



Full Length Article

# On weighted approximation with Jacobi weights

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Received 2 January 2018; received in revised form 27 June 2018; accepted 4 September 2018

Available online 11 September 2018

Communicated by András Kroó

## Abstract

We obtain matching direct and inverse theorems for the degree of weighted  $L_p$ -approximation by polynomials with the Jacobi weights  $(1 - x)^\alpha(1 + x)^\beta$ . Combined, the estimates yield a constructive characterization of various smoothness classes of functions via the degree of their approximation by algebraic polynomials. In addition, we prove Whitney type inequalities which are of independent interest. © 2018 Elsevier Inc. All rights reserved.

*MSC:* 41Axx

*Keywords:* Approximation by polynomials in weighted  $L_p$ -norms; Degree of approximation; Direct and inverse theorems; Jacobi weights; Moduli of smoothness; Characterization of smoothness classes; Whitney-type estimates

## 1. Introduction and main results

In this paper, we are interested in weighted polynomial approximation with the Jacobi weights

$$w_{\alpha,\beta}(x) := (1 - x)^\alpha(1 + x)^\beta, \quad \alpha, \beta \in J_p := \begin{cases} (-1/p, \infty), & \text{if } 0 < p < \infty, \\ [0, \infty), & \text{if } p = \infty. \end{cases}$$

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Let  $L_p^{\alpha,\beta}(I) := \left\{ f \mid \|w_{\alpha,\beta} f\|_{L_p(I)} < \infty \right\}$ , where  $\|\cdot\|_{L_p(I)}$  is the usual  $L_p$  (quasi)norm on the interval  $I \subseteq [-1, 1]$ , and, for  $f \in L_p^{\alpha,\beta}(I)$ , denote by

$$E_n(f, I)_{\alpha,\beta,p} := \inf_{p_n \in \mathbb{P}_n} \|w_{\alpha,\beta}(f - p_n)\|_{L_p(I)}$$

the error of best weighted approximation of  $f$  by polynomials in  $\mathbb{P}_n$ , the set of algebraic polynomials of degree not more than  $n - 1$ . For  $I = [-1, 1]$ , we denote  $\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$ ,  $L_p^{\alpha,\beta} := L_p^{\alpha,\beta}([-1, 1])$ ,  $E_n(f)_{\alpha,\beta,p} := E_n(f, [-1, 1])_{\alpha,\beta,p}$ , etc.

**Definition 1.1** ([10]). For  $r \in \mathbb{N}_0$  and  $0 < p \leq \infty$ , denote  $\mathbb{B}_p^0(w_{\alpha,\beta}) := L_p^{\alpha,\beta}$  and

$$\mathbb{B}_p^r(w_{\alpha,\beta}) := \left\{ f \mid f^{(r-1)} \in AC_{loc}(-1, 1) \text{ and } \varphi^r f^{(r)} \in L_p^{\alpha,\beta} \right\}, \quad r \geq 1,$$

where  $\varphi(x) := \sqrt{1 - x^2}$  and  $AC_{loc}(-1, 1)$  denotes the set of functions which are locally absolutely continuous in  $(-1, 1)$ .

We remark that, in the case  $p < 1$ , our definition of derivatives is understood in the classical sense, i.e., the assumption  $f^{(r-1)} \in AC_{loc}(-1, 1)$  in the case  $r \geq 2$  is understood in the sense that  $f$  is the  $(r - 1)$ st integral of a locally absolutely continuous  $f^{(r-1)}$  plus a polynomial of degree  $r - 2$ .

As is common when dealing with  $L_p$  spaces, we will not distinguish between a function in  $\mathbb{B}_p^r(w_{\alpha,\beta})$  and all functions which are equivalent to it in  $L_p^{\alpha,\beta}$ .

**Definition 1.2** ([10]). For  $k, r \in \mathbb{N}$  and  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ ,  $0 < p \leq \infty$ , define

$$\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} := \sup_{0 \leq h \leq t} \left\| W_{kh}^{r/2+\alpha,r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_p, \tag{1.1}$$

where

$$W_\delta^{\xi,\zeta}(x) := (1 - x - \delta\varphi(x)/2)^\xi (1 + x - \delta\varphi(x)/2)^\zeta,$$

and

$$\Delta_h^k(f, x) := \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - \frac{kh}{2} + ih), & \text{if } [x - \frac{kh}{2}, x + \frac{kh}{2}] \subseteq [-1, 1], \\ 0, & \text{otherwise,} \end{cases}$$

is the  $k$ th symmetric difference.

For  $\delta > 0$ , denote (see [9])

$$\mathfrak{D}_\delta := \{x \mid 1 - \delta\varphi(x)/2 \geq |x|\} \setminus \{\pm 1\} = [-1 + \mu(\delta), 1 - \mu(\delta)],$$

where

$$\mu(\delta) := 2\delta^2/(4 + \delta^2).$$

We note that  $\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2}$  if  $\delta_2 < \delta_1 \leq 2$ , and that  $\mathfrak{D}_\delta = \emptyset$  if  $\delta > 2$ . Also, since  $\Delta_{h\varphi(x)}^k(f, x) = 0$  if  $x \notin \mathfrak{D}_{kh}$ ,

$$\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} = \sup_{0 < h \leq t} \left\| W_{kh}^{r/2+\alpha,r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_{L_p(\mathfrak{D}_{kh})}. \tag{1.2}$$

In particular,  $\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} = \omega_{k,r}^\varphi(f^{(r)}, 2/k)_{\alpha,\beta,p}$ , for all  $t \geq 2/k$ .

Following [10] we also define the weighted averaged moduli.

**Definition 1.3** ([10]). For  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$  and  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ ,  $0 < p < \infty$ , the  $k$ th weighted averaged modulus of smoothness of  $f$  is defined as

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} := \left( \frac{1}{t} \int_0^t \int_{\mathcal{D}_{k\tau}} |W_{k\tau}^{r/2+\alpha,r/2+\beta}(x) \Delta_{\tau\varphi(x)}^k(f^{(r)}, x)|^p dx d\tau \right)^{1/p}.$$

If  $p = \infty$  and  $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$ , we write

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty} := \omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,\infty}. \tag{1.3}$$

Clearly,

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \leq \omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}, \quad t > 0. \tag{1.4}$$

Moreover, it was proved in [10] that if  $r/2 + \alpha, r/2 + \beta \geq 0$ , then the weighted moduli and the weighted averaged moduli are equivalent.

Throughout this paper, all constants  $c$  may depend only on  $k, r, p, \alpha$  and  $\beta$ , unless a specific dependence on an additional parameter is mentioned.

We have the following direct (Jackson-type) theorem.

**Theorem 1.4.** Let  $k \in \mathbb{N}$ ,  $0 < p \leq \infty$ ,  $\alpha \geq 0$  and  $\beta \geq 0$ . If  $f \in L_p^{\alpha,\beta}$ , then for every  $n \geq k$  and  $0 < \vartheta \leq 1$ ,

$$E_n(f)_{\alpha,\beta,p} \leq c \omega_{k,0}^\varphi(f, \vartheta/n)_{\alpha,\beta,p}, \tag{1.5}$$

where the constant  $c$  depends only on  $k, \alpha, \beta, p$  and  $\vartheta$ .

