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On the equivalence of the K-functional and the modulus of continuity on the Morrey spaces

Victor Burenkov[a,b], Arash Ghorbanalizadeh[c,*], Yoshihiro Sawano[d,a]

[a] S. M. Nikol'skii Mathematical Institute, RUDN University, 6 Miklukho-Maklay St., Moscow, 117198, Russia
[b] Steklov Mathematical Institute, Russian Academy of Science Gubkin St. 42, Moscow, GSP1, 117966, Russia
[c] Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran
[d] Department of Mathematics and Information Science, Tokyo Metropolitan University, Tokyo, Japan

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Abstract

In this note, the Morrey spaces and the Sobolev–Morrey spaces are considered. In particular, the K-functional with respect to these spaces is estimated from above and below. As an application, we characterize the Nikol'skii–Besov–Morrey spaces via real interpolation.

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1. Introduction

We aim to describe the K-functional for the pair consisting of the Morrey space and the Sobolev–Morrey spaces via the modulus of continuity of the Morrey spaces over the interval (a, b). As a corollary, we can describe the Nikol'skii–Besov–Morrey spaces over the infinite interval $(a, b) \subset \mathbb{R}$ via real interpolation.

The Morrey spaces were introduced by [22], where C. Morrey studied the local behavior of solutions to elliptic differential equations. Now the Morrey spaces are used in several branches

* Corresponding author.
E-mail addresses: Burenkov@cardiff.ac.uk (V. Burenkov), ghorbanalizadeh@iasbs.ac.ir (A. Ghorbanalizadeh), ysawano@tmu.ac.jp (Y. Sawano).

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of mathematics such as real analysis, PDE and potential theory. Let $1 \le q \le p \le \infty$. Recall that the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all $L_q(\mathbb{R}^n)$-locally integrable functions f for which the norm

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q dy \right)^{\frac{1}{q}},$$

is finite, where Q runs over all cubes in $\mathbb{R}^n$. Clearly if $p = q$, $\mathcal{M}_q^p(\mathbb{R}^n)$ coincides with the Lebesgue space $L_p(\mathbb{R}^n)$. Moreover, $\mathcal{M}_q^\infty(\mathbb{R}^n)$ coincides with the space $L_\infty(\mathbb{R}^n)$. If $1 < q < p < \infty$, there are difficulties in handling the Morrey spaces due to the following reasons:

(1) Unless p/q is fixed, $\mathcal{M}_q^p(\mathbb{R}^n)$ do not interpolate well see [5,13,14,19,20,29,43,44].[1]

(2) The Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is not reflexive; see [36, Example 5.2] and [42, Theorem 1.3].

(3) The space $\mathcal{D}(\mathbb{R}^n)$, the space of all compactly supported infinitely differentiable functions, is not dense in $\mathcal{M}_q^p(\mathbb{R}^n)$; see [40, Proposition 2.16].

(4) The Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is not separable; see [40, Proposition 2.16].

(5) According to the terminology of [3], $\mathcal{M}_q^p(\mathbb{R}^n)$ is not a Banach function space; see [36, Example 3.3].

(6) The Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is not embedded into $L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ as the example of the function

$$\sum_{j=100}^{\infty} [\log_2 \log_2 j]^{1/p} \chi_{[j!, j! + [\log_2 \log_2 j]^{-1}]}$$

implies; see [14, Section 6].

Detailed exposition of various properties of the Morrey spaces and their numerous generalizations can be found in recent books [1,33] and survey papers [6,7,11,26,28,37,38]. Despite the difficulties in handling the Morrey spaces mentioned above, we can still consider the real interpolation between the Morrey spaces $\mathcal{M}_q^p(a,b)$ and the homogeneous and non-homogeneous Sobolev–Morrey spaces $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$, $W^r\left(\mathcal{M}_q^p(a,b)\right)$ respectively, where $r \in \mathbb{N}$.

2. Preliminaries

Let $-\infty \le a < b \le \infty$ and $1 \le q \le p \le \infty$. The Morrey space $\mathcal{M}_q^p(a,b)$ is the set of measurable functions $f \in L_q(a,b)$ for which the norm

$$\|f\|_{\mathcal{M}_q^p(a,b)} = \sup_{(\alpha,\beta) \subset (a,b)} (\beta - \alpha)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\alpha,\beta)}$$

is finite.

Note that for $(c,d) \subset (a,b)$,

$$\|f\|_{\mathcal{M}_q^p(c,d)} \le \|f\|_{\mathcal{M}_q^p(a,b)}; \qquad f \in \mathcal{M}_q^p(a,b), \tag{2.1}$$

and that for any $h \in \mathbb{R}$ and for all $f \in \mathcal{M}_q^p(a+h, b+h)$

$$\|f(\cdot + h)\|_{\mathcal{M}_q^p(a,b)} = \|f\|_{\mathcal{M}_q^p(a+h,b+h)}. \tag{2.2}$$

[1] Note that the local variant $\mathcal{LM}_q^p(\mathbb{R})$ of the spaces $\mathcal{M}_q^p(\mathbb{R}^n)$, defined by the above expression for $\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)}$, where cubes Q should contain the origin, is good for interpolation. The scale $\mathcal{LM}_q^p(\mathbb{R})$ is closed under the procedure of the real method of interpolation; see [8–10,21].

Lemma 2.1. *Let $a, b, c, d \in \mathbb{R}$ satisfy $a \le c < d \le b$, and let $1 \le q \le p \le \infty$. If a measurable function f defined on (a, b) is supported on a subinterval $[c, d]$ of (a, b), then*

$$\|f\|_{\mathcal{M}_q^p(a,b)} = \|f\|_{\mathcal{M}_q^p(c,d)}.$$

Proof. Clearly

$$\|f\|_{\mathcal{M}_q^p(a,b)} = \sup_{(\alpha,\beta)\subset(a,b)} \big(\beta-\alpha\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\alpha,\beta)} = \sup_{(\alpha,\beta)\subset(a,b)} \big(\beta-\alpha\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q((\alpha,\beta)\cap(c,d))}.$$

There are five possibilities to consider:

(1) $a \le \alpha < \beta \le c$ or $d \le \alpha < \beta \le b$;

(2) $a \le \alpha \le c < \beta \le d$;

(3) $a \le \alpha \le c < d < \beta \le b$;

(4) $c < \alpha < \beta \le d$;

(5) $c < \alpha < d \le \beta \le b$.

Hence

$$\|f\|_{L_q\big((\alpha,\beta)\cap(c,d)\big)} = \begin{cases} 0 & \text{in case (1);} \\ \|f\|_{L_q(c,\beta)} & \text{in case (2);} \\ \|f\|_{L_q(c,d)} & \text{in case (3);} \\ \|f\|_{L_q(\alpha,\beta)} & \text{in case (4);} \\ \|f\|_{L_q(\alpha,d)} & \text{in case (5).} \end{cases}$$

Moreover,

$$\beta-\alpha \ge \begin{cases} \beta-c & \text{in case (2);} \\ d-c & \text{in case (3);} \\ d-\alpha & \text{in case (5).} \end{cases}$$

Therefore, since $\frac{1}{p}-\frac{1}{q} \le 0$,

$$\begin{aligned} \|f\|_{\mathcal{M}_q^p(a,b)} &= \max\Bigg\{ \sup_{(c,\beta)\subset(c,d)} \big(\beta-c\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(c,\beta)},\ \sup_{(c,d)\subset(\alpha,\beta)} \big(d-c\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(c,d)}, \\ &\qquad \sup_{\substack{(\alpha,\beta)\subset(c,d)\\ \alpha>c}} \big(\beta-\alpha\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\alpha,\beta)},\ \sup_{\substack{(\alpha,d)\subset(c,d)\\ \alpha>c}} \big(d-\alpha\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\alpha,d)} \Bigg\} \\ &= \sup_{(\gamma,\delta)\subset(c,d)} \big(\delta-\gamma\big)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\gamma,\delta)} = \|f\|_{\mathcal{M}_q^p(c,d)}. \end{aligned}$$

This completes the proof. □

The next lemma shows that the Morrey norm is local in the following sense:

