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The weighted property (A) and the greedy algorithm

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Abstract

We investigate the efficiency of the Thresholding Greedy Algorithm, by comparing it to optimal "weighted" approximations. For a weight w, we describe w-greedy, w-almost-greedy, and w-partially-greedy bases, and examine some properties of w-semi-greedy bases. To achieve these goals, we introduce and study the w-Property (A).

MSC: 46B15; 41A65

Keywords: Thresholding greedy algorithm; Unconditional basis; Property (A); w-greedy bases

1. Introduction

In this paper, we compare the error of the Thresholding Greedy Algorithm (TGA) with the smallest error obtained by a "weighted" adaptive approximation. Throughout, $(\mathbb{X}, \|\cdot\|)$ is a real

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Banach space with a semi-normalized Schauder basis (or simply "basis") $\mathcal{B} = (e_n)_{n=1}^{\infty}$, with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$ (see the precise definition in Section 2.1).

The **Thresholding Greedy Algorithm (TGA)** was introduced in [15]: for $x \in \mathbb{X}$ we produce the sequence of **greedy approximands**

$$\mathcal{G}_m(x) = \sum_{k \in A_m(x)} e_k^*(x) e_k,$$

where $A_m(x)$ is a mth **greedy set** of x, that is, $|A_m(x)| = m$ and $\inf_{k \in A_m(x)} |e_k^*(x)| \geq \sup_{k \notin A_m(x)} |e_k^*(x)|$. A basis $\mathcal{B}$ is **quasi-greedy** [15] if there is a positive constant C such that

$$\|x - \mathcal{G}_m(x)\| \leq C\|x\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{1}$$

The **quasi-greedy constant** of $\mathcal{B}$ is the least C satisfying (1); henceforth it is denoted by C_q. By [17], a basis is quasi-greedy if and only if the TGA converges for any x — that is,

$$\lim_{m \to \infty} \|x - \mathcal{G}_m(x)\| = 0, \ \forall x \in \mathbb{X}.$$

A basis $\mathcal{B}$ is called **greedy** [15] if the TGA produces an "optimal" approximation, that is, there exists a positive constant C so that

$$\|x - \mathcal{G}_m(x)\| \leq C \inf\{\|x - \sum_{n \in A} a_n e_n\| : A \subset \mathbb{N}, |A| = m, a_n \in \mathbb{R}\}, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{2}$$

By [15], a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is greedy if and only if it is democratic and unconditional.

When defining the "greediness" of a basis, one compares the greedy approximation error with the smallest error of m-term approximation – that is, approximation supported on a subset of $\mathbb{N}$ of cardinality m. In some practical situations, certain subsets of $\mathbb{N}$ are "more equal than others". Building on the earlier work [7,14] considers a weight $w = (w_i)_{i=1}^{\infty} \in (0, \infty)^{\mathbb{N}}$. For $A \subset \mathbb{N}$, $w(A) = \sum_{i \in A} w_i$ denotes the w-measure of A. We define the error $\sigma_\delta^w(x)$ as

$$\sigma_\delta^w(x, \mathcal{B})_{\mathbb{X}} = \sigma_\delta^w(x) := \inf\{\|x - \sum_{n \in A} a_n e_n\| : |A| < \infty, w(A) \leq \delta, a_n \in \mathbb{R}\}.$$

Definition 1.1 (*[14]*)**.** A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w**-greedy** if there exists a constant $C \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C\sigma_{w(A_m(x))}^w(x), \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{3}$$

We say that $\mathcal{B}$ is C-w-greedy, and denote by C_g the least constant that satisfies (3).

In this notation, being greedy corresponds to being w-greedy with $w \equiv 1$. Above, we mention that a basis is greedy if and only if it is unconditional and democratic. We therefore need a "weighted" notion of democracy. We use shorthand notation

$$\mathbf{1}_A = \sum_{i \in A} e_i, \quad \text{and} \quad \mathbf{1}_{\varepsilon A} = \sum_{i \in A} \varepsilon_i e_i,$$

where A is a finite set and $\varepsilon = (\varepsilon_n)_n$, with $\varepsilon_n = \pm 1$.

Definition 1.2 (*[14]*)**.** A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w**-democratic** (w**-superdemocratic**) if there exists a constant $C \geq 1$ such that

$$\|\mathbf{1}_A\| \leq C\|\mathbf{1}_B\|, \ \text{(resp. } \|\mathbf{1}_{\varepsilon A}\| \leq C\|\mathbf{1}_{\eta B}\|), \tag{4}$$

for any pair of finite sets A, B with $w(A) \leq w(B)$, and ε, η are sequences of ± 1's. We say that $\mathcal{B}$ is C-w-(super)democratic, and denote by C_d (resp. C_s) the least constant that satisfies (4).

Recently, several researchers became interested in w-greedy bases. A characterization of w-greedy bases in terms of their unconditionality and w-democracy is given in [16]. In [4], the first author and Ó. Blasco characterize w-greedy bases using the best m-term error in the approximation "by polynomials with constant coefficients", generalizing the proven result in [5].

Of further interest are precise estimates on the greedy constant. By [9], even for $w \equiv 1$, the w-democracy and unconditionality alone do not determine whether a given basis is w-greedy with constant 1. In this setting, F. Albiac and P. Wojtaszczyk introduced in [2] the so called Property (A) (defined below) in order to obtain finer estimate for the greedy constant C_g (and, in particular, to characterize bases with $C_g = 1$). The results of [2] were further generalized in [10]. Thus, in our setting, precise estimates for the greedy constant require a weighted version of Property (A):

Definition 1.3. A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ satisfies the w-**Property (A)** if there exists a constant $C \geq 1$ such that

$$\|x + t\mathbf{1}_{\varepsilon A}\| \leq C\|x + t\mathbf{1}_{\eta B}\|, \tag{5}$$

for any $x \in \mathbb{X}$, for any pair of finite sets A, B such that $w(A) \leq w(B)$, $A \cap B = \emptyset$, $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$, for any $\varepsilon, \eta \in \{\pm 1\}$ and $t \geq \sup_j |e_j^*(x)|$. We say that $\mathcal{B}$ has the C-w-Property (A), and denote by C_a the least constant that satisfies (5).

The paper is structured as follows. In Section 2, we collect the necessary definitions and facts (especially concerning Schauder bases). Section 3 is dedicated to examining one of our main tools: the w-Property (A). We connect this property to others, used to study the TGA. Democracy is handled in Section 3.3. Further, in Section 3.4 we introduce the Property (C) (possessed by all quasi-greedy bases). Proposition 3.13 shows that w-superdemocracy and Property (C) imply w-Property (A).

The main results of the paper are contained in Section 4. There, we describe w-greedy and w-almost greedy bases in terms of their unconditionality (or being quasi-greedy) and w-Property (A) (Theorems 4.1 and 4.3). The condition of being w-almost greedy is weaker than w-greedy: instead of comparing the greedy approximand with all approximands of supported on a set of bounded weight, we compare with approximands whose coefficients are identical to those of x (see Definition 4.2).

In Section 5, we consider the w-semi-greedy bases – that is, the bases where the Chebyshev greedy approximands are optimal. Theorem 5.2 shows that such bases necessarily possess the w-Property (A).

In Section 6, we compare the efficiency of greedy approximation with that of the canonical basis projections. This gives rise to the notion of an w-partially-greedy basis; such bases are characterized in Theorem 6.4, using the property of being "w-conservative" (defined in Section 6). The property of being w-conservative turns out to be strictly weaker than being w-democratic (Proposition 6.10).

We freely use the standard "greedy" terminology. The reader can consult e.g. [16] for more information.

2. Definitions and notation

In this section, we review some common definitions, and introduce the necessary notation.

