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The N-Widths of Spaces of Holomorphic Functions on Bounded Symmetric Domains of Tube Type

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Let D be a bounded symmetric domain of tube type and Σ be the Shilov boundary of D. Denote by $H_2(D)$ and $A_2(D)$ the Hardy and Bergman spaces, respectively, of holomorphic functions on D; and let $B(H_2(D))$ and $B(A_2(D))$ denote the closed unit balls in these spaces. For an integer $l \geqslant 0$ we define the notion $\mathscr{R}^l f$ of the lth radial derivative of a holomorphic function f on D, and we prove the following results: Let $0 < \rho < 1$. Denote by W the class of holomorphic functions f on D for which $\mathscr{R}^l f \in B(H_2(D))$ and set $X = C(\rho\Sigma)$. Then we show that the linear and Gelfand N-widths of W in X coincide, and we compute the exact value. We do the same for the case in which W is the class of holomorphic functions f for which $\mathscr{R}^l f \in B(A_2(D))$, and $X = C(\rho\Sigma)$. Next, let $X = L^p(\rho\Sigma)$ (respectively, $L^p(\rho D)$) for $1 \leqslant p \leqslant \infty$, and let W be a class of holomorphic functions f on D for which $\mathscr{R}^l f \in B(H_p(D))$ (respectively, $B(A_p(D))$). We show that the Kolmogorov, linear, Gelfand, and Bernstein N-widths all coincide, we calculate the exact value, and we identify optimal subspaces or optimal linear operators. These results extend work of Yu. A. Farkov (1993, *J. Approx. Theory* **75**, 183–197) and K. Yu. Osipenko (1995, *J. Approx. Theory* **82**, 135–155), and initiate the study of N-widths of spaces of holomorphic functions on bounded symmetric domains. © 2000 Academic Press

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1. INTRODUCTION

Suppose that X is a normed linear space and W is a subset of X. The concept of an N-*width of* W *in* X was introduced as a means of measuring the degree to which W can be approximated by N-dimensional subspaces of X. The following four kinds of N-widths have been studied extensively, both from the abstract point of view as well as for specific choices of X and W. The *Kolmogorov* N-*width* is defined by the formula

$$d_N(W, X) = \inf_{X_N} \sup_{v \in W} \inf_{w \in X_N} \|v - w\|$$

in which the infimum is taken over all N-dimensional subspaces X_N of X. The *linear* N-*width* is given by

$$\delta_N(W, X) = \inf_{P_N} \sup_{v \in W} \|v - P_N v\|, \tag{1.1}$$

the infimum being taken over all bounded linear operators P_N on X of rank at most N. The *Bernstein* N-*width* is defined as

$$b_N(W, X) = \sup_{X_{N+1}} \sup\{t \in \mathbb{R} : tB(X_{N+1}) \subseteq W\},$$

where the first supremum is taken with respect to all $(N+1)$-dimensional subspaces X_{N+1} of X, and $B(X_{N+1})$ denotes the closed unit ball in X_{N+1}. Finally, the *Gelfand* N-*width* is given by

$$d^N(W, X) = \inf_{X^N} \sup_{v \in W \cap X^N} \|v\| \tag{1.2}$$

with the infimum taken over all subspaces X^N of X of codimension N. For a comprehensive treatment of these N-widths we refer to the book [8] by Pinkus.

In this paper we begin the study of these N-widths for some spaces of holomorphic functions defined on bounded symmetric domains. Our interest in this subject was generated by the papers [6] of Osipenko and

[3] of Farkov, and our results can be viewed as extending some of the results in these papers to the setting of bounded symmetric domains of tube type. We refer to the book [2] by Faraut and Korányi for the structure of such domains.

We briefly outline the results in this paper. Let D be an irreducible bounded symmetric domain of tube type. We realize D in terms of its associated simple Euclidean Jordan algebra V as the open unit ball in the complexification $V^{\mathbb{C}}$, the metric on $V^{\mathbb{C}}$ being given by the spectral norm [2, p. 198]. Let Σ be the Shilov boundary of D. For $0<\rho<1$ set $\Sigma_\rho = \rho\Sigma = \{\rho z \in V^{\mathbb{C}} : z \in \Sigma\}$ and $D_\rho = \rho D = \{\rho z \in V^{\mathbb{C}} : z \in D\}$. For a holomorphic function f on D we define its *radial derivative* $\mathscr{R}^l f$ *of order* l by (2.11) below. Now we can describe our results.

(1) Let $H_2(D)$ be the Hardy space and $A_2(D)$ be the Bergman space of D [2, Chaps. X and XIII], and denote by $B(H_2(D))$ and $B(A_2(D))$ the corresponding closed unit balls. Let X be the Banach space $C(\Sigma_\rho)$ of continuous functions on Σ_ρ, and let W be either the class $H_{2,l}(D)$ of all holomorphic functions f on D for which $\mathscr{R}^l f \in B(H_2(D))$, or the class $A_{2,l}(D)$ for which $\mathscr{R}^l f \in B(A_2(D))$. *We prove that the linear and Gelfand* N*-widths of* W *in* X *coincide, and we calculate the exact value.*

(2) Let $p \geqslant 1$ and denote by $H_p(D)$ and $A_p(D)$ the Hardy and Bergman spaces of D, respectively. Let $B(H_p(D))$ and $B(A_p(D))$ denote the corresponding closed init balls. *If* $X = L^p(\Sigma_\rho)$ *and* $W = H_{p,l}(D)$ *is the Hardy–Sobolev space of holomorphic functions* f *on* D *for which* $\mathscr{R}^l f \in B(H_p(D))$, *then we prove that all four* N*-widths of* W *in* X *coincide and we compute the exact value. We do the same for the case in which* $X = L^p(D_\rho)$ *and* $W = A_{p,l}(D)$ *is the Bergman–Sobolev space of holomorphic functions* f *on* D *for which* $\mathscr{R}^l f \in B(A_p(D))$.

This paper is organized as follows. In Section 2 we give the classification of bounded symmetric domains of tube type, provide some prerequisite analysis on these domains, and define the radial derivatives. In Section 3 we derive exact values of the N-widths for $H_{2,l}(D)$ and $A_{2,l}(D)$ in $C(\Sigma_\rho)$. In Sections 4–6 we develop some preliminary material on the Sobolev spaces on D and obtain upper bounds for linear N-widths and lower bounds for Bernstein N-widths. The results in these sections prepare the way for the main result, Theorem 7.1 in Section 7, in which we present the exact formulas for all four types of N-widths of $H_{p,l}(D)$ in $L^p(\Sigma_\rho)$ and $A_{p,l}(D)$ in $L^p(D_\rho)$. Additionally, we identify the corresponding optimal subspaces or optimal operators.

We remark that our work in this paper treats N-widths for bounded symmetric domains of tube type. N-widths for the non-tube type domains will be studied in [1].

2. PRELIMINARIES ON BOUNDED SYMMETRIC DOMAINS

For the reader less familiar with the theory of bounded symmetric domains, we begin with some general background. In brief, a bounded symmetric domain is an open domain in a finite-dimensional complex vector space having a transitive group of biholomorphic transformations with respect to which the domain is a Riemannian symmetric space. A bounded symmetric domain is a product of *irreducible* such domains, which leads one to study the irreducible domains. These are of two types. Those irreducible domains which can be realized, in analogy to the ordinary upper half-plane, as a tube domain over a cone are said to be *of tube type*. Otherwise an domain is said to be *not of tube type*. The irreducible bounded symmetric domains are classified completely. Those of tube type fall into four so-called *classical* families of domains, together with a single *exceptional* domain. We list the four classical families, the first, three of which are said to be *matrix domains*, and the fourth of which is known as the *Lie spheres*. For the matrix domains we use the notation $a<b$ to mean that $b-a$ is positive-definite, and we denote by I_r the $r\times r$ identity matrix.