It follows from [10, Lemma 1.11] that, if  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $r/2 + \alpha \geq 0$ ,  $r/2 + \beta \geq 0$ ,  $1 \leq p \leq \infty$  and  $f \in \mathbb{B}_p^{r+1}(w_{\alpha,\beta})$ , then

$$\omega_{k+1,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} \leq ct \omega_{k,r+1}^\varphi(f^{(r+1)}, t)_{\alpha,\beta,p}, \quad t > 0.$$

Hence, (1.5) implies that, for  $f \in B_p^r(w_{\alpha,\beta})$ ,  $1 \leq p \leq \infty$ ,

$$E_n(f)_{\alpha,\beta,p} \leq c \omega_{k+r,0}^\varphi(f, 1/n)_{\alpha,\beta,p} \leq cn^{-r} \omega_{k,r}^\varphi(f^{(r)}, 1/n)_{\alpha,\beta,p}, \quad n \geq k+r,$$

provided  $\alpha, \beta \geq 0$ . We strengthen this result by showing that the last estimate is, in fact, valid for all  $\alpha, \beta \geq -r/2$ . Namely,

**Theorem 1.5.** Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ , and  $\alpha, \beta \in J_p$  be such that  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ . If  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ , then

$$E_n(f)_{\alpha,\beta,p} \leq \frac{c}{n^r} \omega_{k,r}^\varphi(f^{(r)}, 1/n)_{\alpha,\beta,p}, \quad n \geq k+r. \tag{1.6}$$

We remark that Theorem 1.5 is not valid if  $r \geq 1$  and  $0 < p < 1$  (one can show this using the same construction that was used in the proof of [5, Theorem 3 and Corollary 4]).

Jackson type estimates of the form (1.5) and (1.6) frequently appear with the inequalities being valid for  $n$  sufficiently large. In order to have these inequalities for small  $n$ , we need certain Whitney type results. We devote Section 3 to Whitney type estimates, and we feel that the results in this section are of independent interest by themselves.

Next, we have the following inverse result in the case  $1 \leq p \leq \infty$ .

**Theorem 1.6.** Suppose that  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ ,  $\alpha, \beta \in J_p$  are such that  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ , and  $f \in L_p^{\alpha, \beta}$ . If

$$\sum_{n=1}^{\infty} r n^{r-1} E_n(f)_{\alpha, \beta, p} < +\infty \tag{1.7}$$

(i.e., if  $r = 0$  then this condition is vacuous), then  $f \in \mathbb{B}_p^r(w_{\alpha, \beta})$ , and for  $k \in \mathbb{N}$  and  $N \in \mathbb{N}$ ,

$$\begin{aligned} \omega_{k,r}^{\phi}(f^{(r)}, t)_{\alpha, \beta, p} \leq c & \sum_{n > \max\{N, 1/t\}} r n^{r-1} E_n(f)_{\alpha, \beta, p} \\ & + c t^k \sum_{N \leq n \leq \max\{N, 1/t\}} n^{k+r-1} E_n(f)_{\alpha, \beta, p} \\ & + c(N)t^k E_{k+r}(f)_{\alpha, \beta, p}, \quad t > 0. \end{aligned} \tag{1.8}$$

In particular, if  $N \leq k + r$ , then

$$\begin{aligned} \omega_{k,r}^{\phi}(f^{(r)}, t)_{\alpha, \beta, p} \leq c & \sum_{n > \max\{N, 1/t\}} r n^{r-1} E_n(f)_{\alpha, \beta, p} \\ & + c t^k \sum_{N \leq n \leq \max\{N, 1/t\}} n^{k+r-1} E_n(f)_{\alpha, \beta, p}, \quad t > 0. \end{aligned}$$

**Remark 1.7.** (i) Note that the first term in (1.8) disappears if  $r = 0$ .

(ii) If  $\alpha = \beta = 0$ , Theorem 1.6 was proved in [9].

(iii) The case  $\alpha, \beta \geq 0$ ,  $N = 1$  and  $r = 0$  follows from [3, Theorem 8.2.4] by virtue of (4.2).

Denote by  $\Phi$  the set of nondecreasing functions  $\phi : [0, \infty) \rightarrow [0, \infty)$ , satisfying  $\lim_{t \rightarrow 0^+} \phi(t) = 0$ . The following is an immediate corollary of Theorem 1.6 (in fact, it is a restatement of Theorem 1.6 in terms of  $\phi$ ).

**Corollary 1.8.** Suppose that  $r \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $\alpha, \beta \in J_p$  are such that  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ , and  $\phi \in \Phi$  is such that

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}} du < +\infty$$

(i.e., if  $r = 0$  then this condition is vacuous). Then, if  $f \in L_p^{\alpha, \beta}$  is such that

$$E_n(f)_{\alpha, \beta, p} \leq \phi\left(\frac{1}{n+1}\right), \quad \text{for all } n \geq N,$$

then  $f \in \mathbb{B}_p^r(w_{\alpha, \beta})$ , and for  $k \in \mathbb{N}$  and  $0 < t \leq 1/2$ ,

$$\omega_{k,r}^{\phi}(f^{(r)}, t)_{\alpha, \beta, p} \leq c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du + c(N)t^k E_{k+r}(f)_{\alpha, \beta, p}.$$

In particular, if  $N \leq k + r$ , then

$$\omega_{k,r}^{\phi}(f^{(r)}, t)_{\alpha, \beta, p} \leq c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du.$$

**Remark 1.9.** We take this opportunity to correct an inadvertent misprint in three of our earlier papers where the inverse theorems of this type were proved in the case  $\alpha = \beta = 0$ . Namely, the

inequality  $E_n(f)_p \leq \phi(1/n)$  in [7, Theorem 3.2] (the case  $p = \infty$ ), and in [9, Theorem 9.1] and [8, Theorem  $I_r$ ] (the case  $1 \leq p \leq \infty$ ), should be replaced by  $E_n(f)_p \leq \phi(1/(n + 1))$ . Otherwise, the last estimates in these results are not justified/valid if  $N = 1, k = 1$  and  $r = 0$  since  $E_{k+r}(f)_p = E_1(f)_p \leq \phi(1)$  cannot be estimated above by  $\int_t^1 \phi(u)u^{-2}du$  without any extra assumptions on the function  $\phi$ .

It immediately follows from Theorem 1.4 that if  $\alpha, \beta \in J_p, r/2 + \alpha \geq 0, r/2 + \beta \geq 0$  and  $\omega_{k,r}^\phi(f^{(r)}, t)_{\alpha,\beta,p} \leq t^\gamma$ , then  $E_n(f)_{\alpha,\beta,p} \leq cn^{-r-\gamma}$ . Conversely, an immediate consequence of Theorem 1.6 (Corollary 1.8) is the following result which, for  $\alpha, \beta \geq 0$ , was proved by a different method in [8, Theorem 5.3].

**Corollary 1.10.** *Suppose that  $r \in \mathbb{N}_0, N \in \mathbb{N}, 1 \leq p \leq \infty$ , and  $\alpha, \beta \in J_p$  are such that  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ . If  $f \in L_p^{\alpha,\beta}$  is such that, for some  $N \in \mathbb{N}$  and  $r < \gamma < k + r$ ,*

$$E_n(f)_{\alpha,\beta,p} \leq n^{-\gamma}, \quad n \geq N, \tag{1.9}$$

then  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ , and

$$\omega_{k,r}^\phi(f^{(r)}, t)_{\alpha,\beta,p} \leq ct^{\gamma-r} + c(N)t^k E_{k+r}(f)_{\alpha,\beta,p}, \quad t > 0.$$

In particular, if  $N \leq k + r$ , then

$$\omega_{k,r}^\phi(f^{(r)}, t)_{\alpha,\beta,p} \leq ct^{\gamma-r}, \quad t > 0.$$

Finally, we have the following inverse theorem for  $0 < p < 1$  which is an immediate corollary of [6, Theorem 10.1] and [7, Lemma 4.5].

