hold.

We also analyze the interlacing of the zeros of Hahn polynomials with respect to the parameter β. Observe that

$$d\mu(\alpha,\beta+1,N;x) = \left(\frac{\beta+1+N-x}{\beta+1}\right) d\mu(\alpha,\beta,N;x) = \frac{(\alpha+1)_x}{x!}\frac{(\beta+2)_{N-x}}{(N-x)!},$$

and

$$\begin{aligned} d\mu(\alpha,\beta+2,N;x) &= \left(\frac{\beta+2+N-x}{\beta+2}\right)\left(\frac{\beta+1+N-x}{\beta+1}\right) d\mu(\alpha,\beta,N;x) \\ &= \frac{(\alpha+1)_x}{x!}\frac{(\beta+3)_{N-x}}{(N-x)!}, \end{aligned}$$

for $x = 0, 1, \ldots, N$. Then, by Theorem 2, we have the following.

Corollary 6. *For $\alpha, \beta > -1$, $n \le N$ and $0 < \tau \le 2$, the inequalities*

$$\begin{aligned} q_{n+1,1}(\alpha,\beta,N) &< q_{n,1}(\alpha,\beta+\tau,N) < q_{n,1}(\alpha,\beta,N) < \cdots < q_{n+1,n}(\alpha,\beta,N) \\ &< q_{n,n}(\alpha,\beta+\tau,N) < q_{n,n}(\alpha,\beta,N) < q_{n+1,n+1}(\alpha,\beta,N) \end{aligned}$$

hold.

It is worth mentioning that the latter statement differs from the result in [13, Theorem 5.1]. The reason is that in that theorem as well as in [12, Theorem 7.1.2] the monotonicity properties of the zeros of Hahn polynomials, as defined in the present paper, and in these references, are not stated correctly. These zeros are increasing functions of α and decreasing functions of β, while the zeros of Jacobi polynomials decrease with α and increase with β. Both results can be established via Markov's theorem and the fact that their zeros possess opposite monotonicity behavior with respect to the parameters α and β is rather clear from the limit relation (see [12, (6.2.10)])

$$\lim_{N\to\infty} Q(Nx;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

4.5. Interlacing of zeros of Askey–Wilson polynomials

The Askey–Wilson polynomials (see [12, p. 381])

$$P_n(x;t|q) = t_1^{-n}(t_1t_2,t_1t_3,t_1t_4;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n},\ t_1t_2t_3t_4q^{n-1},\ t_1e^{i\theta},\ t_1e^{-i\theta} \\ t_1t_2,\ t_1t_3,\ t_1t_4 \end{matrix}\,\middle|\, q,q\right),$$

where t denotes the ordered tuple (t_1,t_2,t_3,t_4) and $\max\{t_1,t_2,t_3,t_4\} < 1$, are orthogonal on $(-1,1)$ with respect to the measure

$$d\mu(x;t_1,t_2,t_3,t_4|q) := \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{\prod_{j=1}^{4}(t_je^{i\theta},t_je^{-i\theta};q)_\infty}\frac{1}{\sqrt{1-x^2}}dx,$$

where $x=\cos\theta$. Let $t_1 = q^\alpha$. Then obviously

$$\begin{aligned} d\mu(x;q^{\alpha+1},t_2,t_3,t_4|q) &= (1-2xq^\alpha+q^{2\alpha})\,d\mu(x;q^\alpha,t_2,t_3,t_4|q),\\ d\mu(x;q^{\alpha+1},q^{\beta+1},t_3,t_4|q) &= (1-2xq^\alpha+q^{2\alpha})(1-2xq^\beta+q^\beta)\,d\mu(x;q^\alpha,q^\beta,t_3,t_4|q) \end{aligned}$$

and

$$\begin{aligned} d\mu(x;q^{\alpha+2},t_2,t_3,t_4|q) &= (1-2xq^\alpha+q^{2\alpha})(1-2xq^{\alpha+1}+q^{2\alpha+2})\\ &\quad\times d\mu(x;q^\alpha,t_2,t_3,t_4|q). \end{aligned}$$

Let $\kappa_{n,k}(\alpha)$ be the zeros of $P_n(x;q^\alpha,t_2,t_3,t_4|q)$ and $\kappa_{n,k}(\alpha,\beta)$ be the zeros of $P_n(x;q^\alpha,q^\beta,t_3,t_4|q)$. Then we have the following.

