

## An Algebraic Cell Decomposition of the Nonnegative Part of a Flag Variety

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We study the nonnegative part  $\mathcal{B}_{\geq 0}$  of the flag variety  $\mathcal{B}$  of a reductive algebraic group, as defined by Lusztig. Using positivity properties of the canonical basis it is shown that  $\mathcal{B}_{\geq 0}$  has an algebraic cell decomposition indexed by pairs of elements  $w \leq w'$  of the Weyl group. This result was conjectured by G. Lusztig (1994, in “Progr. Math.,” Vol. 123, pp. 531–568, Birkhäuser, Basel). © 1999 Academic Press

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The theory of total positivity for reductive algebraic groups  $G$  was introduced by Lusztig in the fundamental paper [Lu1]. While the definitions of the totally positive parts of  $G$  and other related varieties can be stated in elementary ways, many of their properties are only proved using the deep positivity properties of canonical bases. The cell decomposition of the totally nonnegative part  $\mathcal{B}_{\geq 0}$  of a flag variety proved here has been conjectured in [Lu1] and is another example of this phenomenon. The idea behind this cell decomposition is the following. By [Lu1], the totally positive part  $\mathcal{B}_{> 0}$  of a real flag variety is a connected component of the intersection of two opposed big cells. Here we generalize this result by identifying a “totally positive” connected component in any intersection of two opposed Bruhat cells and showing that it is topologically a cell. These connected components are then the cells in the proposed cell decomposition of  $\mathcal{B}_{\geq 0}$ . In particular,  $\mathcal{B}_{> 0}$  becomes the (unique) open cell, and the 0-dimensional cells are simply  $\{\dot{w}B^+\dot{w}^{-1}\}$  for Weyl group elements  $w$ . For general intersections of opposed Bruhat cells it is not obvious that a

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positive part exists, that is, that their intersection with  $\mathcal{B}_{\geq 0}$  is nonempty. Showing this is in some sense the heart of the proof, and this is where we require the use of canonical bases (see Proposition 2.6).

The parameterization of this decomposition comes from indexing the Bruhat cells and opposite Bruhat cells by Weyl group elements in an appropriate way. Then pairs of opposed Bruhat cells with nonempty intersection (of pure dimension  $l(w') - l(w)$ ) are labelled precisely by pairs  $(w, w')$  with  $w \leq w'$  (see [KL, D]).

## 1. PRELIMINARIES

Let  $G$  be a reductive linear algebraic group split over  $\mathbb{R}$  with fixed pinning  $(T, B^+, B^-, x_i, y_i; i \in I)$  (see [Lu1]). In the following all varieties will be identified with their  $\mathbb{R}$ -valued points. Furthermore the topology considered will be the Hausdorff topology coming from the standard topology on  $\mathbb{R}$ . Let  $\mathcal{B}$  be the variety of all Borel subgroups of  $G$ , and  $W = N_G(T)/T$  denote the Weyl group with longest element  $w_0 \in W$ . Let  $\{s_i | i \in I\}$  be the set of simple reflections in  $W$  corresponding to the pinning. Write  $l(w)$  for the length of  $w \in W$ , and  $\dot{w}$  for a representative of  $w$  in  $N_G(T)$ . We start by recalling some results of Lusztig's from [Lu1].

**1.1. Total Positivity in  $U^+$ .** Let  $U^+$  and  $U^-$  denote the unipotent radicals of  $B^+$  respectively  $B^-$ . The totally nonnegative part  $U_{\geq 0}^+$  of  $U^+$  is the semigroup generated by the set  $\{x_i(a) | a \in \mathbb{R}_{\geq 0}, i \in I\}$ . Analogously,  $U_{\geq 0}^-$  is the semigroup generated by  $\{y_i(a) | a \in \mathbb{R}_{\geq 0}, i \in I\}$ .

For every  $w \in W$  with reduced expression  $w = s_{i_1} \cdots s_{i_k}$  consider the map

$$\begin{aligned} (\mathbb{R}^*)^k &\rightarrow U^+ \\ (a_1, \dots, a_k) &\mapsto x_{i_1}(a_1) \cdots x_{i_k}(a_k), \end{aligned}$$

depending on  $\mathbf{i} := (i_1, \dots, i_k) \in I^k$ , where  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ . The image of the connected component  $\mathbb{R}_{>0}^k$  under this map clearly lies in  $U_{>0}^+$ . We denote it by  $U^+(w)$ . By [Lu1, 2.7] it is independent of the choice of reduced expression  $\mathbf{i}$ . In fact,  $U^+(w)$  can also be characterized as

$$U^+(w) = B^- \dot{w} B^- \cap U_{\geq 0}^+.$$

In particular, the  $U^+(w)$  form a cell decomposition of  $U_{\geq 0}^+$

$$U_{\geq 0}^+ = \bigsqcup_{w \in W} U^+(w).$$

For  $w = w_0$  the component  $U^+(w_0)$  is itself a semigroup. It is called the “totally positive submonoid” of  $U^+$  and denoted by  $U^+_{>0}$ . The closure of  $U^+_{>0}$  turns out to coincide with  $U^+_{\geq 0}$  (see [Lu1, 4.2]).