**Theorem 1.11.** *Let  $k \in \mathbb{N}, \alpha \geq 0, \beta \geq 0$  and  $f \in L_p^{\alpha,\beta}, 0 < p < 1$ . Then there exists a positive constant  $\vartheta \leq 1$  depending only on  $k, p, \alpha$  and  $\beta$  such that, for any  $n \in \mathbb{N}$ ,*

$$\omega_{k,0}^\phi(f, \vartheta n^{-1})_{\alpha,\beta,p}^p \leq cn^{-kp} \sum_{m=1}^n m^{kp-1} E_m(f)_{\alpha,\beta,p}^p.$$

### 2. Auxiliary lemma

**Lemma 2.1.** *Let  $0 < \delta \leq 2$ , and let  $y := y(x), y : [-1, 1] \rightarrow \mathbb{R}$  be such that*

$$y(x) + \delta\varphi(y(x))/2 = x, \quad x \in [-1, 1].$$

Then,

- (i)  $y$  is strictly increasing on  $[-1, 1]$ , and  $y'(x) \leq 2, x \in [-1, 1]$ ,
- (ii)  $y([-1 + 2\mu(\delta), 1]) = \mathfrak{D}_\delta$ ,
- (iii)  $y'(x) \geq 2/3, x \in [-1 + 2\mu(\delta), 1]$ ,
- (iv) if  $y_\lambda(x) := y(x) + \lambda\varphi(y(x))$ , then  $1/3 \leq y'_\lambda(x) \leq 3$ , for all  $|\lambda| \leq \delta/2$  and  $x \in [-1 + 2\mu(\delta), 1]$ ,
- (v) for all  $x \in [-1 + 2\mu(\delta), 1]$ ,

$$\mu(\delta) + 2(1 - x)/3 \leq 1 - y(x) \leq \mu(\delta) + 2(1 - x) \tag{2.1}$$

and

$$(1 + x)/2 \leq 1 + y(x) \leq 1 + x. \tag{2.2}$$

Note that it is not difficult to see that the function  $y = y(x)$  in the statement of [Lemma 2.1](#) is well defined for all  $x \in [-1, 1]$  and, in fact,

$$y(x) = \frac{4x - \delta\sqrt{4 - 4x^2 + \delta^2}}{4 + \delta^2}, \quad -1 \leq x \leq 1.$$

However, we will not be using this explicit formula in this paper.

**Proof.** Since  $x \leq 1$ , we have  $y + \delta\varphi(y)/2 \leq 1$  which can be rewritten as  $\delta/(2\varphi(y)) \leq 1/(1 + y)$ , and so, if  $y \geq 0$ , then

$$1 - \frac{\delta y}{2\varphi(y)} \geq \frac{1}{1 + y} \geq \frac{1}{2},$$

and, clearly,  $1 - \delta y/(2\varphi(y)) \geq 1/2$  if  $y < 0$  as well.

Therefore, since

$$\frac{dy}{dx} = \left(1 - \frac{\delta y}{2\varphi(y)}\right)^{-1},$$

we immediately conclude that [\(i\)](#) holds.

Now, since  $y$  is nondecreasing,  $y([-1 + 2\mu(\delta), 1]) = [y(-1 + 2\mu(\delta)), y(1)]$ , and [\(ii\)](#) follows because  $y(1) = 1 - \mu(\delta)$  and  $y(-1 + 2\mu(\delta)) = -1 + \mu(\delta)$ .

It follows from [\(ii\)](#) that, for  $x \in [-1 + 2\mu(\delta), 1]$ , we have  $y - \delta\varphi(y)/2 \geq -1$ , which can be rewritten as  $\delta/(2\varphi(y)) \leq 1/(1 - y)$ , and so, if  $y \leq 0$ , then

$$1 - \frac{\delta y}{2\varphi(y)} \leq \frac{1 - 2y}{1 - y} \leq \frac{3}{2},$$

and, clearly,  $1 - \delta y/(2\varphi(y)) \leq 3/2$  if  $y > 0$  as well. This implies [\(iii\)](#).

Now, it follows from the above estimates that  $\delta/(2\varphi(y)) \leq 1/(1 + |y|)$ , for  $x \in [-1 + 2\mu(\delta), 1]$ , which implies

$$y'_\lambda(x) = \left(1 - \frac{\lambda y}{\varphi(y)}\right) y'(x) \leq 2 \left(1 + \frac{\delta|y|}{2\varphi(y)}\right) \leq \frac{2 + 4|y|}{1 + |y|} \leq 3,$$

and

$$y'_\lambda(x) \geq \frac{2}{3} \left(1 - \frac{\delta|y|}{2\varphi(y)}\right) \geq \frac{2}{3(1 + |y|)} \geq \frac{1}{3},$$

and so [\(iv\)](#) is verified.

Now, by

$$\frac{dy}{dx}(\xi) = \frac{y(1) - y(x)}{1 - x} = \frac{1 - \mu(\delta) - y(x)}{1 - x}, \quad \xi \in (x, 1),$$

[\(i\)](#) and [\(iii\)](#) imply [\(2.1\)](#), for  $x \in [-1 + 2\mu(\delta), 1]$ . Finally, the second inequality in [\(2.2\)](#) is obvious, and the first one immediately follows from [\(ii\)](#) which implies

$$1 + x = 1 + y + \delta\varphi(y)/2 \leq 2(1 + y).$$

Thus, [\(v\)](#) is verified.  $\square$

### 3. Whitney-type estimates

In this section, we prove Whitney-type estimates, which we feel are of independent interest, and which we need in order to prove the direct (Jackson-type) theorem ([Theorem 1.4](#)) for small  $n$ .

Recall that the celebrated Whitney inequalities for the ordinary moduli of smoothness were first proved by Whitney [16] for functions in  $C[a, b]$ . Later Brudnyi [1] extended the inequalities to  $L_p[a, b]$ ,  $1 \leq p < \infty$  and, finally, Storozhenko [15] proved the inequalities for  $L_p[a, b]$ ,  $0 < p < 1$ .

**Theorem 3.1.** *Let  $k \in \mathbb{N}$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ ,  $0 < p \leq \infty$ ,  $f \in L_p^{\alpha,\beta}$ ,  $0 < h \leq 2$  and  $x_0 \in \mathfrak{D}_h$ . Then, for any  $\theta \in (0, 1]$ , we have*

$$E_k(f, [x_0 - h\varphi(x_0)/2, x_0 + h\varphi(x_0)/2])_{\alpha,\beta,p} \leq c\omega_{k,0}^{*\varphi}(f, \theta h)_{\alpha,\beta,p} \leq c\omega_{k,0}^\varphi(f, \theta h)_{\alpha,\beta,p},$$

where  $c$  depends only on  $k, \alpha, \beta, p$  and  $\theta$ .

Choosing  $x_0 = 0$  and  $h = 2$  in Theorem 3.1 we immediately get the following corollary.

**Corollary 3.2.** *Let  $k \in \mathbb{N}$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ ,  $0 < p \leq \infty$  and  $f \in L_p^{\alpha,\beta}$ . Then, for any  $\theta \in (0, 1]$ , we have*

$$E_k(f)_{\alpha,\beta,p} \leq c\omega_{k,0}^{*\varphi}(f, \theta)_{\alpha,\beta,p} \leq c\omega_{k,0}^\varphi(f, \theta)_{\alpha,\beta,p}, \tag{3.1}$$

where  $c$  depends only on  $k, \alpha, \beta, p$  and  $\theta$ .

Also, if  $x_0 \pm h\varphi(x_0)/2 = \pm 1$ , Theorem 3.1 immediately gives the following result (by letting  $h := t\sqrt{4A/(4 - At^2)}$ ,  $x_0 := \pm(1 - \mu(h))$ ,  $\theta := \min\{1, 1/\sqrt{2A}\}$ , and using monotonicity of the moduli with respect to  $t$ ).