2.1. Properties of bases

Here we recall definitions pertaining to, and common properties of, bases. More specifically, we equip our Banach space $\mathbb{X}$ with a sequence of its elements $(e_n)_{n\in\mathbb{N}}$, so that there exists a sequence of **biorthogonal** elements $e_n^* \in \mathbb{X}^*$ — that is, $e_i^*(e_j) = \delta_{i,j}$. Further, we impose the **semi-normalization** condition:

$$0 < c_1 := \inf_n \min\{\|e_n\|, \|e_n^*\|\} \leq \sup_n \max\{\|e_n\|, \|e_n^*\|\} =: c_2 < \infty \tag{6}$$

If A is a finite subset of $\mathbb{N}$, we define the **coordinate projection** P_A: for $x \in \mathbb{X}$, $P_A(x) = \sum_{j\in A} e_j^*(x)e_j$. $P_A^c = I - P_A$ is the complementary projection. For $m \in \mathbb{N}$, let $S_m = P_{\{1,\ldots,m\}}$ be the mth **partial sum**. We are assuming the basis is **Schauder**: $\lim_m \|S_m(x) - x\| = 0$ for any $x \in \mathbb{X}$. The **basis constant** $K_b = \sup_m \|S_m\|$ is known to be finite.

The **support** of $x \in \mathbb{X}$ is defined via supp $(x) = \{i \in \mathbb{N} : e_i^*(x) \neq 0\}$.

Recall that a basis $\mathcal{B}$ in $\mathbb{X}$ is **unconditional** if any rearrangement of $\sum_n e_n^*(x)e_n$ converges in norm to x for any $x \in \mathbb{X}$. This is equivalent to the existence of the constant K so that $\|x - P_A(x)\| \leq K\|x\|$ for any $x \in \mathbb{X}$, and any finite $A \subset \mathbb{N}$ (actually, in this situation P_A makes sense even for infinite sets A). We denote by K_u the least constant K above, and call it the **(suppression) unconditionality constant**. The basis $\mathcal{B}$ is then said to be K_u**-(suppression) unconditional**.

The bases (e_i) and (f_i) are called **equivalent** if there exists a constant C so that the inequality $C^{-1}\|\sum_i a_i e_i\| \leq \|\sum_i a_i f_i\| \leq C\|\sum_i a_i e_i\|$ holds for any finite sequence of scalars (a_i).

2.2. Other notation: sets and functions

The cardinality of a set A is denoted by $|A|$. If A and B are subsets of $\mathbb{N}$, $A < B$ means that $\max_{j\in A} j < \min_{j\in B} j$. If a and b are functions of some variable, $a \lesssim b$ means that there exists a constant $c > 0$ such that $a \leq c \cdot b$.

3. The w-Property (A)

The w-Property (A) (defined in Section 1) will be used throughout this paper. We start by examining it closer.

3.1. Equivalent reformulations of the w-Property (A)

For further use, we need the following reformulation of the w-Property (A) (inspired by [1]).

Proposition 3.1. *A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ has the C_a-w-Property (A) if and only if*

$$\|x\| \leq C_a\|x - P_A(x) + \mathbf{1}_{\eta B}\|, \tag{7}$$

for any $x \in \mathbb{X}$ with $\sup_j |e_j^*(x)| \leq 1$, $A, B \subset \mathbb{N}$ finite sets, $w(A) \leq w(B)$, $B \cap$ supp $(x) = \emptyset$ and $\eta \in \{\pm 1\}$.

The proof requires a technical result.

Lemma 3.2. *Suppose D is a finite subset of $\mathbb{N}$, and $x \in \mathbb{X}\setminus\{0\}$ satisfies supp $(x)\cap D = \emptyset$. Then, for any $\varepsilon > 0$ there exists a finitely supported $y \in \mathbb{X}$, so that $\|x - y\| < \varepsilon$, supp $(y) \cap D = \emptyset$, and* $\max_j |e_j^*(x)| = \max_j |e_j^*(y)|$.

Proof. It suffices to consider $\varepsilon < 1/(2c_2)$. By scaling, we can assume that $\max_j |e_j^*(x)| = 1$ (then $\|x\| \geq 1/c_2$). Clearly $P_D(x) = 0$, and $P_D^c(x) = x$. Now set $\delta = \varepsilon/(3c_2^2\|x\|)$. As span $[e_j : j \in \mathbb{N}]$ is dense in $\mathbb{X}$, there exists a finitely supported $z \in \mathbb{X}$ so that $\|x - z\| < \delta/\|P_D^c\|$. Let $u = P_D^c(z)$, then $\|x - u\| = \|P_D^c(x - z)\| < \delta$. For every j, $|e_j^*(x - u)| < c_2\delta$, hence $C = \max_j |e_j^*(u)| \in (1 - c_2\delta, 1 + c_2\delta)$. Now let $y = u/C$. Then $\max_j |e_j^*(y)| = 1$, and

$$\|x - y\| \leq \|x - u\| + |1 - C^{-1}|\|u\| < \delta + \frac{c_2\delta}{1 - c_2\delta}(\|x\| + \delta) < \varepsilon. \quad \square$$

Proof of Proposition 3.1. By Lemma 3.2, it suffices to restrict our attention to finitely supported vectors $x \in \mathbb{X}$ only. So, throughout this proof, we assume $|\text{supp } (x)| < \infty$.

Suppose that $\mathcal{B}$ has the C_a-w-Property (A), and $x, A, B, \varepsilon, \eta$ are as in the statement of the proposition with $\sup_j |e_j^*(x)| \leq 1$. Applying the definition of w-Property (A) to $P_{A^c}x$, A, and B, we obtain

$$\|P_{A^c}(x) + \mathbf{1}_{\varepsilon A}\| \leq C_a\|P_{A^c}(x) + \mathbf{1}_{\eta B}\| = C_a\|x - P_A(x) + \mathbf{1}_{\eta B}\|.$$

To finish the proof, observe that x belongs to the convex hull of the set $\left\{P_{A^c}(x) + \mathbf{1}_{\varepsilon A}\right\}_{\varepsilon\in\{\pm 1\}}$.

Now, suppose (7), and prove that the basis $\mathcal{B}$ has the w-Property (A) with the same constant. Take $x \in \mathbb{X}$ with $\sup_j |e_j^*(x)| \leq 1$, A, B finite sets such that $w(A) \leq w(B)$, $A \cap B = \emptyset$, supp $(x) \cap (A \cup B) = \emptyset$ and $\varepsilon, \eta \in \{\pm 1\}$. Define $x' = x + \mathbf{1}_{\varepsilon A}$. Using (7),

$$\|x + \mathbf{1}_{\varepsilon A}\| = \|x'\| \leq C_a\|x' - P_A(x') + \mathbf{1}_{\eta B}\| = C_a\|x + \mathbf{1}_{\eta B}\|. \quad \square$$

Remark 3.3. In this paper, we focus on the situation when $\mathcal{B}$ is a Schauder basis. However, the w-Property (A) can be defined for a more general context, as for example Markushevich systems; the proof of Proposition 3.1 goes through as well. Also, the following four statements are equivalent:

(a) $\mathcal{B}$ satisfies the w-Property (A) (see Definition 1.3).
(b) There exists a constant C so that $\|x\| \leq C\|x - P_A(x) + t\mathbf{1}_{\eta B}\|$ for any $\eta \in \{\pm 1\}$, $x \in \mathbb{X}$, A, B finite sets such that $B \cap \text{supp } (x) = \emptyset$, $w(A) \leq w(B)$ and $t \geq \sup_j |e_j^*(x)|$.
(c) There exists a constant C' so that $\|x + s\mathbf{1}_{\varepsilon A}\| \leq C'\|x + s\mathbf{1}_{\eta B}\|$ for any $x \in \mathbb{X}$, A, B finite sets such that $w(A) \leq w(B)$, $A \cap B = \emptyset$, supp $(x) \cap (A \cup B) = \emptyset$, $\varepsilon, \eta \in \{\pm 1\}$ and $s = \sup_j |e_j^*(x)|$.
(d) There exists a constant C'' so that $\|x\| \leq C''\|x - P_A(x) + s\mathbf{1}_{\eta B}\|$ for any $x \in \mathbb{X}$, $\eta \in \{\pm 1\}$, A, B finite sets such that $B \cap \text{supp } (x) = \emptyset$, $w(A) \leq w(B)$ and $s = \sup_j |e_j^*(x)|$.

Indeed, the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) (with $C'' = C$) are immediate. The equivalence (a) $\Leftrightarrow$ (b) (with the same constant) has been established in Proposition 3.1. Minor adjustments to that argument give us (c) $\Leftrightarrow$ (d).