(1) The domain of all complex $r\times r$ matrices z such that $zz^* < I_r$.

(2) The domain of all complex symmetric $r\times r$ matrices z such that $zz^* < I_r$.

(3) The domain of all complex skew-symmetric $2r\times 2r$ matrices z such that $zz^* < I_{2r}$.

(4) The domain consisting of all column vectors $z\in\mathbb{C}^n$, $n>2$, such that

$$1 - z^*z > \sqrt{(z^*z)^2 - |z^tz|^2}.$$

The exceptional domain has complex dimension 27, and can be realized in terms of 3×3 matrices over the Cayley algebra.

All of these domains can be studied simultaneously via the use of Jordan algebras, and it is such a realization of a bounded symmetric domain of tube type that we use in our work on N-widths. Thus, let V be a simple Euclidean Jordan algebra [2, Chap. III], denote by n its dimension as a real vector space, denote its rank by r, and let ε be the identity element in V. Endow the complexification $V^{\mathbb{C}} = V + iV$ with the spectral norm [2, p. 198], and set

$$D = \{w \in V^{\mathbb{C}} : |w| < 1\}, \tag{2.1}$$

where $|w|$ is the spectral norm of w. The domain D is called *the generalized unit disk in* $V^{\mathbb{C}}$, and is an irreducible bounded symmetric domain of tube type. Moreover, for any irreducible bounded symmetric domain of tube type there exists a simple Euclidean Jordan algebra V such that the domain is biholomorphically equivalent to (2.1).

In the remainder of this paper we let D be an irreducible bounded symmetric domain of tube type realized in terms of the Jordan algebra V as the domain (2.1).

Denote by Σ the *Shilov boundary* of D, consisting of all invertible elements $z \in V^{\mathbb{C}}$ such that $z^{-1} = \bar{z}$ [2, p. 190]. Set $G(\Sigma) = \{g \in GL(V^{\mathbb{C}}) : g\Sigma = \Sigma\}$, and let U denote the connected component of the identity element in $G(\Sigma)$. The group U acts transitively on Σ. We denote by σ the normalized (unit volume) U-invariant measure on Σ.

Let $\mathscr{P}(V^{\mathbb{C}})$ denote the space of all polynomials on $V^{\mathbb{C}}$, and $\mathscr{P}^k(V^{\mathbb{C}})$ the subspace of polynomials homogeneous of degree k. By a *partition* we mean an r-tuple $\mathbf{m} = (m_1, ..., m_r)$ of integers such that $m_1 \geqslant \cdots \geqslant m_r \geqslant 0$. We refer to the number $|\mathbf{m}| = m_1 + \cdots + m_r$ as the *length* of the partition $\mathbf{m}$. The *natural representation* τ of $GL(V^{\mathbb{C}})$ on $\mathscr{P}(V^{\mathbb{C}})$ is defined by

$$(\tau(g)\, p)(x) = p(g^{-1}x), \tag{2.2}$$

for all $x \in V^{\mathbb{C}}$, where $g \in GL(V^{\mathbb{C}})$ and $p \in \mathscr{P}(V^{\mathbb{C}})$. Given a partition $\mathbf{m}$, let

$$\Delta_{\mathbf{m}}(x) = \Delta_1(x)^{m_1 - m_2}\, \Delta_2(x)^{m_2 - m_3} \cdots \Delta_{r-1}(x)^{m_{r-1} - m_r}\, \Delta_r(x)^{m_r},$$

where $\Delta_j(x)$ is the jth *principal minor* of $x \in \Omega$ [2, p. 114]. The function $\Delta_{\mathbf{m}}$ is called the *power function* of $\mathbf{m}$.

Let $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ denote the subspace of $\mathscr{P}(V^{\mathbb{C}})$ generated by the polynomials $\tau(g)\, \Delta_{\mathbf{m}}$, $g \in GL(V^{\mathbb{C}})$. By [2, Theorem XI.2.4], $\mathscr{P}(V^{\mathbb{C}})$ decomposes as the orthogonal direct sum

$$\mathscr{P}(V^{\mathbb{C}}) = \bigoplus_{\mathbf{m}} \mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}}), \tag{2.3}$$

where the summation is over all partitions $\mathbf{m}$. The polynomials in $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ are homogeneous of degree $|\mathbf{m}|$, so $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}}) \subseteq \mathscr{P}^{|\mathbf{m}|}(V^{\mathbb{C}})$ and (2.3) is equivalent to the decomposition

$$\mathscr{P}^k(V^{\mathbb{C}}) = \bigoplus_{|\mathbf{m}| = k} \mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}}), \tag{2.4}$$

where the sum is over all partitions of length k. Let $d_{\mathbf{m}}$ denote the dimension of $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$. Then, by (2.4) and a well-known formula for the dimension of $\mathscr{P}^k(V^{\mathbb{C}})$,

$$\sum_{|\mathbf{m}|=k} d_{\mathbf{m}} = \dim \mathscr{P}^k(V^{\mathbb{C}}) = \binom{n+k-1}{k}. \tag{2.5}$$

For any positive integer μ set

$$N_\mu = \sum_{k=0}^{\mu-1} \dim \mathscr{P}^k(V^{\mathbb{C}}) = \sum_{k=0}^{\mu-1} \binom{n+k-1}{k}. \tag{2.6}$$

By a standard property of the binomial coefficients,

$$N_\mu = \binom{n+\mu-1}{n}. \tag{2.7}$$

The number N_μ is the dimension of the space of all polynomials in $\mathscr{P}(V^{\mathbb{C}})$ homogeneous of degree strictly less than μ.

It is important to note the group theoretic significance of the decomposition (2.3). Associated to the Jordan algebra V is a symmetric cone Ω, defined as the connected component of ε in the set of invertible elements of V [2, Chap. III]. Let $G(\Omega) = \{g \in GL(V) : g\Omega \subseteq \Omega\}$, the automorphism group of Ω, and denote by G the connected component of the identity in $G(\Omega)$. Then G acts transitively on Ω, the isotropy subgroup $K = \{k \in G : k\varepsilon = \varepsilon\}$ of ε is a maximal compact subgroup of G, and $\Omega \simeq G/K$. Thus Ω is a symmetric space; hence, the term "symmetric cone."

Now consider $\mathscr{P}(V^{\mathbb{C}})$ as a G-module under the action (2.2) defined by τ. Each of the subspaces $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ is irreducible, distinct values of $\mathbf{m}$ correspond to inequivalent representations, and (2.3) is the multiplicity-free decomposition of $\mathscr{P}(V^{\mathbb{C}})$ as a G-module into its irreducible constituents [2, Chap. XI]. In particular, in each subspace $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ there is a non-zero K-invariant polynomial $\Phi_{\mathbf{m}}$, unique up to constant multiples, which when normalized so that $\Phi_{\mathbf{m}}(\varepsilon) = 1$ is called the *spherical polynomial of order* $\mathbf{m}$. Note that $\Delta_{\mathbf{m}} \in \mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ and

$$\Phi_{\mathbf{m}}(z) = \int_K \Delta_{\mathbf{m}}(kz)\, dk$$

for all $z \in V^{\mathbb{C}}$, where dk is normalized Haar measure on K.