Lemma 2.2. *Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$, and let $f \in \mathcal{M}_q^p(a,b) \cap \mathcal{M}_q^p(b,2b-a)$. Then $f \in \mathcal{M}_q^p(a,2b-a)$ and*

$$\|f\|_{\mathcal{M}_q^p(a,2b-a)} \le \|f\|_{\mathcal{M}_q^p(a,b)} + \|f\|_{\mathcal{M}_q^p(b,2b-a)}. \tag{2.3}$$

Proof. If we invoke Lemma 2.1, then we have

$$\begin{aligned}\|f\|_{\mathcal{M}_q^p(a,2b-a)} &\le \|\chi_{(a,b)} f\|_{\mathcal{M}_q^p(a,2b-a)} + \|\chi_{(b,2b-a)} f\|_{\mathcal{M}_q^p(a,2b-a)}\\ &= \|f\|_{\mathcal{M}_q^p(a,b)} + \|f\|_{\mathcal{M}_q^p(b,2b-a)}. \quad \square\end{aligned}$$

We move on to the Sobolev–Morrey spaces. Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$ and $r \in \mathbb{N}$. The homogeneous Sobolev–Morrey space $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ is defined as the space of all functions $f \in L_1^{\mathrm{loc}}(a,b)$ for which the weak derivative $f^{(r)}$ exists on (a,b) and

$$\|f\|_{\dot{W}^r(\mathcal{M}_q^p(a,b))} := \|f^{(r)}\|_{\mathcal{M}_q^p(a,b)} < \infty.$$

(Recall that the weak derivative $f^{(r)}$ of a function $f \in L_1^{\mathrm{loc}}(a,b)$ exists on (a,b) if and only if f is equivalent to a function $\widetilde{f}$ such that $\widetilde{f}^{(r-1)}$ exists and is locally absolutely continuous on (a,b). Moreover, $f^{(r)}$ is equivalent to the derivative $\widetilde{f}^{(r)}$ which exists almost everywhere on (a,b).)

The non-homogeneous Sobolev–Morrey space $W^r\mathcal{M}_q^p(a,b)$ is a subset of $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ consisting of all functions f, for which

$$\|f\|_{W^r(\mathcal{M}_q^p(a,b))} := \sum_{k=0}^{r} \|f^{(k)}\|_{\mathcal{M}_q^p(a,b)} < \infty.$$

For any complex valued function f on (a,b) and $h \in \mathbb{R}$, $T(h)f$ is defined by

$$T(h)f(x) := f(x+h), \qquad x \in (a-h, b-h).$$

Let $r \in \mathbb{N}$. The rth difference of $f : (a,b) \to \mathbb{C}$ with step length $h \in \mathbb{R}$, which is a function defined on $(a,b) \cap (a-rh, b-rh)$, is defined by

$$\Delta_h^r f := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} T(hk)f = (T(h) - T(0))^r f. \tag{2.4}$$

Definition 2.3. Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$, and let $r \in \mathbb{N}$. The $\mathcal{M}_q^p(a,b)$-*modulus of continuity* of order r of $f \in \mathcal{M}_q^p(a,b)$ is defined for $t > 0$ by

$$\begin{aligned}\omega_r(f,t;\mathcal{M}_q^p(a,b)) &:= \sup_{0\le|h|\le t} \left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p\left((a,b)\cap(a-rh,b-rh)\right)}\\ &= \max\left\{\sup_{0\le h\le t}\left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p(a,b-rh)}, \sup_{-t\le h\le 0}\left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p(a-rh,b)}\right\}.\end{aligned} \tag{2.5}$$

Lemma 2.4. *Let $-\infty \le a < b \le \infty$ and $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, and $f \in \mathcal{M}_q^p(a,b)$. Then for any $m \in \mathbb{N}$*

$$\omega_r(f,mt;\mathcal{M}_q^p(a,b)) \le m^r \omega_r(f,t;\mathcal{M}_q^p(a,b)). \tag{2.6}$$

Proof. Recall that for all $h \in \mathbb{R}$

$$\Delta^r_{mh} f(x) = \sum_{k_1=0}^{m-1} \cdots \sum_{k_r=0}^{m-1} T\big((k_1 + \cdots + k_r)h\big) \Delta^r_h f(x)$$

for all $x \in (a, b) \cap (a - rmh, b - rmh)$ (see, for example, [4], §16).

By (2.2) it follows that for any interval $(\alpha, \beta) \subset (a, b) \cap (a - rmh, b - rmh)$

$$\begin{aligned}
\|\Delta^r_{mh} f\|_{L_q(\alpha,\beta)} &\leq \sum_{k_1=0}^{m-1} \cdots \sum_{k_r=0}^{m-1} \|T\big((k_1 + \cdots + k_r)h\big)\Delta^r_h f\|_{L_q(\alpha,\beta)} \\
&= \sum_{k_1=0}^{m-1} \cdots \sum_{k_r=0}^{m-1} \|\Delta^r_h f\|_{L_q(\alpha+(k_1+\cdots+k_r)h, \beta+(k_1+\cdots+k_r)h)} \\
&\leq \sup_{\substack{(\gamma,\delta)\subset(a,b)\cap(a-rh,b-rh) \\ \delta-\gamma=\beta-\alpha}} \left\|\Delta^r_h f\right\|_{L_q(\gamma,\delta)} \left(\sum_{k_1=0}^{m-1} \cdots \sum_{k_r=0}^{m-1} 1\right), \\
&= m^r \sup_{\substack{(\gamma,\delta)\subset(a,b)\cap(a-rh,b-rh) \\ \delta-\gamma=\beta-\alpha}} \left\|\Delta^r_h f\right\|_{L_q(\gamma,\delta)}.
\end{aligned}$$

For $h > 0$,

$$\alpha + (k_1 + \cdots + k_r)h \geq a$$

and

$$\beta + (k_1 + \cdots + k_r)h \leq b - mrh + (m-1)rh = b - rh,$$

hence

$$(\alpha + (k_1 + \cdots + k_r)h, \beta + (k_1 + \cdots + k_r)h) \subset (a, b - rh),$$

and similarly for each $h < 0$,

$$(\alpha + (k_1 + \cdots + k_r)h, \beta + (k_1 + \cdots + k_r)h) \subset (a - rh, b).$$

Therefore

$$\begin{aligned}
\omega_r(f, mt; \mathcal{M}^p_q(a, b)) &\leq \sup_{0<|h|\leq mt} \|\Delta^r_h f\|_{\mathcal{M}^p_q\left((a,b)\cap(a-rh,b-rh)\right)} \\
&= \sup_{0<|\eta|\leq t} \|\Delta^r_{m\eta} f\|_{\mathcal{M}^p_q\left((a,b)\cap(a-mr\eta,b-mr\eta)\right)} \\
&= \sup_{0<|\eta|\leq t} \sup_{(\alpha,\beta)\subset(a,b)\cap(a-mr\eta,b-mr\eta)} \big(\beta - \alpha\big)^{\frac{1}{p}-\frac{1}{q}} \|\Delta^r_{m\eta} f\|_{L_q(\alpha,\beta)} \\
&\leq m^r \sup_{(\gamma,\delta)\subset(a,b)\cap(a-r\eta,b-r\eta)} \big(\delta - \gamma\big)^{\frac{1}{p}-\frac{1}{q}} \left\|\Delta^r_h f\right\|_{L_q\left(\gamma,\delta\right)} \\
&= m^r \omega_r(f, t; \mathcal{M}^p_q(a, b)). \quad \square
\end{aligned}$$

Corollary 2.5. *Let $-\infty \leq a < b \leq \infty$ and $1 \leq q \leq p \leq \infty$, $r \in \mathbb{N}$, and $f \in \mathcal{M}^p_q(a, b)$. Then for any $m \geq 1$*

$$\omega_r(f, mt; \mathcal{M}^p_q(a, b)) \leq 2^r m^r \omega_r(f, t; \mathcal{M}^p_q(a, b)). \tag{2.7}$$

Proof. It suffices to note that $m \le [m]+1$, $\big([m]+1\big)^r \le 2^r m^r$, and to use the monotonicity of ω_r and inequality (2.6). $\square$

Corollary 2.6. *Let $-\infty \le a < b \le \infty$ and $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, and $f \in \mathcal{M}_q^p(a,b)$. Then for any $0 < \nu_1 < \nu_2 < \infty$, for all $0 < t \le 1$*

$$t^{-\nu_1 r}\omega_r(f, t^{\nu_1}; \mathcal{M}_q^p(a,b)) \le 2^r t^{-\nu_2 r} m^r \omega_r(f, t^{\nu_2}; \mathcal{M}_q^p(a,b)). \tag{2.8}$$

Proof. It suffices to note that using Corollary 2.5

$$\omega_r(f, t^{\nu_1}; \mathcal{M}_q^p(a,b)) = \omega_r(f, t^{\nu_1-\nu_2}t^{\nu_2}; \mathcal{M}_q^p(a,b)) \le 2^r t^{(\nu_1-\nu_2)r}\omega_r(f, t^{\nu_2}; \mathcal{M}_q^p(a,b)). \quad \square \tag{2.9}$$

Lemma 2.7. *Let $-\infty \le a < b \le \infty$ and $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, and let $f \in \mathcal{M}_q^p(a,b)$. Then*

$$\omega_r(f, t; \mathcal{M}_q^p(a,b)) \le 2^r \left\| f \right\|_{\mathcal{M}_q^p(a,b)}$$

for all $t > 0$.