To establish (d) $\Rightarrow$ (b), take x, A, B, η as in (b) and $t \geq \sup_j |e_j^*(x)|$. As before, we can assume that x is finitely supported. Find k so that $|e_k^*(x)| = \sup_j |e_j^*(x)|$. By replacing x by $-x$ if necessary, we can assume $s = e_k^*(x) \geq 0$. Let $c = t - s$, and consider

$$x' = x + ce_k = \sum_{j\in\text{supp } (x)\setminus\{k\}} e_j^*(x)e_j + te_k.$$

Note that $\|x - x'\| \leq cc_2 \leq tc_2$. Furthermore, $x' - P_A(x')$ equals either $x - P_A(x)$ (if $k \in A$), or $x - P_A(x) + ce_k$ (if $k \notin A$). In either case,

$$\|x - P_A(x) + t\mathbf{1}_{\eta B}\| \geq \|x' - P_A(x') + t\mathbf{1}_{\eta B}\| - tc_2.$$

By (d), we have $\|x'\| \leq C''\|x' - P_A(x') + t\mathbf{1}_{\eta B}\|$. By the above,

$$\|x\| - tc_2 \leq C''(\|x - P_A(x) + t\mathbf{1}_{\eta B}\| + tc_2)$$

As $\|x - P_A(x) + t\mathbf{1}_{\eta B}\| \geq tc_2^{-1}$, we conclude that $\|x\| \leq (C'' + 2c_2^2)\|x - P_A(x) + t\mathbf{1}_{\eta B}\|$.

3.2. Equivalent weights

We show that the w-Property (A) is preserved under a "bounded perturbation" of the weight w.

Definition 3.4. We say that the weights $v = (v_n)_{n=1}^{\infty}$ and $w = (w_n)_{n=1}^{\infty}$ are **equivalent** ($v \approx w$) if there exist $0 < a \leq b < \infty$ so that $av_n \leq w_n \leq bv_n$ for any $n \in \mathbb{N}$.

Proposition 3.5. *Let v, w be weights and suppose that $v \approx w$. Then every basis in a Banach space $\mathbb{X}$ with the w-Property (A) also has the v-Property (A).*

Proof. Let $x \in \mathbb{X}$ with $|\text{supp}\,(x)| < \infty$ and $\sup_j |e_j^*(x)| \leq 1$, A and B finite satisfying $v(A) \leq v(B)$, $A \cap B = \emptyset$, $\text{supp}\,(x) \cap (A \cup B) = \emptyset$ and $\varepsilon, \eta \in \{\pm 1\}$. We set

$$\Gamma = \{n \in A : w_n \geq w(B)\}.$$

Observe that

$$w(A) \leq b \cdot v(A) \leq b \cdot v(B) \leq \frac{b}{a} \cdot w(B),$$

which gives us

$$w(A) \geq w(\Gamma) \geq |\Gamma| \cdot w(B) \geq |\Gamma| \cdot \frac{a}{b} \cdot w(A),$$

and hence $|\Gamma| \leq b/a$. Next, we give the following partition of $A \setminus \Gamma$: $A_1 < \cdots < A_m$, so that for each $i = 1, \ldots, m$, the set A_i is a maximal such that $w(A_i) \leq w(B)$. Due to maximality,

$$w(B) < w(A_i) + w(A_{i+1}) \text{ for all } i = 1, \ldots, m-1.$$

Thus,

$$(m-1) \cdot w(B) < \sum_{i=1}^{m-1} [w(A_i) + w(A_{i+1})] < 2 \cdot w(A \setminus \Gamma) \leq 2 \cdot w(A) \leq \frac{2b}{a} \cdot w(B).$$

This gives us

$$m \leq \frac{2b}{a} + 1.$$

Hence, using the bounds of $|\Gamma|$, m and the condition of the w-Property (A) with constant C_a,

$$\|x + \mathbf{1}_{\varepsilon A}\| \leq \|\mathbf{1}_{\varepsilon \Gamma}\| + \|x + \sum_{i=1}^{m} \mathbf{1}_{\varepsilon A_i}\| \leq \sum_{n \in \Gamma} \|e_n\| + \sum_{i=1}^{m} \|\frac{x}{m} + \mathbf{1}_{\varepsilon A_i}\|$$

$$
\begin{aligned}
&\leq c_2^2|\Gamma|\|x+\mathbf{1}_{\eta B}\| + C_a m\left\|\frac{x}{m}+\mathbf{1}_{\eta B}\right\| \leq \frac{c_2^2 b}{a}\|x+\mathbf{1}_{\eta B}\| + C_a\|x+m\mathbf{1}_{\eta B}\| \\
&\leq \frac{c_2^2 b}{a}\|x+\mathbf{1}_{\eta B}\| + C_a m\|x+\mathbf{1}_{\eta B}\| + C_a(m-1)\|x\| \\
&\leq \frac{c_2^2 b}{a}\|x+\mathbf{1}_{\eta B}\| + C_a m\|x+\mathbf{1}_{\eta B}\| + C_a^2(m-1)\|x+\mathbf{1}_{\eta B}\| \\
&\leq \left(\frac{c_2^2 b + (2b+a)C_a^2}{a}\right)\|x+\mathbf{1}_{\eta B}\|. \quad \square
\end{aligned}
$$

Remark 3.6. In a similar fashion, one can show that, if the weights w and v are equivalent, then any w-democratic (w-superdemocratic, w-conservative – for the definitions, see below) basis is also v-democratic (resp. v-superdemocratic or v-conservative).

Remark 3.7. The converse to Proposition 3.5 does not hold in general. For example, suppose the weights w, v belong to ℓ_1. By [13], the family of w-democratic (or v-democratic) bases consists precisely of those bases which are equivalent to the canonical basis of c_0. However, w and v need not be equivalent.

3.3. Democracy-like properties

We show that the w-Property (A) is stronger than w-super-democracy (which, in turn, is stronger than the w-democracy).

Proposition 3.8. *If a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ has the C_a-w-Property (A), then $\mathcal{B}$ is $2C_a$-w-superdemocratic.*

The converse is not true: Example 3.15 presents a basis which is superdemocratic but fails the Property (A).

Proof. Take two finite sets A, B with $w(A) \leq w(B)$, and show that, for any choice of signs, $\|\mathbf{1}_{\eta A}\| \leq 2C_a\|\mathbf{1}_{\varepsilon B}\|$. As in [6, Subsection 4.4], it is enough to prove our inequality for $\varepsilon \equiv 1$ (otherwise, replace $\mathcal{B} = \{e_n : n \in \mathbb{N}\}$ by $\{\varepsilon_n e_n : n \in \mathbb{N}\}$). Since $\mathbf{1}_{\eta A} \in 2S$, where $S = \{\sum_{A' \subset A} \theta_{A'}\mathbf{1}_{A'} : \sum_{A' \subset A}|\theta_{A'}| \leq 1\}$ (see [11, Lemma 6.4]), it suffices to show that

$$\|\mathbf{1}_{A'}\| \leq C_a\|\mathbf{1}_B\|, \ \forall A' \subset A.$$

Take $A' \subset A$. Obviously, $\mathbf{1}_{A'} = \mathbf{1}_{A' \setminus B} + \mathbf{1}_{B \cap A'}$. Then, using the w-Property (A),

$$\|\mathbf{1}_{A'}\| = \|\mathbf{1}_{A' \setminus B} + \mathbf{1}_{B \cap A'}\| \leq C_a\|\mathbf{1}_{B \setminus A'} + \mathbf{1}_{B \cap A'}\| = C_a\|\mathbf{1}_B\|.$$

We can apply the w-Property (A) because

$$
\begin{aligned}
&w(A') = w(A' \setminus B) + w(A' \cap B) \leq w(A) \leq w(B) = w(B \setminus A') + w(B \cap A') \\
&\quad \Rightarrow w(A' \setminus B) \leq w(B \setminus A').
\end{aligned}
$$

This completes the proof. $\square$

Remark 3.9. It is unknown if the factor of 2 can be removed in the above result.

Improving [13, Proposition 4.5], we prove that, in certain cases, any w-superdemocratic basis has to contain a subsequence equivalent to the canonical basis of c_0 (c_0-basis henceforth), or even to be equivalent to such basis.

