Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 142 (2006) 83–115
Journal of Approximation Theory
www.elsevier.com/locate/jat

Construction of multivariate compactly supported prewavelets in L_2 space and pre-Riesz bases in Sobolev spaces ☆

Ming-Jun Lai*

Department of Mathematics, The University of Georgia, Athens, GA30602, USA

Received 14 December 2004; accepted 21 March 2006

Communicated by Rong-Qing Jia
Available online 18 May 2006

Abstract

We give a new constructive method for finding compactly supported prewavelets in L_2 spaces in the multivariate setting. This method works for any dimensional space. When this method is generalized to the Sobolev space setting, it produces a pre-Riesz basis for $H^s(\mathbb{R}^d)$ which can be useful for applications.

MSC: primary 42C15; secondary 42C30

Keywords: Box splines; Compactly supported; Multivariate; Multi-resolution analysis; Prewavelets; Pre-Riesz bases; Sobolev spaces

1. Introduction

In the last fifteen years, there are many methods available to construct prewavelets in L_2 space in the literature. We refer the reader to [20,3,8,12–15,5,17,19,24]. These generate an active and healthy research atmosphere to promote the theory of wavelets and their applications in various areas such as geometric design (cf. [13]) Several constructions mentioned above have a restriction on the dimension, i.e., $d \leqslant 3$. Some of them are ad hoc methods which work only for piecewise linear splines. Although the construction given in [3] is a simple and general method, there are two

☆ This research is partly supported by the National Science Foundation under Grant EAR-0327577 and the Army Research Office under Grant DAAD19-03-1-0203.

* Fax: +1 706 542 5907.
E-mail address: mjlai@math.uga.edu.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2006.03.010

kinds of conditions to check if the constructed functions span a L_2 stable basis. When applying to box spline functions, one condition requires that the direction set of a box spline satisfy a "parity" property which excludes the continuous piecewise linear box spline. The other one is similar to the one in [21] which works for $d \leqslant 3$ only. In [15], a general method is given to obtain compactly supported prewavelets in the multivariate setting. However, the method failed to be constructive due to the fact that it uses the well-known Quillen–Suslin theorem. The construction of prewavelets in Sobolev spaces was attempted. The results have been negative so far. Indeed, in [19], the nonexistence of compactly supported box spline prewavelets in Sobolev spaces was proved. In a recent paper [16], the researchers constructed such wavelet functions whose derivatives generate a Riesz basis in L_2 norm instead of the prewavelets under a Sobolev norm. Although their Riesz wavelets have very short support, the orthogonalities among translations and/or among dilations are lost.

In this paper, I shall provide a new constructive method which yields compactly supported prewavelets in $L_2(\mathbb{R}^d)$ for any $d \geqslant 1$. A simple condition on the mask of refinable functions is given to ensure that the functions obtained from our constructive method generate a L_2 stable basis. The new construction improves the existing ones in various senses which will be detailed later. The method has an immediate generalization in the Sobolev setting. Thus I formula the constructive procedure in Sobolev spaces. However the construction produces only a pre-Riesz basis in $H^s(\mathbb{R}^d)$ for $s > 0$ if one refinable function is used to generate a multiresolution approximation (MRA) of $H^s(\mathbb{R}^d)$. This will give another reason for the nonexistence of prewavelets in Sobolev space as mentioned above. However, such a pre-Riesz basis can be modified to get a Riesz basis for the Sobolev space by sacrificing the compactly supportedness of prewavelets. Nevertheless, a pre-Riesz basis can be useful on its own if the function to be approximated satisfies an additional property.

Next let us introduce some necessary notation and definitions to explain the concept for pre-wavelets. Let s be a nonnegative real number. When s is an integer, we set

$$H^s(\mathbb{R}^d) = \{f, f^{(k)} \in L_2(\mathbb{R}^d), 0 \leqslant k \leqslant s\}$$

to be the usual Sobolev space which is equipped with the following inner product

$$\langle f, g\rangle_s := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\omega|^2)^s \hat{f}(\omega)\overline{\hat{g}(\omega)}\, \mathrm{d}\omega,$$

where $\hat{f}$ denotes the usual Fourier transform of f. We shall use $\|f\|_s := \sqrt{\langle f, f\rangle_s}$ to denote the norm for $H^s(\mathbb{R}^d)$. When s is not an integer, we may simply let

$$H^s(\mathbb{R}^d) := \{f, \langle f, f\rangle_s < \infty\}$$

be the standard Sobolev space. Certainly, when s is an integer, we may use the following equivalent form of inner product

$$\langle f, g\rangle_s \equiv \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^d} D^\alpha f(x) D^\alpha g(x)\, dx,$$

where $\alpha \in \mathbb{Z}_+^d$ is a multi-integer, D^α is a standard partial derivative with order α, and $|\alpha|$ is the sum of all components of α. For any sequence $\{c_k,\ k \in \mathbb{Z}^d\}$, $\mathbf{c}$ is square summable if $\sum_{m\in\mathbb{Z}^d} |c_m|^2$

$< \infty$. In this case, let

$$\|\{c_k,\ k \in \mathbb{Z}^d\}\|_2 := \left(\sum_{m\in\mathbb{Z}^d} |c_m|^2\right)^{1/2}.$$

We now need the definition of MRA of $H^s(\mathbb{R}^d)$. It is a nonstationary multiresolution analysis (cf. [10] for its basic properties).

Definition 1.1. A MRA of $H^s(\mathbb{R}^d)$ is a sequence of subspaces V_j, $j \in \mathbb{Z}$ of $H^s(\mathbb{R}^d)$ such that

(i) $V_j \subset V_{j+1}$;
(ii) $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $H^s(\mathbb{R}^d)$;
(iii) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$;
(iv) For every $j \in \mathbb{Z}$, there is a function $\phi_j \in V_j$ such that the integer translates, $\phi_j(2^j x - m)$, $m \in \mathbb{Z}^d$ form a Riesz basis for $V_j = \text{span}\{2^{jd/2}\phi_j(2^j x - m), m \in \mathbb{Z}^d\}$, i.e., there exist two positive numbers α_j and β_j such that

$$\alpha_j \|\{c_m, m \in \mathbb{Z}^d\}\|_2^2 \leqslant \|\sum_{m\in\mathbb{Z}^d} c_m 2^{jd/2}\phi_j(2^j x - m)\|_2^2 \leqslant \beta_j \|\{c_m, m \in \mathbb{Z}^d\}\|_2^2,$$

for all square summable sequence $\{c_m, m \in \mathbb{Z}^d\}$.

It is easy to see the following

$$\alpha_j = \min_{\omega\in[0,2\pi]^d} \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}(\omega + 2m\pi)^2)^s |\widehat{\phi_j}(\omega + 2m\pi)|^2 \tag{1}$$

and

$$\beta_j = \max_{\omega\in[0,2\pi]^d} \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}(\omega + 2m\pi)^2)^s |\widehat{\phi_j}(\omega + 2m\pi)|^2. \tag{2}$$

Let $\{\phi_j, j \in \mathbb{Z}\}$ be a sequence of refinable functions, i.e., $\phi_j = \sum_{k\in\mathbb{Z}} c_{j,k}\phi_{j+1}(\cdot - k)$ for some coefficients $c_{j,k}$ for all $j \in \mathbb{Z}$. We say $\{\phi_j, j \in \mathbb{Z}\}$ generates a MRA for a Sobolev space $H^s(\mathbb{R}^d)$ if letting $V_j := \text{closure}_{H^s(\mathbb{R}^d)}\{\phi_j(2^j x - m),\quad m \in \mathbb{Z}^d\}$ for $j \in \mathbb{Z}$, then the sequence $\{V_j, j \in \mathbb{Z}\}$ is an MRA of $H^s(\mathbb{R}^d)$.

Definition 1.2. A collection $\psi_{j,k}$, $k = 2, \ldots, 2^d$ of functions in $H^s(\mathbb{R}^d)$ satisfying the following five properties are called prewavelets:

1° the closure $W_{j,k}$ of the linear span of integer translates of $\psi_{j,k}$ is orthogonal to the closure V_j of the linear span of integer translates of ϕ_j;
2° $W_{j,k}$ is orthogonal each other among $k = 2, \ldots, 2^d$,
3° V_{j+1} is the direct sum V_j and $W_{j,k}$, $k = 2, \ldots, 2^d$;
4° the integer translates of $\psi_{j,k}$ form a Riesz basis for $W_{j,k}$;
5° the collection $\{\psi_{j,k}(2^j x - m), m \in \mathbb{Z}^d, j \in \mathbb{Z}, k = 2, \ldots, 2^d\}$ form a Riesz basis for $H^s(\mathbb{R}^d)$. That is, the collection is linearly independent, the linear combinations of the elements in the

collection are dense in $H^s(\mathbb{R}^d)$, and there exist two positive constants A and B such that

$$A\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2 \leqslant \left\|\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{jd/2}\psi_{j,k}(2^j\cdot -m)\right\|_s^2$$
$$\leqslant B\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2$$

for all square-summable sequence $\{c_{j,k,m},\ j\in\mathbb{Z}, k=2,\ldots,2^d, m\in\mathbb{Z}^d\}$.

In the remaining of the paper, we shall also use the following definition of pre-Riesz basis (cf. [1]). It is also called Bessel system (cf. [22]).

Definition 1.3. The collection $\{\psi_{j,k},\ j,k\in\mathbb{Z}\}$ in $H^s(\mathbb{R}^d)$ is a pre-Riesz basis if the collection is a basis for $H^s(\mathbb{R}^d)$ satisfying 1°–4° and
6° there exists a positive constant B such that

$$\left\|\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{jd/2}\psi_{j,k}(2^j\cdot -m)\right\|_s^2 \leqslant B\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2$$

for any square-summable sequence $\{c_{j,k,m},\ j\in\mathbb{Z}, k=2,\cdots,2^d, m\in\mathbb{Z}^d\}$.

Let us point out that a pre-Riesz basis for $H^s(\mathbb{R}^d)$ can be useful. First for any square-summable sequence $\{c_{j,k,m},\ j\in\mathbb{Z}, m\in\mathbb{Z}^d, k=2,\ldots,2^d\}$,

$$f=\sum_{j\in\mathbb{Z}}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{jd/2}\psi_{j,k}(2^j\cdot -m)$$

is in $H^s(\mathbb{R}^d)$ by 6°.

Secondly, we can use a pre-Riesz basis to approximate functions $f\in H^s(\mathbb{R}^d)$ in the following sense. Since it is a basis, any $f\in H^s(\mathbb{R}^d)$ can be written as a linear combination of $2^{jd/2}\psi_{j,k}(2^j\cdot -m)$ with coefficients $c_{j,k,m}$. If the coefficient sequence $\{c_{j,k,m},\ j\in\mathbb{Z}, m\in\mathbb{Z}^d, k=2,\ldots,2^d\}$ of f is square-summable, then we can use the pre-Riesz basis like a Riesz basis to approximate f. More precisely, we have

$$\left\|f-\sum_{j=-N}^{N}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d |m|\leqslant M} c_{j,k,m}2^{jd/2}\psi_{j,k}(2^j\cdot -m)\right\|_s^2$$
$$\leqslant B\sum_{|j|>N}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d |m|>M}|c_{j,k,m}|^2 \longrightarrow 0$$

as $N\to+\infty$ and $M\to+\infty$.

Thirdly, coefficients $c_{j,k,m}$ of a function $f \in H^s(\mathbb{R}^s)$ can be computed by solving the following discrete convolution equation

$$\begin{aligned}\langle f, 2^{jd/2}\psi_{j,k}(2^j\cdot -m)\rangle_s &= \sum_{n\in\mathbb{Z}^d} c_{j,k,n}\langle 2^{jd/2}\psi_{j,k}(2^j\cdot -n), 2^{jd/2}\psi_{j,k}(2^j\cdot -m)\rangle_s \\ &= \sum_{n\in\mathbb{Z}^d} c_{j,k,n}\langle 2^{jd/2}\psi_{j,k}(2^j\cdot), 2^{jd/2}\psi_{j,k}(2^j\cdot -m+n)\rangle_s\end{aligned}$$

for $m \in \mathbb{Z}^d$ by the orthogonality between levels V_j and among groups $W_{j,k}$ due to 1°–3° above. The solution of the discrete convolution equation is guaranteed by 4°.

Fourthly, we can use a pre-Riesz basis for data compression like any prewavelets and wavelets. More precisely, suppose that we have an approximation $\mathbb{P}_j f$ of $f \in H^s(\mathbb{R}^d)$ in V_j. Then we decompose it in several (finitely many) levels:

$$\begin{aligned}\mathbb{P}_j f &= \mathbb{P}_{j-1} f + \sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j-1,k,m}2^{(j-1)d/2}\psi_{j-1,k}(2^{j-1}\cdot -m) \\ &= \cdots \\ &= \mathbb{P}_{j-\ell} f + \sum_{n=1}^{\ell}\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j-n,k,m}2^{(j-n)d/2}\psi_{j-n,k}(2^{j-n}\cdot -m).\end{aligned}$$

We threshold small coefficients off from the collection $\{c_{j-n,k,m}, n = 1,\ldots,\ell, k = 2,\ldots,2^d, k \in \mathbb{Z}^s\}$. That is, for a small number $\varepsilon > 0$, if $|c_{j-n,k,m}| \leqslant \varepsilon$, we set it to be zero. Heuristically, since $\psi_{j,k,m}$ behaviors like a wave, if f is a smooth function and does not have many waves or variations, then many of the coefficients are small. Hence, by thresholding them to be zero, we have a new representation of $\mathbb{P}_j f$ with less number of nonzero coefficients than that in $\mathbb{P}_j f$.