Equip $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ with the inner product

$$\langle f_1 \mid f_2 \rangle_\Sigma = \int_\Sigma f_1(z)\overline{f_2(z)}\, d\sigma(z) \tag{2.8}$$

for $f_1, f_2 \in \mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$. Then $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$ becomes a reproducing kernel Hilbert space, and we denote the reproducing kernel by $H^{\mathbf{m}}(z, w)$ for $z, w \in V^{\mathbb{C}}$. By [2, Proposition XII.2.4], $H^{\mathbf{m}}(z, \varepsilon) = d_{\mathbf{m}} \Phi_{\mathbf{m}}(z)$ for all $z \in V^{\mathbb{C}}$, and hence

$$H^{\mathbf{m}}(\varepsilon, \varepsilon) = d_{\mathbf{m}} \Phi_{\mathbf{m}}(\varepsilon) = d_{\mathbf{m}}.$$

Let $\{\varphi_1^{\mathbf{m}}, ..., \varphi_{d_{\mathbf{m}}}^{\mathbf{m}}\}$ be a basis for $\mathscr{P}_{\mathbf{m}}(V^{\mathbb{C}})$, orthonormal with respect to the inner product (2.8.). By the general theory of reproducing kernel spaces, we have

$$H^{\mathbf{m}}(z, w) = \sum_{j=1}^{d_{\mathbf{m}}} \varphi_j^{\mathbf{m}}(z) \overline{\varphi_j^{\mathbf{m}}(w)} \tag{2.9}$$

for all $z, w \in D$.

We close this section with the definition of the radial derivatives. Let f be a holomorphic function on the domain D, and denote its decomposition into homogeneous parts

$$f(z) = \sum_{k=0}^{\infty} F_k(z), \tag{2.10}$$

for $z \in D$, where F_k is a homogeneous polynomial of degree k. We define the *radial derivative of f of order l* by

$$\mathscr{R}^l f(z) = \sum_{k=l}^{\infty} \frac{k!}{(k-l)!} F_k(z). \tag{2.11}$$

Note that for each positive integer l, $\mathscr{R}^l$ is an invariant differential operator on Ω. That is, $\mathscr{R}^l$ commutes with the natural representation (2.2) of $GL(V^{\mathbb{C}})$ on $\mathscr{P}(V^{\mathbb{C}})$.

3. N-WIDTHS OF $H_{2,l}(D)$ AND $A_{2,l}(D)$ IN $C(\Sigma_\rho)$

The *Hardy space* $H_2(D)$ is the space of holomorphic functions f on D such that

$$\|f\|_{H_2(D)}^2 = \sup_{0<\rho<1} \int_{\Sigma} |f(\rho z)|^2 \, d\sigma(z) < \infty.$$

By [2, p. 270], if $f \in H_2(D)$ then the function f_ρ, defined by $f_\rho(z) = f(\rho z)$, has a limit as $\rho \to 1-$. This limit is also denoted by f, and

$$\|f\|_{H_2(D)} = \|f\|_{\Sigma}, \tag{3.1}$$

where $\|f\|_\Sigma$ is the norm derived from (2.8). By means of (3.1), we identify the Hilbert spaces $H_2(D)$ and $L^2(\Sigma)$.

Fix a positive integer l, and denote by $\mathscr{H}_2^l(D)$ the space of holomorphic functions $f\colon D \to \mathbb{C}$ that satisfy the following two properties:

(i) In the homogeneous decomposition (2.10) of f, $F_k(z) \equiv 0$ for all $k = 0, 1, ..., l-1$; and

(ii) $R^l f \in H_2(D)$.

For $l = 0$ we set $\mathscr{H}_2^0(D) = H_2(D)$. Then the space $\mathscr{H}_2^l(D)$ is a Hilbert space with inner product

$$(f, g)_l = (\mathscr{R}^l f, \mathscr{R}^l g)_{H_2(D)}. \tag{3.2}$$

Let

$$\mathscr{B}^0 = \bigcup_{|\mathbf{m}| \geqslant l} \{\varphi_1^{\mathbf{m}}, ..., \varphi_{d_\mathbf{m}}^{\mathbf{m}}\}$$

denote the basis of $\mathscr{H}_2^l(D)$ in which $\{\varphi_1^{\mathbf{m}}, ..., \varphi_{d_\mathbf{m}}^{\mathbf{m}}\}$ is the basis for $\mathscr{P}_\mathbf{m}(V^\mathbb{C})$ that appears in (2.9). Write the Fourier expansions of $f, g \in \mathscr{H}_2^l(D)$ with respect to the basis $\mathscr{B}^0$ as

$$f(z) = \sum_{|\mathbf{m}| \geqslant l} \sum_{j=1}^{d_\mathbf{m}} c_{j,\mathbf{m}} \varphi_j^{\mathbf{m}}(z)$$

and

$$g(z) = \sum_{|\mathbf{m}| \geqslant l} \sum_{j=1}^{d_\mathbf{m}} d_{j,\mathbf{m}} \varphi_j^{\mathbf{m}}(z).$$

Because the basis $\mathscr{B}^0$ is complete and orthonormal for the subspace of $H_2(D)$ (or $L^2(\Sigma)$) spanned by $\mathscr{P}_\mathbf{m}(V^\mathbb{C})$ with $|\mathbf{m}| \geqslant l$, by (2.11) and (3.2),

$$(f, g)_l = \sum_{|\mathbf{m}| \geqslant l} \left(\frac{|\mathbf{m}|!}{(|\mathbf{m}| - l)!}\right)^2 \sum_{j=1}^{d_\mathbf{m}} c_{j,\mathbf{m}} \overline{d_{j,\mathbf{m}}}.$$

Therefore the set of functions

$$\left\{\frac{(|\mathbf{m}| - l)!}{|\mathbf{m}|!} \varphi_j^{\mathbf{m}}(z) : |\mathbf{m}| \geqslant l\right\} \tag{3.3}$$

forms a complete orthonormal basis for $\mathscr{H}_2^l(D)$.

For $0<\rho<1$ we set $\Sigma_\rho=\rho\Sigma=\{\rho z: z\in\Sigma\}$, and we denote by σ_ρ normalized U-invariant measure on Σ_ρ. Then the functions (3.3) are also orthogonal in $L^2(\Sigma_\rho, \sigma_\rho)$, and

$$\|\varphi_j^{\mathbf{m}}\|^2_{L^2(\Sigma_\rho,\sigma_\rho)}=\int_{\Sigma_\rho}|\varphi_j^{\mathbf{m}}(\psi)|^2\,d\sigma_\rho(\psi)=\int_{\Sigma}|\varphi_j^{\mathbf{m}}(\rho z)|^2\,d\sigma(z)=\rho^{2\,|\mathbf{m}|}, \tag{3.4}$$

where the last equality follows from the homogeneity and orthonormality of $\varphi_j^{\mathbf{m}}$ in $L^2(\Sigma)$.