Proposition 3.10. *Suppose that $\mathcal{B} = (e_n)_{n=1}^{\infty}$ is a C_s-w-superdemocratic basis in a Banach space $\mathbb{X}$.*

(i) *If A is finite and $w(A) \le \limsup_{n\to\infty} w_n$, then $\max_{\varepsilon\in\{\pm 1\}} \|\mathbf{1}_{\varepsilon A}\| \le c_2 C_s$.*
(ii) *If $\sup_n w_n = \infty$, then $\mathcal{B}$ is equivalent to the c_0-basis.*
(iii) *If $\inf_n w_n = 0$, then there exist $i_1 < i_2 < \cdots$ so that the sequence $(e_{i_k})_{k\in\mathbb{N}}$ is equivalent to the c_0-basis. Moreover, if $\lim_n w_n = 0$, then for any infinite set $A \subset \mathbb{N}$ we can select $i_1, i_2, \ldots \in A$ with the properties described above.*
(iv) *If $\sum_n w_n < \infty$, then $\mathcal{B}$ is equivalent to the c_0-basis.*

Proof. (i) Find $n \in \mathbb{N}\setminus A$ so that $w_n > w(A)$, then $\|\mathbf{1}_{\varepsilon A}\| \le C_s \|1_{\{n\}}\| \le c_2 C_s$.

(ii) By (i), $\|\mathbf{1}_{\varepsilon A}\| \le c_2 C_s$ for all choices of signs, which yields the desired equivalence.

(iii) Suppose E is an infinite subset of $\mathbb{N}$ so that $\inf_{i\in\mathbb{N}} w_i = 0$ (if $\lim_i w_i = 0$, then any infinite subset of $\mathbb{N}$ has this property). Find positive integers $N < i_1 < i_2 < \cdots$ so that $i_1, i_2, \ldots \in E$, and $\sum_{j=1}^{N} w_j > \sum_{k=N+1}^{\infty} w_k$. Let $B = \{1, \ldots, N\}$ and suppose A is a finite subset of $\{i_1, i_2, \ldots\}$. Then $w(B) > w(A)$, hence, by the w-superdemocracy, $\|\mathbf{1}_{\varepsilon A}\| \le C_s \|\mathbf{1}_B\|$ holds for any choice of signs ε. By convexity, the sequence $e_{i_1}, e_{i_2}, \ldots$ is equivalent to the c_0-basis.

(iv) The proof proceeds as in (iii); the key difference is that now, we find $N \in \mathbb{N}$ so that $\sum_{j=1}^{N} w_j > \sum_{k=1}^{\infty} w_k$ (that is, $i_k = k - N$). □

From this we immediately obtain:

Corollary 3.11. *If the weight w is unbounded, then a basis has the w-Property (A) if and only if it is equivalent to the canonical basis of c_0.*

3.4. Property (C)

The Property (C) (defined below) arises naturally in the study of quasi-greedy bases. It can also be used to determine whether a given basis has the w-Property (A).

Definition 3.12. A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ satisfies the **Property (C)** if there exists a positive constant C such that

$$\min_{j\in\Lambda} |e_j^*(x)| \|\mathbf{1}_{\varepsilon\Lambda}\| \le C\|x\|, \tag{8}$$

for any $x \in \mathbb{X}$, any greedy set Λ of x and $\varepsilon \in \{\pm 1\}$. We denote by C_u the smallest constant C that satisfies (8) and we say that $\mathcal{B}$ has the C_u-Property (C).

It is well known any quasi-greedy basis has Property (C) (see [6, Lemma 2.3]). Generalizing [6, Lemma 2.2], we prove that any w-superdemocratic basis with Property (C) has the w-Property (A).

Proposition 3.13. *If a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is C_s-w-superdemocratic and satisfies Property (C) with constant C_u, then $\mathcal{B}$ has the C_a-w-Property (A) with $C_a \le 1 + 2C_u C_s$.*

Proof. Take $x, A, B, \varepsilon, \eta$ as in the definition of the w-Property (A) and assume that $\sup_j |e_j^*(x)| \leq 1$. Then,

$$\|x + \mathbf{1}_{\varepsilon A}\| \leq \|x + \mathbf{1}_{\eta B}\| + \|\mathbf{1}_{\eta B}\| + \|\mathbf{1}_{\varepsilon A}\|. \tag{9}$$

Using the w-superdemocracy and $w(A) \leq w(B)$, we obtain that $\|\mathbf{1}_{\varepsilon A}\| \leq C_s \|\mathbf{1}_{\eta B}\|$. Now, we only have to estimate $\|\mathbf{1}_{\eta B}\|$. For that, we consider the element $y := x + \mathbf{1}_{\eta B}$. It is clear that $\mathbf{1}_{\eta B}$ is a greedy sum for y, so

$$\min_{j \in B} |e_j^*(y)| \|\mathbf{1}_{\eta B}\| = \|\mathbf{1}_{\eta B}\| \leq C_u \|y\| = C_u \|x + \mathbf{1}_{\eta B}\|. \tag{10}$$

Then, using (9) and (10),

$$\|x + \mathbf{1}_{\varepsilon A}\| \leq \|x + \mathbf{1}_{\eta B}\| + 2C_s C_u \|x + \mathbf{1}_{\eta B}\| = (1 + 2C_s C_u) \|x + \mathbf{1}_{\eta B}\|.$$

Hence, the basis has the w-Property (A) with constant $C_a \leq 1 + 2C_u C_s$. □

Remark 3.14. We do not know whether the converse of Proposition 3.13 holds. More specifically, the w-Property (A) implies the w-superdemocracy (see Proposition 3.8), however, we do not know whether it necessarily implies the Property (C). The following example shows that superdemocracy alone does not imply Property (C).

Example 3.15. Let $\mathbb{X} = \ell_1 \oplus c_0$ and $\|(x, y)\| = \|x\|_{\ell_1} + \|y\|_\infty$. Let $(e_n)_n$ be the canonical basis in ℓ_1 and $(f_m)_m$ the canonical basis in c_0. We define

$$E_{2n-1} = \left(\frac{1}{2}e_n, \frac{-1}{2}f_n\right), \quad E_{2n} = \left(\frac{1}{4}e_n, \frac{3}{4}f_n\right), n = 1, 2, \ldots,$$

and consider $\mathcal{B} = \{E_n\}_n = \{E_{2n-1}, E_{2n}\}_n$. This basis is normalized. To establish superdemocracy, we need a suitable lower estimate for $\|\mathbf{1}_{\varepsilon A}\|$. To this end, given a finite $A \subset \mathbb{N}$, we write

$$\begin{aligned} A_1 &= \{k \in \mathbb{N} : 2k \in A \text{ and } 2k-1 \in A\}, \\ A_2 &= \{k \in \mathbb{N} : 2k \in A \text{ and } 2k-1 \notin A\}, \\ A_3 &= \{k \in \mathbb{N} : 2k \notin A \text{ and } 2k-1 \in A\}. \end{aligned}$$

Observe that the sets A_1, A_2, A_3 are mutually disjoint, and $2|A_1| + |A_2| + |A_3| = |A|$. For any choice of signs,

$$\begin{aligned} \|\mathbf{1}_{\varepsilon A}\| &= \|\sum_{k \in A_1} \varepsilon_{2k} E_{2k} + \varepsilon_{2k-1} E_{2k-1} + \sum_{k \in A_2} \varepsilon_{2k} E_{2_k} + \sum_{k \in A_3} \varepsilon_{2k-1} E_{2k-1}\| \\ &= \|\sum_{k \in A_1} \left([\frac{1}{4}\varepsilon_{2k} + \frac{1}{2}\varepsilon_{2k-1}]e_k, [\frac{3}{4}\varepsilon_{2k} - \frac{1}{2}\varepsilon_{2k-1}]f_k\right) \\ &+ \sum_{k \in A_2} \varepsilon_{2k}(\frac{1}{4}e_k, \frac{3}{4}f_k) + \sum_{k \in A_3} \varepsilon_{2k-1}(\frac{1}{2}e_k, -\frac{1}{2}f_k) \| \\ &\geq \sum_{k \in A_1} |\frac{1}{4}\varepsilon_{2k} + \frac{1}{2}\varepsilon_{2k-1}| + \sum_{k \in A_2} \frac{1}{4} + \sum_{k \in A_3} \frac{1}{2}. \end{aligned}$$

Therefore,

$$\|\mathbf{1}_{\varepsilon A}\| \geq \frac{1}{4}|A_1| + \frac{1}{4}|A_2| + \frac{1}{2}|A_3| \geq \frac{1}{8}|A|,$$

establishing the superdemocracy.