Finally, suppose that we use V_j to approximate the solution of the following partial differential equation (PDE) (a Cauchy problem and in the weak formulation)

$$\langle u, v\rangle_s = \langle f, v\rangle_0 \quad \forall v \in H^s(\mathbb{R}^d)$$

with appropriate boundary conditions at infinity which are not included here just for simplicity, where $\langle\cdot,\cdot\rangle_s$ is the bilinear form associated with the PDE. Let $A_{f,j} \in V_j$ be an approximation of u in V_j satisfying

$$\langle A_{f,j}, 2^{jd/2}\phi(2^j\cdot -m)\rangle_s = \langle f, 2^{jd/2}\phi(2^j\cdot -m)\rangle,$$

for all $m \in \mathbb{Z}^d$. Suppose that A_{f,j_0} has been computed for an integer j_0. To compute a better approximation A_{f,j_0+1} in V_{j_0+1}, we may use the integer translates of the functions $\psi_{j_0,k}$, $k = 2,\ldots,2^d$ to find $B_{f,j_0} \in \bigoplus_{k=2}^{2^d} W_{j_0,k}$ such that $A_{f,j_0} + B_{f,j_0} = A_{f,j_0+1}$, even though integer translates of $\psi_{j,k}$'s form a pre-Riesz basis for $H^s(\mathbb{R}^d)$ satisfying 1°–4° and 6°. By 3°, we know that B_{f,j_0} can be stably computed. Indeed, letting $B_{f,j_0} = \sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j_0,k,m}2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)$, the coefficients $c_{j_0,k,m}$ are the solution of the discrete convolution equations

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d} c_{j_0,k,n}\langle 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot), 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m+n)\rangle_s \\ &\quad = \langle f, 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)\rangle_0 \qquad (3)\end{aligned}$$

for $k = 2, \ldots, 2^d$ since

$$\begin{aligned}\langle f, 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)\rangle &= \langle A_{f,j_0+1}, 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)\rangle_s \\ &= \langle A_{f,j_0}, 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)\rangle_s \\ &\quad +\langle B_{f,j_0}, 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m)\rangle_s \\ &= \sum_{n\in\mathbb{Z}^d} c_{j_0,k,n}\langle 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot), 2^{j_0 d/2}\psi_{j_0,k}(2^{j_0}\cdot -m+n)\rangle_s\end{aligned}$$

by the orthogonalities $\psi_{j_0,k} \perp A_{f,j}$ and $\psi_{j_0,k} \perp \psi_{j_0,k'}$ for $k' \neq k, k' = 2, \ldots, 2^d$ and

$$\begin{aligned}\langle f, 2^{j_0 d/2}\phi_{j_0}(2^{j_0}\cdot -m)\rangle &= \langle A_{f,j_0+1}, 2^{j_0 d/2}\phi_{j_0}(2^{j_0}\cdot -m)\rangle_s \\ &= \langle A_{f,j_0}, 2^{j_0 d/2}\phi_{j_0}(2^{j_0}\cdot -m)\rangle_s\end{aligned}$$

which is already valid by the computation of A_{f,j_0}.

With our pre-Riesz basis, we solve several subsystems (3) to get B_{f,j_0} and add to the previous solution A_{f,j_0} to obtain A_{f,j_0+1}. This is completely different than the finite element method which solves a whole new system of equations in V_{j_0+1} and throws off the previous solution A_{f,j_0} completely. This method is also different than a multi-grid method where the solution A_{f,j_0} is used to get a good initial guess for solving A_{f,j_0+1} iteratively. Certainly, we can continue applying this method to get a multi-level method. However, since the $\psi_{j,k}$ are only a pre-Riesz basis, the Riesz constants are deteriorate as $j \to +\infty$. The accuracy of the computation in (3) will lose eventually as $j \to +\infty$. Nevertheless, in practice, we are only interested in A_{f,j_0+j} for some small integer j, e.g., $j = 1$ or $j = 2$. For example, with the given computer resource, we can compute A_{f,j_0}, but not A_{f,j_0+1}. Since the subsystems in (3) have the same size of the linear system as that of A_{f,j_0}, we solve (3) and add the sub-solutions together to get A_{f,j_0+1}.

These account for the usefulness of the pre-Riesz bases. The rest of the paper is organized as follows. We first need a necessary and sufficient condition for the orthogonality between two subspaces of V_{j+1} in any Sobolev space. Such orthogonal conditions can be found in, e.g., [19]. For convenience, we include a short proof. In §3, we suppose that ϕ_j generate an MRA for a Sobolev space. We then compute 2^d compactly supported functions in V_{j+1} which are orthogonal to V_j under a condition on ϕ. Among them, we show that there are $2^d - 1$ of them which are linearly independent under another condition. These two conditions will be detailed later. Then we use a technique like the Gram–Schmidt orthonormalization procedure to obtain the desired orthogonality among $W_{j,k}$ The construction yields prewavelets in $L_2(\mathbb{R}^d)$ or a pre-Riesz basis for $H^s(\mathbb{R}^d)$. In §4 we shall use multivariate box splines to verify that many box spline functions satisfy these two conditions. Two examples of B-spline prewavelets and one example of box spline prewavelets are given to illustrate the constructive procedure. In §5 we continue to analyse the Riesz bound property. When $s > 0$, we show that using one refinable function ϕ, the functions so constructed do not satisfy the Riesz bound condition. This explains another reason for the nonexistence of compactly supported prewavelets. We shall explain how to modify one scaling function into nonstationary scaling functions so that the functions so constructed in §3 are indeed prewavelets. Also in this section, we show many box spline functions can be used to construct pre-Riesz bases for Sobolev spaces and an example of orthogonal decomposition in $H^1(\mathbb{R}^2)$ will be demonstrated.

2. Preliminaries

In this paper, we assume that there exists a sequence of functions ϕ_j which generate an MRA of $H^s(\mathbb{R}^d)$. Suppose that all ϕ_j are compactly supported and the mask P_j are defined by

$$\widehat{\phi_j}(2\omega) = P_j(z)\widehat{\phi_{j+1}}(\omega),$$

where $\widehat{\phi_j}$ denotes the Fourier transform of ϕ_j and similar for $\widehat{\phi_{j+1}}$. Here $z = exp(i\omega)$. Note that $P_j(z)$ is a Laurent polynomial for each $j \in \mathbb{Z}$.

We are looking for compactly supported functions $\psi_{j,k}$, $k = 2, \ldots, 2^d$ in V_{j+1} such that

$$V_{j+1} = V_j \bigoplus \bigoplus_{k=2}^{2^d} W_{j,k},$$

where $W_{j,k}$ is the closure of the linear span of integer translates of $\psi_{j,k}(2^j x - m)$, $m \in \mathbb{Z}^d$; that is,

$$W_{j,k} := \text{closure}_{H^s(\mathbb{R}^d)}\{\psi_{j,k}(2^j x - m),\ m \in \mathbb{Z}^d\}$$

such that $\phi_j(\cdot - m), \psi_{j,k}(\cdot - m), m \in \mathbb{Z}^d, k = 2, \ldots, 2^d$ form a stable basis for V_{j+1}.

To do so, we first introduce a Laurent polynomial

$$\Phi_j^s(z) := \sum_{m\in\mathbb{Z}^d} \langle 2^{dj/2}\phi_j(2^j x), 2^{dj/2}\phi_j(2^j x - m)\rangle_s z^m.$$

This function Φ_j^s may be called the generalized Euler–Frobenius polynomial since when $s = 0$, $j = 0$, and ϕ_j is a B-spline function, Φ_j^s is the well-known Euler–Frobenius polynomial. When Φ_j^s is independent of *j*, we shall denote it by Φ^s. When $s = 0$, we let $\Phi = \Phi^0$.

Next we need a necessary and sufficient condition for the orthogonality. Writing

$$2^{dj/2}g_{j,k}(2^j x) = \sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - m) \in V_{j+1},$$

and

$$G_{j,k}(z) = \frac{1}{2^{d/2}}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}z^m,$$

the Fourier transforms of $g_{j,k}$ and ϕ_{j+1} are related by

$$\widehat{g_{j,k}}(2\omega) = G_{j,k}(z)\widehat{\phi_{j+1}}(\omega).$$

Let

$$\mathcal{G}_{j,k} = \text{closure}_{H^s(\mathbb{R}^d)}\{g_{j,k}(2^j x - m),\ m \in \mathbb{Z}^d\}$$

be the closure of the linear span of integer translates of $g_{j,k}$.

In this paper we introduce a special operator E which maps any Laurent polynomial f into such a Laurent polynomial which contains all the even index terms of f. For example, when $d = 2$ and $z = (z_1, z_2)$,

$$E(f(z)) = \frac{1}{4}(f(z_1, z_2) + f(-z_1, z_2) + f(z_1, -z_2) + f(-z_1, -z_2)).$$

Then we have the following

Theorem 2.1. *$\mathcal{G}_{j,k}$ is orthogonal to $\mathcal{G}_{j,k'}$ for $k' \neq k$ if and only if*

$$E(G_{j,k}(z)\overline{G_{j,k'}(z)}\Phi^s_{j+1}(z)) = 0. \tag{4}$$

Proof. The orthogonality condition $\mathcal{G}_{j,k} \perp \mathcal{G}_{j,k'}$ if and only if

$$\langle 2^{dj/2} g_{j,k}(2^j x), 2^{dj/2} g_{j,k'}(2^j x - m)\rangle_s = 0 \quad \forall m \in \mathbb{Z}^d.$$

These are equivalent to the following one condition

$$\sum_{m\in\mathbb{Z}^d} \langle 2^{dj/2} g_{j,k}(2^j x), 2^{dj/2} g_{j,k'}(2^j x - m)\rangle_s z^{2m} = 0 \quad \forall z \in T^d,$$

where T^d denotes the torus in $\mathbb{C}^d$. Expanding the left-hand side of the above equation, we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} c_{j,k,\ell} c_{j,k',n} \\
&\quad \times \langle 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - \ell), 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - 2m - n)\rangle_s z^{2m} \\
&\quad = \sum_{m\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} c_{j,k,\ell} c_{j,k',n} \\
&\qquad \times \langle 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x), 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - 2m - n + \ell)\rangle_s z^{2m}.
\end{aligned} \tag{5}$$

On the other hand, we have

$$\begin{aligned}
G_{j,k}(z)\overline{G_{j,k'}(z)}\Phi^s_{j+1}(z) = &\sum_{m\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} c_{j,k,\ell} c_{j,k',n} \\
&\times \langle 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x), 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - m)\rangle_s z^{m+\ell-n}.
\end{aligned}$$

Thus, applying the operator E, we have

$$\begin{aligned}
&E(G_{j,k}(z)\overline{G_{j,k'}(z)}\Phi^s_{j+1}(z)) \\
&\quad = \sum_{\substack{\ell,m,n\in\mathbb{Z}^d \\ m+\ell-n=2i, i\in\mathbb{Z}^d}} c_{j,k,\ell} c_{j,k',n} \\
&\qquad \times \langle 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x), 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - m)\rangle_s z^{2i} \\
&\quad = \sum_{i\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} c_{j,k,\ell} c_{j,k',n} \\
&\qquad \times \langle 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x), 2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - 2i - n + \ell)\rangle_s z^{2i}.
\end{aligned} \tag{6}$$

Comparing with (5) and (6), we conclude that (4) is necessary and sufficient. This completes the proof. $\square$

3. Our constructive procedure

We divide the construction of compactly supported prewavelets into two steps. The first step is to construct compactly supported $g_{j,k} \in V_{j+1}, k = 1, \ldots, 2^d$ such that the closure $\mathcal{G}_{j,k}$ of the linear span of integer translates $g_{j,k}$ is orthogonal to V_j for each k. The second step is to use a technique like Gram–Schmidt orthonormal procedure to orthogonalize these $g_{j,k}$'s. Such a procedure is standard in the literature on constructive theory of wavelets and has been used by many researchers in their papers, e.g., in [15].

To be more precise, we let $\{n_1, \ldots, n_{2^d}\} = \{0,1\}^d$ with $n_k \in \mathbb{Z}^{2^d}$ and $g_{j,k} \in V_{j+1}$ satisfy

$$g_{j,k}(2^j x - m) \perp V_j, \quad m \in \mathbb{Z}^d$$

and

$$\begin{aligned}&2^{d(j+1)/2}\phi_{j+1}(2^{j+1}x - n_k)\\&\quad = \sum_{m\in\mathbb{Z}^d}\left(a_{j,k,m}2^{dj/2}\phi_j(2^j x - m) + b_{j,k,m}2^{dj/2}g_{j,k}(2^j x - m)\right)\end{aligned}$$

for each $k \in \{1, \ldots, 2^d\}$. That is, we want to have $\mathcal{G}_{j,k}$ is orthogonal to V_j and $V_{j+1} = V_j \bigoplus \left(\mathcal{G}_{j,1} + \cdots + \mathcal{G}_{j,2^d}\right)$. In terms of Fourier transform, the above equation can be rewritten as

$$\begin{aligned}\frac{1}{2^{d/2}}e^{in_k\omega/2^{j+1}}\widehat{\phi}_{j+1}\left(\frac{\omega}{2^{j+1}}\right) &= A_{j,k}\left(\frac{\omega}{2^j}\right)\widehat{\phi}_j\left(\frac{\omega}{2^j}\right) + B_{j,k}\left(\frac{\omega}{2^j}\right)\widehat{g_{j,k}}\left(\frac{\omega}{2^j}\right)\\&= A_{j,k}\left(\frac{\omega}{2^j}\right)P_j(z^{1/2^j})\widehat{\phi_{j+1}}\left(\frac{\omega}{2^{j+1}}\right) + B_{j,k}\left(\frac{\omega}{2^j}\right)G_{j,k}(z^{\frac{1}{2^j}})\widehat{\phi_{j+1}}\left(\frac{\omega}{2^{j+1}}\right),\end{aligned}$$

where

$$A_{j,k}(\omega) = \sum_{m\in\mathbb{Z}^d} 2^{jd/2}a_{j,k,m}e^{im\omega} \quad \text{and} \quad B_{j,k}(\omega) = \sum_{m\in\mathbb{Z}^d} 2^{jd/2}b_{j,k,m}e^{im\omega}.$$

It follows that

$$A_{j,k}(2\omega)P_j(z) + B_{j,k}(2\omega)G_{j,k}(z) = \frac{e^{in_k\omega}}{2^{d/2}}, \tag{7}$$

for $k = 1, \ldots, 2^d$. Using Theorem 2.1, the solution of $A_{j,k}$, $B_{j,k}$ and $G_{j,k}$ can be easily found as shown in the following.