Proof. Using (2.2), (2.4), (2.5) and the triangle inequality, we have

$$\begin{aligned}
&\omega_r(f, t; \mathcal{M}_q^p(a,b)) \\
&\le \max\left\{ \sup_{0\le h\le t} \sum_{k=0}^{r} \binom{r}{k} \|T(hk)f\|_{\mathcal{M}_q^p(a,b-rh)}, \sup_{-t\le h\le 0} \sum_{k=0}^{r} \binom{r}{k} \|T(hk)f\|_{\mathcal{M}_q^p(a-rh,b)} \right\} \\
&= \max\left\{ \sup_{0\le h\le t} \sum_{k=0}^{r} \binom{r}{k} \|f\|_{\mathcal{M}_q^p(a+kh,b-(r-k)h)}, \sup_{-t\le h\le 0} \sum_{k=0}^{r} \binom{r}{k} \|f\|_{\mathcal{M}_q^p(a-(r-k)h,b+kh)} \right\} \\
&\le 2^r \left\| f \right\|_{\mathcal{M}_q^p(a,b)}. \quad \square
\end{aligned}$$

Let $g \in \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$. For all $t \in \left(\frac{a-b}{r}, \frac{b-a}{r}\right)$, for almost all $x \in (a,b)$, such that $x + rt \in (a,b)$ we have

$$\Delta_t^r g(x) = \int_0^t \cdots \int_0^t g^{(r)}(x + t_1 + \cdots + t_r)dt_1 \cdots dt_r. \tag{2.10}$$

We will use (2.10) to prove the following fact:

Lemma 2.8. *Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$, $r \in \mathbb{N}$ and $f \in W^r\left(\mathcal{M}_q^p(a,b)\right)$. Then*

$$\omega_r(f, t; \mathcal{M}_q^p(a,b)) \le t^r \; \|f^{(r)}\|_{\mathcal{M}_q^p(a,b)}, \qquad t > 0.$$

Proof. Fix $h \in (0,t]$. Using (2.10) and the generalized Minkowski inequality (see [12, Lemma 3.3]), we obtain

$$\begin{aligned}
&\|\Delta_h^r f\|_{\mathcal{M}_q^p\left((a,b)\cap(a-rh,b-rh)\right)} \\
&= \left\| \int_0^h \cdots \int_0^h f^{(r)}(\cdot + t_1 + \cdots + t_r)dt_1 \cdots dt_r \right\|_{\mathcal{M}_q^p\left((a,b)\cap(a-rh,b-rh)\right)}
\end{aligned}$$

$$\le \int_0^h \cdots \int_0^h \left\| f^{(r)}(\cdot + t_1 + \cdots + t_r) \right\|_{\mathcal{M}_q^p\left((a,b)\cap(a-rh,b-rh)\right)} dt_1 \cdots dt_r$$
$$= \int_0^h \cdots \int_0^h \left\| f^{(r)} \right\|_{\mathcal{M}_q^p((a,b)\cap(a-rh+t_1+\cdots+t_r,b-rh+t_1+\cdots+t_r))} dt_1 \cdots dt_r$$
$$\le h^r \| f^{(r)} \|_{\mathcal{M}_q^p(a,b)}.$$

The proof for the case $h \in [-t, 0)$ is similar. □

Now we define the Peetre K-functionals related to the Morrey space $\mathcal{M}_q^p(a,b)$.

Definition 2.9. Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$ and $r \in \mathbb{N}$. Let $f \in \mathcal{M}_q^p(a,b)$. For $t > 0$, the Peetre K-functional with respect to the pair $\mathcal{M}_q^p(a,b)$ and $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ is defined as

$$K(f,t;\mathcal{M}_q^p(a,b), \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)) := \inf_{g\in \dot{W}^r(\mathcal{M}_q^p(a,b))} \left\{ \| f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g^{(r)}\|_{\mathcal{M}_q^p(a,b)} \right\};$$

and for pair of $\mathcal{M}_q^p(a,b)$ and $W^r\mathcal{M}_q^p(a,b)$ is defined as

$$K(f,t;\mathcal{M}_q^p(a,b), W^r(\mathcal{M}_q^p(a,b))) := \inf_{g\in W^r\mathcal{M}_q^p(a,b)} \left\{ \| f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g\|_{W^r(\mathcal{M}_q^p(a,b))} \right\}$$
$$= \inf_{g\in W^r\mathcal{M}_q^p(a,b)} \left\{ \| f-g\|_{\mathcal{M}_q^p(a,b)} + t\sum_{k=0}^r \|g^{(k)}\|_{\mathcal{M}_q^p(a,b)} \right\}.$$

In the proof of the next theorem, we need the *Steklov-type function* $S_{r,t}^{\pm} f$ for $f \in \mathcal{M}_q^p(a,b)$: Let $f \in \mathcal{M}_q^p(a,b)$ and $0 < t < \left(\frac{b-a}{r^2}\right)^r$. Set

$$S_{r,t}^{+}(f)(x) := \frac{1}{t}\int_{[0,\sqrt[r]{t}]^r} \Big[\sum_{l=1}^r (-1)^{l+1}\binom{r}{l} f\Big(x+l(h_1+\cdots+h_r)\Big)\Big] dh_1\cdots dh_r \tag{2.11}$$
$$= \frac{1}{t}\int_{[0,\sqrt[r]{t}]^r} \Delta_{h_1+\cdots+h_r}^r f(x) dh_1\cdots dh_r + f(x)$$

for $x \in (a, b - r^2\sqrt[r]{t})$ and

$$S_{r,t}^{-}(f)(x) := \frac{1}{t}\int_{[-\sqrt[r]{t},0]^r} \Big[\sum_{l=1}^r (-1)^{l+1}\binom{r}{l} f\Big(x+l(h_1+\cdots+h_r)\Big)\Big] dh_1\cdots dh_r \tag{2.12}$$
$$= \frac{1}{t}\int_{[-\sqrt[r]{t},0]^r} \Delta_{h_1+h_2+\cdots+h_r}^r f(x) dh_1\cdots dh_r + f(x)$$

for $x \in (a + r^2\sqrt[r]{t}, b)$.