**Corollary 3.3.** *Let  $k \in \mathbb{N}$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ ,  $A > 0$ ,  $0 < p \leq \infty$  and  $f \in L_p^{\alpha,\beta}$ . Then, for any  $0 < t \leq \sqrt{2/A}$ , we have*

$$E_k(f, [1 - At^2, 1])_{\alpha,\beta,p} \leq c\omega_{k,0}^{*\varphi}(f, t)_{\alpha,\beta,p} \leq c\omega_{k,0}^\varphi(f, t)_{\alpha,\beta,p},$$

and

$$E_k(f, [-1, -1 + At^2])_{\alpha,\beta,p} \leq c\omega_{k,0}^{*\varphi}(f, t)_{\alpha,\beta,p} \leq c\omega_{k,0}^\varphi(f, t)_{\alpha,\beta,p},$$

where  $c$  depends on  $k, p, \alpha, \beta$  and  $A$ .

**Proof of Theorem 3.1.** Theorem 3.1 follows from the classical (non-weighted) Whitney’s inequality (see [2, Theorem 6.4.2 and Theorem 12.5.5]), which readily implies (see e.g. [13, Sections 3.1 and 7.1]), for each interval  $J \subset [-1, 1]$ , the existence of a polynomial  $p_k \in \mathbb{P}_k$ , such that

$$\begin{aligned} \|f - p_k\|_{L_p(J)} &\leq c\omega_k(f, |J|; J)_p \\ &\leq c \frac{|J|^{k-1+1/\min\{1,p\}}}{\delta^{k-1+1/\min\{1,p\}}} \omega_k(f, \delta; J)_p, \quad 0 < \delta \leq |J|, \end{aligned} \tag{3.2}$$

where  $|J|$  is the length of the interval  $J$ .

In order to prove Theorem 3.1, we assume, without loss of generality, that  $x_0 \geq 0$ , and denote

$$[a, b] := [x_0 - h\varphi(x_0)/2, x_0 + h\varphi(x_0)/2], \quad W_p := \omega_{k,0}^{*\varphi}(f, \theta h)_{\alpha,\beta,p},$$

Note that

$$1 - x \leq 2(1 - x_0) \quad \text{and} \quad 1 + x \leq 2(1 + x_0), \quad x \in [a, b], \tag{3.3}$$

since  $x_0$  is the middle of  $[a, b]$ , and so

$$\varphi(b) \leq \varphi(x) \leq 2\varphi(x_0), \quad \text{for all } x \in [a, b], \tag{3.4}$$

where the first inequality is valid since  $|x| \leq b$  (because  $x_0$  is assumed to be nonnegative).

We will consider two cases: (i)  $\varphi(x_0) \leq 2\varphi(b)$  and (ii)  $\varphi(x_0) > 2\varphi(b)$ .

In the case (i),  $x_0$  and  $[a, b]$  are “far away” from the endpoints of  $[-1, 1]$  and so  $w_{\alpha,\beta}(x) \sim w_{\alpha,\beta}(x_0)$ , for all  $x \in [a, b]$ . This is a simpler case, and we can reduce the proof to the classical non-weighted Whitney’s inequality. The case (ii) is a bit more involved since the right endpoint  $b$  is now “close to 1” (in fact, we will show that it is sufficient to assume that it is equal to 1), and so the weight  $w_{\alpha,\beta}(x)$  is no longer equivalent to a constant over  $[a, b]$ . The idea of the proof in this case is to consider the interval  $J := [a, b] \cap [-\tilde{b}, \tilde{b}]$ , where  $\tilde{b} := 1 - (c\theta h)^2$  with some small constant  $c$  depending only on  $k$ . Then,  $w_{\alpha,\beta}(x) \sim w_{\alpha,\beta}(x_0)$ , for all  $x \in J$ , and the main task remaining in this case is to show that a polynomial  $p_k \in \mathbb{P}_k$  that approximates  $f$  well on  $J$  also approximates it with the right order on  $[a, b] \setminus J$ .

**Case (i):**  $\varphi(x_0) \leq 2\varphi(b)$ .

Then, for all  $x \in [a, b]$ ,

$$1 - x_0 \leq \varphi^2(x_0) \leq 4\varphi^2(b) \leq 4\varphi^2(x) \leq 8(1 - x) \tag{3.5}$$

and

$$1 + x_0 = \frac{\varphi^2(x_0)}{1 - x_0} \leq \frac{4\varphi^2(b)}{1 - x_0} \leq \frac{8\varphi^2(x)}{1 - x} = 8(1 + x). \tag{3.6}$$

Note that (3.3), (3.5) and (3.6) imply that

$$w_{\alpha,\beta}(x_0) \sim w_{\alpha,\beta}(x) \leq c\mathcal{W}_{k\tau}^{\alpha,\beta}(x), \quad \text{if } x \pm k\tau\varphi(x)/2 \in [a, b]. \tag{3.7}$$

Now, let  $J := [a, b]$  and  $\delta := \theta h\varphi(b)$ , and note that

$$\frac{\theta}{2}|J| = \frac{\theta}{2}h\varphi(x_0) \leq \delta \leq \theta h\varphi(x_0) \leq |J|. \tag{3.8}$$

So, for  $p = \infty$ , we have

$$\begin{aligned} \omega_k(f, \delta; J)_\infty &= \sup_{0 < s \leq \delta} \sup_{x \in J} |\Delta_s^k(f, x; J)| = \sup_{0 < \tau \leq \delta/\varphi(b)} \sup_{x \in J} |\Delta_{\tau\varphi(b)}^k(f, x; J)| \\ &= \sup_{0 < \tau \leq \theta h} \sup_{x \in J} |\Delta_{\tau\varphi(b)}^k(f, x; J)| \leq \sup_{0 < \tau \leq \theta h} \sup_{x \in J} |\Delta_{\tau\varphi(x)}^k(f, x; J)| \\ &\leq c w_{\alpha,\beta}^{-1}(x_0) \sup_{0 < \tau \leq \theta h} \sup_{x \in J} \left| \mathcal{W}_{k\tau}^{\alpha,\beta}(x) \Delta_{\tau\varphi(x)}^k(f, x; J) \right| \\ &= c w_{\alpha,\beta}^{-1}(x_0) W_\infty, \end{aligned}$$

where in the last inequality we used (3.7).

If  $p < \infty$ , then it is well known (see e.g. [13, Lemma 7.2]) that

$$\omega_k(f, t; J)_p^p \leq c \frac{1}{t} \int_0^t \int_J |\Delta_s^k(f, x; J)|^p dx ds, \quad 0 < t \leq |J|/k.$$

Hence, using (3.4) and (3.7) we have

$$\begin{aligned}
 c\delta\omega_k(f, \delta; J)_p^p &\leq \int_J \int_0^\delta |\Delta_s^k(f, x, J)|^p ds dx \\
 &= \int_J \int_0^{\delta/\varphi(x)} \varphi(x) |\Delta_{\tau\varphi(x)}^k(f, x, J)|^p d\tau dx \\
 &\leq \int_J \int_0^{\theta h} \varphi(x) |\Delta_{\tau\varphi(x)}^k(f, x, J)|^p d\tau dx \\
 &\leq c w_{\alpha, \beta}^{-p}(x_0) \varphi(b) \int_J \int_0^{\theta h} |\mathcal{W}_{k\tau}^{\alpha, \beta}(x) \Delta_{\tau\varphi(x)}^k(f, x, J)|^p d\tau dx \\
 &\leq c w_{\alpha, \beta}^{-p}(x_0) \varphi(b) \int_0^{\theta h} \int_{\mathfrak{D}_{k\tau}} |\mathcal{W}_{k\tau}^{\alpha, \beta}(x) \Delta_{\tau\varphi(x)}^k(f, x)|^p dx d\tau \\
 &= c w_{\alpha, \beta}^{-p}(x_0) \theta h \varphi(b) W_p^p.
 \end{aligned}$$

Thus, for all  $0 < p \leq \infty$ , we have

$$\omega_k(f, \delta; J)_p \leq c w_{\alpha, \beta}^{-1}(x_0) W_p,$$

which, by virtue of (3.3), yields

$$\|w_{\alpha, \beta}(f - p_k)\|_{L_p(J)} \leq c w_{\alpha, \beta}(x_0) \|f - p_k\|_{L_p(J)} \leq c w_{\alpha, \beta}(x_0) \omega_k(f, \delta; J)_p \leq c W_p,$$

and so the proof is complete in Case (i).