Lemma 3.1. *Suppose that $E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z)) \neq 0$ for all $z \in T^d$. Let*

$$\begin{aligned}A_{j,k}(2\omega) &:= \frac{E(e^{in_k\omega}\overline{P_j(z)}\Phi^s_{j+1}(z))}{E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))},\\B_{j,k}(2\omega) &:= \frac{1}{E(P_j(\omega)\overline{P_j(z)}\Phi^s_{j+1}(z))},\\G_{j,k}(z) &:= \frac{1}{2^{d/2}}E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))e^{in_k\omega} - \frac{1}{2^{d/2}}E(e^{in_k\omega}\overline{P_j(z)}\Phi^s_{j+1}(z))P_j(z).\end{aligned}$$

Then $\mathcal{G}_{j,k}$ is orthogonal to V_j for all $k = 1, \ldots, 2^d$ and

$$V_{j+1} = V_j \bigoplus (\mathcal{G}_{j,1} + \cdots + \mathcal{G}_{j,2^d}).$$

Proof. Using the assumption of Theorem 3.1, we know that $A_{j,k}$, $B_{j,k}$ are well-definted. It is clear that (7) is satisfied. Thus, V_{j+1} are the direct sum of V_j and $\mathcal{G}_{j,k}$, $k = 1, \ldots, 2^d$. To see $\mathcal{G}_{j,k}$ is orthogonal to V_j, we use Theorem 2.1 to see $E(G_{j,k}(z)\overline{P_j(z)}\Phi^s_{j+1}(z)) = 0$. Since $B_{j,k}(2\omega) \neq 0$ and

$$E(B_{j,k}(2\omega)G_{j,k}(z)\overline{P_j(z)}\Phi^s_{j+1}(z)) = B_{j,k}(2\omega)E(G_{j,k}(z)\overline{P_j(z)}\Phi^s_{j+1}(z)),$$

we may consider

$$\begin{aligned}&E(B_{j,k}(2\omega)G_{j,k}(z)\overline{P_j(z)}\Phi^s_{j+1}(z))\\ &\quad = E((e^{in_k\omega} - A_{j,k}(2\omega)P_j(z))\overline{P_j(z)}\Phi^s_{j+1}(z)) = 0\end{aligned}$$

by the construction of $A_{j,k}$. This completes the proof. □

To know more about $E(P_j(\omega)\overline{P_j(\omega)}\Phi^s_{j+1}(z))$, we have the following result. When $s = 0$, the result is known (cf. [8]).

Lemma 3.2. *For any nonnegative number s, we have*

$$E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z)) = 2^{-d}\Phi^s_j(z^2).$$

Proof. We first follow the ideas in [7, p. 47] to get

$$\begin{aligned}\Phi^s_j(z) &= \sum_{m\in\mathbb{Z}^d} \langle 2^{jd/2}\phi_j(2^j\cdot), 2^{jd/2}\phi_j(2^j\cdot - m)\rangle_s e^{im\omega}\\ &= \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}|\omega + 2m\pi|^2)^s |\widehat{\phi_j}(\omega + 2m\pi)|^2\end{aligned}$$

by using the Poisson summation formula. It follows that

$$\begin{aligned}&P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z)\\ &\quad = \sum_{m\in\mathbb{Z}^d} P_j(z)\overline{P_j(z)}(1 + 2^{2j+2}|\omega + 2m\pi|^2)^s|\widehat{\phi_{j+1}}(\omega + 2m\pi)|^2\\ &\quad = \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j+2}|\omega + 2m\pi|^2)^s|P_j(z)\widehat{\phi_{j+1}}(\omega + 2m\pi)|^2\\ &\quad = \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}|2\omega + 4m\pi|^2)^s|\widehat{\phi_j}(2\omega + 4m\pi)|^2.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))\\ &\quad = 2^{-d}\sum_{n\in\{0,1\}^d}\sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}|2\omega + 2n\pi + 4m\pi|^2)^s|\widehat{\phi_j}(2\omega + 2n\pi + 4m\pi)|^2\\ &\quad = 2^{-d}\sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}|2\omega + 2m\pi|^2)^s|\widehat{\phi_j}(2\omega + 2m\pi)|^2\\ &\quad = 2^{-d}\Phi^s_j(z^2)\end{aligned}$$

by using the Poisson summation formula again. This completes the proof. □

Thus, to ensure the condition in Theorem 3.1, we only check if $\Phi_j^s(z^2) \neq 0$.

Next we show that $g_{j,k}$, $k = 1, \ldots, 2^d$ are linearly dependent in the sense of (9), but $2^d - 1$ of them are linearly independent if P_j satisfies another condition. Let us write P_j in its polyphase form, i.e.,

$$P_j(z) = \sum_{k=1}^{2^d} e^{in_k\omega} P_{j,k}(z^2), \tag{8}$$

where n_k, $k = 1, \ldots, 2^d$ denote the multi-integers in the collection $\{0, 1\}^d$.

Theorem 3.3. *Suppose that $\Phi_j^s(z^2) \neq 0$. Suppose that there exists an integer k such that $P_{j,k}(z^2) \neq 0$ for all z on the torus T^d. For simplicity, let us assume that $P_{j,1}(z^2) \neq 0$ for all z on T^d. Then the integer translates of $g_{j,k}$, $k = 2, \ldots, 2^d$ form a Riesz basis for $V_{j+1} \ominus V_j$. Hence it follows that there exist a nonzero square summable sequence $\{f_{j,k,m}, k = 2, \ldots, 2^d, m \in \mathbb{Z}^d\}$ such that*

$$g_{j,1}(x) = \sum_{m \in \mathbb{Z}^d} \sum_{k=2}^{2^d} f_{j,k,m} g_{j,k}(x - m). \tag{9}$$

Proof. To show that integer translates of $g_{j,k}$, $k = 2, \ldots, 2^d$ form a Riesz basis for $V_{j+1} \ominus V_j$, we will show that integer translates of ϕ_j together with integer translates of $g_{j,k}$, $k = 2, \ldots, 2^d$ form a Riesz basis for V_{j+1}. Recall that integer translates of ϕ_{j+1} form a Riesz basis for V_{j+1}. By using Lemma 3.4 below we need to show that the following matrix is nonsingular

$$\begin{bmatrix} P_j((-1)^{n_\ell} z) \\ E(P_j(z)\overline{P_j(z)}\Phi_{j+1}^s(z))e^{in_2(\omega+n_\ell\pi)} - E(e^{in_2(\omega+n_\ell\pi)}\overline{P_j(z)}\Phi_{j+1}^s(z))P_j((-1)^{n_\ell} z) \\ \vdots \\ E(P_j(z)\overline{P_j(z)}\Phi_{j+1}^s(z))e^{in_{2^d}(\omega+n_\ell\pi)} - E(e^{in_{2^d}(\omega+n_\ell\pi)}\overline{P_j(z)}\Phi_{j+1}^s(z))P_j((-1)^{n_\ell} z) \end{bmatrix}_{\ell=1,\ldots,2^d}.$$

Let us simplify the above matrix by row reductions. We first multiply

$$E(e^{in_\ell(\omega+n_\ell\pi)}\overline{P_j(z)}\Phi_{j+1}^s(z))$$

to the first row and add the resulting first row to the ℓth row for $\ell = 2, \ldots, 2^d$. Next, we simply factor $E(P_j(z)\overline{P_j(z)}\Phi_{j+1}^s(z))$ out of each row from the second row to the last row. After the row reductions and factorization, the resulting matrix is

$$\begin{bmatrix} P_j((-1)^{n_\ell} z) \\ e^{in_2(\omega+n_\ell\pi)} \\ \vdots \\ e^{in_{2^d}(\omega+n_\ell\pi)} \end{bmatrix}_{\ell=1,\ldots,2^d}. \tag{10}$$

Thus, we need to verify that the above new matrix is nonsingular. Using the polyphase form (8) of $P_j(z)$, we use row reductions to simplify the new matrix again by multiplying $-P_{j,k}(z^2)$ to the kth row and adding it to the first row for $k = 2, \ldots, 2^d$ and then factoring the nonzero factor

$P_{j,1}(z^2)$ from the first row. The new matrix is simplified to

$$\begin{bmatrix} 1 \\ e^{in_2(\omega+n_\ell\pi)} \\ \vdots \\ e^{in_{2^d}(\omega+n_\ell\pi)} \end{bmatrix}_{\ell=1,\dots,2^d}$$

which apparently is not singular.

By Lemma 3.1, we have $\mathcal{G}_{j,k}$ is orthogonal to V_j for $k = 2, \dots, 2^d$. That is,

$$\langle g_{j,k}(\cdot - m), \phi_j(\cdot - m)\rangle_s = 0$$

for all $m \in \mathbb{Z}^d$, $k = 2, \dots, 2^d$. Thus, the integer translates of $g_{j,k}$, $k = 2, \dots, 2^d$ form a Riesz basis for $V_{j+1} \ominus V_j$.

Since integer translates of $g_{j,1}$ are in $V_{j+1} \ominus V_j$, $g_{j,1}$ is linearly dependent on $g_{j,k}$, $k = 2, \dots, 2^d$. Thus, we complete the proof. $\square$

The fact that $g_{j,1}$ is linearly dependent on $g_{j,k}$, $k = 2, \dots, 2^d$ in the sense of (9) has a computational proof of independent interest. In terms of Fourier transform, the linear dependence (9) is

$$\widehat{g_{j,1}}(\omega) = \sum_{k=2}^{2^d} F_{j,k}(z)\widehat{g_{j,k}}(\omega),$$

where $F_{j,k}$ is the discrete Fourier transform of the sequence of $f_{j,k,m}$'s. We claim that $F_{j,k} = -P_{j,k}/P_{j,1}$. Indeed, using the dilation relations, the linear dependence can be rewritten as

$$G_{j,1}(z)\widehat{\phi_{j+1}}(\omega) = \sum_{k=2}^{2^d} F_{j,k}(z^2)G_{j,k}(z)\widehat{\phi_{j+1}}(\omega).$$

By the formula for $G_{j,k}$ in Theorem 3.1, we have

$$\begin{aligned} &E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))e^{in_1\omega} - E(e^{in_1\omega}\overline{P_j(z)}\Phi^s_{j+1}(z))P_j(z) \\ &= \sum_{k=2}^{2^d} F_{j,k}(z^2)\left(E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))e^{in_k\omega} - E(e^{in_k\omega}\overline{P_j(z)}\Phi^s_{j+1}(z))P_j(z)\right). \end{aligned}$$

Putting in $F_{j,k} = -P_{j,k}/P_{j,1}$ in the above equation, we can verify that the above equation is equivalent to the following

$$\begin{aligned} &\sum_{k\in\{0,1\}^d} E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))e^{in_k\omega}P_{j,k}(z^2) \\ &= \sum_{k\in\{0,1\}^d} P_{j,k}(z^2)E(e^{in_k\omega}\overline{P_j(z)}\Phi^s_{j+1}(z))P_j(z). \end{aligned}$$

Now it is easy to see that the above equation is valid. This shows that $g_{j,k}$, $k = 1, \dots, 2^d$ are linearly dependent in the sense of (9). $\square$

The following lemma was used in the proof of Theorem 3.3. Similar results in the L_2 setting may be found in [15]. A detail proof is included here since a part of its proof will be needed later.

Lemma 3.4. *Suppose that integer translates of ϕ_{j+1} form a Riesz basis for V_{j+1}. Let $h_1, \ldots, h_{2^d}$ be compactly supported functions in V_{j+1}, i.e.,*

$$\widehat{h_k}(2\omega) = H_k(z)\widehat{\phi_{j+1}}(\omega)$$

for $k = 1, \ldots, 2^d$. Then integer translates of h_k, $k = 1, \ldots, 2^d$ form a Riesz basis for V_{j+1} if and only if the matrix $[H_k((-1)^{n_\ell} z)]_{1 \leqslant k,\ell \leqslant 2^d}$ is nonsingular for all $z \in T^d$.

Proof. It is easy to see that the linear span of integer translates of $\{h_k, k = 1, \ldots, 2^d\}$ is a subspace of V_{j+1}. For any function f in V_{j+1}, we can write $f(x) = \sum_{\ell \in \mathbb{Z}^d} f_\ell \phi_{j+1}(2x - \ell)$ for a square summable sequence $\{f_\ell, \ell \in \mathbb{Z}^d\}$. Let us show that there exist square summable coefficients $c_{k,\ell}$, $\ell \in \mathbb{Z}^d$, $k = 1, \ldots, 2^d$ such that

$$f(x) = \sum_{k=1}^{2^d} \sum_{\ell \in \mathbb{Z}^d} c_{k,\ell} h_k(x - \ell).$$

Indeed, in terms of Fourier transform, we have

$$F(z)\widehat{\phi}_{j+1}(\omega) = 2^d \sum_{k=1}^{2^d} C_k(z^2)\widehat{h}_k(2\omega),$$

where $F(z)$ stands for the discrete Fourier transform of sequence $\{f_\ell, \ell \in \mathbb{Z}^d\}$ and $C_k(z)$ is the discrete Fourier transform of a sequence $\{c_{k,\ell}, \ell \in \mathbb{Z}^d\}$. That is, we have

$$F(z) = 2\sum_{k=1}^{2^d} H_k(z)C_k(z^2) \quad \text{or} \quad F((-1)^{n_\ell} z) = 2\sum_{k=1}^{2^d} H_k((-1)^{n_\ell} z)C_k(z^2), \quad \ell = 1, \ldots, 2^d.$$

Since $[H_k((-1)^{n_\ell} z)]_{1 \leqslant k,\ell \leqslant 2^d}$ is nonsingular, the above system of equations has a unique solution $[C_1(z^2), \ldots, C_{2^d}(z^2)]$. Thus, the linear span of integer translates of h_k, $k = 1, \ldots, 2^d$ is V_{j+1}.