Theorem 2.10. *Let $I \subset \mathbb{R}$ be an infinite interval, $r \in \mathbb{N}$ and $1 \le q \le p \le \infty$. Then there exist $c_1(r), c_2(r) > 0$ depending only on r such that for all $f \in \mathcal{M}_q^p(I) \cap \dot{W}^r\left(\mathcal{M}_q^p(I)\right)$ and for all $t > 0$, we have*

$$c_1(r)\,\omega_r\big(f,\sqrt[r]{t};\mathcal{M}_q^p(I)\big) \le K\left(f,t;\mathcal{M}_q^p(I), \dot{W}^r\left(\mathcal{M}_q^p(I)\right)\right) \le c_2(r)\,\omega_r\big(f,\sqrt[r]{t};\mathcal{M}_q^p(I)\big). \tag{2.13}$$

Proof. Let $g \in \dot{W}^r\left(\mathcal{M}^p_q(I)\right)$. Then $g^{(r)} \in \mathcal{M}^p_q(I)$. Therefore, thanks to Lemmas 2.7 and 2.8, (2.10) and the definition of $K(f, t; \mathcal{M}^p_q(I), \dot{W}^r\left(\mathcal{M}^p_q(I)\right))$, we obtain

$$\begin{aligned}\omega_r(f, \sqrt[r]{t}; \mathcal{M}^p_q(I)) &\leq \omega_r(f-g, \sqrt[r]{t}; \mathcal{M}^p_q(I)) + \omega_r(g, \sqrt[r]{t}; \mathcal{M}^p_q(I))\\ &\leq 2^r\|f-g\|_{\mathcal{M}^p_q(I)} + t\|g^{(r)}\|_{\mathcal{M}^p_q(I)}\\ &\leq 2^r(\|f-g\|_{\mathcal{M}^p_q(I)} + t\|g^{(r)}\|_{\mathcal{M}^p_q(I)}).\end{aligned}$$

Since $g \in \dot{W}^r(\mathcal{M}^p_q(I))$ is arbitrary, we have

$$c_1(r)\,\omega_r(f, \sqrt[r]{t}; \mathcal{M}^p_q(I)) \leq K\left(f, t; \mathcal{M}^p_q(I), \dot{W}^r\left(\mathcal{M}^p_q(I)\right)\right)$$

for any $f \in \mathcal{M}^p_q(I)$, where $c_1(r) := 2^{-r}$. Thus, the left-hand-side inequality in (2.13) is proved.

In order to prove the right-hand-side inequality, we consider two cases.

Case 1: Let $I = (a, \infty)$, and $a \geq -\infty$. By (2.11), applying the generalized Minkowski inequality for the Morrey spaces and Lemma 2.4, we get

$$\begin{aligned}\left\|S^+_{r,t}(f) - f\right\|_{\mathcal{M}^p_q(a,\infty)} &= \frac{1}{t}\left\|\int_{[0,\sqrt[r]{t}]^r} \Delta^r_{h_1+\cdots+h_r} f(\cdot)dh_1\cdots dh_r\right\|_{\mathcal{M}^p_q(a,\infty)}\\ &\leq \frac{1}{t}\int_{[0,\sqrt[r]{t}]^r}\left\|\Delta^r_{h_1+\cdots+h_r} f(\cdot)\right\|_{\mathcal{M}^p_q(a,\infty)} dh_1\cdots dh_r\\ &\leq \sup_{0\leq h\leq r\sqrt[r]{t}}\left\|\Delta^r_h f(\cdot)\right\|_{\mathcal{M}^p_q(a,\infty)}\frac{1}{t}\int_{[0,\sqrt[r]{t}]^r} dh_1\cdots dh_r\\ &\leq \omega_r(f, r\sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)) \qquad (2.14)\\ &\leq r^r\omega_r(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)). \qquad (2.15)\end{aligned}$$

Since for almost all $x \in (a, \infty)$

$$\frac{d}{dx}\Big(\int_0^t f(x+h_1)dh_1\Big) = \frac{d}{dx}\Big(\int_x^{x+t} f(u_1)du_1\Big) = f(x+t) - f(x) = \Delta_t f(x)$$

and for $r > 1$

$$\begin{aligned}&\frac{d^k}{dx^k}\Big(\int_{[0,\sqrt[r]{t}]^r} f(x+l(h_1+\cdots+h_r))dh_1\cdots dh_r\Big)\\ &= \frac{d^{k-1}}{dx^{k-1}}\frac{1}{l}\frac{d}{dx}\left(\int_x^{x+r\sqrt[r]{t}}\left(\int_{[0,\sqrt[r]{t}]^{r-1}} f(u_1+l(h_2+\cdots+h_r))dh_1\cdots dh_r\right)du_1\right)\\ &= \frac{1}{l}\frac{d^{k-1}}{dx^{k-1}}\left(\int_{[0,\sqrt[r]{t}]^{r-1}}\Delta_{r\sqrt[r]{t}} f(x+l(h_2+\cdots+h_r))dh_1\cdots dh_r\right),\end{aligned}$$

it follows that for any natural $k \leq r$

$$\begin{aligned}&\frac{d^k}{dx^k}\Big(\int_{[0,\sqrt[r]{t}]^r} f(x+l(h_1+\cdots+h_r))dh_1\cdots dh_r\Big)\\ &= \begin{cases}\dfrac{1}{l^k}\left(\displaystyle\int_{[0,\sqrt[r]{t}]^{r-k}}\Delta^k_{r\sqrt[r]{t}} f(x+l(h_{k+1}+\cdots+h_r))dh_{k+1}\cdots dh_r\right) & \text{if } k < r,\\ \dfrac{1}{l^r}\Delta^r_{r\sqrt[r]{t}} f(x) & \text{if } k = r.\end{cases} \qquad (2.16)\end{aligned}$$

By applying formula (2.16) with $k = r$, we get that for almost all $x \in (a, \infty)$

$$S^{+}_{r,t}(f)^{(r)}(x) = \frac{1}{t}\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l}\frac{d^r}{dx^r}\left(\int_{[0,\sqrt[r]{t}]^r} f(x + l(h_1 + \cdots + h_r))dh_1\cdots dh_r\right)$$
$$= \frac{1}{t}\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l} l^{-r}\Delta^r_{l\sqrt[r]{t}} f(x).$$

Hence by Minkowski's inequality and Lemma 2.4

$$\begin{aligned}
\left\|S^{+}_{r,t}(f)\right\|_{\dot{W}^r(\mathcal{M}^p_q(a,\infty))} &= \left\|S^{+}_{r,t}(f)^{(r)}\right\|_{\mathcal{M}^p_q(a,\infty)} \\
&\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-r}\left\|\Delta^r_{l\sqrt[r]{t}} f\right\|_{\mathcal{M}^p_q(a,\infty)} \\
&\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-r}\omega_r\left(f, l\sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)\right) \\
&\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)\right) \\
&= \frac{2^r - 1}{t}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)\right).
\end{aligned} \tag{2.17}$$

Since $S^{+}_{r,t}(f) \in \dot{W}^r\left(\mathcal{M}^p_q(a,\infty)\right)$,

$$\begin{aligned}
K\left(f, t; \mathcal{M}^p_q(a,\infty), \dot{W}^r\left(\mathcal{M}^p_q(a,\infty)\right)\right) &\le \left\|f - S^{+}_{r,t}(f)\right\|_{\mathcal{M}^p_q(a,\infty)} + t\left\|S^{+}_{r,t}(f)^{(r)}\right\|_{\mathcal{M}^p_q(a,\infty)} \\
&\le (r^r + 2^r - 1)\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)\right) \\
&\le 2r^r\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,\infty)\right)
\end{aligned}$$

and the right-hand side inequality in (2.13) follows with $c_2(r) = 2r^r$.

Case 2. If $I = (-\infty, b)$ for any $b \le \infty$, then the above arguments work if the operator $S^{+}_{r,t}(f)$ is replaced by the operator $S^{-}_{r,t}(f)$. ☐

Theorem 2.11. *Let $-\infty \le a < b \le \infty$, $r \in \mathbb{N}$ and $1 \le q \le p \le \infty$. Then there exist $c_3(r), c_4(r) > 0$ depending only on r such that for all $f \in W^r\left(\mathcal{M}^p_q(a,b)\right)$, we have*

$$c_3(r)\left[[t]_1\|f\|_{\mathcal{M}^p_q(a,b)} + \sum_{k=1}^{r}\omega_k\left(f, \sqrt[k]{t}; \mathcal{M}^p_q(a,b)\right)\right] \le K\left(f, t; \mathcal{M}^p_q(a,b), W^r\left(\mathcal{M}^p_q(a,b)\right)\right) \tag{2.18}$$

for all $t > 0$, where $[t]_1 := \min\{t, 1\}$; and

$$\begin{aligned}
&K\left(f, t; \mathcal{M}^p_q(a,b), W^r\left(\mathcal{M}^p_q(a,b)\right)\right) \\
&\le c_4(r)\left(1 + (b-a)^{-r}\right)\left[t\|f\|_{\mathcal{M}^p_q(a,b)} + \sum_{k=1}^{r}\omega_k\left(f, \sqrt[k]{t}; \mathcal{M}^p_q(a,b)\right)\right];
\end{aligned} \tag{2.19}$$

for all $0 < t \le \min\left\{\left(\frac{b-a}{3r^2}\right)^r, 1\right\}$.