An important consequence of (3.4) is the monotonicity of these norms. That is, if $\mathbf{m}$ and $\mathbf{m}'$ are two partitions such that $|\mathbf{m}|\leqslant|\mathbf{m}'|$ then

$$\|\varphi_j^{\mathbf{m}}\|_{L^2(\Sigma_\rho,\sigma_\rho)}\geqslant\|\varphi_{j'}^{\mathbf{m}'}\|_{L^2(\Sigma_\rho,\sigma_\rho)} \tag{3.5}$$

for all j and j'.

Finally we define the class $H_{2,l}(D)$ of functions whose Gelfand and linear N-widths in $C(\Sigma_\rho)$ we will determine. Let $l\geqslant 1$. A holomorphic function f on D will lie in $H_{2,l}(D)$ if $\mathcal{R}^l f\in B(H_2(D))$, where $B(H_2(D))$ is the unit ball in the Hilbert space $H_2(D)$. Note that $\mathcal{H}_2^l(D)$ is orthogonal to

$$\bigoplus_{k=0}^{l-1}\mathcal{P}_k(V^{\mathbb{C}})$$

in the Hilbert space $L^2(\Sigma_\rho,\sigma_\rho)$. Hence, we can write

$$H_{2,l}(D)=B(\mathcal{H}_2^l(D))+\left(\bigoplus_{k=0}^{l-1}\mathcal{P}_k(V^{\mathbb{C}})\right)$$

in $L^2(\Sigma_\rho,\sigma_\rho)$, or in the Banach space $C(\Sigma_\rho)$, where $B(\mathcal{H}_2^l(D))$ is the unit ball in the Hilbert space $H_2^l(D)$. When $l=0$, we set $H_{2,l}(D)=B(H_2(D))$.

We are now in a position to apply a result due to Osipenko [6, Theorem 2], which can be described as follows. Let H be a reproducing kernel Hilbert space of continuous functions on a set Ω and E a compact subset of Ω with a probability measure ξ such that the restriction map $f\mapsto f|_E$ is a bounded linear operator from H to the Banach space $C(E)$. View a function in H as an element of $L^2(E,\xi)$, and suppose that we are given an orthonormal basis $\{\varphi_1,\varphi_2,...\}$ of H that is also an orthogonal set in $L^2(E,\xi)$ ordered so that

$$\|\varphi_1\|_E\geqslant\|\varphi_2\|_E\geqslant\cdots, \tag{3.6}$$

where $\|\cdot\|_E$ denotes the norm in $L^2(E,\xi)$. Finally, suppose we are also given an r-dimensional subspace X_r of $C(E)$ with the property that X_r is orthogonal to H in the inner product on $L^2(E,\xi)$. Let $A=B(H)+X_r$ where $B(H)$ is the unit ball in H viewed as a subset of $C(E)$. Then

Osipenko's theorem provides the following estimates on the linear and Gelfand $(N+r)$-widths of A in $C(E)$:

$$\sqrt{\sum_{j>N} \|\varphi_j\|_E^2} \leqslant \delta_{N+r}(A, C(E)) = d^{N+r}(A, C(E)) \leqslant \sup_{z \in E} \sqrt{\sum_{j>N} |\varphi_j(z)|^2}. \tag{3.7}$$

The preceding calculations show that the hypotheses of Osipenko's theorem hold with $H = \mathscr{H}_2^l(D)$, $E = \Sigma_\rho$, and $A = H_{2,l}(D)$ (cf. (3.5) and (3.6)). Then the estimate in (3.7) becomes

$$\begin{aligned}
&\left(\sum_{k=\mu}^{\infty} \left(\frac{(k-l)!}{k!} \right)^2 \binom{n+k-1}{n-1} \rho^{2k} \right)^{1/2} \\
&\qquad \leqslant d^{N_\mu}(H_{2,l}(D), C(\Sigma_\rho)) \\
&\qquad = \delta_{N_\mu}(H_{2,l}(D), C(\Sigma_\rho)) \\
&\qquad \leqslant \sup_{z \in \Sigma_\rho} \left(\sum_{|\mathbf{m}| \geqslant \mu} \left(\frac{(|\mathbf{m}|-l)!}{|\mathbf{m}|!} \right)^2 \sum_{j=1}^{d_{\mathbf{m}}} |\varphi_j^{\mathbf{m}}(z)|^2 \right)^{1/2},
\end{aligned} \tag{3.8}$$

where $0 < \rho < 1$, $\mu \geqslant l \geqslant 0$, N_μ is given by (2.7), and δ_{N_μ} and d^{N_μ} denote the linear and Gelfand N_μ-widths, (1.1) and (1.2), respectively.

3.1. THEOREM. *Let $0 < \rho < 1$ and $\mu \geqslant l \geqslant 0$ as above. Then the Gelfand and linear N_μ-widths $d^{N_\mu}(H_{2,l}(D), C(\Sigma_\rho))$ and $\delta_{N_\mu}(H_{2,l}(D), C(\Sigma_\rho))$ both are equal to*

$$\rho^\mu \left(\frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{((\mu+k-l)!)^2 \, (n+\mu-1+k)!}{((\mu+k)!)^3} \rho^{2k} \right)^{1/2}. \tag{3.9}$$

Proof. It is enough to show that the upper bound in (3.8) coincides with the lower bound, since the expression in (3.9) is just a simplified version of the lower bound in (3.8).

By (2.9)

$$\sum_{j=1}^{d_{\mathbf{m}}} |\varphi_j^{\mathbf{m}}(z)|^2 = H^{\mathbf{m}}(z, z). \tag{3.10}$$

By [2, Theorem XII.1.1; 2, Proposition XII.2.4(iii)],

$$|H^{\mathbf{m}}(z, z)| \leqslant \rho^{2|\mathbf{m}|} \, d_{\mathbf{m}} \tag{3.11}$$

for $z \in \Sigma_\rho$. By (3.10) and (3.11), we find that the upper bound in (3.8) is bounded above by

$$\begin{aligned}\left(\sum_{|\mathbf{m}| \geqslant \mu} \left(\frac{(|\mathbf{m}|-l)!}{|\mathbf{m}|!}\right)^2 \rho^{2|\mathbf{m}|} d_{\mathbf{m}}\right)^{1/2} &= \left(\sum_{k=\mu}^{\infty} \left(\frac{(k-l)!}{k!}\right)^2 \rho^{2k} \sum_{|\mathbf{m}|=k} d_{\mathbf{m}}\right)^{1/2} \\ &= \left(\sum_{k=\mu}^{\infty} \left(\frac{(k-l)!}{k!}\right)^2 \rho^{2k} \binom{n+k-1}{k}\right)^{1/2},\end{aligned} \tag{3.12}$$

where the last equality follows from (2.5). Since (3.12) and the lower bound in (3.8) are identical, the proof is complete.

Now we turn to the case of the Bergman space $A_2(D)$, which is the set of all holomorphic functions f on D satisfying the condition

$$\|f\|^2_{A_2(D)} = \int_D |f(z)|^2 \, dv(z) < \infty,$$

where v is the normalized Lebesgue measure on D. In analogy to the preceding case, for $l \geqslant 1$ we let $\mathscr{A}^l_2(D)$ be the space of holomorphic functions f on D for which $F_k(z)$ vanishes for $0 \leqslant k \leqslant l-1$ and $\mathscr{R}^l f \in A_2(D)$; and we denote by $A_{2,l}(D)$ the class of holomorphic functions f on D for which $\mathscr{R}^l f \in B(A_2(D))$. For $l=0$ we set $\mathscr{A}^0_2(D) = A_2(D)$ and $A_{2,0}(D) = B(A_2(D))$. Also, for a complex number s and a partition $\mathbf{m}$ we use the notation [2, p. 129]

$$(s)_{\mathbf{m}} = \prod_{j=1}^{r} \left(s - \tfrac{1}{2}(j-1)\, d\right)_{m_j}$$

for the generalized shifted factorial for the Jordan algebra V, where $(s)_j = s(s+1) \cdots (s+j-1)$ is the usual shifted factorial and $d = 2(n-r)/r(r-1)$ [2, p. 71].