**Case (ii):**  $\varphi(x_0) > 2\varphi(b)$ .

We first note that, in this case, it suffices to assume that  $b = 1$ . Indeed, suppose that the theorem is proved for all  $\hat{x}_0$  and  $\hat{h}$  such that  $\hat{x}_0 + \hat{h}\varphi(\hat{x}_0)/2 = 1$ , and let  $x_0$  and  $h$  be such that  $\varphi(x_0) > 2\varphi(b)$  (recall that  $b = x_0 + h\varphi(x_0)/2$ ). We let  $\hat{x}_0 := x_0$  and  $\hat{h} := 2(1 - x_0)/\varphi(x_0)$  so that  $x_0 + \hat{h}\varphi(x_0)/2 = 1$ . Now, since

$$1 - x_0 = \frac{\varphi^2(x_0)}{1 + x_0} > \frac{4\varphi^2(b)}{1 + x_0} = 4(1 - b) \frac{1 + b}{1 + x_0} \geq 4(1 - b),$$

we have

$$h\varphi(x_0) = 2(b - x_0) = 2(1 - x_0) - 2(1 - b) > 3(1 - x_0)/2.$$

Therefore,  $h \leq \hat{h} \leq 4h/3$ , and so

$$\begin{aligned}
 E_k(f, [x_0 - h\varphi(x_0)/2, x_0 + h\varphi(x_0)/2])_{\alpha, \beta, p} \\
 \leq E_k(f, [x_0 - \hat{h}\varphi(x_0)/2, x_0 + \hat{h}\varphi(x_0)/2])_{\alpha, \beta, p} \\
 \leq c \omega_{k, 0}^{*\varphi}(f, \theta_1 \hat{h})_{\alpha, \beta, p} \leq c W_p,
 \end{aligned}$$

where  $\theta_1 := 3\theta/4$ .

Hence, for the rest of this proof, we assume that  $b = 1$ . Note that

$$b - a = h\varphi(x_0) = 2(1 - x_0) = 2\mu(h) \in [h^2/2, h^2]. \quad (3.9)$$

Define

$$\tilde{h} := \frac{\theta h}{10k}, \quad \tilde{b} := 1 - \tilde{h}^2 \quad \text{and} \quad J := [a, b] \cap [-\tilde{b}, \tilde{b}].$$

Then  $x_0 \in J$ , and, for all  $x \in J$ , we have

$$\frac{1 - x_0}{1 - x} \leq \frac{\mu(h)}{\tilde{h}^2} < c, \quad \frac{1 + x_0}{1 + x} \leq \frac{2}{\max\{\tilde{h}^2, 1 + a\}} \leq \frac{c}{\max\{h^2, 4 - h^2\}} < c, \tag{3.10}$$

and, recalling (3.4),

$$\varphi(\tilde{b}) \leq \varphi(x) \leq 2\varphi(x_0) \leq c\varphi(\tilde{b}).$$

We now let  $\delta := \theta h\varphi(\tilde{b})$ , note that

$$c|J| \leq c(b - a) \leq \delta \leq b - a \leq c|J|,$$

and conclude using the same argument that was used in Case (i) and using (3.10) instead of (3.5) and (3.6), that there is a polynomial  $p_k \in \mathbb{P}_k$ , such that

$$\|w_{\alpha,\beta}(f - p_k)\|_{L_p(J)} \leq cW_p. \tag{3.11}$$

So, to finish the proof in Case (ii) we have to show that, for the function  $g := f - p_k$ , the inequalities

$$\|w_{\alpha,\beta}g\|_{L_p[\tilde{b},1]} \leq cW_p. \tag{3.12}$$

and, if  $a < -\tilde{b}$ ,

$$\|w_{\alpha,\beta}g\|_{L_p[a,-\tilde{b}]} \leq cW_p. \tag{3.13}$$

hold. We prove (3.12), the proof of (3.13) being similar and simpler, since  $a < -\tilde{b}$  only holds for “large”  $h$  (i.e., those  $h$  that are close to 2). More precisely,

$$a < -\tilde{b} \quad \text{if and only if} \quad \frac{\theta^2 h^2}{100k^2} + \frac{4h^2}{4 + h^2} > 2.$$

We let  $t \in [2\tilde{h}/\sqrt{k}, 4\tilde{h}/\sqrt{k}]$  be fixed for now, and denote by  $y = y(x)$  and  $y_i = y_i(x)$ ,  $1 \leq i \leq k$ , the functions such that

$$y(x) + kt\varphi(y(x))/2 = x \quad \text{and} \quad y_i(x) := x - it\varphi(y(x)) = y(x) + (k/2 - i)t\varphi(y(x)).$$

Note that functions  $y$  and  $y_i$  are well defined (see remark after the statement of Lemma 2.1).

We now note that  $[\tilde{b}, 1] \subset [-1 + 2\mu(kt), 1]$ , since

$$-1 + 2\mu(kt) \leq -1 + k^2 t^2 \leq -1 + 16k\tilde{h}^2 \leq 1 - \tilde{h}^2 = \tilde{b}$$

and so Lemma 2.1 with  $\delta = kt$  implies that, for all  $x \in [\tilde{b}, 1]$ ,  $2/3 \leq y'(x) \leq 2$ ,  $1/3 \leq y'_i(x) \leq 3$ , and

$$\varphi^2(y(x)) \leq (1 + x)(\mu(kt) + 2(1 - x)) \leq 2(\mu(kt) + 2\tilde{h}^2) \leq k^2 t^2 + 4\tilde{h}^2 \leq 25k\tilde{h}^2. \tag{3.14}$$

Additionally, note that

$$y_i(x) \in J, \quad x \in [\tilde{b}, 1] \quad \text{and} \quad 1 \leq i \leq k. \tag{3.15}$$

Indeed, since  $y(1) = 1 - \mu(kt)$ , we have, for  $x \in [\tilde{b}, 1]$ ,

$$y_i(x) \leq y_1(x) \leq y_1(1) = 1 - t\varphi(y(1)) = 1 - 2\mu(kt)/k \leq 1 - kt^2/2 \leq 1 - 2\tilde{h}^2 < \tilde{b},$$

and, using (3.14) and (3.9),

$$\begin{aligned} y_i(x) &\geq y_k(x) \geq y_k(\tilde{b}) = \tilde{b} - kt\varphi(y(\tilde{b})) \geq 1 - \tilde{h}^2 - 5k^{3/2}t\tilde{h} \geq 1 - 21k\tilde{h}^2 \\ &\geq \max\{-1 + \tilde{h}^2, a\}, \end{aligned}$$

which yields (3.15). Note also that (3.14) and inequalities  $t \leq 4\tilde{h}/\sqrt{k}$  and  $\tilde{h} \leq (5k)^{-1}$  imply that  $1 + y(x) \geq 3kt\varphi(y(x))/2$ , for  $x \in [\tilde{b}, 1]$ .