Next, in order to see that integer translates of $\{h_k, k = 1, \ldots, 2^d\}$ form a Riesz basis for V_{j+1} we need to show that there exist two positive constants A_j and B_j such that

$$A_j \sum_{k=1}^{2^d} \|C_k(z)\|_2^2 \leqslant \left\| \sum_{k=1}^{2^d} C_k(z)\widehat{h}_k(\omega) \right\|_s^2 \leqslant B_j \sum_{k=1}^{2^d} \|C_k(z)\|_2^2, \tag{11}$$

for two positive constants A_j and B_j, where

$$\|C_k(z)\|_2^2 = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} |C_k(z)|^2 \, dz.$$

The middle term in the above inequalities gives

$$\begin{aligned} \left\| \sum_{k=1}^{2^d} C_k(z)\widehat{h}_k(\omega) \right\|_s^2 &= \left\| \sum_{k=1}^{2^d} C_k(z)\widehat{H}_k(e^{i\omega/2})\widehat{\phi_{j+1}}(\omega/2) \right\|_s^2 \\ &= \int_{\mathbb{R}^d} (1+\omega^2)^s \left| \sum_{k=1}^{2^d} C_k(z)\widehat{H}_k(e^{i\omega/2})\widehat{\phi_{j+1}}(\omega/2) \right|^2 d\omega. \end{aligned}$$

It follows that

$$\left\| \sum_{k=1}^{2^d} C_k(z)\widehat{h_k}(\omega) \right\|_s^2 = \sum_{\ell=1}^{2^d} \int_{[0,2\pi]^d} \left| \sum_{k=1}^{2^d} C_k(z)\widehat{H_k}(e^{i(\omega/2+n_\ell\pi)}) \right|^2 R_\ell(\omega)\, d\omega, \tag{12}$$

where

$$R_\ell(\omega) = \sum_{n\in\mathbb{Z}^d} (1+(\omega+2(2n+n_\ell)\pi)^2)^s |\widehat{\phi_{j+1}}(\omega/2+(2n+n_\ell)\pi)|^2.$$

It is clear that

$$R_\ell(\omega) \geqslant \sum_{n\in\mathbb{Z}^d} (1+(\omega/2+(2n+n_\ell)\pi)^2)^s |\widehat{\phi_{j+1}}(\omega/2+(2n+n_\ell)\pi)|^2 \geqslant \alpha_{j+1}$$

since integer translates of ϕ_{j+1} form a Riesz basis for V_{j+1}. It follows that

$$\begin{aligned}\left\| \sum_{k=1}^{2^d} C_k(z)\widehat{h_k}(\omega) \right\|_s^2 &\geqslant \sum_{\ell=1}^{2^d} \int_{[0,2\pi]^d} \alpha_{j+1} \left| \sum_{k=1}^{2^d} C_k(z)\widehat{H_k}(e^{i(\omega/2+n_\ell\pi)} \right|^2 d\omega \\ &= \alpha_{j+1} \int_{[0,2\pi]^d} \mathcal{C}(z)^T \mathcal{H}(\omega)^T \mathcal{H}(\omega)\mathcal{C}(z)\, d\omega,\end{aligned}$$

where $\mathcal{C}(z) = (C_1(z), \ldots, C_{2^d}(z))^T$ is a vector of length 2^d and

$$\mathcal{H}(\omega) = \begin{bmatrix} H_1(e^{i(\omega/2+n_\ell\pi)}) \\ \vdots \\ H_{2^d}(e^{i(\omega/2+n_\ell\pi)}) \end{bmatrix}_{\ell=1,\ldots,2^d}^T .$$

That is, $\mathcal{H}(2\omega) = [H_k((-1)^\ell z)]_{1\leqslant k,\ell\leqslant 2^d}$. Since $\mathcal{H}(\omega)$ is nonsingular, $\mathcal{H}(\omega)^T\mathcal{H}(\omega)$ is positive definite and hence there is a positive number $\lambda_1 > 0$ such that

$$\mathcal{C}(z)^T \mathcal{H}(\omega)^T \mathcal{H}(\omega)\mathcal{C}(z) \geqslant \lambda_1 \mathcal{C}(z)^T \mathcal{C}(z)$$

and equivalently,

$$\left\| \sum_{k=1}^{2^d} C_k(z)\widehat{h_k}(\omega) \right\|_s^2 \geqslant \alpha_{j+1}\lambda_1 \int_{[0,2\pi]^d} \mathcal{C}(z)^T \mathcal{C}(z)\, d\omega$$

which gives the first inequality in (11) with $A_j = \alpha_{j+1}\lambda_1$ using (1). The second inequality in (11) can be shown in the similar way by noticing

$$R_\ell(\omega) \leqslant \sum_{n\in\mathbb{Z}^d} (4+(\omega+2(2n+n_\ell)\pi)^2)^s |\widehat{\phi_{j+1}}(\omega/2+(2n+n\ell)\pi)|^2 \leqslant 4^s\beta_{j+1}$$

and $\mathcal{H}(\omega)$ is a matrix with trigonometric polynomial entries and hence, the largest eigenvalue λ_{2^d} of $\mathcal{H}(\omega)^T\mathcal{H}(\omega)$ is bounded above. Thus using (2), $B_j = 4^s\beta_{j+1}\lambda_{2^d}$.

The necessary part is trivial. Indeed, if $\mathcal{H}(\omega)$ is singular for some $\omega = \omega_0$, there exists a nonzero vector C_0 such that $\mathcal{H}(\omega_0)C_0 = 0$. Thus we can choose $C_k(z)$ such that $C_k(z) = C_0$

when $z = e^{i\omega_0}$ and $C_k(z) \approx 0$ for other z which is not in a neighborhood of $z = e^{i\omega_0}$. Then, from (12), we can see that the first inequality in (11) cannot hold for any fixed positive number A_j. This completes the proof. □

The second step is to use a technique like the well-known Gram–Schmidt orthonormalization to construct $\psi_{j,k}$ from $g_{j,k}$ such that $\psi_{j,k}$ are orthogonal among each other. It is a standard technique (cf. e.g., [15] for the L_2 setting). For completeness, we outline this technique in the Sobolev space setting here. For convenience, let

$$W_{j,k} := \text{closure}_{H^s(\mathbb{R}^s)}\{\psi_{j,k}(2^j x - m), m \in \mathbb{Z}^d\}, \quad k = 2, \ldots, 2^d.$$

We first choose $\psi_{j,2} = g_{j,2}$. Let

$$\psi_{j,3}(2^j x) = \sum_{m\in\mathbb{Z}^d} (c_{1,j,m}\psi_{j,2}(2^j x - m) + c_{2,j,m} g_{j,3}(2^j x - m))$$

for some coefficients $c_{1,j,m}$ and $c_{2,j,m}$. To define these coefficients, we write them in terms of Fourier transform

$$\begin{aligned}\widehat{\psi_{j,3}}(2\omega) &= C_1(z^2)\widehat{\psi_{j,2}}(2\omega) + C_2(z^2)\widehat{g_{j,3}}(2\omega)\\ &= (C_1(z^2)G_{j,2}(z) + C_2(z^2)G_{j,3}(z))\widehat{\phi}_{j+1}(\omega),\end{aligned}$$

where C_1 and C_2 are discrete Fourier transform of sequences $c_{1,j,m}$'s and $c_{2,j,m}$'s. For convenience, we let $Q_{j,2}(z) = G_{j,2}(z)$ and

$$Q_{j,3}(z) = C_1(z^2)G_{j,2}(z) + C_2(z^2)G_{j,3}(z).$$

In order to have $W_{j,3} \perp W_{j,2}$, the orthogonal condition (2.1) implies that

$$C_1(z^2)E(G_{j,2}(z)\overline{G_{j,2}(z)}\Phi^s_{j+1}) + C_2(z^2)E(G_{j,3}(z)\overline{G_{j,2}(z)}\Phi^s_{j+1}(z)) = 0. \tag{13}$$

By choosing

$$\begin{aligned}C_1(z^2) &= E(G_{j,3}(z)\overline{G_{j,2}(z)}\Phi^s_{j+1}(z)),\\ C_2(z^2) &= -E(G_{j,2}(z)\overline{G_{j,2}(z)}\Phi^s_{j+1}(z)),\end{aligned}$$

we know that Eq. (13) holds and $W_{j,3}$ is perpendicular to $W_{j,2}$. We continue this procedure above. To be more precise, let us show how to construct $\psi_{j,4}$. That is, let

$$\psi_{j,4}(2^j x) = \sum_{m\in\mathbb{Z}^d} (d_{1,j,m}\psi_{j,2}(2^j x - m) + d_{2,j,m}\psi_{j,3}(2^j x - m)) + d_{3,j,m} g_{j,4}(2^j x - m)).$$

In terms of Fourier transform, we have

$$\begin{aligned}\widehat{\psi_{j,4}}(2\omega) &= D_1(z^2)\widehat{\psi_{j,2}}(\omega) + D_2(z^2)\widehat{\psi_{j,3}}(\omega) + D_3(z^2)\widehat{g_{j,4}}(\omega)\\ &= (D_1(z^2)Q_{j,2}(z) + D_2(z^2)Q_{j,3}(z) + D_3(z^2)G_{j,4}(z)))\widehat{\phi_{j+1}}(\omega).\end{aligned}$$

In order to have $W_{j,4} \perp W_{j,2}$ and $W_{j,4} \perp W_{j,3}$, we have the following two equations with three unknowns:

$$\begin{aligned}&D_1(z^2)E(Q_{j,2}(z)\overline{Q_{j,2}(z)}\Phi^s_{j+1}(z)) + D_3(z^2)E(G_{j,4}(z)\overline{Q_{j,2}(z)}\Phi^s_{j+1}(z)) = 0,\\ &D_2(z^2)E(Q_{j,3}(z)\overline{Q_{j,3}(z)}\Phi^s_{j+1}(z)) + D_3(z^2)E(G_{j,4}(z)\overline{Q_{j,3}(z)}\Phi^s_{j+1}(z)) = 0\end{aligned} \tag{14}$$

which is an upper triangular homogeneous linear system. It can be solved easily. A solution may be given below. Let

$$D_1(z^2) = E(Q_{j,3}(z)\overline{Q_{j,3}(z)}\Phi^s_{j+1}(z))E(G_{j,4}(z)\overline{Q_{j,2}(z)}\Phi^s_{j+1}(z)),$$
$$D_2(z^2) = E(Q_{j,2}(z)\overline{Q_{j,2}(z)}\Phi^s_{j+1}(z))E(G_{j,4}(z)\overline{Q_{j,3}(z)}\Phi^s_{j+1}(z)),$$
$$D_3(z^2) = -E(Q_{j,2}(z)\overline{Q_{j,2}(z)}\Phi^s_{j+1})(z))E(Q_{j,3}(z)\overline{Q_{j,3}(z)}\Phi^s_{j+1}(z)).$$

With these Laurent polynomials D_1, D_2, D_3, the two equations in (14) are satisfied simultaneously. Thus, we obtain the desired function $\psi_{j,4}$. Repeating the above constructive steps when $d > 2$, we find $\psi_{j,k}$, $k = 2, \ldots, 2^d$. It is easy to see that $\psi_{j,k}$'s are compactly support when ϕ_j are compactly supported and s is an integer. The above construction shows that the integer translates of $\psi_{j,k}$ form a Riesz basis for $W_{j,k}$ for $k = 2, \ldots, 2^d$ and $W_{j,k}$'s are mutually orthogonal.

Let us write

$$\widehat{\psi_{j,k}}(2\omega) = H_{j,k}(z)\widehat{\phi_{j+1}}(\omega)$$

for $k = 2, \ldots, 2^d$. As in the proof of Lemma 3.4., let

$$\mathcal{H}_j(\omega) = [H_{j,k}(e^{i(\omega+n_\ell\pi)})]_{\substack{2\leqslant k\leqslant 2^d \\ 1\leqslant \ell\leqslant 2^d}}.$$

Since $\psi_{j,k}$ are orthogonal each other, the matrix $\mathcal{H}_j(\omega)^T\mathcal{H}_j(\omega)$ is a diagonal matrix with entries

$$\sum_{\ell=1}^{2^d} |H_{j,k}(e^{i(\omega+n_\ell\pi)})|^2, k = 2, \ldots, 2^d. \tag{15}$$

Let $\lambda_{j,1}$ and $\lambda_{j,2^d-1}$ be the smallest and largest eigenvalues of $\mathcal{H}_j(\omega)^T\mathcal{H}_j(\omega)$. That is,

$$\lambda_{j,1} = \min_{k=2,\ldots,2^d} \min_{\omega\in[0,2\pi]^d} \sum_{\ell=1}^{2^d} |H_{j,k}(e^{i(\omega+n_\ell\pi)})|^2$$

and

$$\lambda_{j,2^d-1} = \max_{k=2,\ldots,2^d} \max_{\omega\in[0,2\pi]^d} \sum_{\ell=1}^{2^d} |H_{j,k}(e^{i(\omega+n_\ell\pi)})|^2.$$

The construction above shows that $\lambda_{j,1} > 0$ and $\lambda_{j,2^d-1} < \infty$. Let

$$\tilde{A}_j = \lambda_{j,1}\alpha_{j+1} \quad \text{and} \quad \tilde{B}_j = \lambda_{j,2^d-1}\beta_{j+1}. \tag{16}$$

Hence, $\tilde{A}_j > 0$ and $\tilde{B}_j < \infty$.

Before we continue to pursue compactly supported prewavelets, let us pause and make an observation. Using proof of Lemma 3.4, we can show that $\phi_j(\cdot)$, $\phi_{j+1}(2\cdot -n_k)$, $k = 2, \ldots, 2^d$ and their integer translates form a Riesz basis for V_{j+1} since the associated $\mathcal{H}(\omega)$ is the matrix in (10) which is nonsingular as in the proof of Theorem 3.3 under the assumption that $P_{j,1}(z) \neq 0$. Therefore, the above Gram–Schmidt procedure can be applied to this set of functions. If we start with ϕ_j and then construct $\psi_{j,k}$, $k = 2, \ldots, 2^d$, then all $\psi_{j,k}$ will be orthogonal to each other and orthogonal to V_j. Hence we have

Proposition 3.5. *Suppose that $P_{j,1}(z) \neq 0$ for all z on the torus T^d. There there exist a set of functions $\psi_{j,k}$ which are a linear combination of ϕ_j and $\phi_{j+1}(2\cdot -n_k)$, $k = 2, \ldots, 2^d$ such that integer translates of $\psi_{j,k}$ form a Riesz basis for $V_{j+1} \ominus V_j$.*

This observation was made during the review process by one of reviewers. This provides a simpler method to construct prewavelets by skipping our first computational step completely. However, all $\psi_{j,k}$ will contain some integer translates of ϕ_j which may not be good for some applications, e.g., image decomposition. Indeed, highpass parts from these $\psi_{j,k}$ may mix with some aliasing terms of the lowpass part from ϕ_j.

