Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 181 (2014) 1–5

Journal of Approximation Theory

www.elsevier.com/locate/jat

Notes

A note on strong asymptotics of weighted Chebyshev polynomials

András Kroó[*,1]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary
Budapest University of Technology and Economics, Department of Analysis, Budapest, Hungary

Received 20 July 2013; received in revised form 30 December 2013; accepted 19 January 2014
Available online 31 January 2014

Communicated by Vilmos Totik

Abstract

Denote by $T_n(., w)$ the nth degree Chebyshev polynomial with respect to the positive weight w. It is shown in this note that if w is C^{1+} then $T_n(\cos\phi, w) = \Re\{e^{-in\phi}\pi^2(e^{i\phi})\} + o(1)$ where $\pi(z)$ stands for the Szegő function corresponding to w. This extends an earlier result proved by Kroó and Peherstorfer who verified this asymptotic relation for $C^{2+\alpha}$, $\alpha > 0$ weights.

Keywords: Weighted Chebyshev polynomials; Asymptotics; Strong unicity constant

Let w be a positive continuous weight function on $[-1, 1]$ and denote by

$$T_n(x, w) = a_n x^n + \cdots + a_0, \quad a_n > 0$$

the normalized Chebyshev polynomial of degree n with respect to w uniquely defined by

$$1 = \|T_n(., w)w\| \le \|(a_n x^n + b_{n-1}x^{n-1} + \cdots + b_0)w(x)\|, \quad \forall b_{n-1}, \ldots, b_0 \in \mathbb{R}.$$

Here $\|\cdots\|$ stands for the usual supremum norm on $[-1, 1]$. By the classical Chebyshev alternation theorem $T_n(., w)$ is characterized by the property that function $T_n(., w)w$ equioscillates $n + 1$ times on $[-1, 1]$ between 1 and -1.

* Correspondence to: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Hungary.
E-mail address: kroo.andras@renyi.mta.hu.

[1] Written during the author's visit as Chair of Excellence at the Carlos III University of Madrid.

0021-9045/$ - see front matter
http://dx.doi.org/10.1016/j.jat.2014.01.005

In case when $w = \frac{1}{\rho_m}$ with ρ_m a polynomial of degree $m < n$ positive on $[-1, 1]$ the extremal polynomial $T_n(., w)$ can be found explicitly; this result goes back to Chebyshev (see [1]). Namely, let g_m be the well-known Fejer–Riesz representation of ρ_m, that is g_m is the unique polynomial of degree m with real coefficients and all zeros in $|z| > 1$ such that $|g_m(e^{i\phi})|^2 = \rho_m(\cos\phi)$, $g_m(0) > 0$. Then with $z = e^{i\phi}$, $x = \cos\phi$ we have

$$T_n(x, 1/\rho_m) = \Re\{z^{-n} g_m^2(z)\}. \tag{1}$$

Clearly, $T_n(., 1/\rho_m)/\rho_m$ equioscillates $n + 1$ times on $[-1, 1]$ between 1 and -1. Since

$$\Re\{z^{-n} g_m^2(z)\}^2 + \Im\{z^{-n} g_m^2(z)\}^2 = \rho_m^2, \quad z = e^{i\phi} \tag{2}$$

the $n + 1$ equioscillations of $T_n(., 1/\rho_m)/\rho_m$ are zeros of $\Im\{z^{-n} g_m^2(z)\}^2$.

Finding asymptotic representation for the weighted Chebyshev polynomials has been one of the basic problems in Approximation Theory for many years. The nth root asymptotic of the weighted Chebyshev polynomials was given by Fekete and Walsh [3]; strong asymptotics outside $[-1, 1]$ was found by Lubinsky and Saff (see [7,6]).

The first general result on strong asymptotics for weighted Chebyshev polynomials on $[-1, 1]$ was given in [4] where it was shown that if w is of the smoothness class $C^{2+\alpha}$, $\alpha > 0$ (that is the *second derivative* of $w(\cos\phi)$ is Lipα) then $T_n(x, w) = \Re\{z^{-n}\pi^2(z)\} + o(1)$, $z = e^{i\phi}$, $x = \cos\phi$ uniformly for $\phi \in [0, \pi]$. Here $\pi(z)$ stand for the classical Szegő function corresponding to the weight w, that is $\pi(z)$ is the analytic and nonzero function on $|z| < 1$ satisfying the boundary condition $|\pi(z)| = w(\cos\phi)^{-1/2}$, $z = e^{i\phi}$.