Hence,

$$w_{\alpha,\beta}(x) = w_{\alpha,\beta}(y(x) + kt\varphi(y(x))/2) \leq 2^\beta w_{\alpha,\beta}(y_i(x)), \quad x \in [\tilde{b}, 1],$$

and using (3.15) and (3.11) we get, for  $0 < p < \infty$ ,

$$\begin{aligned} \|w_{\alpha,\beta}g(y_i)\|_{L_p[\tilde{b},1]} &\leq 2^\beta \|w_{\alpha,\beta}(y_i)g(y_i)\|_{L_p[\tilde{b},1]} \leq c \|w_{\alpha,\beta}(y_i)g(y_i)(y_i')^{1/p}\|_{L_p[\tilde{b},1]} \\ &\leq c \|w_{\alpha,\beta}g\|_{L_p(J)} \leq cW_p. \end{aligned}$$

If  $p = \infty$ , then similar (and, in fact, simpler) arguments yield

$$\|w_{\alpha,\beta}g(y_i)\|_{L_\infty[\tilde{b},1]} \leq cW_\infty, \quad 1 \leq i \leq k.$$

Now, for  $x \in [\tilde{b}, 1]$ ,

$$\begin{aligned} g(x) &= \Delta_{t\varphi(y(x))}^k(g, y(x)) - \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} g\left(y(x) + \left(i - \frac{k}{2}\right)t\varphi(y(x))\right) \\ &= \Delta_{t\varphi(y(x))}^k(g, y(x)) - \sum_{i=1}^k (-1)^i \binom{k}{i} g(y_i(x)), \end{aligned}$$

and so

$$\begin{aligned} \|w_{\alpha,\beta}g\|_{L_p[\tilde{b},1]} &\leq c \|w_{\alpha,\beta}\Delta_{t\varphi(y)}^k(g, y)\|_{L_p[\tilde{b},1]} + c \sum_{i=1}^k \binom{k}{i} \|w_{\alpha,\beta}g(y_i)\|_{L_p[\tilde{b},1]} \\ &\leq c \|w_{\alpha,\beta}\Delta_{t\varphi(y)}^k(g, y)\|_{L_p[\tilde{b},1]} + cW_p \\ &\leq c \left\| \mathcal{W}_{tk}^{\alpha,\beta}(y)\Delta_{t\varphi(y)}^k(g, y) \right\|_{L_p[\tilde{b},1]} + cW_p \\ &\leq c \left\| \mathcal{W}_{tk}^{\alpha,\beta}\Delta_{t\varphi}(g, \cdot) \right\|_{L_p(\mathfrak{D}_{kt})} + cW_p. \end{aligned}$$

This completes the proof in the case  $p = \infty$ . If  $p < \infty$ , then integrating with respect to  $t$  over  $[2\tilde{h}/\sqrt{k}, 4\tilde{h}/\sqrt{k}]$  we get

$$\|w_{\alpha,\beta}g\|_{L_p[\tilde{b},1]}^p \leq \frac{c}{\tilde{h}} \int_{2\tilde{h}/\sqrt{k}}^{4\tilde{h}/\sqrt{k}} \left\| \mathcal{W}_{tk}^{\alpha,\beta}\Delta_{t\varphi}(g, \cdot) \right\|_{L_p(\mathfrak{D}_{kt})}^p dt + cW_p^p \leq cW_p^p.$$

The proof is now complete.  $\square$

We now prove a Whitney-type result for functions from  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ ,  $r \in \mathbb{N}$ .

**Theorem 3.4.** *Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ , and let  $\alpha, \beta \in J_p$  be such that  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ . If  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ , then for any  $\theta \in (0, 1]$ ,*

$$E_{k+r}(f)_{\alpha,\beta,p} \leq c\omega_{k,r}^\varphi(f^{(r)}, \theta)_{\alpha,\beta,p}. \tag{3.16}$$

**Proof.** Note that  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$  implies that  $f^{(r)} \in L_p^{r/2+\alpha,r/2+\beta}$ , and so it follows from (3.1) that

$$E_k(f^{(r)})_{r/2+\alpha,r/2+\beta,p} \leq c\omega_{k,0}^\varphi(f^{(r)}, \theta)_{r/2+\alpha,r/2+\beta,p} = cW_{r,p},$$

where  $W_{r,p} := \omega_{k,r}^\varphi(f^{(r)}, \theta)_{\alpha,\beta,p}$ .

Let  $\tilde{P}_k \in \mathbb{P}_k$  be a polynomial such that

$$\left\| w_{r/2+\alpha, r/2+\beta} (f^{(r)} - \tilde{P}_k) \right\|_p < cW_{r,p},$$

and define  $P_{k+r} \in \mathbb{P}_{k+r}$  by

$$P_{k+r}(x) := f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(r-1)}(0)}{(r-1)!}x^{r-1} + \frac{1}{(r-1)!} \int_0^x (x-u)^{r-1} \tilde{P}_k(u) du.$$

Assuming that  $x \geq 0$  (for  $x < 0$  all estimates are analogous), we have by Hölder’s inequality

$$\begin{aligned} & (r-1)! |f(x) - P_{k+r}(x)| \\ & \leq \int_0^x (x-u)^{r-1} |f^{(r)}(u) - \tilde{P}_k(u)| du \\ & = \int_0^x \frac{(x-u)^{r-1}}{w_{r/2+\alpha, r/2+\beta}(u)} w_{r/2+\alpha, r/2+\beta}(u) |f^{(r)}(u) - \tilde{P}_k(u)| du \\ & \leq cA_q(x)W_{r,p}, \end{aligned}$$

where  $q := p/(p-1)$ ,

$$A_q(x) := \left( \int_0^x \left( \frac{(x-u)^{r-1}}{w_{r/2+\alpha, r/2+\beta}(u)} \right)^q du \right)^{1/q}, \quad \text{if } q < \infty,$$

and

$$A_\infty(x) := \sup_{u \in [0,x]} \left( \frac{(x-u)^{r-1}}{w_{r/2+\alpha, r/2+\beta}(u)} \right).$$

Now, since

$$\frac{(x-u)^{r-1}}{w_{r/2+\alpha, r/2+\beta}(u)} \leq \frac{(x-u)^{r-1}}{(1-u)^{r/2+\alpha}} \leq (1-u)^{r/2-\alpha-1},$$

we have

$$A_q^q(x) \leq \int_0^x (1-u)^{q(r/2-\alpha-1)} du \quad \text{and} \quad A_\infty(x) \leq \sup_{u \in [0,x]} (1-u)^{r/2-\alpha-1}.$$

If  $q < \infty$  and  $q(r/2 - \alpha - 1) > -1$ , then

$$A_q^q(x) \leq \int_0^1 (1-u)^{q(r/2-\alpha-1)} du = c,$$

which yields

$$\|f - P_{k+r}\|_{L_\infty[0,1]} \leq cW_{r,p},$$

and hence

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[0,1]} \leq cW_{r,p} \|w_{\alpha,\beta}\|_{L_p[0,1]} \leq cW_{r,p}, \tag{3.17}$$

where we used the fact that  $\alpha \in J_p$ . Similarly, (3.17) holds if  $q = \infty$  ( $p = 1$ ) and  $r/2 - \alpha - 1 \geq 0$ .

If  $q < \infty$  and  $q(r/2 - \alpha - 1) < -1$ , then

$$A_q^q(x) \leq c(1-x)^{q(r/2-\alpha-1)+1},$$

and so, recalling that  $x \geq 0$ , we have

$$w_{\alpha,\beta}(x)A_q(x) \leq c(1-x)^{r/2-1/p}.$$

Hence,

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[0,1]} \leq c \|w_{\alpha,\beta} A_q\|_{L_p[0,1]} W_{r,p} \leq c W_{r,p}. \tag{3.18}$$

Similarly, one shows that (3.18) holds if  $q = \infty$  ( $p = 1$ ) and  $r/2 - \alpha - 1 < 0$ .