In all of the above,  $U^+$  can be replaced by  $U^-$ , with the  $x_i$ 's replaced by the  $y_i$ 's. In this way  $U^+_{>0}$ ,  $U^-_{\geq 0}$ , and the  $U^-(w)$ 's are defined.

1.2. *Total Positivity in  $\mathcal{B}$ .* We denote the action of  $g \in G$  on  $\mathcal{B}$  by

$$\begin{aligned} g &: \mathcal{B} \rightarrow \mathcal{B} \\ B &\mapsto g \cdot B, \end{aligned}$$

where  $g \cdot B$  stands for the conjugate Borel subgroup  $gBg^{-1}$ .

The totally positive part  $\mathcal{B}_{>0}$  of the flag variety is then defined as the orbit  $U^+_{>0} \cdot B^-$  of  $B^- \in \mathcal{B}$  under conjugation by elements of  $U^+_{>0}$ . Lusztig showed in [Lu1, 8.7] that

$$U^+_{>0} \cdot B^- = U^-_{>0} \cdot B^+.$$

So the definition of  $\mathcal{B}_{>0}$  is symmetric in  $U^+$  and  $U^-$ . It is interesting to note that the proof of this elementary statement actually already uses canonical bases, and is therefore not at all elementary.

Finally, the totally nonnegative part  $\mathcal{B}_{\geq 0}$  of  $\mathcal{B}$  is defined to be the closure of  $\mathcal{B}_{>0}$  in  $\mathcal{B}$  (with respect to the Hausdorff topology). The proposed cell decomposition of  $\mathcal{B}_{\geq 0}$  comes from the two opposed Bruhat decompositions of  $\mathcal{B}$ . We will sometimes use the notation

$$B \xrightarrow{w} B' \quad \text{for } B, B' \in \mathcal{B} \text{ and } w \in W$$

if  $(B, B')$  is conjugate under  $G$  (acting diagonally) to the pair  $(B^-, \dot{w} \cdot B^-)$ . Note that our conventions differ from the ones in [Lu1] in this point. Let  $w, w' \in W$ . We use the following notation for Bruhat cells with respect to  $B^-$  and  $B^+$ ,

$$\begin{aligned} \mathcal{E}_{w'}^- &:= \left\{ B \mid B^- \xrightarrow{w'} B \right\} = B^- \dot{w}' \cdot B^- \\ \mathcal{E}_w^+ &:= \left\{ B \mid B^+ \xrightarrow{w_0 w} B \right\} = B^+ \dot{w} \cdot B^+. \end{aligned}$$

Note that  $\mathcal{E}_{w'}^-$  has dimension  $l(w')$  while  $\mathcal{E}_w^+$  has codimension  $l(w)$  in  $\mathcal{B}$ . The intersection  $\mathcal{E}_{w'}^- \cap \mathcal{E}_w^+$  is nonempty precisely if  $w \leq w'$ , in which case it is a smooth variety of pure dimension  $l(w') - l(w)$ . For a proof of this statement see [Lu2]. It can also be found implicitly in the Appendix of [KL].

We write

$$\mathcal{R}_{w, w'} := \mathcal{E}_{w'}^- \cap \mathcal{E}_w^+.$$

Our aim is to prove that  $\mathcal{R}_{w,w';>0} := \mathcal{R}_{w,w'} \cap \mathcal{B}_{\geq 0}$  is a cell of dimension  $l(w') - l(w)$ . More precisely we propose that it is homeomorphic to  $\mathbb{R}_{>0}^{l(w')-l(w)}$  by a homeomorphism that extends to a (real) algebraic morphism  $(\mathbb{R}^*)^{l(w')-l(w)} \rightarrow \mathcal{R}_{w,w'}$  (see Theorem 2.8). This was conjectured by Lusztig (see [Lu1, 8.15; Lu2]).

**1.3. KEY EXAMPLE.** If  $w = 1$  in  $\mathcal{R}_{w,w'}$  then the conjecture follows directly from Lusztig's cell decomposition of  $U_{\geq 0}^+$ . In this case

$$\mathcal{R}_{1,w';>0} = U^+(w') \cdot B^-.$$

The desired algebraic map  $(\mathbb{R}^*)^{l(w')} \rightarrow \mathcal{R}_{1,w'}$  is given by

$$(a_1, \dots, a_k) \mapsto x_{i_1}(a_1) \cdots x_{i_k}(a_k) \cdot B^-,$$

where  $s_{i_1} \cdots s_{i_k} = w'$  is a reduced expression. We will refer to this as the key example.

Using the identity  $U_{>0}^+ \cdot B^- = U_{>0}^- \cdot B^+$  one can also construct the positive part for  $\mathcal{R}_{w,w_0}$ , this time from the cell decomposition of  $U_{\geq 0}^-$ . One gets

$$\mathcal{R}_{w,w_0;>0} = U^-(ww_0) \cdot B^+,$$

and for the desired map  $(\mathbb{R}^*)^{l(w_0)-l(w)} \rightarrow \mathcal{R}_{w,w_0}$ ,

$$(b_1, \dots, b_m) \mapsto y_{j_1}(b_1) \cdots y_{j_m}(b_m) \cdot B^+,$$

where  $s_{j_1} \cdots s_{j_m} = ww_0$  is a reduced expression. These last statements are no longer elementary consequences of the definitions, though, since we had to use the deep symmetry of  $\mathcal{B}_{>0}$  with regard to  $B^+$  and  $B^-$ .