We now resume our discussion on constructing compactly supported wavelets. Let us study the Riesz bounds for the integer translates of $\psi_{j,k}$ for all $j \in \mathbb{Z}$ and $k = 2, \ldots, 2^d$. That is, we need to have $\tilde{A}_j$ bounded from below and $\tilde{B}_j$ is bounded from the above.

If we normalize $\psi_{j,k}$ by letting $\psi^*_{j,k} = \psi_{j,k}/\sqrt{\tilde{B}_j}$, then the collection $\{\psi^*_{j,k}, k = 2, \ldots, 2^d, j \in \mathbb{Z}\}$ forms a pre-Riesz basis for $H^s(\mathbb{R}^d)$. Indeed, using the assumptions (i)–(iii) of the MRA and the orthogonal decomposition constructed above, we have

$$H^s(\mathbb{R}^d) = \bigoplus_{j=-\infty}^{\infty} \bigoplus_{k=2}^{2^d} W_{j,k}.$$

For every $f \in H^s(\mathbb{R}^d)$ given in the following form

$$f = \sum_{j\in\mathbb{Z}} \sum_{k=2,\ldots,2^d} \sum_{m\in\mathbb{Z}^d} c_{j,k,m} 2^{jd/2} \psi^*_{j,k}(2^j x - m),$$

we have

$$\begin{aligned}
&\left\| \sum_{j\in\mathbb{Z}} \sum_{k=2,\ldots,2^d} \sum_{m\in\mathbb{Z}^d} c_{j,k,m} 2^{jd/2} \psi^*_{j,k}(2^j x - m) \right\|_s^2 \\
&\quad = \sum_{j\in\mathbb{Z}} \sum_{k=2,\ldots,2^d} \left\| \sum_{m\in\mathbb{Z}^d} c_{j,k,m} 2^{jd/2} \psi^*_{j,k}(2^j x - m) \right\|_s^2 \\
&\quad = \sum_{j\in\mathbb{Z}} \sum_{k=2,\ldots,2^d} \int_{\mathbb{R}^d} (1+\omega^2)^s \left| \sum_{m\in\mathbb{Z}^d} c_{j,k,m} e^{-im\omega/2^j} \frac{1}{2^{jd/2}} \widehat{\psi^*_{j,k}}\left(\frac{\omega}{2^j}\right) \right|^2 d\omega \\
&\quad = \int_{[0,2\pi]^d} \left| C_{j,k}(e^{i\omega}) \right|^2 \sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}(\omega + 2m\pi)^2)^s |\widehat{\psi^*_{jk}}(\omega + 2m\pi)|^2 \, d\omega \\
&\quad \leqslant \sum_{j\in\mathbb{Z}} \sum_{k=2,\ldots,2^d} \sum_{m\in\mathbb{Z}^d} |c_{j,k,m}|^2
\end{aligned}$$

because of the orthogonality and the Riesz upper bounds for $W_{j,k}$, $k = 2, \ldots, 2^d$.

Next we note that the collection $\{2^{jd/2}\psi^*_{j,k}(2^j\cdot -m),\ j\in\mathbb{Z}, k=2,\ldots,2^d, m\in\mathbb{Z}^d\}$ has a property of the linear independence in the following sense that if

$$\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{jd/2}\psi^*_{j,k}(2^jx-m)=0 \tag{17}$$

and $\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2<\infty$, then $c_{j,k,m}=0$ for all j,k and m. Indeed, by (17) and the orthogonality of $W_{j,k}$, we know that

$$\sum_{k=2}^{2^d}\sum_{m\in\mathbb{Z}^d} c_{j,k,m}2^{jd/2}\psi^*_{j,k}(2^jx-m)=0.$$

Then, we use Theorem 3.3 to conclude that $c_{j,k,m}=0$.

Thus, the collection $\{2^{jd/2}\psi^*_{j,k}(2^jx-m),\ j\in\mathbb{Z}, k=2,\ldots,2^d, m\in\mathbb{Z}^d\}$ forms a pre-Riesz basis for $H^s(\mathbb{R}^d)$.

On the other hand, for every $f\in H^s(\mathbb{R}^d)$, we have

$$\begin{aligned}\|f\|_s^2&=\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\left\|\sum_{m\in\mathbb{Z}^d}c_{j,k,m}2^{jd/2}\psi^*_{j,k}(2^j\cdot-m)\right\|_s^2\\&\geqslant\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\tilde{A}_j/\tilde{B}_j\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2.\end{aligned}$$

Hence, under the assumption that $\tilde{A}_j/\tilde{B}_j$ is bounded away from zero for $j\in\mathbb{Z}$, i.e., $\tilde{A}_j/\tilde{B}_j\geqslant A>0$ for all $j\in\mathbb{Z}$, we have

$$\begin{aligned}\|f\|_s^2&=\left\|\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\right\|\sum_{m\in\mathbb{Z}^d}c_{j,k,m}2^{jd/2}\psi^*_{j,k}(2^jx-m)\|_s^2\\&\geqslant A\sum_{j\in\mathbb{Z}}\sum_{k=2,\ldots,2^d}\sum_{m\in\mathbb{Z}^d}|c_{j,k,m}|^2.\end{aligned}$$

Then the collection $\{2^{jd/2}\psi^*_{j,k}(2^j\cdot-m),\ j\in\mathbb{Z}, k=2,\ldots,2^d, m\in\mathbb{Z}^d\}$ is a Riesz basis. This completes the proof of the following

Theorem 3.6. *Suppose that a sequence $\{\phi_j,\ j\in\mathbb{Z}^d\}$ generates an MRA for Sobolev space $H^s(\mathbb{R}^d)$. Suppose that ϕ_j are compactly supported. Let Φ^s_j be the generalized Euler–Frobenius polynomial associated with ϕ_j. Denote $\widehat{\phi_j}(2\omega)=P_j(z)\widehat{\phi_{j+1}}(\omega)$. Write P_j in the polyphase form*

$$P_j(z)=\sum_{\ell=1}^{2^d}e^{in_\ell\omega}P_{j,\ell}(z^2).$$

Suppose that for an integer ℓ between 1 and 2^d, $P_{j,\ell}(z)\neq 0$ for $z\in\mathcal{T}^d$ and

$$E(P_j(z)\overline{P_j(z)}\Phi^s_{j+1}(z))\neq 0\quad\forall z\in\mathcal{T}^d.$$

Then there exist functions $\psi_{j,k}, k = 2, \ldots, 2^d$ *such that the closure* $W_{j,k}$ *of the linear span of integer translates* $\psi_{j,k}(2^j x - m), m \in \mathbb{Z}^d$ *is orthogonal to* V_j *for* $k = 2, \ldots, 2^d$, $V_{j+1} = V_j \bigoplus W_j$ *with*

$$W_j := W_{j,2} \oplus \cdots \oplus W_{j,2^d}$$

and the integer translates of $\psi_{j,k}$*'s form a Riesz basis for* W_j. *All of them forms a pre-Riesz basis for* $H^s(\mathbb{R}^d)$ *after a normalization. Furthermore, if the Riesz bound condition* $\tilde{A}_j/\tilde{B}_j \geqslant A$ *for all* $j \in \mathbb{Z}$, *where* $\tilde{A}_j$ and $\tilde{B}_j$ *defined in* (16), *then the functions* $\psi^*_{j,k}$ *are prewavelets. When s is an integer and* ϕ_j *are compactly supported,* $\psi^*_{j,k}$ *are compactly supported.*

4. Pre-wavelets in $L_2(\mathbb{R}^d)$

Let us first consider $s = 0$ and the standard $L_2(\mathbb{R}^d)$. We usually choose $\phi_j = \phi$ for all *j*. It follows that P_j is independent of *j* and so is Φ^0_j. Hence, $G_{j,k}$'s as constructed in Theorem 3.1 are independent of *j*. From (16) we can see that $\tilde{A}_j$ and $\tilde{B}_j$ are independent of *j* and hence, $\tilde{A}_j/\tilde{B}_j$ is a constant. This demonstrates that the $\psi^*_{j,k}$ constructed above are indeed prewavelets for $L_2(\mathbb{R}^d)$. Thus we have

Theorem 4.1. *If a refinable function* ϕ *generates an MRA for* $L_2(\mathbb{R}^d)$. *If* $\Phi(z) \neq 0$ *for all* $z = e^{i\omega}$ *and one of the polyphase of the mask* $P(\omega)$ *of* ϕ *is not zero for all* $e^{i\omega}$. *Then the functions* $\psi_{j,k}$ *constructed in the previous section are prewavelets for* $L_2(\mathbb{R}^d)$ *satisfying the conditions* 1°–5°.

The construction of prewavelets in $L_2(\mathbb{R}^d)$ improves the constructions given in [21,5,13] in the sense that our method works for any dimension $d \geqslant 1$. From Lemma 3.1. we can see that our construction of prewavelets is straightforward while the method in [15] requires the well-known Quillen–Suslin theorem which has an algorithmic proof based on Gröbner basis approach. Our method is a systematic treatment of the construction of prewavelets while the methods in [17,14,12] are ad hoc one which works only for piecewise linear spline functions. In the following, we shall show that many box spline functions satisfy the two conditions in Theorem 3.4. Thus, our method improve the construction in [8,24] in the sense that our method is a general method. Also, our method provides an explicit condition for the stability of the integer translations of $\psi_{j,k}$ which will be shown to include one of the stability conditions in [3] and is simpler to use than the other stability condition in [3].

Next, we show how to use box splines to construct prewavelets in $L_2(\mathbb{R}^d)$ since multivariate box splines are a very important class of refinable functions. Let us recall the definition of box splines. Let *D* be a set of nonzero vectors in $\mathbb{R}^d$ (counting multiple of a same vector) which span $\mathbb{R}^d$. The box spline ϕ_D associates with the direction set *D* is the function whose Fourier transform is defined by

$$\hat{\phi}_D(\omega) = \prod_{y \in D} \frac{1 - e^{-i\xi\cdot\omega}}{iy \cdot \omega}.$$

It is well-known that box spline ϕ_D is a piecewise polynomial function of degree $\leqslant \#D - d$, where $\#D$ denotes the cardinality of *D*. For more properties of box splines, see [6,4]. In particular, for $d = 2$, $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$, and

$$D = \{\underbrace{e_1, \ldots, e_1}_{\ell}, \underbrace{e_2, \ldots, e_2}_{m}, \underbrace{e_1 + e_2, \ldots, e_1 + e_2}_{n}\},$$

the box spline $\phi_{\ell mn}$ based on such direction set D is called three-direction box spline whose Fourier transform is

$$\hat{\phi}_{\ell mn}(\omega_1, \omega_2) = \left(\frac{1 - e^{-i\omega_1}}{i\omega_1}\right)^{\ell} \left(\frac{1 - e^{-i\omega_2}}{i\omega_2}\right)^{m} \left(\frac{1 - e^{-i(\omega_1+\omega_2)}}{i(\omega_1 + \omega_2)}\right)^{n}.$$

(For computation of the Bézier coefficients of three-direction box splines, see [18].)

It is well-known that box spline ϕ_D generates a bonafide MRA of $L_2(\mathbb{R}^d)$ (cf. [20]) when the direction set D is unimodular, i.e., the determinant of any d directions which span $\mathbb{R}^d$ is 1 or -1 (cf. [4]). The unimodularity also implies $\Phi(\omega) > 0$ by the result in [11]. Let P_D be the mask associated with ϕ_D, i.e., $\widehat{\phi_D}(2\omega) = P_D(z)\widehat{\phi_D}(\omega)$.

Next, let us check that first polyphase component of $P_D(z)$ is not zero, equivalently, $E(P_D(\omega)) \neq 0$. We look at some examples first.

Example 4.2. Consider $\phi_{1,1,1}$. Since $P_{1,1,1}(z) = (1+z_1)(1+z_2)(1+z_1z_2)/8$, we have $E(P_{1,1,1}(z)) = (1 + z_1^2z_2^2)/8$ which is zero when, e.g., $z_1 = \sqrt{-1}$ and $z_2 = \pm 1$. However, if we let $D = \{e_1, e_2, -e_1 - e_2\}$, then D is unimodular and $P_D(z) = (1 + z_1)(1 + z_2)(1 + 1/(z_1z_2))/8$. In this case, $E(P_D(z)) = 1/4$. Therefore, we can apply the procedure in §3 to construct compactly supported prewavelets in $L_2(\mathbb{R}^2)$ as shown at the end of this section.

This motivates us the following:

Theorem 4.3. *Consider the linear box spline in $\mathbb{R}^d$. That is, let*

$$D = \{e_1, \ldots, e_d, -(e_1 + \cdots + e_d)\},$$

where e_i denotes the standard unit vector in $\mathbb{R}^d$ which is 1 *in the ith component while zero in the rest of the components for $i = 1, \ldots, d$. Then $E(P_D(z)) = \frac{1}{2^d}$.*

Proof. It is easy to see that

$$P_D(z) = \prod_{i=1}^{d} \left(\frac{1 + z_i}{2}\right) \left(\frac{1 + 1/(z_1 \cdots z_d)}{2}\right).$$

Then we can see that the even index term $E(P_D(z))$ is only the constant term which is $2/2^{d+1} = 1/2^d$. This completes the proof. □

Example 4.4. Consider $\phi_{2,2,1}$. Since $P_{2,2,1}(z) = (1 + z_1)^2(1 + z_2)^2(1 + z_1z_2)/32$, it is easy to check that

$$E(P_{2,2,1}(z)) = \frac{1}{32}(5z_1^2z_2^2 + z_1^2 + z_2^2 + 1).$$

Since

$$\begin{aligned}32|E(P_{2,2,1}(z)| &= |5 + (z_1z_2)^{-2} + (z_1)^{-2} + (z_2)^{-2}| \\ &> 5 - |(z_1z_2)^{-2}| - |(z_1)^{-2}| - |(z_2)^{-2}| = 2,\end{aligned}$$

we know that $E(P_{2,2,1}(z) \neq 0$ for $z = (z_1, z_2)$ with $|z_1| = |z_2| = 1$.