Corollary 7. *The inequalities*

$$\begin{aligned} \kappa_{n+1,1}(\alpha) &< \kappa_{n,1}(\alpha+\tau) < \kappa_{n,1}(\alpha) < \cdots < \kappa_{n+1,n}(\alpha)\\ &< \kappa_{n,n}(\alpha+\tau) < \kappa_{n,n}(\alpha) < \kappa_{n+1,n+1}(\alpha) \end{aligned} \tag{21}$$

hold for $0<\tau\le 2$, *and*

$$\begin{aligned} \kappa_{n+1,1}(\alpha,\beta) &< \kappa_{n,1}(\alpha+\tau,\beta+\gamma) < \kappa_{n,1}(\alpha,\beta) < \cdots < \kappa_{n+1,n}(\alpha,\beta)\\ &< \kappa_{n,n}(\alpha+\tau,\beta+\gamma) < \kappa_{n,n}(\alpha,\beta) < \kappa_{n+1,n+1}(\alpha,\beta) \end{aligned} \tag{22}$$

hold for $0<\tau,\gamma\le 1$.

In order to prove these statements, it suffices to observe that the extra factors which appear in the above relations between the weight functions of Askey–Wilson polynomials have zeros greater than one when the parameters are shifted by one or two. Then Theorems 2 and 3, for these particular cases, imply that (21) holds for $\tau = 2$, while (22) holds for $\tau = \gamma = 1$. Finally, Markov's theorem and straightforward calculations, as those on p. 154 in Jordaan and Toókos [14], show that, for fixed α, the zeros $\kappa_{n,j}(\alpha+\tau)$ are decreasing functions of τ. Similarly, for fixed α and β, the zeros $\kappa_{n,j}(\alpha+\tau, \beta+\gamma)$ are decreasing functions of τ and γ. This completes the proof of (21) for the whole range $0 < \tau \le 2$ of the parameter τ and the proof of (22) for all τ and λ with $0 < \tau \le 1$ and $0 < \gamma \le 1$.

Using similar arguments, we can establish the results on interlacing of zeros of q-orthogonal polynomials, given in [14] as well as some extensions in the spirit of Corollary 7.

5. Further open questions

The first natural question which arises in connection with Theorem 2 is if the interval $(0, 2]$ is the largest possible one, such that, if τ varies in it, the interlacing holds. A straightforward observation is that this interval can be extended to $[-2, 2]$ provided $d\mu_\tau(c; x)$, given by (2) is well defined. Moreover, it is of interest to see the role the points c and d play when one considers the latter question.

In most of the above mentioned recent contributions the authors considered this question for the entire range of the parameters and in the classical cases they found proper numerical examples which show that in these situations the interval $(0, 2]$ is indeed the largest possible one. In what follows we try to interpret these examples, especially the ones concerning the zeros of Jacobi polynomials, using our proof of Theorem 2. Recall the identity

$$\frac{d^2}{dx^2}\left\{(1-x^2)P_n^{(1,1)}(x)\right\} = c_n P_n^{(1,1)}(x),$$

where c_n is a nonzero constant. If we consider the family of Jacobi polynomials, when these are defined as hypergeometric functions, they exist beyond the range of orthogonality, so that

$$(1-x^2)P_n^{(1,1)}(x) = d_n P_{n+2}^{(-1,-1)}(x), \quad d_n \neq 0.$$

Let us perform the well known procedure of separating a family of polynomials with respect to an even measure, in a symmetric with respect to the origin interval, into two families, via quadratic transformation, as described in [2] in the general setting or in [5] for the ultraspherical polynomials. Then, denoting by $\mathcal{P}_n^{(\alpha,\beta)}(x)$ the Jacobi polynomials for the interval $[0, 1]$, orthogonal on it with respect to $x^\alpha(1-x)^\beta$, we see that the above identity reads as

$$(1-x)\mathcal{P}_n^{(\alpha,1)}(x) = \delta_n \mathcal{P}_{n+1}^{(\alpha,-1)}(x), \quad \delta_n \neq 0,$$

where either $\alpha = -1/2$ or $\alpha = 1/2$. Then, if one chooses $\alpha = \pm 1/2$, $\beta > -1$, but very close to -1 and $\tau = 2$, the zeros of $\mathcal{P}_n^{(\alpha,\beta+\tau)}(x)$ and $\mathcal{P}_{n+1}^{(\alpha,\beta)}(x)$ will interlace but will be very close to each other. Thus, if we increase τ a bit, beyond the interval $(0, 2]$, the zeros of the corresponding polynomials will not interlace any longer. This is what happens indeed and it is not a surprise that the examples given in [10] use parameters when either α or β is very close to -1.