Finally, the 0-dimensional cells are known. By [Lu1, 8.13] the  $W$ -conjugates of  $B^-$  all lie in  $\mathcal{B}_{\geq 0}$ . Therefore  $\mathcal{R}_{w,w} = \mathcal{R}_{w,w;>0} = \{\dot{w} \cdot B^-\}$ .

## 2. PROOF OF THE CELL DECOMPOSITION

The general idea of our proof of the proposed cell decomposition is to use the recursive procedure of Kazhdan and Lusztig [KL] for studying the variety  $\mathcal{R}_{w,w'}$  and to examine what happens to the positive part at every step. For  $B \in \mathcal{B}$ , we will sometimes write  $B \geq 0$  instead of  $B \in \mathcal{B}_{\geq 0}$ . We begin with some preliminary definitions.

Let  $w, v \in W$  with  $l(wv) = l(w) + l(v)$ . Then there are algebraic maps

$$\phi_{w,v}: \mathcal{C}_{wv}^- \rightarrow \mathcal{C}_w^- \quad \text{and} \quad \phi^{w,v}: \mathcal{C}_w^+ \rightarrow \mathcal{C}_{wv}^+$$

such that

$$\begin{aligned} B^- &\xrightarrow{w} \phi_{w,v}(B) \xrightarrow{v} B \\ B^+ &\xrightarrow{w_0 w v} \phi^{w,v}(B) \xrightarrow{v^{-1}} B. \end{aligned}$$

These are uniquely determined as can be checked using properties of the Bruhat decomposition. Also, writing down  $\phi_{w,v}$  (and similarly  $\phi^{w,v}$ ) explicitly one sees that they are algebraic. For example, consider  $\mathcal{E}_{wv}^-$  and  $\mathcal{E}_w^-$  as parameterized by

$$\begin{aligned} \mathbb{R}^{l(w)+l(v)} &\xrightarrow{\sim} \mathcal{E}_{wv}^- \\ (a_1, \dots, a_m) &\mapsto y_{i_1}(a_1) \dot{s}_{i_1} \cdots y_{i_m}(a_m) \dot{s}_{i_m} \cdot B^- \\ \mathbb{R}^{l(w)} &\xrightarrow{\sim} \mathcal{E}_w^- \\ (a_1, \dots, a_k) &\mapsto y_{i_1}(a_1) \dot{s}_{i_1} \cdots y_{i_k}(a_k) \dot{s}_{i_k} \cdot B^-, \end{aligned}$$

where  $s_{i_1} \cdots s_{i_k} \cdots s_{i_m} = wv$  is a reduced expression for  $wv$  such that  $s_{i_1} \cdots s_{i_k}$  is a reduced expression for  $w$ . Then  $\phi_{w,v}$  is given in these coordinates simply by projection onto the first  $l(w)$  components.

We start by checking that  $\phi_{w,v}$  preserves positivity in the key example.

**2.1. LEMMA.** *Let  $w' \in W$ ,  $B \in \mathcal{B}_{1,w'; >0}$ , and  $v \in W$  such that  $l(w'v^{-1}) = l(w') - l(v)$ . Then  $\phi_{w'v^{-1},v}(B) \in \mathcal{B}_{1,w'v^{-1}; >0}$ .*

*Proof.* By the key example,  $B = x \cdot B^-$  for some  $x \in U^+(w')$ . We write  $x$  as  $x_{i_1}(a_1) \cdots x_{i_m}(a_m)$  where  $a_1, \dots, a_m \in \mathbb{R}_{>0}$  and  $s_{i_1} \cdots s_{i_m} = w'$  is a reduced expression with  $s_{i_{k+1}} \cdots s_{i_m} = v$ . Then it is easy to check that  $\phi_{w'v^{-1},v}(B) = x_{i_1}(a_1) \cdots x_{i_k}(a_k) \cdot B^-$  and therefore lies in  $U^+(w'v^{-1}) \cdot B^- = \mathcal{B}_{1,w'v^{-1}; >0}$ . ■

To try to reduce the study of  $\mathcal{B}_{w,w'; >0}$  to the key example  $\mathcal{B}_{1,w'; >0}$ , we conjugate by an element of  $U^-(w^{-1})$ . Some elementary properties of the resulting map are listed in Lemma 2.2.