Let $0 < \rho < 1$, and l be a nonnegative integer. In order to apply the estimate (3.7) in the proof of the theorem that follows, we reenumerate the set of all partitions $\mathbf{m}$ satisfying $|\mathbf{m}| \geqslant l$ as $\{\mathbf{m}^{(j)}: j=0, 1, ...\}$ in such a way that,

$$\left\{\left(\frac{(|\mathbf{m}^{(j)}|-l)!}{|\mathbf{m}^{(j)}|!}\right)^2 \frac{(2n/r)_{\mathbf{m}^{(j)}}}{(n/r)_{\mathbf{m}^{(j)}}} \rho^{2|\mathbf{m}^{(j)}|} : j=0, 1, ...\right\} \tag{3.13}$$

forms a non-increasing sequence. For any nonnegative integer μ, set

$$M_\mu = \begin{cases} N_l, & \mu = 0, \\ N_l + \sum_{j=0}^{\mu-1} d_{\mathbf{m}^{(j)}}, & \mu \geqslant 1, \end{cases} \tag{3.14}$$

where N_l is given by (2.7).

3.2. THEOREM. *Under the above assumptions, $d^{M_\mu}(A_{2,l}(D), C(\Sigma_\rho))$ and $\delta_{M_\mu}(A_{2,l}(D), C(\Sigma_\rho))$, the Gelfand and linear N-widths of $A_{2,l}(D)$ in $C(\Sigma_\rho)$, both are equal to*

$$\left(\sum_{j=\mu}^{\infty} \left(\frac{(|\mathbf{m}^{(j)}|-l)!}{|\mathbf{m}^{(j)}|!} \right)^2 \frac{(2n/r)_{\mathbf{m}^{(j)}}}{(n/r)_{\mathbf{m}^{(j)}}} d_{\mathbf{m}^{(j)}} \rho^{2|\mathbf{m}^{(j)}|} \right)^{1/2},$$

where $\mu \geqslant 0$.

Proof. By [2, Proposition XI.4.1; 2, Corollary XI.4.2],

$$\|\varphi_j^{\mathbf{m}}\|^2_{A_2(D)} = \frac{(n/r)_{\mathbf{m}}}{(2n/r)_{\mathbf{m}}} \|\varphi_j^{\mathbf{m}}\|^2_{\Sigma}.$$

Therefore the set of functions

$$\left\{ \frac{(|\mathbf{m}|-l)!}{|\mathbf{m}|!} \left(\frac{(2n/r)_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \right)^{1/2} \varphi_j^{\mathbf{m}} : j = 1, ..., d_{\mathbf{m}}, |\mathbf{m}| \geqslant l \right\}$$

forms an orthonormal basis for $\mathscr{A}_2^l(D)$ and an orthogonal system in $L^2(\Sigma_\rho)$. Moreover, by (3.4)

$$\left\| \frac{(|\mathbf{m}|-l)!}{|\mathbf{m}|!} \left(\frac{(2n/r)_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \right)^{1/2} \varphi_j^{\mathbf{m}} \right\|^2_{L^2(\Sigma_\rho)} = \left(\frac{(|\mathbf{m}|-l)!}{|\mathbf{m}|!} \right)^2 \frac{(2n/r)_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \rho^{2|\mathbf{m}|}. \tag{3.15}$$

Notice that the numbers on the right-hand side of (3.15) are the same as those in (3.13). Thus, when the partitions $\mathbf{m}$ are reenumerated as the sequence $\{\mathbf{m}^{(j)}: j = 0, 1, ...\}$ as above, then the monotonicity condition (3.6) holds, and then the remainder of the proof of Theorem 3.2 is now completely to the proof of Theorem 3.1.

3.3. *Remarks.* (1) Let $H_{\infty,l}(D)$ denote the Bergman–Sobolev space of holomorphic functions f on D for which $\mathscr{R}^l f \in B(H_\infty(D))$. The referee has pointed out that, for $\mu \geqslant l+1$, the same method of calculation can be used to obtain the exact values of the Bernstein width

$$b_{N_\mu - 1}(H_{\infty, l}(D), L^2(\Sigma_\rho)) = \left(\frac{1}{(n-1)!} \sum_{k=l}^{\mu-1} \frac{k!(n+k-1)!}{((k-l)!)^2} \rho^{-2k} \right)^{-1/2},$$

where N_μ is given by (2.7). We also refer to [7] for the case of the unit ball in $\mathbb{C}^n$.

(2) As in Theorem 3.2, we may reenumerate partitions $\mathbf{m}$ satisfying $|\mathbf{m}| \geqslant l$ as $\{\mathbf{m}^{(j)}: j=0, 1, ...\}$ in such a way that

$$\left\{ \left(\frac{(|\mathbf{m}^{(j)}|-l)!}{|\mathbf{m}^{(j)}|!} \right)^2 \frac{(n/r)_{\mathbf{m}^{(j)}}}{(2n/r)_{\mathbf{m}^{(j)}}} \rho^{2\,|\mathbf{m}^{(j)}|} : j=0, 1, ... \right\} \tag{3.16}$$

forms a non-increasing sequence. Note that (3.16) is different from (3.13) so that this reenumeration is different from the reenumeration in Theorem 3.2. By defining M_μ as in (3.14) we obtain, for $\mu > 0$,

$$b_{M_\mu - 1}(H_{\infty, l}(D), L^2(D_\rho))$$
$$= \left(\sum_{j=0}^{\mu-1} \left(\frac{|\mathbf{m}^{(j)}|!}{(|\mathbf{m}^{(j)}|-l)!} \right)^2 d_{\mathbf{m}^{(j)}} \frac{(2n/r)_{\mathbf{m}^{(j)}}}{(n/r)_{\mathbf{m}^{(j)}}} \rho^{-2\,|\mathbf{m}^{(j)}|} \right)^{-1/2}.$$

4. PRELIMINARIES ON SOBOLEV SPACES

For $p \geqslant 1$, the Hardy space $H_p(D)$ is the class of functions $f: D \to \mathbb{C}$ such that

$$\|f\|_{H_p(D)}^p = \sup_{0<\alpha<1} \int_\Sigma |f(\alpha z)|^p \, d\sigma(z) < \infty.$$

Let $H_{p,l}(D)$ denote the Hardy–Sobolev space of all holomorphic functions f on D for which $\mathscr{R}^l f \in B(H_p(D))$. Similarly, let $A_{p,l}(D)$ denote the Bergman–Sobolev space of all holomorphic functions f on D for which $\mathscr{R}^l f \in B(A_p(D))$ where $A_p(D)$ is the class of all functions $f: D \to \mathbb{C}$ such that,

$$\|f\|_{A_p(D)}^p = \int_D |f(z)|^p \, dv(z) < \infty.$$

Recall from Section 2 that for each partition $\mathbf{m}$ the set of functions $\{\varphi_1^{\mathbf{m}}, ..., \varphi_{d_{\mathbf{m}}}^{\mathbf{m}}\}$ is a basis for $P_{\mathbf{m}}(V^{\mathbb{C}})$, orthonormal with respect to the norm on $L^2(\Sigma)$. Reenumerate the set $\{\varphi_j^{\mathbf{m}}: \mathbf{m}$ any partition and $1 \leqslant j \leqslant d_{\mathbf{m}}\}$ as