However, this phenomenon permits another interpretation which is obvious from the proof of Theorem 2. If one analyzes the arguments in that proof, it is clear that the positivity of the constant $e_n - \gamma_n$, given by (8) is crucial for the interlacing. The closer to zero it is, the closer the zeros of $p_n(2, c; x)$ and $p_{n+1}(x)$ will be. On the other hand, because of the (4) of the kernel polynomials, this constant is close to zero when $p_{n+1}(c)$ is. This explains why the above example and those provided in [10] use values of α or β near -1. Similar effect occurs for all families of orthogonal polynomials.

Observe that when either the measure, or the location of the point c is such that the quantity in (8) remains large, then we might expect that interlacing of the zeros of $p_n(\tau, c; x)$ with those of $p_n(x)$ and $p_{n+1}(x)$ still holds for all n when τ varies beyond $(0, 2]$. Thus, we conjecture, for instance that, if α and β are fixed large positive numbers, then there is $\varepsilon = \varepsilon(\alpha, \beta) > 0$, such that the zeros of $P_n^{(\alpha,\beta+\tau)}(x)$ interlace with the zeros of both $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ for all $n \in \mathbb{N}$ and for every $\tau \in (0, 2 + \varepsilon)$.

References

[1] R. Askey, Graphs as an aid to understanding special functions, in: Asymptotic and Computational Analysis, Winnipeg 1989, in: Lectures and Notes in Pure and Applied Mathematics, vol. 124, 1990, pp. 3–33.

[2] T.S. Chihara, Mathematics and its Applications Series, in: An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] L. Chihara, D. Stanton, Zeros of generalized Krawtchouk polynomials, J. Approx. Theory 60 (1990) 43–57.

[4] E.B. Christoffel, Über die Gaussische Quadratur und eine Verallgemeinerung derselben, J. Reine Angew. Math. 55 (1858) 61–82.

[5] D.K. Dimitrov, On a conjecture concerning monotonicity of zeros of ultraspherical polynomials, J. Approx. Theory 85 (1996) 88–97.

[6] D.K. Dimitrov, R. Rodrigues, On the behaviour of zeros of Jacobi polynomials, J. Approx. Theory 116 (2002) 224–239.

[7] K. Driver, K. Jordaan, Separation theorems for the zeros of certain hypergeometric polynomials, J. Comput. Appl. Math. 199 (2007) 48–55.

[8] K. Driver, K. Jordaan, Interlacing of zeros of shifted sequences of one-parameter orthogonal polynomials, Numer. Math. 107 (2007) 615–624.

[9] K. Driver, K. Jordaan, Zeros of linear combinations of Laguerre polynomials from different sequences, J. Comput. Appl. Math. 233 (2009) 719–722.

[10] K. Driver, K. Jordaan, N. Mbuyi, Interlacing of the zeros of Jacobi polynomials with different parameters, Numer. Algorithms 49 (2008) 143–152.

[11] K. Driver, K. Jordaan, N. Mbuyi, Interlacing of zeros of linear combinations of classical orthogonal polynomials from different sequences, Appl. Numer. Math. 59 (2009) 2424–2429.

[12] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, in: Encyclopedia of Mathematics and its Applications, vol. 98, Cambrige University Press, 2009.

[13] K. Jordaan, F. Toókos, Interlacing theorems for the zeros of some orthogonal polynomials from different sequences, Appl. Numer. Math. 59 (2009) 2015–2022.

[14] K. Jordaan, F. Toókos, Mixed recurrence relations and interlacing of the zeros of some q-orthogonal polynomials from different sequences, Acta Math. Hungar. 128 (2010) 150–164.

[15] R.J. Levit, The zeros of the Hahn polynomials, SIAM Rev. 9 (2) (1967) 191–203.

[16] F. Peherstorfer, Linear combination of orthogonal polynomials generating positive quadrature formulas, Math. Comp. 55 (191) (1990) 231–241.

[17] J. Segura, Interlacing of the zeros of contiguous hypergeometric functions, Numer. Algorithms 49 (2008) 387–407.

[18] J.A. Shohat, On mechanical quadratures, in particular, with positive coefficients, Trans. Amer. Math. Soc. 42 (1937) 461–496.

[19] G. Szegő, Orthogonal Polynomials, fourth ed., in: Amer. Math. Soc. Coll. Publ, vol. 23, Amer. Math. Soc, Providence, RI, 1975.