**2.2. LEMMA.** *Let  $w \in W$  and  $y \in U^-(w^{-1})$ .*

- (a) *If  $B \in \mathcal{B}_{\geq 0}$  then  $y \cdot B \in \mathcal{B}_{\geq 0}$ .*
- (b) *Conjugation by  $y$  induces embeddings*

$$\mathcal{E}_w^+ \rightarrow \mathcal{E}_1^+, \quad \mathcal{E}_w^- \rightarrow \mathcal{E}_{w'}^-, \quad \text{and} \quad \mathcal{B}_{w,w'} \rightarrow \mathcal{B}_{1,w'}.$$

- (c) *For  $B \in \mathcal{E}_{w'v}^-$  with  $l(w'v) = l(w') + l(v)$ , we have*

$$\phi_{w',v}(y \cdot B) = y \cdot \phi_{w',v}(B).$$

*Proof.* Conjugation by  $y$  preserves  $U_{\geq 0}^- \cdot B^+$  and therefore also its closure which is  $\mathcal{B}_{\geq 0}$ . This implies (a). To see (b) note that  $y \in B^+ \dot{w}^{-1} B^+$ . Hence if  $B \in \mathcal{E}_w^+ = B^+ \dot{w} \dot{w}_0 \cdot B^+$ , then  $y \cdot B \in B^+ \dot{w}_0 \cdot B^+ = \mathcal{E}_1^+$ . The remaining statements in (b) follow since  $y \in U^-$ . Finally, recall that  $B^- \xrightarrow{w'} \phi_{w',v}(B) \xrightarrow{v} B$ , by definition. Then conjugation by  $y$  gives  $B^- \xrightarrow{w'} y \cdot \phi_{w',v}(B) \rightarrow y \cdot B$  which implies (c). ■

In the next lemma we begin to study the maps  $\phi_{w,v}$  and  $\phi^{w,v}$  in general. The proof of this lemma will no longer be elementary, in that we use Lusztig's deep result about the symmetry of  $\mathcal{B}_{>0}$  with respect to  $B^+$  and  $B^-$  to deduce properties for  $\phi^{w,v}$ .

**2.3. LEMMA.** *Suppose  $v, w, w' \in W$ , such that  $l(wv) = l(w) + l(v)$  and  $l(w'v) = l(w') + l(v)$ .*

(a) *Let  $B \in \mathcal{E}_{w'v}^-$ . If  $B \geq 0$ , then  $\phi_{w',v}(B) \geq 0$ .*

(b) *Let  $B \in \mathcal{E}_w^+$ . If  $B \geq 0$ , then  $\phi^{w,v}(B) \geq 0$ .*

(c) *If  $w \leq w'$  then  $\phi_{w',v}$  gives rise to an isomorphism  $\phi: \mathcal{R}_{wv, w'v} \rightarrow \mathcal{R}_{w, w'}$  that restricts to a bijection*

$$\phi_{>0}: \mathcal{R}_{wv, w'v; >0} \rightarrow \mathcal{R}_{w, w'; >0}.$$

*Proof.* Note that statement (a) is true for  $B \in \mathcal{R}_{1, w'v} \subset \mathcal{E}_{w'v}^-$ , by Lemma 2.1. Now suppose  $B \in \mathcal{R}_{w, w'v; >0}$  where  $w \neq 1$ . In that case consider the curve  $\mathbb{R}_{>0} \rightarrow U^-(w^{-1})$  given by

$$t \mapsto y(t) := y_{j_1}(t) y_{j_2}(t) \cdots y_{j_l}(t)$$

for some reduced expression  $s_{j_1} s_{j_2} \cdots s_{j_l}$  of  $w^{-1}$ . By Lemma 2.2(a) and (b) we have  $y(t) \cdot B \in \mathcal{R}_{1, w'v; >0}$ . Hence  $\phi_{w',v}(y(t) \cdot B) \geq 0$  and by continuity as  $t$  goes to 0, also  $\phi_{w',v}(B) \geq 0$ . This implies (a), since  $\mathcal{E}_{w'v}^- = \bigsqcup_w \mathcal{R}_{w, w'v}$ . Assertion (b) follows by symmetry. Statement (c) is an immediate consequence of (a) and (b), since the inverse of  $\phi$  is given by  $\phi^{w,v}|_{\mathcal{R}_{w, w'}}$ . ■

**2.4.** Let  $w < w' \in W$  and  $s$  be a simple reflection such that  $w < ws$  and  $w's < w'$ . By properties of the Bruhat decomposition,  $\phi_{w's, s}$  restricts to

$$\pi = \pi_{w, w', s}: \mathcal{R}_{w, w'} \rightarrow \mathcal{R}_{w, w's} \sqcup \mathcal{R}_{ws, w's}.$$

In the following we study the behaviour of  $\pi_{w, w', s}$  (denoted  $\pi$  if the  $w, w'$ , and  $s$  are clear from context) with respect to total positivity.