It remains to consider the case  $q < \infty$  and  $q(r/2 - \alpha - 1) = -1$ . We have

$$A_q^q(x) \leq \int_0^x (1-u)^{-1} du = c |\ln(1-x)|,$$

and so

$$w_{\alpha,\beta}(x) A_q(x) \leq c(1-x)^\alpha |\ln(1-x)|^{1/q}.$$

For  $p < \infty$ , since  $\alpha p > -1$ , we have

$$\|w_{\alpha,\beta} A_q\|_{L_p[0,1]}^p \leq c \int_0^1 (1-x)^{\alpha p} |\ln(1-x)|^{p-1} dx < c.$$

Finally, if  $p = \infty$ , then  $q = 1$  and  $\alpha = r/2 > 0$ , and so  $\|w_{\alpha,\beta} A_1\|_{L_\infty[0,1]} < c$ . Hence, (3.18) holds in this case as well.

Similarly, one shows that

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[-1,0]} \leq c W_{r,p},$$

and the proof is complete.  $\square$

#### 4. Direct estimates: proof of Theorems 1.4 and 1.5

The following lemma is [10, Corollary 4.4] with  $r = 0$ .

**Lemma 4.1.** *Let  $k \in \mathbb{N}$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$  and  $f \in L_p^{\alpha,\beta}$ ,  $0 < p \leq \infty$ . Then, there exists  $N \in \mathbb{N}$  depending on  $k$ ,  $p$ ,  $\alpha$  and  $\beta$ , such that for every  $n \geq N$  and  $0 < \vartheta \leq 1$ , there is a polynomial  $P_n \in \mathbb{P}_n$  satisfying*

$$\|w_{\alpha,\beta}(f - P_n)\|_p \leq c \omega_{k,0}^{*\varphi}(f, \vartheta/n)_{\alpha,\beta,p} \leq c \omega_{k,0}^\varphi(f, \vartheta/n)_{\alpha,\beta,p},$$

and

$$n^{-k} \|w_{\alpha,\beta} \varphi^k P_n^{(k)}\|_p \leq c \omega_{k,0}^{*\varphi}(f, \vartheta/n)_{\alpha,\beta,p} \leq c \omega_{k,0}^\varphi(f, \vartheta/n)_{\alpha,\beta,p},$$

where constants  $c$  depend only on  $k$ ,  $p$ ,  $\alpha$ ,  $\beta$  and  $\vartheta$ .

**Proof of Theorem 1.4.** Estimate (1.5) immediately follows from Lemma 4.1 for  $n \geq N$ . For  $k \leq n < N$ , (1.5) follows from Corollary 3.2 with  $\theta := \vartheta/N$ , since

$$E_n(f)_{\alpha,\beta,p} \leq E_k(f)_{\alpha,\beta,p} \leq c \omega_{k,0}^\varphi(f, \vartheta/N)_{\alpha,\beta,p} \leq c \omega_{k,0}^\varphi(f, \vartheta/n)_{\alpha,\beta,p}. \quad \square$$

**Remark 4.2.** In the case  $1 \leq p \leq \infty$ , it was shown by Ky [11, Theorem 4] (see also Luther and Russo [12]) that if  $\alpha, \beta \geq 0$ , then

$$E_n(f)_{\alpha,\beta,p} \leq c \omega_\varphi^k(f, 1/n)_{w_{\alpha,\beta,p}}, \quad n \geq n_0. \tag{4.1}$$

By virtue of [10, Corollary 1.7 and (5.2)], we have, for  $1 \leq p \leq \infty$ ,

$$\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega_\varphi^k(f^{(r)}, t)_{w_{\alpha,\beta} \varphi^r, p}, \quad 0 < t \leq t_0. \tag{4.2}$$

Thus, in the case  $1 \leq p \leq \infty$ , (1.5) with  $n \geq n_0$  follows from (4.1). We also remark that, even though (4.1) was stated with  $n_0 = k$  in [11], the proof used [3, Theorem 6.1.1] where  $0 < t \leq t_0$ , and so was only justified for sufficiently large  $n$ .

**Proof of Theorem 1.5.** The case  $r = 0$  is Theorem 1.4. Thus we may assume that  $r \geq 1$ . It follows by [3, Theorem 8.2.1 and (6.3.2)] that, for  $n \geq n_0$ ,

$$\begin{aligned}
 E_n(f)_{\alpha,\beta,p} &\leq c \int_0^{1/n} (\Omega_\varphi^{k+r}(f, t)_{w_{\alpha,\beta,p}/t}) dt \\
 &\leq c \int_0^{1/n} t^{r-1} \Omega_\varphi^k(f^{(r)}, t)_{w_{\alpha,\beta,\varphi^r,p}} dt \\
 &\leq \frac{c}{n^r} \Omega_\varphi^k(f^{(r)}, 1/n)_{w_{\alpha,\beta,\varphi^r,p}} \leq \frac{c}{n^r} \omega_\varphi^k(f^{(r)}, 1/n)_{w_{\alpha,\beta,\varphi^r,p}},
 \end{aligned}
 \tag{4.3}$$

where the main-part modulus  $\Omega_\varphi^m$  is defined in [3, (8.1.2)]. Hence, (1.5) follows by (4.2). For  $k + r \leq n < n_0$ , (1.5) immediately follows from Theorem 3.4 with  $\theta := 1/n_0$ , as above. This completes the proof.  $\square$

**5. Inverse theorem: proof of Theorem 1.6**

We first prove this theorem in the case  $r \geq 1$ .

For the proof we need the following fundamental inequality (see [4,14] as well as [3, (8.1.3)]): given  $\alpha, \beta \in J_p$ ,  $1 \leq p \leq \infty$ , we have

$$\|w_{\alpha,\beta}\varphi^r p_n^{(r)}\|_p \leq c(r, p, \alpha, \beta)n^r \|w_{\alpha,\beta} p_n\|_p, \quad p_n \in \mathbb{P}_n.
 \tag{5.1}$$

Let  $f \in L_p^{\alpha,\beta}$  and let  $P_n \in \mathbb{P}_n$  be a polynomial of best approximation of  $f$  in  $L_p^{\alpha,\beta}$ . That is,  $E_n(f)_{\alpha,\beta,p} = \|w_{\alpha,\beta}(f - P_n)\|_p$ ,  $n \geq 1$ .

Throughout the proof, we often use the estimate

$$\begin{aligned}
 &\sum_{j=l}^m (2^j N)^\nu E_{2^j N}(f)_{\alpha,\beta,p} \\
 &\leq (1 + 2^\nu) \sum_{j=l}^{m-1} (2^j N)^\nu E_{2^j N}(f)_{\alpha,\beta,p} \\
 &\leq (1 + 2^\nu) 2^\nu \sum_{j=l}^{m-1} \sum_{n=2^{j-1}N+1}^{2^j N} n^{\nu-1} E_n(f)_{\alpha,\beta,p} \\
 &= (1 + 2^\nu) 2^\nu \sum_{n=2^{l-1}N+1}^{2^{m-1}N} n^{\nu-1} E_n(f)_{\alpha,\beta,p},
 \end{aligned}
 \tag{5.2}$$

where  $\nu \geq 1$  and  $1 \leq l < m$ , which is also valid if  $m = \infty$ .