Proof. Step 1: Let $g \in W^r\left(\mathcal{M}_q^p(a,b)\right)$. By Lemmas 2.7 and 2.8 for any $k = 1, 2, \dots, r$

$$\begin{aligned}\omega_k\big(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\big) &\le \omega_k\big(f-g, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\big) + \omega_k\big(g, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\big)\\ &\le 2^k \|f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g^{(k)}\|_{\mathcal{M}_q^p(a,b)}.\end{aligned}$$

Also,

$$[t]_1\|f\|_{\mathcal{M}_q^p(a,b)} \le [t]_1\|f-g\|_{\mathcal{M}_q^p(a,b)} + [t]_1\|g\|_{\mathcal{M}_q^p(a,b)} \le \|f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g\|_{\mathcal{M}_q^p(a,b)}.$$

So,

$$\begin{aligned}[t]_1\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \omega_k\big(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\big) &\le 2^{r+1}\|f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g\|_{W^r\mathcal{M}_q^p(a,b)}\\ &\le 2^{r+1}\left(\|f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g\|_{W^r\mathcal{M}_q^p(a,b)}\right).\end{aligned}$$

Since $g \in W^r\left(\mathcal{M}_q^p(a,b)\right)$ is arbitrary, we have

$$2^{-r-1}\left([t]_1\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \omega_k\big(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\big)\right) \le K\left(f, t; \mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right);$$

for all $t \ge 0$. Thus, inequality (2.18) is proved with $c_3(r) = 2^{-r-1}$.

Step 2: In order to prove (2.19) for the interval (a, ∞), $a \ge -\infty$, we use estimate (2.14):

$$\left\|S_{r,t}^+(f) - f\right\|_{\mathcal{M}_q^p(a,\infty)} \le r^r\omega_r(f, \sqrt[r]{t}; \mathcal{M}_q^p(a,\infty)). \tag{2.20}$$

and the analogous estimate for the interval $(-\infty, b)$, where $b \le \infty$:

$$\left\|S_{r,t}^-(f) - f\right\|_{\mathcal{M}_q^p(-\infty,b)} \le r^r\omega_r(f, \sqrt[r]{t}; \mathcal{M}_q^p(-\infty,b)). \tag{2.21}$$

In the case of a finite interval (a, b), following the arguments in the proof of Theorem 2.10, we get

$$\left\|S_{r,t}^+(f) - f\right\|_{\mathcal{M}_q^p\left(a,b-\frac{b-a}{3}\right)} \le \sup_{0<h\le r\sqrt[r]{t}}\left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p\left(a,b-\frac{b-a}{3}\right)} \le \sup_{0<h\le r\sqrt[r]{t}}\left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p(a,b-rh)}.$$

because $\left(a, b - \frac{b-a}{3}\right) \subset (a, b-rh)$ for any $0 < t \le (\frac{b-a}{3r^2})^r$. Hence by Lemma 2.4

$$\left\|S_{r,t}^+(f) - f\right\|_{\mathcal{M}_q^p\left(a,b-\frac{b-a}{3}\right)} \le \omega_r\left(f, r\sqrt[r]{t}; \mathcal{M}_q^p\,(a,b)\right) \le r^r\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}_q^p\,(a,b)\right). \tag{2.22}$$

Analogously

$$\left\|S_{r,t}^-(f) - f\right\|_{\mathcal{M}_q^p\left(a+\frac{b-a}{3},b\right)} \le r^r\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}_q^p\,(a,b)\right). \tag{2.23}$$

Step 3: If the interval (a, b) is infinite, similarly to the proof of Theorem 2.10 (see formula (2.17))

$$\left\|S_{r,t}^+(f)^{(r)}\right\|_{\mathcal{M}_q^p(a,\infty)} \le \frac{2^r}{t}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}_q^p\,(a,\infty)\right) \tag{2.24}$$

for any $a \geq -\infty$ and

$$\left\| S^{-}_{r,t}(f)^{(r)} \right\|_{\mathcal{M}^p_q(-\infty,b)} \leq \frac{2^r}{t}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(-\infty,b)\right) \tag{2.25}$$

for any $b \leq \infty$.

If the interval (a,b) is finite, then by the arguments used in the proof of formula (2.17)

$$\begin{aligned}\left\| S^{+}_{r,t}(f)^{(r)} \right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})} &\leq \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-r}\left\| \Delta^r_{l\sqrt[r]{t}} f\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})}\\ &\leq \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-r} \sup_{0<h\leq r\sqrt[r]{t}} \|\Delta^r_h f\|_{\mathcal{M}^p_q(a,b-rh)}.\end{aligned}$$

Hence

$$\left\| S^{+}_{r,t}(f)^{(r)} \right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})} \leq \frac{2^r}{t}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,b)\right). \tag{2.26}$$

Analogously,

$$\left\| S^{-}_{r,t}(f)^{(r)} \right\|_{\mathcal{M}^p_q(a+\frac{b-a}{3},b)} \leq \frac{2^r}{t}\omega_r\left(f, \sqrt[r]{t}; \mathcal{M}^p_q(a,b)\right). \tag{2.27}$$

Step 4: Consider the interval (a,∞) where $a \geq -\infty$. First of all, by (2.11) we have

$$\begin{aligned}\left\| S^{+}_{r,t}(f) \right\|_{\mathcal{M}^p_q(a,\infty)} &\leq \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\left\| \int_{[0,\sqrt[r]{t}]^r} f\Big(\cdot + l(h_1+\cdots+h_r)\Big)dh_1\cdots dh_r \right\|_{\mathcal{M}^p_q(a,\infty)}\\ &\leq \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\int_{[0,\sqrt[r]{t}]^r}\left\| f\Big(\cdot + l(h_1+\cdots+h_r)\Big)\right\|_{\mathcal{M}^p_q(a,\infty)} dh_1\cdots dh_r\\ &\leq \sum_{l=1}^{r}\binom{r}{l}\|f\|_{\mathcal{M}^p_q(a,\infty)} = 2^r\|f\|_{\mathcal{M}^p_q(a,\infty)} \leq \frac{2^r}{t}[t]_1\|f\|_{\mathcal{M}^p_q(a,\infty)}.\end{aligned} \tag{2.28}$$

Let next $k \in \mathbb{N}$, $1 \leq k \leq r-1$. By applying formula (2.16), we get

$$\begin{aligned}&S^{+}_{r,t}(f)^{(k)}(x)\\ &= \frac{1}{t}\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l}\frac{d^k}{dx^k}\left(\int_{[0,\sqrt[r]{t}]^r} f(x+l(h_1+\cdots+h_r))dh_1\cdots dh_r\right)\\ &= \frac{1}{t}\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l} l^{-k}\int_{[0,\sqrt[r]{t}]^{r-k}} \Delta^r_{l\sqrt[r]{t}} f(x+l(h_{k+1}+\cdots+h_r))dh_{k+1}\cdots dh_r.\end{aligned}$$

Hence, by Minkowski's inequalities for sums and integrals

$$\begin{aligned}&\left\| S^{+}_{r,t}(f)^{(k)} \right\|_{\mathcal{M}^p_q(a,\infty)}\\ &\leq \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-k}\left\| \frac{d^k}{dx^k}\left(\int_{[0,\sqrt[r]{t}]^r} f(\cdot+l(h_1+\cdots+h_r))dh_1\cdots dh_r\right)\right\|_{\mathcal{M}^p_q(a,\infty)}\\ &= \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-k}\int_{[0,\sqrt[r]{t}]^{r-k}}\left\| \Delta^k_{l\sqrt[r]{t}} f(\cdot+l(h_{k+1}+\cdots+h_r))\right\|_{\mathcal{M}^p_q(a,\infty)} dh_{k+1}\cdots dh_r\end{aligned}$$