**LEMMA.** *For  $w, w', s \in W$  as above, the map  $\pi = \pi_{w, w', s}$  restricts to a map*

$$\pi_{>0}: \mathcal{R}_{w, w'; >0} \rightarrow \mathcal{R}_{w, w's; >0}.$$

*Proof.* Suppose  $B$  is an element of  $\mathcal{R}_{w, w'; > 0}$ . By Lemma 2.3(a) we know that  $B' := \pi(B)$  lies in  $\mathcal{B}_{\geq 0}$ . It remains to show that  $B'$  lies in  $\mathcal{R}_{w, w's}$ , as opposed to in  $\mathcal{R}_{w's, w's}$ . Choose some  $y \in U^-(w^{-1})$ . Then by Lemma 2.2 we have  $y \cdot B' = y \cdot \phi_{w's, s}(B) = \phi_{w's, s}(y \cdot B)$  and  $y \cdot B \in \mathcal{R}_{1, w'; > 0}$ . Therefore, by Lemma 2.1,  $y \cdot B' \in \mathcal{R}_{1, w's; > 0}$ . But this can only be the case if  $B' \in \mathcal{R}_{w, w's}$ . If  $B'$  were in  $\mathcal{R}_{w's, w's}$  then we would have  $y \cdot B' \in \mathcal{R}_{s, w's}$ , which is a contradiction. ■

**2.5. Total Positivity and Canonical Bases.** We have not yet shown that  $\mathcal{R}_{w, w'; > 0}$  is nonempty. Our proof of this fact requires the deep positivity properties of Lusztig's canonical basis. Let  $\rho$  be the sum of all fundamental weights,  $V$  the irreducible representation of  $G$  with highest weight  $\rho$ , and  $\xi \in V$  a lowest weight vector. Then Lusztig's canonical basis of the (quantized) universal enveloping algebra of  $U^+$  (with respect to the chosen pinning, or equivalently the corresponding set of Chevalley generators) gives rise to a basis  $\mathbb{B}$  of  $V$  uniquely determined by the choice of  $\xi$  (see [Lu3]). Using this basis,  $\mathcal{B}_{\geq 0}$  can be characterized as follows.

**THEOREM (Lusztig [Lu1, 8.17]).** *Let  $G$  be of simply laced type and  $B \in \mathcal{B}$ . Then  $B$  lies in  $\mathcal{B}_{\geq 0}$  if and only if the unique line in  $V$  stabilized by  $B$  is spanned by a vector  $v \in V_{\geq 0} = \sum_{b \in \mathbb{B}} \mathbb{R}_{\geq 0} b$ .*

We apply this theorem in the proof of the following proposition to explicitly construct elements in  $\mathcal{R}_{w, w'; > 0}$ .

**2.6. PROPOSITION.** *Let  $y \in U^-(w^{-1})$  and  $B \in \mathcal{C}_w^+$ . If  $y \cdot B \in \mathcal{B}_{\geq 0}$ , then  $B \in \mathcal{B}_{\geq 0}$ .*

*Proof.* Note that it suffices to prove this proposition for simply laced groups. The non-simply laced case then follows by arguments as in [Lu1, 8.8]. So we assume that  $G$  is simply laced.

$\mathcal{C}_w^+$  is decomposed as

$$\mathcal{C}_w^+ = \bigsqcup_{\{w' | w' \geq w\}} \mathcal{R}_{w, w'}.$$

We prove the proposition stratum by stratum by induction on  $l(w') - l(w)$  (for all  $w \in W$  simultaneously). In the starting case of the induction a stratum consists of just a single element,  $\mathcal{R}_{w, w} = \{\dot{w} \cdot B^-\}$ . By [Lu1, 8.13] this element lies in  $\mathcal{B}_{\geq 0}$ . Thus the statement of the lemma holds trivially.

Consider  $\mathcal{R}_{w, w'}$  with  $w < w'$ . First we show that we can assume the existence of a simple reflection  $s$  such that  $w < ws$  and  $w's < w'$ . Let  $v \in W$  be maximal such that  $l(wv) = l(w) + l(v)$  and  $l(w'v) = l(w') + l(v)$ . We reduce the case  $B \in \mathcal{R}_{w, w'}$  to  $B' \in \mathcal{R}_{wv, w'v}$ . Let  $B \in \mathcal{R}_{w, w'}$  and  $y \in U^-(w^{-1})$  such that  $y \cdot B \geq 0$ . Then  $B' := \phi_{w', v}(B) \in \mathcal{R}_{wv, w'v}$  and  $y \cdot B' = \phi_{w', v}(y \cdot B) \geq 0$ , by Lemmas 2.2 and 2.3. Choose  $y' \in U^-(v^{-1})$ ,

so that  $y'y \in U^-(v^{-1}w^{-1})$ . Then also  $y'y \cdot B' \geq 0$ . Thus if the proposition holds for  $y'y \in U^-(v^{-1}w^{-1})$  and  $B' \in \mathcal{R}_{wv, w'v}$ , then by Lemma 2.3(c) it also holds for  $y \in U^-(w^{-1})$  and  $B \in \mathcal{R}_{w, w'}$ . Therefore we can replace  $w, w'$  by  $wv, w'v$ . By construction, any simple reflection that increases the length of this new  $w$  must decrease the length of the new  $w'$ . And such a simple reflection exists, since  $w < w' \leq w_0$ .