We represent  $f$  as the telescopic series

$$f = P_{k+r} + (P_N - P_{k+r}) + \sum_{j=0}^{\infty} (P_{2^{j+1}N} - P_{2^j N}) =: P_{k+r} + Q + \sum_{j=0}^{\infty} Q_j.
 \tag{5.3}$$

Since

$$\|w_{\alpha,\beta} Q_j\|_p \leq \|w_{\alpha,\beta}(P_{2^{j+1}N} - f)\|_p + \|w_{\alpha,\beta}(f - P_{2^j N})\|_p \leq c E_{2^j N}(f)_{w_{\alpha,\beta,p}},
 \tag{5.4}$$

we have by virtue of (5.1) and (1.7), for each  $1 \leq \nu \leq r$ ,

$$\begin{aligned} \sum_{j=0}^{\infty} \left\| w_{\alpha,\beta} \varphi^{\nu} Q_j^{(\nu)} \right\|_p &\leq c \sum_{j=0}^{\infty} (2^{j+1}N)^{\nu} E_{2^j N}(f)_{\alpha,\beta,p} \\ &\leq c N^{\nu} E_N(f)_{\alpha,\beta,p} + c \sum_{n=N+1}^{\infty} n^{\nu-1} E_n(f)_{\alpha,\beta,p} < \infty. \end{aligned}$$

By the same argument as in the proof of [9, Theorem 9.1], it follows that almost everywhere  $f(x)$  is identical with a function possessing an absolutely continuous derivative of order  $(r - 1)$  and  $f^{(r)} \in L_p[-1 + \varepsilon, 1 - \varepsilon]$ , for any  $\varepsilon > 0$ . In particular, differentiation of (5.3) is justified, and  $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ .

By [10, Lemma 4.1], since  $r/2 + \alpha \geq 0$  and  $r/2 + \beta \geq 0$ , we have

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,p} \leq c \left\| w_{\alpha,\beta} \varphi^r Q_j^{(r)} \right\|_p$$

and

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,p} \leq ct^k \left\| w_{\alpha,\beta} \varphi^{r+k} Q_j^{(r+k)} \right\|_p.$$

Hence, by (5.1) and (5.4) we obtain

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,p} \leq c(2^{j+1}N)^r \left\| w_{\alpha,\beta} Q_j \right\|_p \leq c(2^j N)^r E_{2^j N}(f)_{\alpha,\beta,p}$$

and

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,p} \leq ct^k (2^{j+1}N)^{r+k} \left\| w_{\alpha,\beta} Q_j \right\|_p \leq ct^k (2^j N)^{r+k} E_{2^j N}(f)_{\alpha,\beta,p}.$$

Denoting  $J := \min\{j \in \mathbb{N}_0 : 2^{-j} \leq Nt\}$  (note that  $2^{-J} \leq Nt < 2^{-J+1}$  if  $J \geq 1$ , and  $Nt \geq 1$  if  $J = 0$ ) we now have by (5.2)

$$\begin{aligned} \omega_{k,r}^{\varphi} \left( \sum_{j=J+1}^{\infty} Q_j^{(r)}, t \right)_{\alpha,\beta,p} &\leq c \sum_{j=J+1}^{\infty} \omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,p} \tag{5.5} \\ &\leq c \sum_{j=J+1}^{\infty} (2^j N)^r E_{2^j N}(f)_{\alpha,\beta,p} \\ &\leq c \sum_{n=2^J N+1}^{\infty} n^{r-1} E_n(f)_{\alpha,\beta,p} \\ &\leq c \sum_{n>\max\{N, 1/t\}} n^{r-1} E_n(f)_{\alpha,\beta,p}, \end{aligned}$$

since  $2^J N + 1 > \max\{N, 1/t\}$ . Now, if  $J \geq 2$ , then (5.2) implies

$$\begin{aligned} \omega_{k,r}^{\varphi} \left( \sum_{j=0}^J Q_j^{(r)}, t \right)_{\alpha,\beta,p} &\leq ct^k \sum_{j=0}^J (2^j N)^{r+k} E_{2^j N}(f)_{\alpha,\beta,p} \tag{5.6} \\ &\leq ct^k N^{r+k} E_N(f)_{\alpha,\beta,p} + ct^k \sum_{n=N+1}^{2^{J-1}N} n^{r+k-1} E_n(f)_{\alpha,\beta,p} \\ &\leq ct^k N^{r+k} E_N(f)_{\alpha,\beta,p} + ct^k \sum_{N \leq n \leq \max\{N, 1/t\}} n^{k+r-1} E_n(f)_{\alpha,\beta,p}, \end{aligned}$$

where we used the fact that  $2^{J-1}N \leq \max\{N, 1/t\}$ . If  $J = 0$  or  $1$ , then we have

$$\omega_{k,r}^\varphi\left(\sum_{j=0}^J Q_j^{(r)}, t\right)_{\alpha,\beta,p} \leq ct^k N^{r+k} E_N(f)_{\alpha,\beta,p},$$

and so the last estimate in (5.6) is valid in this case as well.

Finally, if  $N \geq k+r$ , then

$$\begin{aligned} \omega_{k,r}^\varphi(P_{k+r}^{(r)} + Q^{(r)}, t)_{\alpha,\beta,p} &= \omega_{k,r}^\varphi(Q^{(r)}, t)_{\alpha,\beta,p} \leq ct^k \|w_{\alpha,\beta} \varphi^{k+r} Q^{(k+r)}\|_p \\ &\leq ct^k N^{r+k} \|w_{\alpha,\beta} Q\|_p \leq ct^k N^{r+k} E_{k+r}(f)_{\alpha,\beta,p}, \end{aligned} \quad (5.7)$$

and if  $N < k+r$ , then  $\omega_{k,r}^\varphi(P_{k+r}^{(r)} + Q^{(r)}, t)_{\alpha,\beta,p} = 0$ , so that we do not need (5.7).

Combining (5.5)–(5.7) and using the fact that, if  $N \geq k+r$ , then  $E_N(f)_{\alpha,\beta,p} \leq E_{k+r}(f)_{\alpha,\beta,p}$  and, if  $N < k+r$ , then the first term in the last inequality in (5.6) can be absorbed by the second term in that inequality, we obtain (1.8), and our proof is complete in the case  $r \geq 1$ .

Suppose now that  $r = 0$ . We represent  $f$  as

$$f = P_k + Q + \sum_{j=0}^J Q_j + (f - P_{2^{J+1}N}), \quad (5.8)$$

where  $Q := P_N - P_k$  and  $Q_j := P_{2^{j+1}N} - P_{2^jN}$ , and estimate the last term. We have

$$\|w_{\alpha,\beta}(f - P_{2^{J+1}N})\|_p \leq c E_{2^{J+1}N}(f)_{\alpha,\beta,p},$$

and in the case  $J = 0$  or  $1$ , we use the fact that  $Nt \geq c$ , to conclude

$$E_{2^{J+1}N}(f)_{\alpha,\beta,p} \leq E_N(f)_{\alpha,\beta,p} = N^k t^k (Nt)^{-k} E_N(f)_{\alpha,\beta,p} \leq c(N)t^k E_N(f)_{\alpha,\beta,p}.$$

If  $J \geq 2$ , we recall that  $2^{J-1}N < 1/t \leq 2^JN$ , so that  $\max\{N, 1/t\} = 1/t$ , and write

$$\begin{aligned} E_{2^{J+1}N}(f)_{\alpha,\beta,p} &\leq (2^{J-2}N)^{-1} \sum_{n=2^{J-2}N+1}^{2^{J-1}N} E_n(f)_{\alpha,\beta,p} \\ &\leq (2^{J-2}N)^{-k} \sum_{n=2^{J-2}N+1}^{2^{J-1}N} n^{k-1} E_n(f)_{\alpha,\beta,p} \\ &\leq 4^k t^k \sum_{N \leq n < 1/t} n^{k-1} E_n(f)_{\alpha,\beta,p}. \end{aligned}$$

It now remains to apply (5.6) and (5.7) with  $r = 0$ , in order to complete the proof of (1.8) in the case  $r = 0$ .  $\square$

## Acknowledgment

First author was supported by NSERC of Canada Discovery Grant RGPIN 04215-15.

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