To witness the lack of Property(A), take

$$z = \sum_{i=1}^{N} 2E_{2i} - \sum_{i=1}^{N} E_{2i-1} \text{ and } z' = \sum_{i=1}^{N} 2E_{2i+4N} - \sum_{i=1}^{N} E_{2i-1}.$$

Then $\|z\| = \| \sum_{i=1}^{N} (0, 2f_i)\| = 2$, and

$$\|z'\| = \left\| \sum_{i=1}^{N} \left(\frac{1}{2} e_{2i+4N}, \frac{3}{2} f_i \right) - \sum_{i=1}^{N} \left(\frac{1}{2} e_i, \frac{-1}{2} f_i \right) \right\| = \frac{1}{2}(2N+3).$$

Thus, our basis fails Property(A). In light of Proposition 3.13, it fails Property (C) as well.

4. Characterization of w-greedy and w-almost-greedy bases

First we describe w-greedy bases in terms of their unconditionality and w-Property (A).

Theorem 4.1. *Let $\mathcal{B}$ be a basis of a Banach space $\mathbb{X}$.*

(a) *If $\mathcal{B}$ is C_g-w-greedy, then the basis is K_u-unconditional and has the C_a-w-Property (A) with constants $K_u \le C_g$ and $C_a \le C_g$.*
(b) *If $\mathcal{B}$ is K_u-unconditional and has the C_a-w-Property (A), then the basis is C_g-w-greedy with $C_g \le K_u C_a$.*

We next compare the rate of greedy approximation to x with that of approximation by vectors whose coefficients “come from x”.

Definition 4.2 (*[13]*)**.** A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w-**almost-greedy** if there exists a constant $C \ge 1$ such that

$$\|x - \mathcal{G}_m(x)\| \le C \tilde{\sigma}^{w}_{w(A_m(x))}, \quad \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \tag{11}$$

where

$$\tilde{\sigma}^{w}_{\delta}(x, \mathcal{B})_{\mathbb{X}} = \tilde{\sigma}^{w}_{\delta}(x) := \inf\{\|x - P_A(x)\| : |A| < \infty, w(A) \le \delta\}.$$

We say that $\mathcal{B}$ is C-w-almost-greedy, and denote by C_{al} the least constant that satisfies (11).

In the classical setting ($w \equiv 1$), [9] characterizes almost-greedy bases in terms of the quasi-greediness and democracy. Recently, [13] proved that a basis is w-almost-greedy if and only if it is quasi-greedy and w-democratic. However, these results leave open the question of describing bases with $C_{al} = 1$. For instance, [2, Example 5.3] presents a (1-unconditional) basis with quasi-greedy and superdemocracy constants equal to 1, yet $C_{al} > 1$. To remedy this, we describe the w-almost-greediness of a basis in terms of its w-Property (A) and quasi-greediness.

Theorem 4.3. *Let $\mathcal{B}$ be a basis of a Banach space $\mathbb{X}$.*

(a) *If $\mathcal{B}$ is C_{al}-w-almost-greedy, then the basis is C_q-quasi-greedy and has the C_a-w-Property (A) with constants $C_q \le C_{al}$ and $C_a \le C_{al}$.*
(b) *If $\mathcal{B}$ is C_q-quasi-greedy and has the C_a-w-Property (A), then the basis is C_{al}-w-almost-greedy with $C_{al} \le C_q C_a$.*

Remark 4.4. The theorems above show that $\max\{K_u, C_a\} \leq C_g \leq K_u C_a$ and $\max\{C_q, C_a\} \leq C_{al} \leq C_q C_a$. One can observe that there is a gap between the upper and lower estimates for C_g and C_{al}. It is an open problem whether this gap can be made more narrow.

Proof of Theorem 4.1. Assume that $\mathcal{B}$ is C_g-w-greedy.

Unconditionality: Let $x \in \mathbb{X}$ and $A \subset \text{supp}\,(x)$. Define $y := P_{A^c}(x) + \sum_{n\in A}(\alpha + e_n^*(x))e_n$, where

$$\alpha > \sup_{j\in A} |e_j^*(x)| + \sup_{j\in A^c} |e_j^*(x)|.$$

As A is a greedy set of y,

$$\|x - P_A(x)\| = \|y - P_A(y)\| \leq C_g \sigma^w_{w(A)}(y) \leq C_g \|y - \alpha \mathbf{1}_A\| = C_g \|x\|.$$

Thus, the basis is unconditional with constant $K_u \leq C_g$.

w-Property (A): Fix $x \in \mathbb{X}$ and take $t \geq \sup_n |e_n^*(x)|$. Consider $\varepsilon, \eta \in \{\pm 1\}$ and finite sets A, B such that $A \cap B = \emptyset$, $w(A) \leq w(B)$, and $(A \cup B) \cap \text{supp}\,(x) = \emptyset$. Set $y := x + t\mathbf{1}_{\varepsilon A} + (t + \delta)\mathbf{1}_{\eta B}$ with $\delta > 0$. Hence,

$$\|x + t\mathbf{1}_{\varepsilon A}\| = \|y - \mathcal{G}_{|B|}(y)\| \leq C_g \sigma^w_{w(B)}(y) \leq C_g \|y - t\mathbf{1}_{\varepsilon A}\| = C_g \|x + (t + \delta)\mathbf{1}_{\eta B}\|.$$

Taking $\delta \to 0$, we obtain that the basis satisfies the w-Property (A) with constant $C_a \leq C_g$.

Next we prove that if $\mathcal{B}$ is K_u-unconditional and has the C_a-w-Property (A), then it is w-greedy.

Take $x \in \mathbb{X}$ and suppose that A is a greedy set of cardinality m for $x \in \mathbb{X}$ – that is, $P_A(x) = \mathcal{G}_m(x)$. For $\varepsilon > 0$ find $y \in \mathbb{X}$ such that $\|x - y\| < \sigma^w_{w(A)}(x) + \varepsilon$, with $\text{supp}\,(y) = B$ and $w(B) \leq w(A)$. Then, taking $t := \min\{|e_j^*(x)| : j \in A\}$ and $\eta \equiv \text{sgn}\,\{e_j^*(x)\}$, using the reformulation of the w-Property (A) and [6, Lemma 2.5], we obtain that

$$\begin{aligned}\|x - \mathcal{G}_m(x)\| &\leq C_a \|x - P_A(x) - P_{B\setminus A}(x) + t\mathbf{1}_{\eta(A\setminus B)}\| = C_a \|P_{(A\cup B)^c}(x - y) + t\mathbf{1}_{\eta(A\setminus B)}\| \\ &= C_a \|T_t(I - P_B)(x)\| = C_a \|T_t(I - P_B)(x - y)\| \leq K_u C_a \|x - y\|.\end{aligned}$$

Consequently, for any greedy set A we have $\|x - P_A(x)\| \leq K_u C_a \sigma^w_{w(A)}(x)$. □

Proof of Theorem 4.3. Assume that $\mathcal{B}$ is C_{al}-w-almost-greedy.

Quasi-greedy: Since

$$\|x - \mathcal{G}_m(x)\| \leq C_{al} \inf\{\|x - \sum_{n\in B} e_n^*(x)e_n\| : w(B) \leq w(A_m(x)), |B| < \infty\},$$

we can select $B = \emptyset$. Then, we obtain that $\|x - \mathcal{G}_m(x)\| \leq C_{al}\|x\|$, hence the basis is quasi-greedy with constant $C_q \leq C_{al}$.

w-Property (A): We can use the same argument as in Theorem 4.1.

Now, we will prove that if $\mathcal{B}$ is C_q-quasi-greedy and has the C_a-w-Property (A), then it is w-almost-greedy.