So we are reduced to considering strata  $\mathcal{R}_{w, w'} \subset \mathcal{C}_w^+$  for  $w < w'$  with a simple reflection  $s$  such that  $w < ws$  and  $w's < w'$ . This is the case in which we can apply the induction hypothesis. Let  $B \in \mathcal{R}_{w, w'}$  and  $y \in U^-(w^{-1})$  such that  $y \cdot B \geq 0$ . Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{R}_{w, w'} & \xrightarrow{y} & \mathcal{R}_{1, w'} \\ \pi \downarrow & & \downarrow \pi_1 \\ \mathcal{R}_{w, w's} \sqcup \mathcal{R}_{ws, w's} & \xrightarrow{y} & \mathcal{R}_{1, w's} \sqcup \mathcal{R}_{s, w's}, \end{array}$$

where  $\pi$  and  $\pi_1$  are restrictions of  $\phi_{w's, s}$  as in Subsection 2.4, and the horizontal maps refer to conjugation by  $y$ . Then  $y \cdot \pi(B) = \pi_1(y \cdot B)$  and lies in  $\mathcal{R}_{1, w's; > 0}$ , by Lemma 2.1. Therefore  $\pi(B)$  lies in  $\mathcal{R}_{w, w's}$  and we can apply the induction hypothesis to  $\pi(B)$  and  $y$  to get that  $\pi(B) \in \mathcal{R}_{w, w's; > 0}$  (note that  $y$  takes  $\mathcal{R}_{ws, w's}$  to  $\mathcal{R}_{s, w's}$ ). Consider the fibers  $F = \pi^{-1}(\pi(B))$  and  $F_1 = \pi_1^{-1}(y \cdot \pi(B))$  and their nonnegative parts  $F_{> 0} := F \cap \mathcal{B}_{\geq 0}$  and  $F_{1; > 0} := F_1 \cap \mathcal{B}_{\geq 0}$ . Then  $B$  is an element of  $F$  such that  $y \cdot B \in F_{1; > 0}$ . It remains to show that  $B \geq 0$ . This follows from the following claim.

*Claim.* Let  $B'$  be an element of  $\mathcal{R}_{w, w's; > 0}$ , and let  $F$  and  $F_1$  be the fibers  $\pi^{-1}(B')$ , respectively,  $\pi_1^{-1}(y \cdot B')$ .

(a) There is an isomorphism  $\mathbb{R}^* \rightarrow F_1$  that restricts to a homeomorphism  $\mathbb{R}_{> 0} \rightarrow F_{1; > 0}$ .

(b) The isomorphism  $y: F \rightarrow F_1$  given by conjugation with  $y$  restricts to a bijection  $F_{> 0} \rightarrow F_{1; > 0}$  of the positive parts.

Part (a) of this claim follows immediately from our analysis of the key example. Since  $y \cdot B' \in \mathcal{R}_{1, w's; > 0}$  it is of the form  $x \cdot B^-$  for some  $x \in U^+(w')$ . The fiber  $F_1$  is then given by  $F_1 = \{xx_i(a) \cdot B^- | a \in \mathbb{R}^*\}$  where  $s = s_i$  (see the proof of Lemma 2.1). The map  $a \mapsto xx_i(a) \cdot B^-$  from  $\mathbb{R}^* \rightarrow F_1$  has the desired properties.

We now turn to assertion (b). Note that by Lemma 2.2 we already know that conjugation by  $y$  restricts to a map  $F_{> 0} \rightarrow F_{1; > 0}$ . The main point of (b) is to show that  $F_{> 0}$  is nonempty, and in fact equals to the whole preimage of  $F_{1; > 0}$ . To do this we use the characterization of  $\mathcal{B}_{\geq 0}$  in terms of canonical bases to construct elements of  $F_{> 0}$ . Let  $g \in B^+ \dot{w}T \cap B^- \dot{w}'sB^-$  such that  $g \cdot B^- = B'$ . We apply  $g$  to the lowest weight  $\xi$  of the



$\rho$ -representation. Since  $B' \geq 0$  we can assume, by Lusztig's theorem, that

$$g \cdot \xi \in \sum_{b \in \mathbb{B}} \mathbb{R}_{\geq 0} b = V_{\geq 0}.$$

Now written down explicitly,  $F = \{gx_i(a) \cdot B^- | a \in \mathbb{R}^*\}$  where  $s = s_i$ . To determine which elements of  $F$  lie in  $F_{>0}$  we need to study the  $gx_i(a) \cdot \xi$  for  $a \in \mathbb{R}^*$ . Let  $e_i$  be the Chevalley generator of the Lie algebra of  $U^+$  such that  $x_i(a) = \exp(ae_i)$ . Then in the  $\rho$ -representation  $x_i(a) \cdot \xi = \xi + ae_i \cdot \xi$  and  $e_i \cdot \xi \in \mathbb{B}$  is the unique canonical basis element in the  $(-s_i, \rho)$ -weight space. We have therefore  $g \cdot (e_i \cdot \xi) = g s_i \cdot \xi$  for a suitable choice of  $s_i$ . It is easily verified that  $g s_i \cdot B^- = \phi^{w, s}(g \cdot B^-)$  and thus lies in  $\mathcal{B}_{\geq 0}$ . So  $g \cdot (e_i \cdot \xi) \in \sigma V_{\geq 0}$  for some sign  $\sigma \in \{\pm 1\}$ . Therefore