$\{\varphi_{(i)} : i=0, 1, 2, ...\}$ in such a way that $\deg(\varphi_{(i)}) \leqslant \deg(\varphi_{(i+1)})$ for all $i=0, 1, 2, ...$ For a given positive integer N, let

$$N' = \min\{i : \deg(\varphi_{(i)}) = \deg(\varphi_{(N)})\}$$

and

$$\tilde{N} = \deg(\varphi_{(N)}).$$

Note that $N' \leqslant N$ and $\deg(\varphi_{(N'-1)}) = \tilde{N} - 1$. Also, it follows from (2.5) and (2.6) that if $N_k \leqslant N < N_{k+1}$ then $\tilde{N} = k$ and $N' = N_k$.

We set

$$\begin{aligned} \mathscr{P}^{(N)} &= \operatorname{Span}\{\varphi_{(i)} : \deg(\varphi_{(i)}) \leqslant N\}, \\ \Pi^{(N)} &= \operatorname{Span}\{\varphi_{(i)} : i=0, 1, ..., N\}, \end{aligned} \tag{4.1}$$

and

$$\pi^{(N)} = \operatorname{Span}\{\varphi_{(i)} : i=0, 1, ..., N'-1\}.$$

From these definitions we have $\pi^{(N)} = \Pi^{(N'-1)}$, $\Pi^{(N)} \subseteq \mathscr{P}^{(\tilde{N})}$, and $\mathscr{P}^{(k)} = \Pi^{(N_{k+1}-1)}$.

Let

$$X_p^N(D) = \{f \in H_p(D) : F_k = 0, k=0, 1, ..., \tilde{N}-1\}, \tag{4.2}$$

and

$$Y_p^N(D) = \{f \in A_p(D) : F_k = 0, k=0, 1, ..., \tilde{N}-1\}, \tag{4.3}$$

where F_k is the kth term in the homogeneous decomposition (2.10) of f.

For integers N and l with $0 \leqslant l < N$, set

$$\alpha_{N,l} = \frac{(N-l)!}{N!}.$$

For any holomorphic function $f \in H_p(D)$ and $0 < \rho \leqslant 1$, define

$$(P_N f)(z) = \sum_{k=0}^{\tilde{N}-1} (1 - \rho^{2(\tilde{N}-k)}) F_k(z), \tag{4.4}$$

and for $l \geqslant 1$ define

$$(P_N^l f)(z) = \sum_{k=0}^{l-1} F_k(z) + \sum_{k=l}^{\tilde{N}-1} \left(1 - \frac{\alpha_{2\tilde{N}-k,l}}{\alpha_{k,l}} \rho^{2(\tilde{N}-k)}\right) F_k(z). \tag{4.5}$$

When $l=0$, set $P_N^0 = P_N$.

5. UPPER BOUNDS FOR LINEAR N-WIDTHS

5.1. LEMMA. *Let* $1 \leqslant p \leqslant \infty$, $0 < \rho \leqslant 1$, *and* $0 < l < \tilde{N}$.

(a) *If* $f \in H_{p,l}(D)$ *then*

$$\|f - P^l_N f\|_{L^p(\Sigma_\rho)} \leqslant \alpha_{\tilde{N},l}\,\rho^{\tilde{N}}. \tag{5.1}$$

(b) *If* $f \in A_{p,l}(D)$ *then*

$$\|f - P^l_N f\|_{L^p(D_\rho)} \leqslant \alpha_{\tilde{N},l}\,\rho^{\tilde{N}+2np^{-1}}. \tag{5.2}$$

Proof. For $0 < \rho \leqslant 1$ and all real numbers t, let

$$K_{l,N}(\rho, t) = \alpha_{\tilde{N},l} + 2 \sum_{k=\tilde{N}+1}^{\infty} \rho^{k-\tilde{N}} \alpha_{k,l} \cos(k - \tilde{N})\, t. \tag{5.3}$$

it is well known (cf. Pinkus [8, p. 251]) that

$$K_{l,N}(\rho, t) \geqslant 0 \tag{5.4}$$

for all $0 < \rho \leqslant 1$ and all real numbers t.

Let f be holomorphic in D, $0 < s < 1$, $\zeta \in V^{\mathbb{C}}$, with $|\zeta| = 1$, and f_s be defined by $f_s(z) = f(sz)$, $z \in D$. Then

$$\begin{aligned} &f_s(z) - (P^l_N f_s)(z) \\ &\quad = \frac{\lambda^l}{2\pi} \int_{-\pi}^{\pi} \rho^{\tilde{N}-l} \exp(i(l - \tilde{N})(\theta - \phi))\, K_{l,N}(\rho, \theta - \phi)\, f^{(l)}_{s\zeta}(e^{i\theta})\, d\theta, \end{aligned} \tag{5.5}$$

where $z = \lambda\zeta$, $\lambda = \rho e^{i\phi}$, $0 < \rho \leqslant 1$, $f_{s\zeta}(\lambda) = f_s(\lambda\zeta)$, and $f^{(l)}_{s\zeta}$ is the lth derivative in λ of $f_{s\zeta}$. By (2.11) we have

$$\mathcal{R}^l f_s(\lambda\zeta) = \lambda^l f^{(l)}_{s\zeta}(\lambda). \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\begin{aligned} &\int_\Sigma |f_s(\lambda\zeta) - (P^l_N f_s)(\lambda\zeta)|^p\, d\sigma(\zeta) \\ &\qquad \leqslant \rho^{\tilde{N}p} \int_\Sigma \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} K_{l,N}(\rho, \theta - \phi)\, |\mathcal{R}^l f_s(e^{i\theta}\zeta)|\, d\theta \right)^p d\sigma(\zeta). \end{aligned}$$

By (5.3) and (5.4),

$$\|K_{l,N}(\rho, \cdot)\|_{L^1(-\pi,\pi)} = \alpha_{\tilde{N},l}.$$

By a well-known property of convolution,

$$\|f_s(\lambda\cdot)-(P^l_N f_s)(\lambda\cdot)\|_{L^p(\Sigma)} \leqslant \alpha_{\tilde{N},l}\rho^{\tilde{N}} \|\mathscr{R}^l f_s\|_{L^p(\Sigma)}. \tag{5.7}$$

Applying the technique utilized in the proof of Theorem 2 of Graham [5], we deduce that if $f \in H_{p,l}(D)$ then $f \in H_p(D)$. By the Lebesgue dominated convergence theorem [9, Theorem 5.6.6],

$$\lim_{s\to 1-} \|f(\lambda\cdot)-f_s(\lambda\cdot)\|_{L^p(\Sigma)}=0. \tag{5.8}$$

By (4.4) and (4.5), $P^l_N f$ is a polynomial; therefore

$$\lim_{s\to 1-} \|P^l_N f_s(\lambda\cdot)-P^l_N f(\lambda\cdot)\|_{L^p(\Sigma)}=0. \tag{5.9}$$

If $f \in H_{p,l}(D)$ then $\|\mathscr{R}^l f\|_{L^p(\Sigma)} \leqslant 1$. Letting $\phi=0$ in (5.7) and $s \to 1-$, we see that (5.1) follows from (5.8) and (5.9) for $1 \leqslant p < \infty$. In the case in which $p=\infty$, (5.1) follows directly from (5.5) and (5.6).