In this note we will show that this result can be improved substantially by relaxing the smoothness requirement on the weight. In fact we will prove that the above asymptotic relation holds for weights whose *first derivatives are of the Dini–Lipschitz class*. For the positive and differentiable on $[-1, 1]$ weight w denote by $\omega_1(h)$ the modulus of continuity of the *first derivative* of $w(\cos\phi)$. We shall say that $w \in C^{1+}$ if $\omega_1(h)$ satisfies the Dini–Lipschitz property

$$\omega_1\left(\frac{1}{n}\right) = o\left(\frac{1}{\log n}\right), \quad n \to \infty. \tag{3}$$

Theorem. *Let $w \in C^{1+}$ be a positive weight on $[-1, 1]$. Then as $n \to \infty$*

$$T_n(\cos\phi, w) = \Re\{e^{-in\phi}\pi^2(e^{i\phi})\} + o(1), \tag{4}$$

uniformly for $\phi \in [0, \pi]$.

Denote by P_n the space of polynomials with real coefficients of degree at most $n - 1$. Given $f \in C[-1, 1]$ having 0 as its best approximant out of P_n, let $\gamma_n(f, w)$ be the nth weighted *strong unicity constant* of f defined as

$$\gamma_n(f, w) := \sup_{p_n \in P_n \setminus \{0\}} \frac{\|w p_n\|}{\|w(f - p_n)\| - \|wf\|}. \tag{5}$$

It is shown in [4], Lemma 3.2 that given positive weights $w_1, w_2 \in C[-1, 1]$ we have that

$$\|T_n(., w_1) - T_n(., w_2)\| \le c\gamma_n(T_n(., w_2), w_2)\|w_1 - w_2\| \tag{6}$$

where the positive constant c depends only on the minima and maxima of weights w_1, w_2 on $[-1, 1]$.

Now in order to derive asymptotics for $T_n(., w)$ one could approximate weight w by reciprocals of polynomials $1/\rho_m$, and use the explicit representation (1) for $T_n(., 1/\rho_m)$ and (6) with $w_2 := 1/\rho_m$. This approach will be effective provided that the growth to infinity of the quantity $\gamma_n(T_n(., 1/\rho_m), 1/\rho_m)$ as $n \to \infty$ is compensated by a proper rate of approximation of w by reciprocals of polynomials $1/\rho_m$. This method was used in [4] where it was shown that

$$\gamma_n(T_n(., 1/\rho_m), 1/\rho_m) = O(n^2)$$

and a proper approximation result holds if w is $C^{2+\alpha}, \alpha > 0$. In this note we will improve this technique. First a sharper estimate $\gamma_n(T_n(., 1/\rho_m), 1/\rho_m) = O(n)$ will be verified. Then we will provide a more delicate approximation for the Szegő function by reciprocals of polynomials using the de La Vallée Poussin sums. This will result in the replacement of $C^{2+\alpha}, \alpha > 0$ by the substantially wider class of weights $w \in C^{1+}$. We will also show that the estimate $\gamma_n(T_n(., 1/\rho_m), 1/\rho_m) = O(n)$ is sharp in general (see the Remark after the proof of the theorem), which means that the weight must be C^1 in order for the above method to work. This makes a substantial further extension of (4) to wider classes of weights rather unlikely.

First we need to verify a lemma which improves the estimate of Lemma 3.3 from [4].

Lemma. *If ρ_m is a polynomial of degree $m < n$ such that* $0 < a \le \rho_m \le b$ *on* $[-1, 1]$, *then*

$$\gamma_n(T_n(., 1/\rho_m), 1/\rho_m) \le \frac{b}{a}(2n + 1). \tag{7}$$

Proof. With $z = e^{i\phi}, x = \cos\phi$ denote

$$Q_{n+1}(x) := \sin\phi\,\Im\{z^{-n} g_m^2(z)\}, \qquad p_k(x) := \frac{(x - y_k)T_n(., 1/\rho_m) + Q_{n+1}(x)}{x - y_k},$$
$$1 \le k \le n + 1 \tag{8}$$

where $-1 =: y_1 < y_2 < \cdots < y_n < y_{n+1} := 1$ are the equioscillation points of $T_n(., 1/\rho_m)/\rho_m$. By (2) the equioscillation points of $T_n(., 1/\rho_m)/\rho_m$ are the zeros of $Q_{n+1} \in P_{n+2}$. It can easily be shown using (1) and the fact that g_m is a polynomial of degree $m < n$ with real coefficients that the leading coefficients of $(x - y_k)T_n(., 1/\rho_m)$ and $Q_{n+1}(x)$ cancel out in (8), i.e., we have $p_k \in P_n, 1 \le k \le n + 1$. Furthermore, $p_k \in P_n, 1 \le k \le n + 1$ are uniquely defined by the interpolation properties

$$(p_k/\rho_m)(y_j) = \mathrm{sgn} T_n(y_j, 1/\rho_m), \quad 1 \le j \le n + 1, j \ne k.$$