If we invoke Lemma 2.4, then we have

$$\begin{aligned}\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,\infty)} &\le \frac{(\sqrt[r]{t})^{r-k}}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-k}\left\|\Delta^k_{l\sqrt[r]{t}}f\right\|_{\mathcal{M}^p_q(a,\infty)}\\ &\le \frac{(\sqrt[r]{t})^{r-k}}{t}\sum_{l=1}^{r}\binom{r}{l} l^{-k}\omega_k(f, l\sqrt[r]{t};\mathcal{M}^p_q(a,\infty))\\ &\le \frac{(\sqrt[r]{t})^{r-k}}{t}\sum_{l=1}^{r}\binom{r}{l}\omega_k(f, \sqrt[r]{t};\mathcal{M}^p_q(a,\infty)).\end{aligned}$$

By Corollary 2.5 with r replaced by k and $m = t^{\frac{1}{r}-\frac{1}{k}}$

$$\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,\infty)} \le \frac{2^r}{t}(\sqrt[r]{t})^{r-k}\omega_k(f, \sqrt[r]{t};\mathcal{M}^p_q(a,\infty)) \le \frac{4^r}{t}\omega_k(f, \sqrt[k]{t};\mathcal{M}^p_q(a,\infty)). \tag{2.29}$$

Analogously,

$$\left\|S^-_{r,t}(f)\right\|_{\mathcal{M}^p_q(-\infty,b)} \le \frac{2^r}{t}[t]_1\,\|f\|_{\mathcal{M}^p_q(-\infty,b)} \tag{2.30}$$

and

$$\left\|S^-_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(-\infty,b)} \le \frac{4^r}{t}\omega_k(f, \sqrt[k]{t};\mathcal{M}^p_q(-\infty,b)) \tag{2.31}$$

for $b \le \infty$, $k \in \mathbb{N}$, $1 \le k \le r-1$.

Step 5: Next consider the case of a finite interval (a,b). First of all

$$\begin{aligned}&\left\|S^+_{r,t}(f)\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})}\\ &\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\int_{[0,\sqrt[r]{t}]^r}\|f(\cdot+l(h_1+\cdots+h_r))\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})}\,dh_1\cdots dh_r\\ &\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\int_{[0,\sqrt[r]{t}]^r}\|f\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}+l(h_1+\cdots+h_r)\right)}\,dh_1\cdots dh_r\\ &\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\int_{[0,\sqrt[r]{t}]^r}\|f\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}+l(h_1+\cdots+h_r)\right)}\,dh_1\cdots dh_r\\ &\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\int_{[0,\sqrt[r]{t}]^r}\|f\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}+r^2\sqrt[r]{t}\right)}\,dh_1\cdots dh_r\\ &\le 2^r\,\|f\|_{\mathcal{M}^p_q(a,b)}\,;\end{aligned}$$

since $t \le \left(\frac{b-a}{3r^2}\right)^r$.

Similarly to the above for any $k = 1, 2, \ldots, r-1$, we get applying inequality (2.2)

$$\begin{aligned}&\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})}\\ &\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\frac{1}{l^k}\int_{[0,\sqrt[r]{t}]^{r-k}}\left\|\Delta^k_{r\sqrt[r]{t}}f(\cdot+l(h_{k+1}+\cdots+h_r))\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})}\,dh_{k+1}\cdots dh_r\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\frac{1}{l^k}\int_{[0,\sqrt[r]{t}]^{r-k}}\left\|\Delta^k_{r\sqrt[r]{t}}f\right\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}+l(h_{k+1}+\cdots+h_r)\right)}dh_{k+1}\cdots dh_r\\
&\le \frac{1}{t}\sum_{l=1}^{r}\binom{r}{l}\frac{1}{l^k}\int_{[0,\sqrt[r]{t}]^{r-k}}\left\|\Delta^k_{r\sqrt[r]{t}}f\right\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}+l(r-k)\sqrt[r]{t}\right)}dh_{k+1}\cdots dh_r\\
&\le \frac{(\sqrt[r]{t})^{r-k}}{t}\sum_{l=1}^{r}\binom{r}{l}\frac{1}{l^k}\left\|\Delta^k_{r\sqrt[r]{t}}f\right\|_{\mathcal{M}^p_q(a,b-l\sqrt[r]{t})}
\end{aligned}
$$

since $(a,b-\frac{b-a}{3}+l(r-k)\sqrt[r]{t})\subset(a,b-l\sqrt[r]{t})$. Hence, by Lemma 2.4 and Corollary 2.5

$$
\begin{aligned}
\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})} &\le \frac{(\sqrt[r]{t})^{r-k}}{t}\sum_{l=1}^{r}\binom{r}{l}\frac{1}{l^k}\omega_k\left(f,l\sqrt[r]{t};\mathcal{M}^p_q(a,b)\right)\\
&\le \frac{(\sqrt[r]{t})^{r-k}}{t}\left(\sum_{l=1}^{r}\binom{r}{l}\right)\omega_k\left(f,\sqrt[r]{t};\mathcal{M}^p_q(a,b)\right)\\
&\le \frac{2^r}{t}(\sqrt[r]{t})^{r-k}\omega_k\left(f,\sqrt[r]{t};\mathcal{M}^p_q(a,b)\right)\\
&\le \frac{4^r}{t}(\sqrt[r]{t})^{r-k}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}^p_q(a,b)\right).
\end{aligned}
\tag{2.32}
$$

Analogously, for any $k=1,2,\ldots,r-1$

$$
\left\|S^-_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,b-\frac{b-a}{3})} \le \frac{4^r}{t}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}^p_q(a,b)\right). \tag{2.33}
$$

Step 6: We deal with the interval (a,∞) where $a\ge-\infty$. Since $S^+_{r,t}(f)\in W^r\left(\mathcal{M}^p_q(a,\infty)\right)$, by inequalities (2.20), (2.24), (2.28) and (2.29), we get

$$
\begin{aligned}
&K\left(f,t;\mathcal{M}^p_q(a,\infty),W^r(\mathcal{M}^p_q(a,\infty))\right)\le\left\|f-S^+_{r,t}(f)\right\|_{\mathcal{M}^p_q(a,\infty)}+t\sum_{k=0}^{r}\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,\infty)}\\
&\le r^r\omega_r\left(f,\sqrt[r]{t};\mathcal{M}^p_q(a,\infty)\right)+2^r\left([t]_1\|f\|_{\mathcal{M}^p_q(a,\infty)}+\sum_{k=1}^{r}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}^p_q(a,\infty)\right)\right)\\
&\le\left(r^r+2^r\right)\left([t]_1\|f\|_{\mathcal{M}^p_q(a,\infty)}+\sum_{k=1}^{r}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}^p_q(a,\infty)\right)\right).
\end{aligned}
$$

Analogously, by inequalities (2.21), (2.25), (2.30) and (2.31), we get

$$
\begin{aligned}
&K\left(f,t;\mathcal{M}^p_q(-\infty,b),W^r(\mathcal{M}^p_q(-\infty,b))\right)\\
&\le\left(r^r+2^r\right)\left([t]_1\|f\|_{\mathcal{M}^p_q(-\infty,b)}+\sum_{k=1}^{r}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}^p_q(-\infty,b)\right)\right).
\end{aligned}
$$

which completes the proof of inequality (2.19) in the case of an infinite interval (a,b).

Step 7: Next we consider the case of a finite interval (a,b). Let

$$
\varphi(x)=\begin{cases}1 & \text{if } \ 0\le x\le\dfrac{1}{3},\\[2mm] 1-(3x-1)^{r+1} & \text{if } \ \dfrac{1}{3}<x<\dfrac{2}{3},\\[2mm] 0 & \text{if } \ \dfrac{2}{3}\le x\le 1.\end{cases}
$$

and $\psi_{a,b}(x)=\varphi\left(\frac{x-a}{b-a}\right)$. Then

$$0\le\psi_{a,b}(x)\le 1\quad\text{for}\quad a\le x\le b,\quad \psi_{a,b}(x)=1\quad\text{on}\quad [a,a+\frac{b-a}{3}],$$
$$\psi_{a,b}(x)=0\quad\text{on}\quad [b-\frac{b-a}{3},b],$$

and

$$\|\psi_{a,b}^{(k)}\|_{L_\infty(a,b)}\le\frac{c(r)}{(b-a)^k},\qquad k=0,1,\dots,r, \tag{2.34}$$

where $c(r)=3^r(r+1)!$.