Example 4.5. Consider $\phi_{2,2,2}$. Similar to the examples above, we have

$$E(P_{2,2,2}(z)) = \frac{1}{64x^2y^2}(10x^2y^2 + x^2 + y^2 + 1 + y^4x^2 + x^4y^4 + x^4y^2).$$

We can easily see that $E(P_{2,2,2}(z)) \neq 0$.

Example 4.6. Consider $\phi_{3,3,3}$. Similar to the examples above, we have

$$E(P_{3,3,3}(z)) = \frac{4(14x^2y^2 + 3 + 3y^4x^2 + 3x^4y^4 + 3x^4y^2 + 3x^2 + 3y^2)}{8^3x^2y^2}.$$

Note that

$$\begin{aligned}&\frac{14x^2y^2 + 3 + 3y^4x^2 + 3x^4y^4 + 3x^4y^2 + 3x^2 + 3y^2)}{8^3x^2y^2} \\ &\quad = 14 + 6\cos(2\omega + 2\eta) + 6\cos(2\omega) + 6\cos(2\eta) \\ &\quad = 2 + 6(1 + \cos(2\omega))(1 + \cos(2\eta)) + 3(1 + \cos(2\omega + 2\eta) + 1 - \cos(2\omega - 2\eta)) \\ &\quad > 0.\end{aligned}$$

This shows that $E(P_{3,3,3}(z)) \neq 0$.

The following theorem is motivated by one of the stability conditions from [3].

Theorem 4.7. *Let D be a direction set which is unimodular. Suppose that d satisfies the so-called the "parity" condition: D can be partitioned into pairs such that each pair (y_1, y_2) satisfies $y_1 = -y_2$. Then ϕ_D satisfies $E(\phi_D(z)) \neq 0$ for $z \in T^d$.*

Proof. Because of the parity property, $D = D_1 \cup D_2$ such that $\#(D_1) = \#(D_2)$ and for each direction $y_1 \in D_1$, there is a direction $d_2 \in D_2$ with $y_1 = -y_2$. Let us write

$$P_{D_1}(z) = \sum_{j \in \{0,1\}^d} e^{ij\omega} P_j(z^2)$$

in polyphase form. Then since $P_D(z) = P_{D_1}(z)P_{D_1}(1/z)$, we have

$$E(P_D(z)) = \sum_{j \in \{0,1\}^d} |P_j(z^2)|^2.$$

Suppose that $E(P_D(z)) = 0$ for some z. Then it follows that $P_j(z^2) = 0$ for all $j \in \{0, 1\}^d$ and hence, $P_{D_1}((-1)^j z) = 0$ for all $j \in \{0, 1\}^d$. That is, in the abused notation, we have

$P_{D_1}(\omega + j\pi) = 0$'s. It follows that

$$0 = \sum_{j\in\{0,1\}^d} |P_{D_1}(\omega + j\pi)|^2 \sum_{m\in\mathbb{Z}^d} |\widehat{\phi_{D_1}}(\omega + j\pi + 2m\pi)|^2$$
$$= \sum_{m\in\mathbb{Z}^d} |\widehat{\phi_{D_1}}(2\omega + 2m\pi)|^2. \tag{18}$$

However, since D is unimodular, D_1 is unimodular by the definition. That is, the right-hand side of Eq. (18) is not zero and hence, we obtain a contradiction. □

Next, we consider the construction of prewavelets based on tensor products of box splines.

Theorem 4.8. *Suppose that the both masks P_{D_1} and P_{D_2} associated with box spline ϕ_{D_1} and ϕ_{D_2}, respectively, satisfy the two conditions in Theorem* 3.4. *Then the mask $P_{D_1}(\omega)P_{D_2}(\eta)$ of the tensor product of the box spline $\phi_{D_1}(x)\phi_{D_2}(y)$ satisfies the two conditions too.*

Proof. We note that

$$E(P_{D_1}(\omega)P_{D_2}(\eta)) = E(P_1(\omega))E(P_2(\eta)) \quad \text{and} \quad \Phi_{D_1\cup D_2}(z_1, z_2) = \Phi_{D_1}(z_1)\Phi_{D_2}(z_2).$$

The result follows. That is, if we can use the method in §3 to construct compactly supported prewavelets in $L_2(\mathbb{R}^d)$ spaces based on box splines ϕ_{D_1} and ϕ_{D_2}, then we can construct prewavelets based on their tensor product $\phi_{D_1}(x)\phi_{D_2}(y)$ in $L_2(\mathbb{R}^{2d})$. □

Finally, we present some concrete examples of compactly-supported B-spline prewavelets in $L_2(\mathbb{R})$ and box spline prewavelets in $L_2(\mathbb{R}^2)$.

Fix an integer $N > 0$. The Nth order B-spline $\phi^{(N)}$ is the function whose Fourier transform is

$$\widehat{\phi^{(N)}}(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^N.$$

Let us write

$$\widehat{\phi^{(N)}}(2\omega) = P^{(N)}(z)\widehat{\phi^{(N)}}(\omega)$$

with $P^{(N)}(z) = \left(\frac{1+z}{2}\right)^N$.

It is clear that

$$P^{(N)}(z) - P^{(N)}(-z) = \left(\frac{1+z}{2}\right)^N - \left(\frac{1-z}{2}\right)^N \neq 0$$

for any $z = e^{i\omega}$, $\omega \in [0, 2\pi]$. It follows that the polyphase $P_1(z^2)$ associated with $\phi^{(N)}$ is never zero. Next by Lemma 32, we know

$$E(P^{(N)}(z)\overline{P^{(N)}(z)}\Phi(z)) = \tfrac{1}{2}\Phi(z^2).$$

By Poisson summation formula,

$$\Phi(z) = \sum_{m\in\mathbb{Z}} \langle \phi^{(N)}(\cdot), \phi^{(N)}(\cdot - m)\rangle z^m$$
$$= \sum_{m\in\mathbb{Z}} |\widehat{\phi^{(N)}}(\omega + 2m\pi)|^2$$

which is never zero (cf. [23]). This shows that $E(P^{(N)}(z)\overline{P^{(N)}(z)}\Phi(z)) \neq 0$. That is, the two conditions in Theorem 3.3 are satisfied. Hence, the above discussion verifies that all B-spline functions can be used to construct prewavelets for $L_2(\mathbb{R})$. We now present two examples of prewavelets in $L_2(\mathbb{R})$ as follows. Note that our prewavelets have a larger support than those constructed in [9]. The purpose of the examples is to show the detail of our constructive procedure.

Example 4.9. Consider linear B-spline $\phi^{(2)}$ with $P^{(2)}(z) = (1+z)^2/4$. It is easy to see that

$$\Phi(z) = \tfrac{1}{6}z^{-1} + \tfrac{4}{6} + \tfrac{1}{6}z.$$

Indeed, by using the symmetric property of B-splines, i.e., $\phi^{(2)}(x) = \phi^{(2)}(2-x)$ for linear B-spline $\phi^{(2)}$,

$$\begin{aligned}\Phi(z) &= \sum_{m\in\mathbb{Z}} \langle \phi^{(2)}, \phi^{(2)}(\cdot - m)\rangle_0 z^m \\ &= \sum_{m\in\mathbb{Z}} \langle \phi^{(2)}, \phi^{(2)}(2+m-\cdot)\rangle_0 z^m \\ &= \sum_{m\in\mathbb{Z}} \phi^{(4)}(2+m)z^m.\end{aligned}$$

Thus, we know $E(P^{(2)}(z)P^{(2)}(1/z)\Phi(z)) = \frac{1}{2}\Phi(z^2)$ by Lemma 3.2 and

$$E(zP^{(2)}(1/z)\Phi(z))P^{(2)}(z) = \frac{1}{24z^2}(10z^2 + z^4 + 1).$$

Thus, using the formulas in Theorem 3.1, we have

$$G(z) = \frac{1}{96z^2}(z^6 - 6z^5 + 11z^4 - 12z^3 + 11z^2 - 6z + 1)$$

That is, the prewavelet associated with linear B-spline $\phi := \phi^{(2)}$ is

$$\begin{aligned}\psi(x) = \tfrac{1}{96}(&\phi(2x+2) - 6\phi(2x+1) + 11\phi(2x) - 12\phi(2x-1) \\ &+11\phi(2x-2) - 6\phi(2x-3) + \phi(2x-4)).\end{aligned}$$

We can easily verify that

$$\int_{-\infty}^{\infty} \phi(x-m)\psi(x)\,dx = 0$$

for all integer $m \in \mathbb{Z}$.

Example 4.10. Consider cubic B-spline $\phi^{(4)}$ with $P^{(4)}(z) = (1+z)^4/16$. Using the same argument in Example 4.9, we have

$$\Phi(z) = \frac{1}{5040z^3} + \frac{1}{42z^2} + \frac{397}{1680z} + \frac{151}{315} + \frac{397}{1680}z + \frac{1}{42}z^2 + \frac{1}{5040}z^3.$$

Thus, $E(P^{(2)}(z)P^{(2)}(1/z)\Phi(z)) = \frac{1}{2}\Phi(z^2)$ which is

$$\begin{aligned}&E(zP^{(4)}(1/z)\Phi(z)) \\ &\quad= \frac{1}{80640z^6}(18482z^4 + z^{10} + 18482z^6 + 1677z^2 + 1 + 1677z^8).\end{aligned}$$

Hence, by using the formula in Theorem 3.1, $G(z)=\sum_{k=-6}^{8} g_k z^k$ with coefficients g_k as follows:

$$g_{-6}=\frac{-1}{1290240},\quad g_{-5}=\frac{31}{322560},\quad g_{-4}=\frac{-187}{143360},$$

$$g_{-3}=\frac{1081}{161280},\quad g_{-2}=\frac{-1903}{86016},\quad g_{-1}=\frac{17953}{322560},$$

$$g_{0}=\frac{-131051}{1290240},\quad g_{1}=\frac{1441}{11520},\quad g_{2}=\frac{-131051}{1290240},$$

$$g_{3}=\frac{17953}{322560},\quad g_{4}=\frac{-1903}{86016},\quad g_{5}=\frac{1081}{161280},$$

$$g_{6}=\frac{-187}{143360},\quad g_{7}=\frac{31}{322560},\quad g_{8}=\frac{-1}{1290240}.$$

That is, the prewavelet associated with cubic B-spline $\phi := \phi^{(4)}$ is

$$\begin{aligned}\psi(x)=&\frac{1}{1290240}(\phi(2x+6)-124\phi(2x+5)+1683\phi(2x+4)\\ &-8648\phi(2x+3)+28545\phi(2x+2)-71812\phi(2x+1)+131051\phi(2x))\\ &-161392\phi(2x-1)+131051\phi(2x-2)-71812\phi(2x-3)+28545\phi(2x-4)\\ &-8648\phi(2x-5)+1683\phi(2x-6)-124\phi(2x-7)+\phi(2x-8)).\end{aligned}$$

We can easily verify that ψ is orthogonal to the integer translates of ϕ using the computer program MAPLE.

Next we present an example of box spline prewavelets in $L_2(\mathbb{R}^2)$. Note that our prewavelets have a larger support than those constructed in [17,14,15]. The purpose of this example is to show the detail of our constructive procedure.

Example 4.11. We consider box spline $\tilde{B}_{1,1,1}=\phi_D$ based on $D=\{e^1,e^2,-(e^1+e^2)\}$ and construct compactly supported prewavelets in $L_2(\mathbb{R}^2)$. Clearly,

$$P(z)=\frac{1+z_1}{2}\frac{1+z_2}{2}\frac{1+1/(z_1z_2)}{2}.$$

and $\Phi^s(z)=\frac{1}{2}+\frac{1}{12}(z_1+z_2+1/z_1+1/z_2+z_1z_2+1/(z_1z_2))$. It is easy to verify that $E(P(z)\overline{P(z)}\Phi^s(z))=\frac{1}{4}\Phi^s(z^2)$. Using a computer algebra program Maple, we obtain the Laurent polynomials for $G_1,\ldots,G_4$ and Q_2,Q_3,Q_4. They are as follows:

$$768G_1(z_1,z_2)=\begin{bmatrix} z_1^{-3}\\ z_1^{-2}\\ z_1^{-1}\\ 1\\ z_1\\ z_1^2\\ z_1^3\end{bmatrix}^T\begin{bmatrix} -1&-1&-1&-1&0&0&0\\ -1&14&-2&14&-1&0&0\\ -1&-2&-19&-19&-2&-1&0\\ -1&14&-19&60&-19&14&-1\\ 0&-1&-2&-19&-19&-2&-1\\ 0&0&-1&14&-2&14&-1\\ 0&0&0&-1&-1&-1&-1\end{bmatrix}\begin{bmatrix} z_2^{-3}\\ z_2^{-2}\\ z_2^{-1}\\ 1\\ z_2\\ z_2^2\\ z_2^3\end{bmatrix},$$

$$768G_2(z_1,z_2)=\begin{bmatrix} z_1^{-1}\\ 1\\ z_1\\ z_1^2\\ z_1^3\end{bmatrix}^T\begin{bmatrix} -2 & 14 & -10 & 6 & 0 & 0 & 0\\ -2 & -4 & -12 & -20 & -10 & 0 & 0\\ 0 & 14 & -12 & 76 & -12 & 14 & 0\\ 0 & 0 & -10 & -20 & -12 & -4 & -2\\ 0 & 0 & 0 & 6 & -10 & 14 & -2\end{bmatrix}\begin{bmatrix} z_2^{-3}\\ z_2^{-2}\\ z_2^{-1}\\ 1\\ z_2\\ z_2^2\\ z_2^3\end{bmatrix},$$

$$768G_3(z_1,z_2)=\begin{bmatrix} z_1^{-3}\\ z_1^{-2}\\ z_1^{-1}\\ 1\\ z_1\\ z_1^2\\ z_1^3\end{bmatrix}^T\begin{bmatrix} -2 & -2 & 0 & 0 & 0\\ 14 & -4 & 14 & 0 & 0\\ -10 & -12 & -12 & -10 & 0\\ 6 & -20 & 76 & -20 & 6\\ 0 & -10 & -12 & -12 & -10\\ 0 & 0 & 14 & -4 & 14\\ 0 & 0 & 0 & -2 & -2\end{bmatrix}\begin{bmatrix} z_2^{-1}\\ 1\\ z_2\\ z_2^2\\ z_2^3\end{bmatrix},$$

and

$$768G_4(z_1,z_2)=\begin{bmatrix} z_1^{-1}\\ 1\\ z_1\\ z_1^2\\ z_1^3\end{bmatrix}^T\begin{bmatrix} 6 & -10 & 14 & -2 & 0\\ -10 & -20 & -12 & -4 & -2\\ 14 & -12 & 76 & -12 & 14\\ -2 & -4 & -12 & -20 & -10\\ 0 & -2 & 14 & -10 & 6\end{bmatrix}\begin{bmatrix} z_2^{-1}\\ 1\\ z_2\\ z_2^2\\ z_2^3\end{bmatrix}.$$