$$gx_i(a) \cdot \xi = g \cdot \xi + ag \cdot (e_i \cdot \xi) \text{ lies in } V_{\geq 0} \text{ whenever } \sigma a \in \mathbb{R}_{>0},$$

and the connected component

$$F^0 := \{gx_i(a) \cdot B^- | \sigma a \in \mathbb{R}_{>0}\} \subseteq F_{>0}$$

of  $F$  lies in  $F_{>0}$ . Now since  $y: F \rightarrow F_1$  is an isomorphism preserving positivity (2.2(a)), and by the description in (a) above of  $F_{1; >0}$  we see that  $y \cdot F^0 = F_{1; >0}$ . We get inclusions

$$y \cdot F_{>0} \subseteq F_{1; >0} = y \cdot F^0 \subseteq y \cdot F_{>0}.$$

Thus  $F_{>0} = F^0$ , and conjugation by  $y$  induces a bijection  $F_{>0} \rightarrow F_{1; >0}$ .  
■

**2.7. LEMMA.** *Let  $w < w' \in W$  and  $s$  be a simple reflection such that  $w < ws$  and  $w's < w'$ . There exists a real algebraic map  $\psi: \mathcal{R}_{w, w's} \times \mathbb{R}^* \rightarrow \mathcal{R}_{w, w'}$  that restricts to a homeomorphism*

$$\psi_{>0}: \mathcal{R}_{w, w's; >0} \times \mathbb{R}_{>0} \rightarrow \mathcal{R}_{w, w'; >0}$$

and such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{R}_{w, w's} \times \mathbb{R}^* & \xrightarrow{\psi} & \mathcal{R}_{w, w'} \\ \text{pr}_1 \downarrow & & \downarrow \pi \\ \mathcal{R}_{w, w's} & \xrightarrow{\iota} & \mathcal{R}_{w, w's} \sqcup \mathcal{R}_{ws, w's} \end{array} \quad (*)$$

Here  $\text{pr}_1$  is projection onto the first factor,  $\pi = \pi_{w, w', s}$  (as in Subsection 2.4), and  $\iota$  is the obvious inclusion.

*Proof.* First note that this statement is true for  $w = 1$  by the key example. Here we define  $\psi = \psi_1: \mathcal{R}_{1, w's} \times \mathbb{R}^* \rightarrow \mathcal{R}_{1, w'}$  by  $(x \cdot B^-, a) \mapsto xx_i(a) \cdot B^-$ , where  $x \in U^+ \cap B^- \dot{w}' s B^-$  and  $s = s_i$ . It clearly has all the properties required in the lemma.

Let  $y \in U^-(w^{-1})$ . We consider the following commutative diagram, where  $\pi_1 = \pi_{1, w's}$  and  $y$  stands for conjugation by  $y$ .

$$\begin{array}{ccc}
 \mathcal{R}_{w, w's} \times \mathbb{R}^* & & \mathcal{R}_{w, w'} \\
 y \times \text{id} \downarrow & \searrow \tilde{\psi} & \downarrow y \\
 \mathcal{R}_{1, w's} \times \mathbb{R}^* & \xrightarrow{\psi_1} & \mathcal{R}_{1, w'} \\
 \text{pr}_1 \downarrow & & \downarrow \pi_1 \\
 \mathcal{R}_{1, w's} & \xrightarrow{\iota} & \mathcal{R}_{1, w's} \sqcup \mathcal{R}_{s, w's}
 \end{array} \quad (**)$$

By the lower half of this diagram we see that the image of  $\tilde{\psi} := \psi_1 \circ (y \times \text{id})$  lies in  $\pi_1^{-1}(y \cdot \mathcal{R}_{w, w's}) = y \cdot (\pi^{-1}(\mathcal{R}_{w, w's})) \subseteq y \cdot (\mathcal{R}_{w, w'})$ . Therefore  $\psi(B, a) := y^{-1} \cdot \psi_1(y \cdot B, a)$  defines a real algebraic map  $\psi: \mathcal{R}_{w, w's} \times \mathbb{R}^* \rightarrow \mathcal{R}_{w, w'}$  with the property  $\tilde{\psi} = y \circ \psi$ .

Next we study the restriction of  $\psi$  to  $\mathcal{R}_{w, w's; > 0} \times \mathbb{R}_{> 0}$ . Consider an element  $(B, a) \in \mathcal{R}_{w, w's; > 0} \times \mathbb{R}_{> 0}$ . Then  $\psi(B, a) \in \mathcal{R}_{w, w'}$  and, by the properties of  $\psi_1$ ,  $y \cdot \psi(B, a) = \psi_1(y \cdot B, a)$  lies in  $\mathcal{R}_{1, w'; > 0}$  (note that  $y \cdot B \in \mathcal{R}_{1, w's; > 0}$  by Lemma 2.2). Therefore  $\psi(B, a) \geq 0$ , by Proposition 2.6. Thus the restriction of  $\psi$  gives rise to a continuous map  $\psi_{> 0}: \mathcal{R}_{w, w's; > 0} \times \mathbb{R}_{> 0} \rightarrow \mathcal{R}_{w, w'; > 0}$ . Its inverse should be given by  $B' \mapsto (y^{-1} \times \text{id})(\psi_1^{-1}(y \cdot B'))$ . The first component of this map is just  $y^{-1} \circ \pi_1 \circ y = \pi$  and the second component equals the second component of  $\psi_1^{-1} \circ y$ . Thus by Lemmas 2.4 and 2.2(a),  $\psi_{> 0}^{-1}(B') := (y^{-1} \times \text{id})(\psi_1^{-1}(y \cdot B'))$  lies in  $\mathcal{R}_{w, w's; > 0} \times \mathbb{R}_{> 0}$  (for  $B' \in \mathcal{R}_{w, w'; > 0}$ ), and  $\psi_{> 0}$  is a homeomorphism.