For $x \in \mathbb{X}$, let A be a greedy set of cardinality m. For $\varepsilon > 0$, find B such that $\|x - P_B(x)\| < \tilde{\sigma}^w_{w(A)}(x) + \varepsilon$, with $w(B) \leq w(A)$. Then, taking $t := \min\{|e_j^*(x)| : j \in A\}$ and $\eta \equiv \text{sgn}\,\{e_j^*(x)\}$, using the reformulation of the w-Property (A) and [6, Lemma 2.5],

$$\begin{aligned}\|x - \mathcal{G}_m(x)\| &\leq C_a \|P_{(A\cup B)^c}(x - y) + t\mathbf{1}_{\eta(A\setminus B)}\| \\ &= C_a \|T_t(I - P_B)(x)\| \leq C_q C_a \|x - P_B(x)\|.\end{aligned}$$

This gives that, for any greedy set A, $\|x - P_A(x)\| \leq C_q C_a \tilde{\sigma}_{w(A)}(x)$ as desired. □

5. w-semi-greedy bases

In this section we consider the efficiency of the Chebyshev Thresholding Greedy Algorithm (see [8,12]) in the setting of weighted non-linear approximation (pioneered in the recent paper [13]). For $x \in \mathbb{X}$, $m \in \mathbb{N}$, and a greedy set $A_m(x)$, we define the Chebyshev Greedy Approximand of order m as any $\overline{G}_m(x) \in span\{e_i : i \in A_m(x)\}$ such that

$$\|x - \overline{G}_m(x)\| = \min\{\|x - \sum_{n \in A_m(x)} b_n e_n\| : b_n \in \mathbb{R}\}.$$

Definition 5.1 (*[13]*). A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w**-semi-greedy** if there exists a constant $C \geq 1$ such that

$$\|x - \overline{G}_m(x)\| \leq C\sigma^w_{w(A_m(x))}(x), \ \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{12}$$

We say that $\mathcal{B}$ is C-w-semi-greedy, and denote by C_{sg} the least constant that satisfies (12).

Originally, the introduction of the Chebyshev Thresholding Greedy Algorithm was in [8] where the authors wanted to study an enhancement of the TGA to improve the rate of convergence. Also, for $w \equiv 1$, the authors proved that semi-greediness is equivalent to almost-greediness for bases in Banach spaces with finite cotype. Recently, in [13], the authors showed the same result using weights. However, in general, we do not have "if an only if" characterizations of semi-greedy bases, in the spirit of Theorems 4.1 and 4.3. Below, we show that w-semi-greediness implies the w-Property (A) (without any cotype assumptions).

Theorem 5.2. *If a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w-semi-greedy, then $\mathcal{B}$ has the w-Property (A).*

Proof. Assume that $\|x - \overline{G}_m(x)\| \leq C_{sg}\sigma^w_{w(A_m(x))}$ for any $x \in \mathbb{N}$ and $m \in \mathbb{N}$.

We take ε, η, A, B and x in the conditions of the definition of the w-Property (A). In all of the following cases we consider $x \in \mathbb{X}$ such that $|\text{supp }(x)| < \infty$ and $\sup_n |e_n^*(x)| \leq 1$.

Case 1: $\sum_{n=1}^{\infty} w_n = \infty$ and $\sup_n w_n < \infty$.

Case 1.1: $w(B) > \limsup_{n\to\infty} w_n$. Since $\sum_n w_n = \infty$, we can choose E and $n_0 \in \mathbb{N}$ with $\min E > \max(A \cup B \cup \text{supp }(x))$ and $n_0 > \max E$ such that

$$w(E) \leq w(B) < w(E) + w_{n_0} < 2w(B).$$

Set $F := E \cup \{n_0\}$. Then, $w(E) \leq w(B) < w(F) < 2w(B)$.

We define the element $z := x + \mathbf{1}_{\varepsilon A} + (1+\delta)\mathbf{1}_F$. For any scalar sequence $(f_n)_{n\in F}$, we have $\|x + \mathbf{1}_{\varepsilon A}\| \leq K_b\|x + \mathbf{1}_{\varepsilon A} + \sum_{n\in F} f_n e_n\|$. As the basis $\mathcal{B}$ is w-semi-greedy with constant C_{sg}, and $w(A) \leq w(B) < w(F)$, we conclude that

$$\inf_{f_n} \|x + \mathbf{1}_{\varepsilon A} + \sum_{n\in F} f_n e_n\| \leq C_{sg}\sigma^w_{w(F)}(z) \leq C_{sg}\|x + (1+\delta)\mathbf{1}_F\|.$$

Consequently, $\|x + \mathbf{1}_{\varepsilon A}\| \leq K_b C_{sg}\|x + (1+\delta)\mathbf{1}_F\|$. Taking $\delta \to 0$,

$$\begin{aligned}\|x + \mathbf{1}_{\varepsilon A}\| &\leq K_b C_{sg}\|x + \mathbf{1}_F\| \leq K_b C_{sg}\|x + \mathbf{1}_E\| + K_b C_{sg}\|e_{n_0}\| \\ &\leq K_b C_{sg}\|x + \mathbf{1}_E\| + K_b C_{sg} c_2 \\ &\leq K_b C_{sg}(\|x + \mathbf{1}_{\eta B}\| + \|\mathbf{1}_{\eta B}\| + \|\mathbf{1}_E\|) + K_b C_{sg} c_2.\end{aligned} \tag{13}$$

Now, we set $y := \mathbf{1}_{\eta B} + (1+\delta)\mathbf{1}_F$. Reasoning as before, we obtain

$$\|\mathbf{1}_{\eta B}\| \leq K_b \inf_{c_n} \|\mathbf{1}_{\eta B} + \sum_{n\in F} c_n e_n\| \leq K_b C_{sg} \sigma^w_{w(F)}(y) \leq K_b C_{sg} \|(1+\delta)\mathbf{1}_F\|.$$

Sending $\delta \to 0$, we obtain

$$\|\mathbf{1}_{\eta B}\| \leq K_b C_{sg} \|\mathbf{1}_F\| \leq K_b C_{sg} \|\mathbf{1}_E\| + K_b C_{sg} c_2. \tag{14}$$

On the other hand, taking $s := x + (1+\delta)\mathbf{1}_{\eta B} + \mathbf{1}_E$,

$$\|\mathbf{1}_E\| \leq (K_b+1)\|x+\sum_{n\in B} b_n e_n + \mathbf{1}_E\| \leq C_{sg}(K_b+1)\sigma^w_{w(B)}(s) \leq C_{sg}(K_b+1)\|x+(1+\delta)\mathbf{1}_{\eta B}\|.$$

Then, taking $\delta \to 0$,

$$\|\mathbf{1}_E\| \leq C_{sg}(K_b+1)\|x + \mathbf{1}_{\eta B}\|. \tag{15}$$

Finally, using (13)–(15), the basis satisfies the w-Property (A) with constant $K = O(C_{sg}^3 K_b^3 c_2)$.

Case 1.2: $w(A) \leq w(B) \leq \limsup_{n\to\infty} w_n$. Using Proposition 4.5 of [13],

$$\max\{\|\mathbf{1}_{\varepsilon A}\|, \|\mathbf{1}_{\eta B}\|\} \leq 2K_b C_{sg} c_2.$$

Since $1 = |e_j^*(x + \mathbf{1}_{\eta B})| \leq \|e_j^*\|\|x + \mathbf{1}_{\eta B}\| \leq c_2\|x + \mathbf{1}_{\eta B}\|$ for $j \in B$, then

$$\begin{aligned}\|x + \mathbf{1}_{\varepsilon A}\| &\leq \|x + \mathbf{1}_{\eta B}\| + \|\mathbf{1}_{\eta B}\| + \|\mathbf{1}_{\varepsilon A}\| \\ &\leq \|x + \mathbf{1}_{\eta B}\| + 4K_b C_{sg} c_2 \leq (4K_b C_{sg} c_2^2 + 1)\|x + \mathbf{1}_{\eta B}\|.\end{aligned} \tag{16}$$

Case 2: If $\sum_n w_n < \infty$ or $\sup_n w_n = \infty$, using Proposition 4.5 of [13], $\mathcal{B}$ is equivalent to the canonical basis of c_0 and the result is trivial. $\square$

6. w-partially-greedy bases

Another way of estimating the efficiency of greedy approximation is to compare the rate of convergence with straightforward Schauder approximation. To this end we consider w**-partially-greedy bases**. In [9], the authors defined the partially-greedy bases as those satisfying

$$\|x - \mathcal{G}_m(x)\| \leq C\|x - S_m(x)\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N},$$

for some absolute constant C. Moreover, they proved that $\mathcal{B}$ is partially-greedy if and only if $\mathcal{B}$ is quasi-greedy and conservative (that is, $\|\mathbf{1}_A\| \lesssim \|\mathbf{1}_B\|$ for all pair of finite sets A, B such that $A < B$ and $|A| \leq |B|$). Here, we present the notion of w-partially-greedy bases and we characterize these bases using w-conservative bases.