We turn now to the proof of (5.2). Thus, if $f \in A_{p,l}(D)$ then $\|\mathscr{R}^l f\|_{L^p(D)} \leqslant 1$. By (5.5), (5.6), and the argument, as above,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_s(\lambda\zeta)-(P^l_N f_s)(\lambda\eta)|^p \, d\phi \leqslant \alpha^p_{\tilde{N},l}\rho^{\tilde{N}p} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathscr{R}^l f_s(e^{i\theta}\zeta)|^p \, d\theta. \tag{5.10}$$

Integrating both sides of (5.10) over $D(0<s<1, |\zeta|=1)$ and changing variables $\zeta \mapsto z/\rho$, we have

$$\rho^{-2n} \|f-P^l_N f\|^p_{L^p(D_\rho)} \leqslant \alpha^p_{\tilde{N},l}\rho^{\tilde{N}p} \|\mathscr{R}^l f\|^p_{L^p(D)} \leqslant \alpha^p_{\tilde{N},l}\rho^{\tilde{N}p}.$$

Then (5.2) holds for $1 \leqslant p < \infty$. Similarly, (5.2) holds also for $p=\infty$.

5.2. Proposition. *Suppose* $1 \leqslant p \leqslant \infty$, $0 < \rho \leqslant 1$, *and* $0 < l < \tilde{N}$. *Then*

$$\delta_N(H_{p,l}(D); L^p(\Sigma_\rho)) \leqslant \alpha_{\tilde{N},l}\rho^{\tilde{N}}$$

and

$$\delta_N(A_{p,l}(D); L^p(D_\rho)) \leqslant \alpha_{\tilde{N},l}\rho^{\tilde{N}+2np^{-1}}.$$

Proof. The proposition follows directly from Lemma 5.1.

6. LOWER BOUNDS FOR BERNSTEIN N-WIDTHS

6.1. LEMMA. *Let* $0<p\leqslant\infty$, $0<\rho\leqslant 1$, *and* $N\geqslant 0$. *If* $P_N\in\mathscr{P}^{(N)}$ *then*

$$\|P_N(\rho\,\cdot)\|_{L^p(\Sigma)}\geqslant\rho^N\,\|P_N\|_{L^p(\Sigma)} \tag{6.1}$$

and

$$\|P_N\|_{L^p(D_\rho)}\geqslant\rho^{N+2np^{-1}}\,\|P_N\|_{L^p(D)}. \tag{6.2}$$

Proof. By Pinkus [8, p. 252], the inequality (6.1) is valid if V is one-dimensional. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(\rho e^{i\theta}\zeta)|^p\,d\theta\geqslant\frac{\rho^{Np}}{2\pi}\int_{-\pi}^{\pi}|P_N(e^{i\theta}\zeta)|^p\,d\theta \tag{6.3}$$

for all $\zeta\in V^{\mathbb{C}}$, $|\zeta|=1$. Integrating both sides of (6.3) over Σ, we have

$$\begin{aligned}\int_\Sigma|P_N(\rho\zeta)|^p\,d\sigma(\zeta)&=\int_\Sigma\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(\rho e^{i\theta}\zeta)|^p\,d\theta\,d\sigma(\zeta)\\&\geqslant\rho^{Np}\int_\Sigma\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(e^{i\theta}\zeta)|^p\,d\theta\,d\sigma(\zeta)\\&=\rho^{Np}\int_\Sigma|P_N(\zeta)|^p\,d\sigma(\zeta).\end{aligned}$$

Hence (6.1) holds for $0<p<\infty$, and the case in which $p=\infty$ can be proved similarly.

Integrating ζ over D on both sides of (6.3), we have

$$\|P_N(\rho\,\cdot)\|_{L^p(D)}\geqslant\rho^N\,\|P_N\|_{L^p(D)}.$$

Changing variables, we have

$$\|P_N\|_{L^p(D_\rho)}=\rho^{2np^{-1}}\,\|P_N(\rho\,\cdot)\|_{L^p(D)}=\rho^{N+2np^{-1}}\,\|P_N\|_{L^p(D)}.$$

Thus, (6.2) holds for $0<p<\infty$. The case in which $p=\infty$ is established by a similar, but less complicated, argument.

6.2. LEMMA. *Let* $P_N\in\mathscr{P}^{(N)}$, $1\leqslant p\leqslant\infty$, *and* $0\leqslant l<N$. *Then*

$$\|\mathscr{R}^lP_N\|_{H_p(D)}\leqslant\alpha_{N,l}^{-1}\,\|P_N\|_{H_p(D)} \tag{6.4}$$

and

$$\|\mathscr{R}^l P_N\|_{A_p(D)} \leqslant \alpha_{N,l}^{-1} \|P_N\|_{A_p(D)} \tag{6.5}$$

Proof. By Pinkus [8, p. 252], if $Q_N \in \mathscr{P}^{(N)}(\mathbb{C}^1)$ (where $\mathscr{P}^{(N)}(\mathbb{C}^1)$ is defined as in (4.1) but with V of dimension 1), then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |Q_N'(e^{i\theta})|^p \, d\theta \leqslant N^p \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q_N(e^{i\theta})|^p \, d\theta \tag{6.6}$$

and

$$\|Q_N'\|_{C(T)} \leqslant N \|Q_N\|_{C(T)}, \tag{6.7}$$

where $\|\cdot\|_{C(T)}$ is the supremum norm on the circle $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$. By (6.6) and (5.6), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathscr{R}^l P_N(e^{i\theta}\zeta)|^p \, d\theta \leqslant \frac{\alpha_{N,l}^{-p}}{2\pi}\int_{-\pi}^{\pi} |P_N(e^{i\theta}\zeta)|^p \, d\theta. \tag{6.8}$$

Integrating both sides of (6.8) over Σ, we obtain

$$\|\mathscr{R}^l P_N\|_{L^p(\Sigma)} \leqslant \alpha_{N,l}^{-1} \|P_N\|_{L^p(\Sigma)}$$

from which (6.4) follows by (3.1). Integrating ζ over D on both sides of (6.8), we obtain (6.5). Finally, the case in which $p = \infty$ follows from (6.7).

6.3. Proposition. *If* $1 \leqslant p \leqslant \infty$, $0 < \rho \leqslant 1$, *and* $0 \leqslant l < \tilde{N}$, *then*

$$b_N(H_{p,l}(D); L^p(\Sigma_\rho)) \geqslant \alpha_{\tilde{N},l} \rho^{\tilde{N}} \tag{6.9}$$

and

$$b_N(A_{p,l}(D); L^p(D_\rho)) \geqslant \alpha_{\tilde{N},l} \rho^{\tilde{N} + 2np^{-1}}. \tag{6.10}$$

Proof. If $P_N \in \Pi^{(N)}$ then $P_N \in \mathscr{P}^{(\tilde{N})}$. In addition, if $\|P_N\|_{L^p(\Sigma_\rho)} \leqslant \alpha_{\tilde{N},l} \rho^{\tilde{N}}$, then (6.1) implies $\|P_N\|_{L^p(\Sigma)} \leqslant \alpha_{\tilde{N},l}$, and (6.4) implies $\|\mathscr{R}^l P_N\|_{H_p(D)} \leqslant 1$ and $P_N \in H_{p,l}(D)$. Since $\Pi^{(N)}$ is an $(N+1)$-dimensional subspace of $L^p(\Sigma_\rho)$ then (6.9) holds.