It remains to note that the diagram (\*) commutes. This follows since (\*) can be obtained from the lower half of the commutative diagram (\*\*) by conjugating all the maps by  $y^{-1}$  and restricting. ■

**2.8. THEOREM.** *Let  $w < w' \in W$  and  $m := l(w') - l(w)$ . Then there exists a real algebraic map  $\gamma: (\mathbb{R}^*)^m \rightarrow \mathcal{R}_{w, w'}$  such that its restriction defines a homeomorphism  $\gamma_{> 0}: \mathbb{R}_{> 0}^m \rightarrow \mathcal{R}_{w, w'; > 0}$ .*

*Proof.* The theorem is proved by induction on  $l(w') - l(w)$ . For the start of induction we have  $l(w') = l(w) + 1$ . Suppose first that  $w' = ws$  for some simple reflection  $s$ . In this case Lemma 2.7 applies and (since  $\mathcal{R}_{w, w's} = \mathcal{R}_{w, w}$  is a single point) gives the desired map  $\mathbb{R}^* \rightarrow \mathcal{R}_{w, w's}$ . In general,  $w < w'$  and  $l(w') = l(w) + 1$  implies that we can write  $w = xy$

and  $w' = xsy$  for some  $x, y \in W$  with  $l(xy) = l(x) + l(y) = l(w)$ , and  $s$  a simple reflection. But this case reduces to the previous one by Lemma 2.3(c).

Now let  $w < w' \in W$  arbitrary, and let  $v \in W$  be of maximal length such that  $l(wv) = l(w) + l(v)$  and  $l(w'v) = l(w') + l(v)$ . Again by Lemma 2.3(c) we see that it suffices to show the theorem for  $\mathcal{R}_{wv, w'v}$ . In particular we need only prove the theorem for pairs  $w, w'$  for which there exists a simple reflection  $s$  with the property  $w < ws$  and  $w's < w'$ .

In this case we have an algebraic map  $\gamma': (\mathbb{R}^*)^{m-1} \rightarrow \mathcal{R}_{w, w's}$  given by the induction hypothesis. Let  $\gamma: (\mathbb{R}^*)^m \rightarrow \mathcal{R}_{w, w'}$  be defined as the composition

$$\gamma: (\mathbb{R}^*)^m \cong (\mathbb{R}^*)^{m-1} \times \mathbb{R}^* \xrightarrow{\gamma' \times \text{id}} \mathcal{R}_{w, w's} \times \mathbb{R}^* \xrightarrow{\psi} \mathcal{R}_{w, w'},$$

with  $\psi$  as in Lemma 2.7. Then  $\gamma$  is clearly algebraic and restricts to a homeomorphism

$$\gamma_{>0} = \psi_{>0} \circ (\gamma'_{>0} \times \text{id}): \mathbb{R}_{>0}^m \cong \mathbb{R}_{>0}^{m-1} \times \mathbb{R}_{>0} \rightarrow \mathcal{R}_{w, w'; >0}.$$

■

This finishes the proof of the proposed cell decomposition. In conclusion we remark that for each  $w \leq w'$  the cell  $\mathcal{R}_{w, w'; >0}$  is a connected component of  $\mathcal{R}_{w, w'}$ , since it is open and closed in  $\mathcal{R}_{w, w'}$ . The topology of the other connected components of  $\mathcal{R}_{w, w'}$  is not known, but they will in general not all be cells (see, for example, [R]).

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## REFERENCES

- [D] V. Deodhar, On some geometric aspects of Bruhat orderings, *Invent. Math.* **79** (1985), 499–511.
- [KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
- [Lu1] G. Lusztig, Total positivity in reductive groups, in “Lie Theory and Geometry: In Honor of Bertram Kostant,” Progr. in Math., Vol. 123, pp. 531–568, Birkhäuser, Boston, 1994.
- [Lu2] G. Lusztig, Introduction to total positivity, in “Positivity in Lie Theory: Open Problems,” de Gruyter Expositions in Mathematics, Vol. 26, de Gruyter, Berlin, 1998.
- [Lu3] G. Lusztig, “Introduction to Quantum Groups,” Progr. in Math., Vol. 110, Birkhäuser, Boston, 1993.
- [R] K. Rietsch, Intersections of Bruhat cells in real flag varieties, *Internat. Math. Res. Notices* **13** (1997), 623–640.