It was shown by Cline [2] (see [5], Theorem 4.10, p. 28 for a detailed proof) that

$$\gamma_n(T_n(., 1/\rho_m), 1/\rho_m) = \max_{1\le k\le n+1} \|p_k/\rho_m\|, \tag{9}$$

where by (8) and (2)

$$\max_{1\le k\le n+1} \|p_k/\rho_m\| \le \frac{1}{a}(\|T_n(., 1/\rho_m)\| + \|Q'_{n+1}\|) \le \frac{1}{a}(b + \|Q'_{n+1}\|); \tag{10}$$
$$|Q_{n+1}(x)| \le \sqrt{1 - x^2}|\Im\{z^{-n} g_m^2(z)\}| \le b\sqrt{1 - x^2}.$$

The last relation means that the polynomial Q_{n+1} of degree at most $n+1$ has a *circular majorant* on $[-1, 1]$. Then by a known inequality for polynomials with circular majorants (see [8], Theorem 1 on p. 448) we have $\|Q'_{n+1}\| \le 2nb$. Using this together with (9) and (10) we arrive at estimate (7).

Proof of the Theorem. Set $\Omega(\phi) := \log w(\cos\phi)$. Then $|\pi(e^{i\phi})| = e^{-\Omega(\phi)/2}$ and hence

$$\pi(e^{i\phi}) = \exp(-\Omega(\phi)/2 - i\overline{\Omega(\phi)}/2) \tag{11}$$

where the trigonometric conjugate of $\Omega(\phi)$ is denoted by $\overline{\Omega(\phi)}$. Since $\Omega(\phi)$ is differentiable the Jackson theorem and Privalov's theorem for the trigonometric conjugate of $\Omega(\phi)$ yield

$$E_q(\Omega) = O\left(\frac{1}{q}\omega_1(1/q)\right), \qquad E_q(\overline{\Omega}) = O\left(\frac{\log q}{q}\right),$$

where $E_q(\cdot)$ denotes the error of best uniform approximation on $[-\pi, \pi]$ by trigonometric polynomials of degree q, and $\omega_1(h)$ is the modulus of continuity of the first derivative of $w(\cos\phi)$.

Let now $t_q(\phi)$ be the q-th de La Vallée Poussin sum of $\Omega(\phi)$

$$t_q(\phi) := \frac{1}{q-[q/2]}\sum_{k=[q/2]+1}^{q} S_k(\Omega, \phi),$$

where $S_k(\Omega, \phi)$ is its k-th Fourier sum. Then clearly $\overline{t_q(\phi)}$ is the q-th de La Vallée Poussin sum of $\overline{\Omega(\phi)}$. Since the de La Vallée Poussin sums provide asymptotically optimal orders of approximation we obtain by the last two estimates that

$$\|\Omega - t_q\|_{[-\pi,\pi]} = O\left(\frac{1}{q}\omega_1(1/q)\right), \qquad \|\overline{\Omega} - \overline{t_q}\|_{[-\pi,\pi]} = O\left(\frac{\log q}{q}\right). \tag{12}$$

Set now $p_q(z) := -t_q(\phi)/2 - i\overline{t_q(\phi)}/2$, $z = e^{i\phi}$. Then using that $\Omega(\phi)$ is an even function (and hence t_q is a cosine polynomial) it easily follows that p_q is an algebraic polynomial of degree q with real coefficients. Moreover using (12) and (11) we obtain that uniformly on $|z| = 1$

$$\pi(z) = e^{p_q(z)} + O\left(\frac{\log q}{q}\right). \tag{13}$$

Now for arbitrary $n \in \mathbb{N}$ set $q := [n/\log n]$, $k := [n/q] - 1 \sim \log n$, and let $Q_k(z)$ be the $k-1$-st Taylor polynomial of e^z. Furthermore, set $m := kq,\ g_m(z) := Q_k(p_q(z)) \in P_m,\ m < n$. Then by the remainder term estimate for the Taylor expansion of e^z it follows from (13) that as $n \to \infty$ uniformly on $|z| = 1$

$$\pi(z) = g_m(z) + O\left(\frac{\log q}{q} + \frac{c^k}{k!}\right) = g_m(z) + o(1). \tag{14}$$

Since $\pi(z)$ has no zeros in the closed unit disc the polynomials g_m also do not vanish on $|z| \le 1$ for n large enough. Let us show now that $|g_m(e^{i\phi})|$ provides an approximation for $|\pi(e^{i\phi})|$ which is finer than (14). Clearly,

$$|g_m(z)|^2 = |Q_k(p_q(z))|^2 = |e^{p_q(z)}|^2 + O\left(\frac{c^k}{k!}\right) = e^{-t_q(\phi)} + O\left(\frac{c^k}{k!}\right). \tag{15}$$