Let for any $x\in(a,b)$,

$$S_{r,t}(f)(x)=\begin{cases} S^+_{r,t}(f)(x) & \text{if } a<x\le a+\dfrac{b-a}{3},\\ \psi_{a,b}(x)S^+_{r,t}(f)(x)+(1-\psi_{a,b}(x))S^-_{r,t}(f)(x) & \text{if }\ a+\dfrac{b-a}{3}<x<b-\dfrac{b-a}{3},\\ S^-_{r,t}(f)(x) & \text{if }\ b-\dfrac{b-a}{3}\le x<b. \end{cases}$$

By Lemma 2.1, (2.22) and (2.23), we have

$$\begin{aligned}\left\|S_{r,t}(f)-f\right\|_{\mathcal{M}^p_q(a,b)}&=\left\|\psi_{a,b}\left(S^+_{r,t}(f)-f\right)+(1-\psi_{a,b})\left(S^-_{r,t}(f)-f\right)\right\|_{\mathcal{M}^p_q(a,b)}\\ &\le\left\|S^+_{r,t}(f)-f\right\|_{\mathcal{M}^p_q\left(a,b-\frac{b-a}{3}\right)}+\left\|S^-_{r,t}(f)-f\right\|_{\mathcal{M}^p_q\left(a+\frac{b-a}{3},b\right)}\\ &\le 2r^r\omega_r\left(f,\sqrt[r]{t};\mathcal{M}^p_q(a,b)\right)\end{aligned} \tag{2.35}$$

Step 8: Next, by applying Lemmas 2.1 and 2.2, the Leibniz formula, (2.26), (2.27), (2.32)–(2.34), we get that for any $k=0,1,\dots,r$

$$\begin{aligned}&\left\|S_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,b)}=\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,a+\frac{b-a}{3})}\\ &+\left\|\left(\psi_{a,b}(x)S^+_{r,t}(f)+(1-\psi_{a,b})S^-_{r,t}(f)\right)^{(k)}\right\|_{\mathcal{M}^p_q(a+\frac{b-a}{3},b-\frac{b-a}{3})}+\left\|S^-_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(b-\frac{b-a}{3},b)}\\ &\le\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a,a+\frac{b-a}{3})}+\sum_{l=0}^k\binom{k}{l}\left\|S^+_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a+\frac{b-a}{3},b-\frac{b-a}{3})}\|\psi_{a,b}^{(k-l)}\|_{L_\infty(a,b)}\\ &+\sum_{l=0}^k\binom{k}{l}\left\|S^-_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(a+\frac{b-a}{3},b-\frac{b-a}{3})}\|\left(1-\psi_{a,b}\right)^{(k-l)}\|_{L_\infty(a,b)}+\left\|S^-_{r,t}(f)^{(k)}\right\|_{\mathcal{M}^p_q(b-\frac{b-a}{3},b)}\\ &\le 2^{k+1}\max_{0\le m\le k}\|\psi_{a,b}^{(m)}\|_{L_\infty(a,b)}\left(\sum_{l=0}^k\left\|S^+_{r,t}(f)^{(l)}\right\|_{\mathcal{M}^p_q(a,a+\frac{b-a}{3})}+\sum_{l=0}^k\left\|S^-_{r,t}(f)^{(l)}\right\|_{\mathcal{M}^p_q(a+\frac{b-a}{3},b)}\right)\\ &\le\frac{2^{k+2+r}}{t}\left(c(r)\left(1+(b-a)^{-r}\right)\right)\left[[t]_1\|f\|_{\mathcal{M}^p_q(a,b)}+\sum_{l=1}^k\omega_l\left(f,\sqrt[l]{t};\mathcal{M}^p_q(a,b)\right)\right].\end{aligned}$$

Hence

$$\begin{aligned}&\left\|S_{r,t}(f)\right\|_{W^r(\mathcal{M}^p_q(a,b))}\\ &\quad\le\frac{rc(r)2^{2r+2}}{t}\left(\left(1+(b-a)^{-r}\right)\right)\left[[t]_1\|f\|_{\mathcal{M}^p_q(a,b)}+\sum_{l=1}^k\omega_l\left(f,\sqrt[l]{t};\mathcal{M}^p_q(a,b)\right)\right],\end{aligned} \tag{2.36}$$

Since $S_{r,t}(f) \in W^r(\mathcal{M}_q^p(a,b))$ by inequalities (2.35) and (2.36),

$$\begin{aligned}
&K_r(f,t;\mathcal{M}_q^p(a,b), W^r(\mathcal{M}_q^p(a,b)))\\
&\le \left\| f - S_{r,t}(f)\right\|_{\mathcal{M}_q^p(a,b)} + t\left\| S_{r,t}(f)\right\|_{W^r(\mathcal{M}_q^p(a,b))}\\
&\le \tilde{c}(r)\left(\left(1+(b-a)^{-r}\right)\right)\left[[t]_1 \|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^{r} \omega_k\left(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\right)\right]
\end{aligned}$$

where $\tilde{c}(r) = 2r^r + rc(r)4^r$, which completes the proof of inequality (2.19). □

Remark 2.12. Note that if $t > (\frac{b-a}{r})^r$, then

$$\omega_r\big(f, \sqrt[r]{t}; \mathcal{M}_q^p(a,b)\big) = \omega_r\left(f, \frac{b-a}{r}; \mathcal{M}_q^p(a,b)\right),$$

and for all $t > 0$

$$K_r(f,t;\mathcal{M}_q^p(a,b), \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)) \le \|f\|_{\mathcal{M}_q^p(a,b)}.$$

3. Interpolation theorems

Recall that, given two compatible Banach spaces X_0, X_1, $0<\theta<1$ and $1 \le s \le \infty$, the interpolation space $(X_0, X_1)_{\theta,s}$ is defined as the set of all $f \in X_0 + X_1$, for which

$$\|f\|_{(X_0,X_1)_{\theta,s}} = \left\{ \int_0^\infty \left(t^{-\theta} K(f,t;X_0,X_1)\right)^s \frac{dt}{t}\right\}^{\frac{1}{s}} < \infty$$

(with standard modification for $s=\infty$).

For $\lambda > 0$, $r \in \mathbb{N}$, $r > \lambda$, $1 \le s \le \infty$, $1 \le q \le p \le \infty$ the homogeneous Nikol'skii–Besov–Morrey spaces $\dot{B}_s^{\lambda,r}\left(\mathcal{M}_q^p(a,b)\right)$ are defined as the spaces of all measurable functions f defined on (a,b) for which

$$\|f\|_{\dot{B}_s^{\lambda,r}(\mathcal{M}_q^p(a,b))} = \left\{ \int_0^\infty \left(t^{-\lambda}\omega_r(f,t;\mathcal{M}_q^p(a,b))\right)^s \frac{dt}{t}\right\}^{\frac{1}{s}} < \infty.$$

Theorem 2.10 immediately implies the following result.

Theorem 3.1. *Let $0<\theta<1$, $1 \le q \le p \le \infty$, $r \in \mathbb{N}$ and $0 < s \le \infty$. Assume that $(a,b) \subset \mathbb{R}$ is an infinite interval. Then*

$$\left(\mathcal{M}_q^p(a,b), \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s} = \dot{B}_s^{\theta r,r}(\mathcal{M}_q^p(a,b)).$$

Moreover, there exist $c_5(r), c_6(r) > 0$ depending only on r such that

$$c_5(r)\|f\|_{\dot{B}_s^{\theta r,r}(\mathcal{M}_q^p(a,b))} \le \|f\|_{\left(\mathcal{M}_q^p(a,b),\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}} \le c_6(r)\|f\|_{\dot{B}_s^{\theta r,r}(\mathcal{M}_q^p(a,b))}$$

for all $f \in \mathcal{M}_q^p(a,b) \cap \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$.

Proof. It suffices to apply inequality (2.13) and change the variable: $\sqrt[r]{t} = \tau$. □

In its turn Theorem 2.11 implies the following result.