Definition 6.1. We say that $\mathcal{B}$ is w**-partially-greedy** if for all m and r such that $w(\{1,\dots,m\}) \leq w(A_r(x))$, there exists a positive constant C such that

$$\|x - \mathcal{G}_r(x)\| \leq C\|x - S_m(x)\|, \ \forall x \in \mathbb{X}. \tag{17}$$

We say that $\mathcal{B}$ is C-w-partially-greedy, and denote by C_p the least constant C for which (17) holds.

Definition 6.2. We say that $\mathcal{B}$ is w**-conservative** if there exists a positive constant C such that $\|\mathbf{1}_A\| \leq C\|\mathbf{1}_B\|$, for all pair of finite sets A, B such that $A < B$ and $w(A) \leq w(B)$. We say that $\mathcal{B}$ is C-w-conservative, and denote by C_c the least constant that satisfies this inequality.

Remark 6.3. If $w \equiv 1$, we recover the classical definition of partially-greediness (resp. conservativeness), and we will say that $\mathcal{B}$ is partially-greedy (resp. conservative).

It is clear that any conservative basis is democratic; Proposition 6.10 shows the converse need not be true.

Theorem 6.4. *A basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is w-partially-greedy if and only if it is quasi-greedy and w-conservative. Quantitatively,*

$$C_c \leq C_p, \quad C_q \leq C_p + C_p c_2^2 + c_2^2, \quad C_p \leq 2C_q + 8C_q^3 C_c.$$

Before the proof of this result, we need the following lemma.

Lemma 6.5. *If a basis $\mathcal{B}$ in a Banach space $\mathbb{X}$ is C_c-w-conservative and C_q-quasi-greedy, then*

$$\|\sum_{j\in A} a_j e_j\| \leq 4C_q C_c \max_{j\in A} |a_j| \|\mathbf{1}_{\eta B}\|,$$

for any sign η, any finite sets A, B such that $w(A) \leq w(B)$, $A < B$, and any collection of scalars $(a_j)_{j\in A}$.

Proof. We prove that $\|\mathbf{1}_{\varepsilon A}\| \leq 4C_q C_c \|\mathbf{1}_{\eta B}\|$ for any signs ε and η. First, we can decompose $\mathbf{1}_{\varepsilon A} = \mathbf{1}_{A^+} - \mathbf{1}_{A^-}$, where $A^\pm = \{j \in A : \varepsilon_j = \pm 1\}$. Then,

$$\|\mathbf{1}_{\varepsilon A}\| \leq \|\mathbf{1}_{A^+}\| + \|\mathbf{1}_{A^-}\| \leq 2C_c \|\mathbf{1}_B\|.$$

Now, using the condition to be quasi-greedy, it is clear that $\|\mathbf{1}_B\| \leq 2C_q \|\mathbf{1}_{\eta B}\|$, then

$$\|\mathbf{1}_{\varepsilon A}\| \leq 4C_q C_c \|\mathbf{1}_{\eta B}\|.$$

Now, using convexity, we are done. □

Proof of Theorem 6.4. Assume that $\mathcal{B}$ is C_p-w-partially-greedy.

(1) *w-conservative:* take A and B finite sets such that $A < B$ and $w(A) \leq w(B)$. Let $m = \max A$ and define the set $D = [1, \ldots, m] \setminus A$. Of course,

$$w(\{1, \ldots, m\}) = w(A \cup D) \leq w(B \cup D).$$

Define now $x := \mathbf{1}_A + (1+\delta)\mathbf{1}_{B\cup D}$. Then,

$$\|\mathbf{1}_A\| = \|x - \mathcal{G}_{|B\cup D|}(x)\| \leq C_p \|(1+\delta)\mathbf{1}_B\|.$$

Taking $\delta \to 0$, the basis is w-conservative.

(2) *Quasi-greedy:* here, we consider two cases.

(a) Suppose $1 \notin A_r(x)$. Define

$$\tilde{x} = te_1 + \sum_{i=2}^{\infty} e_i^*(x)e_i = x + (t - e_1^*(x))e_1, \text{ where } t > \max |e_i^*(x)|.$$

Then $A_{r+1}(\tilde{x}) = A_r(x) \cup \{1\}$, and $\mathcal{G}_{r+1}(\tilde{x}) = te_1 + \mathcal{G}_r(x)$. Therefore,

$$x - \mathcal{G}_r x = \tilde{x} - \mathcal{G}_{r+1}(\tilde{x}) + e_1^*(x)e_1,$$

hence, by the triangle inequality,

$$\|x - \mathcal{G}_r(x)\| \leq \|\tilde{x} - \mathcal{G}_{r+1}(\tilde{x})\| + |e_1^*(x)|\|e_1\| \leq \|\tilde{x} - \mathcal{G}_{r+1}(\tilde{x})\| + c_2^2\|x\|.$$

We clearly have $w(\{1\}) \leq w(\mathcal{G}_{r+1}(\tilde{x}))$, hence, due to the w-partially greedy property,

$$\|\tilde{x} - \mathcal{G}_{r+1}(\tilde{x})\| \leq C_p \|\sum_{i=2}^{\infty} e_i^*(\tilde{x}) e_i\| = C_p \|x - e_1^*(x) e_1\| \leq C_p(1 + c_2^2)\|x\|.$$

Consequently, $\|x - \mathcal{G}_r x\| \leq (C_p + C_p c_2^2 + c_2^2)\|x\|$.

(b) Now suppose $1 \in A_r(x)$. Define $\tilde{x}$ in the same way as above. Then $A_r(\tilde{x}) = A_r(x)$, and $\tilde{x} - \mathcal{G}_r(\tilde{x}) = x - \mathcal{G}_r(x)$. Consequently,

$$\|x - \mathcal{G}_r(x)\| = \|\tilde{x} - \mathcal{G}_r(\tilde{x})\| \leq C_p \|\sum_{i=2}^{\infty} e_i^*(x) e_i\| = C_p \|x - e_1^*(x) e_1\| \leq C_p(1 + c_2^2)\|x\|.$$

Now, assume that $\mathcal{B}$ is C_c-w-conservative and C_q-quasi-greedy, and show that $\mathcal{B}$ is w-partially-greedy. Take $x \in \mathbb{X}$, m, and r as in the definition of w-partially-greedy, and consider the sets

$$D := \{\rho(j) : j \leq r, \rho(j) \leq m\}, \quad B := \{\rho(j) : j \leq r, \rho(j) > m\}, \quad A := [1, \ldots, m] \setminus D,$$

where ρ is the greedy ordering. Then $A_r(x) = B \cup D$, and $w(A) = w(\{1, \ldots, m\}) - w(D) \leq w(A_r(x)) - w(D) = w(B)$.

$$x - \mathcal{G}_r(x) = \sum_{i=m+1}^{\infty} e_i^*(x) e_i - P_B(x) + P_A(x).$$

On the one hand, $\|P_B(x)\| \leq 2C_q \|x - S_m(x)\|$. On the other hand, using [9, Lemma 2.2] and Lemma 6.5 with $\eta \equiv \text{sgn }(e_j^*(x))$,

$$\begin{aligned}\|P_A(x)\| &\leq 4C_q C_c \max_A |e_i^*(x)| \|\mathbf{1}_{\eta B}\| \leq 4C_q C_c \min_B |e_i^*(x)| \|\mathbf{1}_{\eta B}\| \\ &\leq 8C_q^2 C_c \|P_B(x)\| \leq 8C_q^3 C_c \|x - S_m(x)\|.\end{aligned}$$

Then, $\|x - \mathcal{G}_r(x)\| \lesssim C_q^3 C_c \|x - S_m(x)\|$. $\square$

Remark 6.6. The upper bound for C_p from this theorem improves the estimate given [9, Theorem 3.4] for the case $w \equiv 1$.