On the other hand using the first estimate in (12)

$$\begin{aligned} e^{-t_q(\phi)} &= e^{-\Omega(\phi)} + O\left(\frac{1}{q}\omega_1(1/q)\right) = \frac{1}{w(\cos\phi)} + O\left(\frac{1}{q}\omega_1(1/q)\right) \\ &= |\pi(e^{i\phi})|^2 + O\left(\frac{1}{q}\omega_1(1/q)\right). \end{aligned}$$

Applying the last estimate together with (15) yields

$$|\pi(e^{i\phi})|^2 = |g_m(e^{i\phi})|^2 + O\left(\frac{c^k}{k!} + \frac{1}{q}\omega_1(1/q)\right). \tag{16}$$

Now recall that $q \sim \frac{n}{\log n}$ and hence by the Dini–Lipschitz condition (3) we have that $\frac{1}{q}\omega_1(1/q) \sim \frac{\log n}{n}\omega_1(\frac{\log n}{n}) = o(1/n)$. In addition, since $k \sim \log n$ we also have that $\frac{c^k}{k!} = o(1/n)$. This and (16) evidently yield

$$|\pi(e^{i\phi})|^2 = |g_m(e^{i\phi})|^2 + o(1/n), \qquad w(\cos\phi) = |g_m(e^{i\phi})|^{-2} + o(1/n). \tag{17}$$

Since $g_m \in P_m$ has real coefficients it follows that $|g_m(e^{i\phi})|^2 = \rho_m(\cos\phi)$ with some polynomial $\rho_m \in P_m, m < n$.

Now using (6) with $w_1 := w$, $w_2 := \frac{1}{\rho_m}$ we have by (7) and (17)

$$\begin{aligned}\|T_n(., w) - T_n(., 1/\rho_m)\| &\le c(w)\gamma_n(T_n(., 1/\rho_m), 1/\rho_m)\|w - 1/\rho_m\|\\ &\le c_1(w)n\|w - 1/\rho_m\| = o(1).\end{aligned}$$

In addition, by (14) as $n \to \infty$ uniformly on $|z| = 1$

$$\Re\{z^{-n}\pi^2(z)\} - \Re\{z^{-n}g_m^2(z)\} = o(1).$$

Finally, (1) and the last two estimates yield that uniformly for $\phi \in [0, \pi]$, $z = e^{i\phi}$

$$\Re\{z^{-n}\pi^2(z)\} - T_n(\cos\phi, w) = o(1).$$

Remark. If we set $\rho_m = 1$ in the above lemma then it easily follows from (8) and (9) that

$$\gamma_n(T_n(., 1), 1) \ge \|p_{n+1}\| \ge |p_{n+1}(1)| = Q'_{n+1}(1) + 1 = 2n + 1,$$

i.e., (7) becomes an equality in this case. Hence the $O(n)$ estimate for the strong unicity constant given by the lemma is sharp, in general, yielding that the method used in this note cannot work for non-differentiable weights. This gives rise to a conjecture that the above theorem fails in general for non-differentiable weights, i.e., the C^{1+} assumption imposed above cannot be substantially weakened.

Conjecture. *There exist positive Lip*1 *weights w on* $[-1, 1]$ *non-differentiable at some point of* $(-1, 1)$ *for which asymptotic relation* (4) *fails.*

References

[1] N.I. Akhiezer, Theory of Approximation, Ungar, 1956.

[2] A.K. Cline, Lipschitz conditions on uniform approximation operators, J. Approx. Theory 8 (1973) 160–172.

[3] M. Fekete, J.L. Walsh, On the asymptotic behaviour of polynomials with extremal properties and of their zeros, J. Anal. Math. 4 (1954/55) 49–87.

[4] A. Kroó, F. Peherstorfer, Asymptotic representation of L_∞ and L_1-minimal polynomials, Math. Proc. Cambridge Philos. Soc. 144 (2008) 241–254.

[5] A. Kroó, A. Pinkus, Strong uniqueness, Surv. Approx. Theory 5 (2010) 1–91.

[6] E. Levin, D.S. Lubinsky, Orthogonal Polynomials for Exponential Weights, in: CMS Books in Mathematics, vol. 4, Springer-Verlag, 2001.

[7] D.S. Lubinsky, E.B. Saff, Strong asymptotics for L_p-extremal polynomials ($1 < p$) associated with weights on $[-1, 1]$, in: E.B. Saff (Ed.), Approximation Theory, Tampa Proc. 1985–1986, in: Lecture Notes in Math., vol. 1287, 1987, pp. 83–104.

[8] Q.I. Rahman, On a problem of Turán about polynomials with curved majorants, Trans. Amer. Math. Soc. 163 (1972) 447–455.