Similarly, if $P_N \in \Pi^{(N)}$ and $\|P_N\|_{L^p(D_\rho)} \leqslant \alpha_{\tilde{N},l} \rho^{\tilde{N}+2np^{-1}}$ then (6.2) implies $\|P_N\|_{L^p(D)} \leqslant \alpha_{\tilde{N},l}$, and (6.5) implies $\|\mathscr{R}^l P_N\|_{A_p(D)} \leqslant 1$ and $P_N \in A_{p,l}(D_R)$. It follows that, (6.10) holds.

7. THE MAIN RESULT

7.1. THEOREM. *Let $1 \leqslant p \leqslant \infty$, $0 < \rho \leqslant 1$, and $0 < l < \tilde{N}$. Then*

$$d_N(H_{p,l}(D); L^p(\Sigma_\rho)) = \alpha_{\tilde{N},l}\rho^{\tilde{N}}, \tag{7.1}$$

$$d_N(A_{p,l}(D); L^p(D_\rho)) = \alpha_{\tilde{N},l}\rho^{\tilde{N}+2np^{-1}}, \tag{7.2}$$

and the same equalities hold for the N-widths d^N, δ_N and b_N. Furthermore,

(i) $\pi^{(N)}$ is an optimal subspace for $d_N(H_{p,l}(D); L^p(\Sigma_\rho))$ and $d_N(A_{p,l}(D); L^p(D_\rho))$;

(ii) $X_p^N(D)$ is an optimal subspace for $d^N(H_{p,l}(D); L^p(\Sigma_\rho))$, while $Y_p^N(D)$ is an optimal subspace for $d^N(A_{p,l}(D); L^p(D_\rho))$;

(iii) P_N^l is an optimal operator for $\delta_N(H_{p,l}(D); L^p(\Sigma_\rho))$ and $\delta_N(A_{p,l}(D); L^p(D_\rho))$;

(iv) $\Pi^{(N)}$ is an optimal subspace for $b_N(H_{p,l}(D); L^p(\Sigma_\rho))$ and $b_N(A_{p,l}(D); L^p(D_\rho))$.

Proof. By Pinkus [8, Chap. II], $b_N \leqslant d_N \leqslant \delta_N$ and $b_N \leqslant d^N \leqslant \delta_N$. Thus, (7.1), (7.2) and the other equalities follow from Propositions 5.2 and 6.3; also, parts (iii) and (iv) follow from the proof is of these propositions.

Next, part (i) follows from the fact that $P_N^l f \in \pi^{(N)}$. By (4.2) and (4.5), if $f \in X_p^N(D)$ then $P_N^l f = 0$. Therefore,

$$\begin{aligned}
\sup\{\|f\|_{L^p(\Sigma_\rho)} &: f \in X_p^N(D) \cap H_{p,l}(D)\} \\
&= \sup\{\|f - P_N^l f\|_{L^p(\Sigma_\rho)} : f \in X_p^N(D) \cap H_{p,l}(D)\} \\
&\leqslant \alpha_{\tilde{N},l}\rho^{\tilde{N}},
\end{aligned}$$

proving that $X_p^N(D)$ is an optimal subspace for $d^N(H_{p,l}(D); L^p(\Sigma_\rho))$. The proof that $Y_p^N(D)$ in (4.3) is an optimal subspace for $d^N(A_{p,l}(D); L^p(D_\rho))$ is similar, and this completes the proof of (ii).

7.2. *Remarks.* (1) In Theorem 7.1, $L^p(D_\rho)$ can be replaced by the space of all holomorphic functions with the L^p-norm. This follows by an argument similar to the discussion given by Pinkus [8, pp. 253–254, and remark, p. 256].

(2) The referee has kindly pointed out that Theorem 7.1 admits a convenient form of generalization to certain classes of functions (e.g., the Hardy–Sobolev and Bergman–Sobolev classes for $p = 2$) in the case of a Hilbert space. The one-dimensional case was described in [4].

To explain the generalization to the present context, let $\mathscr{H}$ consist of all holomorphic functions f on D with

$$f(z)=\sum_{\mathbf{m}\geqslant 0}\sum_{j=1}^{d_{\mathbf{m}}} c_{j,\mathbf{m}}\varphi_j^{\mathbf{m}}(z)$$

such that

$$\|f\|_{\mathscr{H}}^2=\sum_{\mathbf{m}\geqslant 0}\gamma_{\mathbf{m}}\sum_{j=1}^{d_{\mathbf{m}}}|c_{j,\mathbf{m}}|^2<\infty, \tag{7.3}$$

where $\{\gamma_{\mathbf{m}}: \mathbf{m}\geqslant 0\}$ is a sequence of non-negative real numbers satisfying

$$\liminf_{|\mathbf{m}|\to\infty}\gamma_{\mathbf{m}}^{1/|\mathbf{m}|}\geqslant 1.$$

Generally, $\mathscr{H}$ is a *semi-Hilbert space* [4, p. 132]. Let $B(\mathscr{H})$ be the unit ball of $\mathscr{H}$ in the semi-norm (7.3). In particular, if $\gamma_{\mathbf{m}}=0$ for $|\mathbf{m}|\leqslant l-1$ and $\gamma_{\mathbf{m}}=(|\mathbf{m}|!/(|\mathbf{m}|-l)!)^2$ for $|\mathbf{m}|\geqslant l$, then $B(\mathscr{H})$ is the space $H_{2,l}(D)$. Also if $\gamma_{\mathbf{m}}=0$ for $|\mathbf{m}|\leqslant l-1$ and

$$\gamma_{\mathbf{m}}=\left(\frac{|\mathbf{m}|!}{(|\mathbf{m}|-l)!}\right)^2\frac{(n/r)_{\mathbf{m}}}{(2n/r)_{\mathbf{m}}}$$

for $|\mathbf{m}|\geqslant l$, then $B(\mathscr{H})$ is the space $A_{2,l}(D)$. For $B(\mathscr{H})=H_{2,l}(D)$ we take $X=L^2(\Sigma_\rho)$, and for $B(\mathscr{H})=A_{2,l}(D)$ we take $X=L^2(D_\rho)$. Then the conditions of [4, Theorem 1] are satisfied for $l<\tilde{N}$. By that result, there exists a self-adjoint, positive semi-definite, operator T_Γ on $L^2(\Sigma_\rho)$ (cf. [4, p. 35]), with eigenvalues $\lambda_0\geqslant\lambda_1\geqslant\lambda_2\geqslant\cdots>0$, such that the Gelfand N-width of $H_{2,l}(D)$ in $L^2(\Sigma_\rho)$ is exactly $\lambda_{N-N_l}^{1/2}$. A direct calculation shows that, $\lambda_{N-N_l}^{1/2}=\alpha_{\tilde{N},l}\rho^{\tilde{N}}$, which establishes (7.1) for the Gelfand N-width of $H_{2,l}(D)$ in $L^2(\Sigma_\rho)$. Further, a similar discussion also applies to (7.2).

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