Remark 6.7. Note that if the inequality $\|x - \mathcal{G}_r(x)\| \leq C\|x - S_m(x)\|$ is satisfied for m and r, then it is automatically satisfied – with a different constant – for any $n < m$ and the same r (since $C\|x - S_m(x)\| \leq (1 + K_b)C\|x - S_n(x)\|$ where K_b is the basis constant). So we only need to check the condition in the definition of w-partially-greedy for the largest m satisfying $w([1, \ldots, m]) \leq w(A_r(x))$.

As we have commented before, using the constant weight $w \equiv 1$, we recover the usual definition of partially-greedy bases. Indeed, for $w \equiv 1$, the largest m satisfying the definition is $m = r$, which recaptures the original definition of partially-greedy given in [9].

Under certain conditions on the weight w, a basis is automatically w-conservative.

Proposition 6.8. *Let w be a weight and set*

$$s_w := \sup \left\{ n \in \mathbb{N}_0 : \text{there exist } |A| = n \text{ and } |B| < \infty \right. \\ \left. \text{such that } A < B \text{ and } w(A) \leq w(B) \right\}.$$

Then $s_w < \infty$ if and only if every seminormalized basis is w-conservative.

Proof. ($\Longrightarrow$): Suppose $s_w < \infty$. Let $(e_n)_{n=1}^\infty$ be a seminormalized basis for a Banach space $\mathbb{X}$, and select two finite sets A, B such that $A < B$ and $w(A) \le w(B)$. Observe that $\|\mathbf{1}_A\| \le c_2 \cdot |A| \le c_2 s_w$. It follows immediately that $\|\mathbf{1}_A\| \le c_2 s_w \le c_2^2 s_w \|\mathbf{1}_B\|$. Hence, $(e_n)_{n=1}^\infty$ is $(c_2^2 s_w)$-w-conservative.

($\Longleftarrow$): Suppose $s_w = \infty$. Let us inductively construct sequences of finite sets $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ satisfying

$$A_1 < B_1 < A_2 < B_2 < A_3 < B_3 < \cdots,$$

and also satisfying $|A_n| \ge n$ and $w(A_n) \le w(B_n)$ for all $n \in \mathbb{N}$. Let us begin by selecting A_1 and B_1 finite sets with $|A_1| = 1$, $A_1 < B_1$, and $w(A_1) \le w(B_1)$, which is possible as $s_w \ge 1$. This is the base case; from now on, we proceed inductively. Since $s_w = \infty$, we may select $\widehat{A}_{n+1} \in \mathbb{N}^{<\infty}$ and $B_{n+1} \in \mathbb{N}^{<\infty}$ with $|\widehat{A}_{n+1}| > n + \max B_n$, $\widehat{A}_{n+1} < B_{n+1}$, and $w(\widehat{A}_{n+1}) < w(B_{n+1})$. Now set $A_{n+1} = \widehat{A}_{n+1} \setminus \{1, \dots, \max B_n\}$ so that we have $|A_{n+1}| > n$, $A_{n+1} < B_{n+1}$, and $w(A_{n+1}) < w(B_{n+1})$. This completes the inductive step, and gives us our intertwining sequences with the desired properties. We may now define a norm on c_{00} via the rule

$$\|(a_n)_{n=1}^\infty\|_{\mathbb{X}} = \|(a_n)_{n=1}^\infty\|_\infty \vee \sup_{k\in\mathbb{N}} \sum_{n\in A_k} |a_n| \quad \forall\, (a_n)_{n=1}^\infty \in c_{00},$$

and denote by $\mathbb{X}$ the completion of c_{00} under this norm. It is clear that the standard canonical basis for this space form a normalized 1-unconditional basis. However, it fails to be w-conservative as $\|\mathbf{1}_{A_k}\|_{\mathbb{X}} = |A_k| \ge k$ whereas $\|\mathbf{1}_{B_k}\|_{\mathbb{X}} = 1$ for all $k \in \mathbb{N}$. $\square$

Proposition 6.9. *Let w be a nonincreasing weight, i.e., $w_{n+1} \le w_n$ for all $n \in \mathbb{N}$. Then every conservative basis in a Banach space is w-conservative with the same constant.*

Proof. Let $(e_n)_{n=1}^\infty$ be a conservative basis in a Banach space $\mathbb{X}$, and select any pair of finite sets A, B satisfying both $A < B$ and $w(A) \le w(B)$. Now,

$$|A| \cdot w_{\max A} \le w(A) \le w(B) \le |B| \cdot w_{\min B} \le |B| \cdot w_{\max A},$$

so that $|A| \le |B|$. $\square$

We finish this section by proving that the condition of being w-conservative is strictly weaker than w-democracy.

Proposition 6.10. *Suppose the weight $w = (w_n)$ satisfies $\sup_n w_n < \infty$, $\sum_n w_n = \infty$. Then there exists a w-conservative unconditional basis which is not w-democratic.*

The construction presented below appears to be new, even in the case of $w \equiv 1$.

Proof. By scaling, we can and do assume that $\sup_n w_n = 1$. For $n \in \mathbb{N}$, let $W_N = w_1 + \cdots + w_N = w(\{1, \dots, N\})$. Define the family

$$\mathcal{S} = \{A \subset \mathbb{N} : w(A) \le \sqrt{W_{\min A - 1}}\}$$

(clearly this family is hereditary: membership passes to subsets). Let $\mathbb{X}$ be the completion of c_{00} under the norm

$$\|(a_n)_n\| = \max\Big\{\sup_n |a_n|, \sup_{C\in\mathcal{S}} \sum_{n\in C} w_n |a_n|\Big\}.$$

Clearly the canonical basis of $\mathbb{X}$ is 1-unconditional and normalized. For future use, observe that, for any finite set A,

$$\|\mathbf{1}_A\| = \max\Big\{1, \sup_{C\in\mathcal{S}, C\subset A} w(C)\Big\}. \tag{18}$$

Next show that our basis is w-conservative. Indeed, suppose $A < B$ and $w(A) \leq w(B)$; show that $\|\mathbf{1}_A\| \leq 2\|\mathbf{1}_B\|$. The inequality is trivial if $\|\mathbf{1}_A\| \leq 2$. Otherwise, find $C \in \mathcal{S}$ so that $w(C) = \|\mathbf{1}_A\| > 2$. Find $D \subset B$ so that $w(D) \geq B$, but no subset of D satisfies this inequality. Find $C' \subset D$ so that $|D\backslash C'| = 1$. Then $C' \in \mathcal{S}$, and $w(C') \geq w(C) - 1$, hence

$$\|\mathbf{1}_B\| \geq w(C') \geq \|\mathbf{1}_A\| - 1 > \frac{1}{2}\|\mathbf{1}_A\|.$$

Finally, suppose, for the sake of contradiction, that our basis is w-democratic. Then there exists a constant κ so that

$$\|\mathbf{1}_B\| \leq \kappa\|\mathbf{1}_A\| \quad \text{whenever} \quad w(B) \leq 2w(A). \tag{19}$$

Take $A = \{1, \ldots, N\}$, where N is so large that $W_N > 1$. If $C \in \mathcal{S}$ is a subset of A, then $w(C) \leq \sqrt{W_N}$, hence $\|\mathbf{1}_A\| \leq \sqrt{W_N}$.

Now find $M > N$ so large that $W_M > (W_N + 1)^2$. Let $B = \{M + 1, \ldots, M + K\}$, where K is selected to satisfy $W_N \leq w(B) \leq W_N + 1$. Then $B \in \mathcal{S}$, and hence, by (18), $\|\mathbf{1}_B\| = w(B) \geq W_N$. Invoke (19) to get $\kappa \geq \|\mathbf{1}_B\|/\|\mathbf{1}_A\| = \sqrt{W_N}$. As W_N can be arbitrarily large, we obtain the desired contradiction. □

Remark 6.11. The notion of being w-conservative strongly depends on the choice of the weight w. [3, Section 6.3] provides an example of a w-greedy (hence w-conservative) basis which is not conservative (hence not greedy).

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