Since $Q_2=G_2$, we now give Q_3 as follows. Let

$$10616832Q_3=[z_1^{-7},\ldots,z_1^{-1},1,z_1,\ldots,z_1^{-7}]\mathcal{Q}[z_2^{-7},\ldots,z_2^{-1},1,z_2,\ldots,z_2^{-9}]^T$$

with matrix $\mathcal{Q}$ being a of size 15×17 defined by $\mathcal{Q}=[\mathcal{Q}_1\mathcal{Q}_2]$ and $\mathcal{Q}_1$ of size 15×9 and $\mathcal{Q}_2$ of size 15×8, where

$$\mathcal{Q}_1=\begin{bmatrix} -1 & 5 & -1 & 9 & 1 & 3 & 1 & -1 & 0\\ 1 & -2 & -26 & -2 & -52 & 2 & -22 & 2 & 3\\ -2 & 7 & 11 & 73 & 71 & 63 & 85 & -17 & 27\\ 2 & -4 & -46 & 24 & -516 & 144 & -628 & 168 & -158\\ -1 & -1 & 33 & 37 & 369 & 271 & 773 & 15 & 480\\ 1 & -2 & -22 & 70 & -672 & 714 & -3110 & 1402 & -2285\\ 0 & -3 & 21 & -45 & 385 & -63 & 1713 & 291 & 2057\\ 0 & 0 & -2 & 44 & -228 & 700 & -2462 & 2712 & -6520\\ 0 & 0 & 0 & -18 & 70 & -174 & 746 & -260 & 2057\\ 0 & 0 & 0 & 0 & -4 & 96 & -340 & 792 & -2285\\ 0 & 0 & 0 & 0 & 0 & -28 & 74 & -150 & 480\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 52 & -158\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14 & 27\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

and

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -14 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 52 & 2 & 0 & 0 & 0 & 0 & 0 & \\ -150 & 74 & -28 & 0 & 0 & 0 & 0 & \\ 792 & -340 & 96 & -4 & 0 & 0 & 0 & \\ -260 & 746 & -174 & 70 & -18 & 0 & 0 & \\ 2712 & -2462 & 700 & -228 & 44 & -2 & 0 & 0 \\ 291 & 1713 & -63 & 385 & -45 & 21 & -3 & 0 \\ 1402 & -3110 & 714 & -672 & 70 & -22 & -2 & 1 \\ 15 & 773 & 271 & 369 & 37 & 33 & -1 & -1 \\ 168 & -628 & 144 & -516 & 24 & -46 & -4 & 2 \\ 2 & -22 & 2 & -52 & -2 & -26 & -2 & 1 \\ -1 & 1 & 3 & 1 & 9 & -1 & 5 & -1 \end{bmatrix}.$$

The expression for Q_4 involves a matrix of size about 51×51. Due to the space limit, we omit the details for Q_4.

5. Pre-Riesz and Riesz bases in $H^s(\mathbb{R}^d)$

We now consider $s > 0$. First, we consider a refinable function ϕ which generates an MRA for $H^s(\mathbb{R}^d)$. ϕ satisfies the refinable equation $\widehat{\phi}(2\omega) = P(z)\widehat{\phi}(\omega)$. Let us examine the boundedness of $\tilde{A}_j/\tilde{B}_j$. In this case, using the expression of Φ_j^s in the proof of Lemma 3.2, we have $\Phi_j^s \longrightarrow \Phi^0$ for $j \to -\infty$. Similarly, using the formula for $G_{j,k}$ in Lemma 3.1, we conclude that $G_{j,k} \longrightarrow G_k$ for $j \to -\infty$ where

$$G_k = \frac{1}{2^{d/2}} E(P(z)\overline{P(z)}\Phi^0(z))e^{in_k\omega} - \frac{1}{2^{d/2}} E(e^{in_k\omega}\overline{P(z)}\Phi^0(z))P(z),$$

for $k = 2, \ldots, 2^d$. Since the Gram–Schmidt orthogonalization procedure is finite, $H_{j,k} \longrightarrow H_k, k = 2, \ldots, 2^d$ which are obtained from $G_k, k = 2, \ldots, 2^d$ by the Gram–Schmidt procedure. The smallest and largest eigenvalues $\lambda_{j,1}$ and $\lambda_{j,2^d-1}$ converge to the λ_1 and λ_{2^d-1} of the corresponding matrix $\mathcal{H}(\omega)^T\mathcal{H}(\omega)$ where

$$\mathcal{H}(\omega) = [H_k(e^{i(\omega+n_\ell\pi)})]_{\substack{2\leqslant k\leqslant 2^d \\ 1\leqslant \ell\leqslant 2^d}}$$

From (1) and (2), we have $\alpha_j \longrightarrow \alpha$ and $\beta_j \longrightarrow \beta$, where

$$\alpha = \min_{\omega\in[0,2\pi]^d} \sum_{m\in\mathbb{Z}^d} |\widehat{\phi}(\omega + 2m\pi)|^2$$

and

$$\beta = \max_{\omega\in[0,2\pi]^d} \sum_{m\in\mathbb{Z}^d} |\widehat{\phi}(\omega + 2m\pi)|^2.$$

Thus, $\tilde{A}_j/\tilde{B}_j \geqslant A > 0$ for $j \to -\infty$. It thus follows that

Proposition 5.1. *Suppose that ϕ generates an MRA for $H^s(\mathbb{R}^d)$. Suppose that $\Phi^0(z) > 0$ and the first polyphase component $E(P) \neq 0$. Then for any fixed integer N, $\psi^*_{j,k}$, $k = 2, \ldots, 2^d$, $j = -\infty, \ldots, N$ constructed in §3 are prewavelets for subspace $H^s_N(\mathbb{R}^d) = \bigoplus_{j=-\infty}^{N} \bigoplus_{k=2}^{2^d} W_{j,k}$ of $H^s(\mathbb{R}^d)$. Hence, the integer translates of $\psi^*_{j,k}$, $k = 2, \ldots, 2^d$, $j = -\infty, \ldots, N$ form a Riesz basis for $H^s_N(\mathbb{R}^s)$.*

Let us further study the case when $j \to +\infty$. First we look at (1) and (2). Note that

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^d} (1 + 2^{2j}(\omega + 2m\pi)^2)^s |\widehat{\phi}(\omega + 2m\pi)|^2\\ &\quad = 2^{2js} \sum_{m\in\mathbb{Z}^d} (2^{-2j} + (\omega + 2m\pi)^2)^s |\widehat{\phi}(\omega + 2m\pi)|^2.\end{aligned}$$

After the factor 2^{2js},

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^d} (2^{-2j} + (\omega + 2m\pi)^2)^s |\widehat{\phi}(\omega + 2m\pi)|^2\\ &\quad \longrightarrow \sum_{m\in\mathbb{Z}^d} (\omega + 2m\pi)^{2s} |\widehat{\phi}(\omega + 2m\pi)|^2 =: \Phi^s_*(z).\end{aligned}$$

That is,

$$\Phi^s_*(z) = \sum_{m\in\mathbb{Z}^d} \langle \phi(x), \phi(x-m)\rangle_{*,s} z^m$$

where

$$\langle f, g\rangle_{*,s} = \frac{1}{(2\pi)^d} \int \omega^{2s} \hat{f}(\omega)\overline{\hat{g}(\omega)}\, d\omega.$$

We have $\Phi^s_* = \lim_{j\to+\infty} \Phi^s_j / 2^{2js}$.

Meanwhile, for $G_{j,k}(\omega)$, $G_{j,k}(\omega)/2^{2js}$ converge to

$$E(P(z)\overline{P(z)}\Phi^s_*(z))e^{in_k\omega} - E(e^{in_k\omega}\overline{P(z)}\Phi^s_*(z))P(\omega) =: G_{*,k}(\omega)$$

using the formula for $G_{j,k}$ in Lemma 3.1. Our main assumption in Lemma 3.1 and Theorem 3.3 requires

$$E(P(z)\overline{P(z)}\Phi^s_*(z)) = \frac{1}{2^d} \sum_{m\in\mathbb{Z}^d} (2\omega + 2m\pi)^{2s} |\hat{\phi}(2\omega + 2m\pi)|^2$$

be nonzero. However, it is zero when $\omega = 0$ if ϕ has a linear accuracy, i.e., $\hat{\phi}(2m\pi) = 0$ for all $m \in \mathbb{Z}^d \backslash \{0\}$. In other words, the integer translates of ϕ reproduce constant functions by the well-known Strang and Fix condition. Especially, for any box spline function ϕ, ϕ satisfies $\widehat{\phi}(2m\pi) = 0$ except for $m = 0$. Thus

$$\tilde{A}_j / \tilde{B}_j = \frac{\tilde{A}_j/(2^{2js})}{\tilde{B}_j/(2^{2js})}$$

may be zero for $j \to +\infty$. Our construction based on one refinable function will not give prewavelets for any Sobolev space $H^s(\mathbb{R}^d)$ with $s > 0$. (see [19] for the first and original reason).

In order to construct a Riesz basis, we have to use nonstationary scaling functions. One approach is to modify a scaling function into nonstationary scaling functions. Suppose ϕ generates an MRA for $H^s(\mathbb{R}^d)$. We define

$$\widehat{\phi_j}(\omega) = \frac{1}{(1+2^{2j}\omega^2)^{s/2}}\widehat{\phi}(\omega)$$

for $j \in \mathbb{Z}$. It is easy to see that $\widehat{\phi_j}$ is refinable and satisfies

$$\widehat{\phi_j}(\omega) = P(e^{i\omega/2})\widehat{\phi_{j+1}}(\omega/2),$$

where P is the mask associated with ϕ, i.e., $\widehat{\phi}(\omega) = P(e^{i\omega/2})\widehat{\phi}(\omega/2)$. Thus, $P_j = P$ for all $j \in \mathbb{Z}^d$. Next it is also easy to see $\widehat{\phi_j} \in H^s(\mathbb{R}^d)$. Let V_j be the span of all $\phi_j(2^j \cdot -m)$, $m \in \mathbb{Z}^d$. We can show that V_j, $j \in \mathbb{Z}^d$ form an MRA for $H^s(\mathbb{R}^d)$ by using a generalized version of results in [2]. To construct prewavelets, let us check

$$\begin{aligned}\Phi_j^s(z) &= \sum_{m\in\mathbb{Z}^d} (1+2^{2j}(\omega+2m\pi)^2)^s |\widehat{\phi_j}(\omega+2m\pi)|^2 \\ &= \sum_{m\in\mathbb{Z}^d} |\widehat{\phi}(\omega+2m\pi)|^2 \neq 0.\end{aligned}$$

Without loss of generality, we may assume that the first component of the polyphase form of P is not zero. That is, $E(P(z)) \neq 0$. Notice that from the computation in Lemma 3.1, $G_{j,k}$'s are independent of j and so are $\tilde{A}_j/\tilde{B}_j$. Therefore, the functions $\psi^*_{j,k}$ are prewavelets for $H^s(\mathbb{R}^d)$ by Theorem 3.6. Only thing that is not satisfactory about this approach is the support of the prewavelets. That is, the support of the prewavelets so constructed may not be compactly supported in general even though ϕ is compactly supported. For the univariate setting, the support of ϕ_j is half-spaced when $s = 1$ and the prewavelets involving some differences of integer translates of ϕ_j may be compactly supported. This issue is still under further investigation. The above discussion can be summarized in the following

Proposition 5.2. *Suppose that ϕ generates an MRA for $H^s(\mathbb{R}^d)$. Then ϕ can be so modified to have ϕ_j, $j \in \mathbb{Z}$ that the construction in §3 produce prewavelets $\psi^*_{j,k}$ for $H^s(\mathbb{R}^d)$.*

Finally, we show that box splines can be used to construct pre-Riesz bases in $H^s(\mathbb{R}^d)$. Recall D is a direction set, box spline ϕ_D and the associated mask P_D from the previous section. Suppose again D is unimodular. The smoothness order $m(D)$ of ϕ_D which can be found by using the following relation

$$m(D) + 1 = \min\{\#(Y), Y \subset D,\ \mathrm{span}(D\backslash Y) \neq \mathbb{R}^d\}.$$

(Cf. [4]). That is, $\phi_D \in C_0^{m(D)-1}(\mathbb{R}^d)$ is $m(D)-1$ continuously differentiable and the derivatives of $m(D)$ order is Lipchitz. Thus, $\phi_D \in H^s(\mathbb{R}^d)$ for all $s \leqslant m(D)$. It is known that ϕ_D generates an MRA for $H^s(\mathbb{R}^d)$ (cf. [19]). Mainly we need to verify the two conditions in Theorem 3.4. We have already explained one of the conditions in the previous section. Only the other one is needed to be shown as follows.

Theorem 5.3. *Suppose that the direction set D is unimodular. Then*

$$E(P_D(z)\overline{P_D(z)}\Phi_D^s(z)) > 0$$

for $0 \leqslant s \leqslant m(D)$.

Proof. Recall the proof of Lemma 3.2. When $s = 0$, we have

$$E(P_D(\omega)\overline{P_D(\omega)}\Phi_D^s(z)) = 2^{-d}\Phi_D^s(z^2).$$

Recall that by Poisson summer formula,

$$\Phi_D^0(z) = \sum_{m\in\mathbb{Z}^d} |\widehat{\phi_D}(\omega + 2m\pi)|^2$$

which is strictly bigger than 0 when D is unimodular (cf. [11]). Thus, $\Phi_D^0(z^2) > 0$.