Theorem 3.2. *Let $0 < \theta < 1$, $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, $0 < s \le \infty$ and $(a, b) \subset \mathbb{R}$. Then*

$$\left(\mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s} = \mathcal{M}_q^p(a,b) \bigcap \left(\cap_{k=1}^r \dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))\right).$$

Moreover, there exist $c_7(r), c_8(r) > 0$ depending only on r such that

$$\begin{aligned} & c_7(r)\left\{s^{-\frac{1}{s}}\left(\theta(1-\theta)\right)^{\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right\} \\ & \le \|f\|_{\left(\mathcal{M}_q^p(a,b), \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}} \\ & \le c_8(r)\left(\left(1+(b-a)^{-r}\right)\right)\left\{s^{-\frac{1}{s}}\left(\theta(1-\theta)\right)^{\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right\} \end{aligned}$$

for all $f \in \mathcal{M}_q^p(a,b) \bigcap \cap_{k=1}^r B_s^{\theta k,k}\left(\mathcal{M}_q^p(a,b)\right)$.

Proof. Step 1: By inequality (2.18), we get

$$\begin{aligned} \|f\|_{\left(\mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}} & \ge c_3(r)\left\{\int_0^\infty \left(t^{-\theta}[t]_1\|f\|_{\mathcal{M}_q^p(a,b)}\right)^s \frac{dt}{t}\right\}^{\frac{1}{s}} \\ & = c_3(r)\left\{\int_0^1 t^{(1-\theta)s-1}dt + \int_1^\infty t^{-\theta s-1}dt\right\}^{\frac{1}{s}} \|f\|_{\mathcal{M}_q^p(a,b)} \\ & = c_3(r)s^{-\frac{1}{s}}(\theta(1-\theta))^{-\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)} \end{aligned}$$

and for $k = 1, 2, \ldots, r$

$$\begin{aligned} \|f\|_{\left(\mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}} & \ge c_3(r)\left\{\int_0^\infty \left(t^{-\theta}\omega_k(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b))\right)^s \frac{dt}{t}\right\}^{\frac{1}{s}} \\ & = c_3(r)k^{\frac{1}{s}}\left(\int_0^\infty \left(\tau^{-\theta k}\omega_k(f, \tau; \mathcal{M}_q^p(a,b))\right)^s \frac{d\tau}{\tau}\right) \\ & \ge c_3(r)\|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}. \end{aligned}$$

Hence

$$\|f\|_{\left(\mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}} \ge \frac{c_3(r)}{r+1}\left(s^{-\frac{1}{s}}(\theta(1-\theta))^{-\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right).$$

Thus,

$$c_7(r) = \frac{c_3(r)}{r+1}$$

does the job.

Step 2: Next, let $\mu = \min\{(\frac{b-a}{3r^2})^r, 1\}$. Since

$$K(f, t; \mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)) \le \|f\|_{\mathcal{M}_q^p(a,b)},$$

and

$$\mu^{-\theta} = \max\left\{(3r^2)^{\theta r}(b-a)^{-\theta r}, 1\right\} \le (3r^2)^r \max\left\{(b-a)^{-\theta r}, 1\right\} \le 3^r r^{2r}\left(1+(b-a)^{-r}\right),$$

it follows from Remark 2.12 that

$$\left\{\int_{\mu}^{\infty}\left(t^{-\theta}K\left(f,t;\mathcal{M}_q^p(a,b),W^r\left(\mathcal{M}_q^p(a,b)\right)\right)\right)^s\frac{dt}{t}\right\}^{\frac{1}{s}}\le(\theta s)^{-\frac{1}{s}}\mu^{-\theta}\|f\|_{\mathcal{M}_q^p(a,b)}$$
$$\le 3^r r^{2r}(\theta s)^{-\frac{1}{s}}\left(1+(b-a)^{-r}\right)\|f\|_{\mathcal{M}_q^p(a,b)}.$$

From Theorem 2.11 we deduce

$$\begin{aligned}
&\left\{\int_0^{\mu}\left(t^{-\theta}K\left(f,t;\mathcal{M}_q^p(a,b),W^r\left(\mathcal{M}_q^p(a,b)\right)\right)\right)^s\frac{dt}{t}\right\}^{\frac{1}{s}}\\
&\le c_4(r)\left(1+(b-a)^{-r}\right)\left\{\int_0^{\mu}\left[t^{-\theta}\left([t]_1\|f\|_{\mathcal{M}_q^p(a,b)}+\sum_{k=1}^r\omega_k\left(f,\sqrt[k]{t};\mathcal{M}_q^p(a,b)\right)\right)\right]^s\frac{dt}{t}\right\}^{\frac{1}{s}}\\
&\le c_4(r)\left(1+(b-a)^{-r}\right)\left\{\int_0^{\mu}\left(t^{-\theta}[t]_1\|f\|_{\mathcal{M}_q^p(a,b)}\right)^s\frac{dt}{t}\right\}^{\frac{1}{s}}\\
&\quad+c_4(r)\left(1+(b-a)^{-r}\right)\sum_{k=1}^r\left\{\int_0^{\mu}\left(t^{-\theta}\omega_k\left(f,\sqrt[k]{t};\mathcal{M}_q^p(a,b)\right)\right)^s\frac{dt}{t}\right\}^{\frac{1}{s}}\\
&\le c_4(r)\left(1+(b-a)^{-r}\right)\\
&\quad\times\left[((1-\theta)s)^{-\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)}+\sum_{k=1}^r k^{\frac{1}{s}}\left\{\int_0^{\infty}\left(\tau^{-\theta k}\omega_k\left(f,\tau;\mathcal{M}_q^p(a,b)\right)\right)^s\frac{d\tau}{\tau}\right\}^{\frac{1}{s}}\right]\\
&\le c_4(r)r\left(1+(b-a)^{-r}\right)\left[((1-\theta)s)^{-\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)}+\sum_{k=1}^r\|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right].
\end{aligned}$$

By the two above inequalities and the obvious inequality

$$\theta^{-\frac{1}{s}}+(1-\theta)^{-\frac{1}{s}}\le 2(\theta^{-1}+(1-\theta)^{-1})^{\frac{1}{s}}=2(\theta(1-\theta))^{-\frac{1}{s}}$$

it follows that

$$\begin{aligned}
&\|f\|_{\left(\mathcal{M}_q^p(a,b),W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}}\\
&\le 2\max\{3^r r^{2r},c_4(r)r\}\left(s^{-\frac{1}{s}}(\theta(1-\theta))^{-\frac{1}{s}}\|f\|_{\mathcal{M}_q^p(a,b)}+\sum_{k=1}^r\|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right).
\end{aligned}$$

Thus,

$$c_8(r)=2\max\{3^r r^{2r},c_4(r)r\}$$

does the job. □

Here we survey the Nikol'skii–Besov–Morrey spaces. In 1984, Netrusov [25] defined the Nikol'skii–Besov–Morrey spaces and obtained some embedding results. In surveys [37,38] Sickel gave a detailed discussion of the real interpolation of the Morrey spaces and Nikol'skii–Besov–Morrey spaces. Further results on interpolation of the Nikol'skii–Besov–Morrey spaces can be found in [43,44]. In 1994, Kozono and Yamazaki [18] shed light on the Besov–Morrey spaces from the point of view of differential equations. In [18] they used the Morrey space $\mathcal{M}_q^p$ to investigate the Cauchy problem for the Navier–Stokes equation. In 2005 Najafov considered the Nikol'skii–Besov–Morrey spaces with dominant mixed derivatives in [23]. Motivated by this, Tang and Xu [39] defined the non-homogeneous Triebel–Lizorkin–Morrey spaces, or equivalently, the non-homogeneous Morrey type Triebel–Lizorkin spaces in words of the paper [39]. Recently this type of function spaces is called a smoothness Morrey spaces. Sawano and Wang obtained the trace theorem independently in [32, Theorem 1.1] and [41, Proposition 1.10], respectively. The wavelet characterization of the smoothness Morrey spaces can be found in [27,30]. Triebel–Lizorkin–Morrey spaces cover Hardy–Morrey spaces; see [31, Theorem 4.2]. We refer to [15–17,34,35,39] for embedding relations of these function spaces. See also [2,24] for more about the generalized Nikol'skii–Besov–Morrey spaces.

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