We now consider the situation when $s > 0$. Using the notation and method in Lemma 3.2, we have

$$\begin{aligned} &E(P_D(z)\overline{P_D(z)}\Phi_D^s(z)) \\ &= \frac{1}{2^d}\sum_{m\in\mathbb{Z}^d} (1 + 4|\omega + 2m\pi|^2)^s |\widehat{\phi_D}(2\omega + 2m\pi)|^2 \\ &\geqslant \frac{1}{2^d}\sum_{m\in\mathbb{Z}^d} |\widehat{\phi_D}(2\omega + 2m\pi)|^2 \end{aligned}$$

which is strictly bigger than 0 when D is unimodular. □

Hence, we can use many box splines to construct compactly supported pre-Riesz bases for $H^s(\mathbb{R}^d)$ since many box splines ϕ_D satisfy $E(P_D) \neq 0$ as shown in the previous section. Finally let us present an example of orthogonal decomposition in $H^1(\mathbb{R}^2)$ to illustrate the constructive steps.

Example 5.4. We consider box spline $\tilde{B}_{1,1,1} = \phi_D$ based on $D = \{e^1, e^2, -(e^1 + e^2)\}$ and construct an orthogonal decomposition in $H^1(\mathbb{R}^2)$. For simplicity, we will construct them in V_1 only. It is easy to verify that

$$\begin{aligned} \Phi^1(z) &= \sum_{m\in\mathbb{Z}^2} \langle 2\phi_D(2x), 2\phi_D(2x-m)\rangle_1 z^m \\ &= \frac{1}{2} + \frac{1}{12}(z_1 + z_2 + 1/z_1 + 1/z_2 + z_1z_2 + 1/(z_1z_2)) \\ &\quad + 4(2 - z_1 - 1/z_1) + 4(2 - z_2 - 1/z_2). \end{aligned}$$

Using Maple, we have

$$E(P(z)\overline{P(z)}\Phi^1(z)) = \tfrac{33}{2} - \tfrac{47}{12}z_1 - \tfrac{47}{12}z_2 - \tfrac{47}{12}z_1^{-1} - \tfrac{47}{12}z_2^{-1} + \tfrac{1}{12}z_1z_2 + \tfrac{1}{12}z_1^{-1}z_2^{-1} > 0.$$

As in Example 4.2, we obtain the Laurent polynomials for $G_1, \ldots, G_4$ and Q_2, Q_3, Q_4. They are as follows:

$$384G_1(z_1,z_2) = \begin{bmatrix} z_1^{-3} \\ z_1^{-2} \\ z_1^{-1} \\ 1 \\ z_1 \\ z_1^2 \\ z_1^3 \end{bmatrix}^T \begin{bmatrix} -1 & -1 & 47 & 47 & 0 & 0 & 0 \\ -1 & 14 & 46 & -658 & 47 & 0 & 0 \\ 47 & 46 & -211 & -163 & 94 & 47 & 0 \\ 47 & -658 & -163 & 2748 & -163 & -658 & 47 \\ 0 & 47 & 94 & -163 & -211 & 46 & 47 \\ 0 & 0 & 47 & -658 & 46 & 14 & -1 \\ 0 & 0 & 0 & 47 & 47 & -1 & -1 \end{bmatrix} \begin{bmatrix} z_2^{-3} \\ z_2^{-2} \\ z_2^{-1} \\ 1 \\ z_2 \\ z_2^2 \\ z_2^3 \end{bmatrix},$$

$$384G_2(z_1,z_2) = \begin{bmatrix} z_1^{-1} \\ 1 \\ z_1 \\ z_1^2 \\ z_1^3 \end{bmatrix}^T \begin{bmatrix} 23 & 31 & -29 & -405 & 0 & 0 & 0 \\ 23 & 46 & -6 & -58 & -29 & 0 & 0 \\ 0 & -353 & -6 & 1526 & -6 & -353 & 0 \\ 0 & 0 & -29 & -58 & -6 & 46 & 23 \\ 0 & 0 & 0 & -405 & -29 & 31 & 23 \end{bmatrix} \begin{bmatrix} z_2^{-3} \\ z_2^{-2} \\ z_2^{-1} \\ 1 \\ z_2 \\ z_2^2 \\ z_2^3 \end{bmatrix},$$

$$384G_3(z_1,z_2) = \begin{bmatrix} z_1^{-3} \\ z_1^{-2} \\ z_1^{-1} \\ 1 \\ z_1 \\ z_1^2 \\ z_1^3 \end{bmatrix}^T \begin{bmatrix} 23 & 23 & 0 & 0 & 0 \\ 31 & 46 & -353 & 0 & 0 \\ -29 & -6 & -6 & -29 & 0 \\ -405 & -58 & 1526 & -58 & -405 \\ 0 & -29 & -6 & -6 & -29 \\ 0 & 0 & -353 & 46 & 31 \\ 0 & 0 & 0 & 23 & 23 \end{bmatrix} \begin{bmatrix} z_2^{-1} \\ 1 \\ z_2 \\ z_2^2 \\ z_2^3 \end{bmatrix},$$

and

$$384G_4(z_1,z_2) = \begin{bmatrix} z_1^{-1} \\ 1 \\ z_1 \\ z_1^2 \\ z_1^3 \end{bmatrix}^T \begin{bmatrix} -45 & -53 & -329 & 47 & 0 \\ -53 & -106 & -6 & 94 & 47 \\ -329 & -6 & 1478 & -6 & -329 \\ 47 & 94 & -6 & -106 & -53 \\ 0 & 47 & -329 & -53 & -45 \end{bmatrix} \begin{bmatrix} z_2^{-1} \\ 1 \\ z_2 \\ z_2^2 \\ z_2^3 \end{bmatrix}.$$

Since $Q_2 = G_2$, we now give Q_3 as follows. Let

$$10616832 Q_3 = [z_1^{-7}, \ldots, z_1^{-1}, 1, z_1, \ldots, z_1^{-7}]\mathcal{Q}[z_2^{-7}, \ldots, z_2^{-1}, 1, z_2, \ldots, z_2^{-9}]^T$$

with matrix $\mathcal{Q}$ being of size 15×17 defined by $\mathcal{Q} = [\mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3]$ and $\mathcal{Q}_1$ of size 15×6, $\mathcal{Q}_2$ of size 15×5, and $\mathcal{Q}_3$ of size 15×7, where

$$\mathcal{Q}_1 = \begin{bmatrix} 23 & -1027 & -6745 & 131001 & 481609 & -3766053 \\ 1081 & 46 & -148514 & -13490 & 5173988 & 963218 \\ -2162 & 135607 & 547283 & -8520383 & -23032993 & 25246911 \\ -139702 & -4324 & 9938450 & 1094520 & -101068020 & -46052496 \\ 50807 & -4199017 & -11095431 & 37245541 & 114451161 & -53158505 \\ 4178065 & 101614 & -66131662 & -22186538 & 843672264 & 227807802 \\ 0 & 4424157 & 12823245 & -35255637 & -125967191 & 34180785 \\ 0 & 0 & 175074718 & 25544876 & -2148644820 & -229747844 \\ 0 & 0 & 0 & 9169230 & 28041046 & -14774238 \\ 0 & 0 & 0 & 0 & 498508508 & 30537216 \\ 0 & 0 & 0 & 0 & 0 & -684508 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{Q}_2 = \begin{bmatrix} -10123847 & -5750497 & 0 & 0 & 0 \\ -30765934 & -20247694 & 160893267 & 0 & 0 \\ 50904199 & -17700717 & -11180078 & 0 & \\ 755212076 & 195925704 & -2063036150 & -15153740 & 488505122 \\ -322886659 & -166746753 & 91661280 & 70952634 & 7691738 \\ -4749939230 & -599720822 & 10035141787 & -12603144 & -4114923076 \\ 345950793 & 231628203 & -67813519 & -159649796 & -121175494 \\ 10153980610 & 464093784 & -19712983960 & 464093784 & 10153980610 \\ -121175494 & -159649796 & -67813519 & 231628203 & 345950793 \\ -4114923076 & -12603144 & 10035141787 & -599720822 & -4749939230 \\ 7691738 & 70952634 & 91661280 & -166746753 & -322886659 \\ 488505122 & -15153740 & -2063036150 & 195925704 & 755212076 \\ 0 & -11180078 & -17700717 & 50904199 & 98444461 \\ 0 & 0 & 160893267 & -20247694 & -30765934 \\ 0 & 0 & 0 & -5750497 & -10123847 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -684508 & 0 & 0 & 0 & 0 & 0 \\ 30537216 & 498508508 & 0 & 0 & 0 & 0 \\ -14774238 & 28041046 & 9169230 & 0 & 0 & 0 \\ -229747844 & -2148644820 & 25544876 & 175074718 & 0 & 0 \\ 34180785 & -125967191 & -35255637 & 12823245 & 4424157 & 0 \\ 227807802 & 843672264 & -22186538 & -66131662 & 101614 & 4178065 \\ -53158505 & 114451161 & 37245541 & -11095431 & -4199017 & 50807 \\ -46052496 & -101068020 & 1094520 & 9938450 & -4324 & -139702 \\ 25246911 & -23032993 & -8520383 & 547283 & 135607 & -2162 \\ 963218 & 5173988 & -13490 & -148514 & 46 & 1081 \\ -3766053 & 481609 & 131001 & -6745 & -1027 & 23 \end{bmatrix}.$$

The matrix associated with Q_4 is more complicatedly involved and the details are left to the interested reader.

Acknowledgment

First of all, the author would like to thank two anonymous reviewers for pointing out a gap in the proof of Theorem 3.3. In particular, one of reviewer is very generous to provide a hint for the proof of Theorem 3.3 and a statement of Lemma 3.4 which enable the author to close the gap. The other reviewer pointed out an observation (Proposition 3.5) which provides another simple method to construct prewavelets. Also the author is grateful to T.N.T Goodman and Alexander Petukhov for carefully reading an old version of the manuscript and for their comments.

References

[1] F. Bastin, C. Boigelot, Biorthogonal wavelets in $H^m(\mathbb{R})$, J. Fourier Anal. Appl. 4 (1998) 749–768.
[2] F. Bastin, P. Laubin, Regular compactly supported wavelets in Sobolev spaces, Duke Math. J. 87 (1997) 481–508.
[3] C. de Boor, R.A. DeVore, A. Ron, On the construction of multivariate (pre)wavelets, Constr. Approx. 9 (1993) 123–166.
[4] C. de Boor, K. Hölig, S. Riememschneider, Box Splines, Springer, New York, 1993.
[5] M.D. Buhmann, O. Davydov, T.N.T. Goodman, Box spline prewavelets of small support, J. Approx. Theory 112 (2001) 16–27.
[6] C.K. Chui, Multivariate Splines, SIAM Publications, Philadelphia, PA, 1988.
[7] C.K. Chui, An Introduction to Wavelets, Academic Press, San Diego, 1992.
[8] C.K. Chui, J. Stöckler, J.D. Ward, On compactly supported box-spline wavelets, Approx. Theory Appl. 8 (1992) 77–100.
[9] C.K. Chui, J.Z. Wang, On compactly supported spline wavelets and a duality principle, Trans. Amer. Math. Sci. 330 (1992) 903–916.
[10] A. Cohen, N. Dyn, Nonstationary subdivision schemes and multiresolution analysis, SIAM J. Math. Anal. 27 (1997) 1745–1769.
[11] W. Dahmen, C.A. Micchelli, Translates of multivariate splines, Linear Algebra Appl. 52 (1983) 217–234.
[12] M.S. Floater, E.G. Quak, Piecewise linear prewavelets on arbitrary triangulations, Numer. Math. 82 (1999) 221–252.
[13] B. Han, Z.W. Shen, Wavelets from the Loop scheme, J. Fourier Anal. Appl. 11 (2005) 615–637.
[14] D. Hong, Y. Mu, Construction of prewavelets with minimum support over triangulations, in: Wavelet Analysis and Multiresolution Methods, Urbana-Champaign, IL, 1999, Dekker, New York, 2000, pp. 145–165.

[15] R.Q. Jia, C.A. Micchelli, Using the refinement equations for the construction of pre-wavelets. II. Powers of two, in: Curves and Surfaces, Chamonix–Mont-Blanc, 1990, Academic Press, Boston, MA, 1991, pp. 209–246.
[16] R.Q. Jia, J.Z. Wang, D.X. Zhou, Compactly supported wavelet bases for Sobolev spaces, Appl. Comput. Harmonic Anal. 15 (2003) 224–241.
[17] U. Kotyczka, P. Oswald, Piecewise linear prewavelets of small support, in: C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory VIII, vol. 2, World Scientific, Singapore, 1995, pp. 235–242.
[18] M.J. Lai, Fortran subroutines for B-nets of box splines on three and four directional meshes, Numer. Algorithm 2 (1992) 33–38.
[19] R.A. Lorentz, P. Oswald, Nonexistence of compactly supported box spline prewavelets in Sobolev spaces, in: Surface Fitting and Multiresolution Methods, Chamonix–Mont-Blanc, 1996, Vanderbilt University Press, Nashville, TN, 1997, pp. 235–244.
[20] S. Riemenschneider, Z.W. Shen, Box splines, cardinal series, and wavelets, in: C.K. Chui (Ed.), Approximation Theory and Functional Analysis, Academic Press, Boston, 1991, pp. 133–149.
[21] S. Riemenschneider, Z.W. Shen, Wavelets and pre-wavelets in low dimensions, J. Approx. Theory 71 (1992) 18–38.
[22] A. Ron, Z.W. Shen, Affine system in $L_2(R^d)$, II. Dual systems, J. Fourier Anal. Appl. 3 (1997) 617–637.
[23] I.J. Schoenberg, Cardinal Spline Interpolation, CBMS, SIAM Publication, Philadelphia, PA, 1973.
[24] H. Yu, S. Shu, S. Zhu, Construction of nontensor products semi-orthogonal wavelet bases with continuity in type 1 triangular partition, Numer. Math. J. Chinese Univ. (2002) 37–44.