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# Artin Level Modules

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We introduce level modules and show that these form a natural class of modules over a polynomial ring. We prove that there exist compressed level modules, i.e., level modules with the expected maximal Hilbert function, given socle type and the number of generators. We also show how to use the theory of level modules to compute minimal free resolutions of the coordinate ring of points from the back, which reveals new examples where random sets of points fail to satisfy the minimal resolution conjecture. 

*Key Words:* graded algebra; graded module; level ring; level algebra; Gorenstein algebra; Cohen–Macaulay ring; compressed algebra; Hilbert function; Betti numbers; unimodality; minimal resolution conjecture; canonical module; Matlis duality.

## 1. INTRODUCTION

The concept of level algebras—algebras whose socle is in one degree—was introduced by Stanley [10] and it turns out that many of the graded algebras encountered in algebraic geometry and in combinatorics are level algebras.

We will introduce level modules that generalize level algebras. A level module over a polynomial ring is a graded module whose set of generators and whose socle are concentrated in single degrees. The theory thus obtained is full of symmetries not enjoyed by level algebras and the class of level modules is closed under several transformations, such as duality and extension. For example, the canonical module of a level algebra is not a level algebra but a level module. Moreover, we show that many of the good properties of level algebras carry over to level modules.



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One interesting consequence of the theory of level modules is that the canonical module $\omega_A$ of an artin level algebra $A$ is a level module. Hence we have that the submodule defining $\omega_A$ as a quotient of a free module is determined by its forms of highest degree. This fact can be used to compute resolutions from the back. In many cases this is computationally much more efficient (see Section 5 for examples). By these methods we have found five new examples—in addition to the four known before—where the minimal resolution conjecture of Lorenzini [7] does not hold for random points.

In the class of level modules there is a subclass consisting of level modules with extremal Hilbert functions—called *compressed* level modules. We prove that a level module is compressed if and only if its graded Betti numbers are concentrated in two degrees.

In [6] Iarrobino defined compressed modules as those modules which have maximal Hilbert series among all modules of given type in analogy with the definition of compressed algebras. Iarrobino proves the existence of compressed modules using the ring of partial differential operators acting on the ring of formal power series and the Matlis duality between them. In his Ph.D. thesis [8] Miri continues the study of compressed modules and is in [9] focusing on compressed modules of type one, i.e., with socle of dimension one.

In our presentation of compressed level modules, influenced by Fröberg and Laksov [5], we do not need to use partial differential operators, but we can work directly in the modules. Moreover, we will be able to prove the existence of compressed level modules over any field, not only over infinite fields. These methods can also be adopted to compressed graded modules in general but here we will only do the case of level modules.

## 2. NOTATIONS AND BASIC RESULTS ON ARTIN LEVEL MODULES

*Setup.* Let $R = k[x_1, x_2, \dots, x_r]$ be the polynomial ring in the $r$ variables $x_1, x_2, \dots, x_r$ over a field $k$ and let $\mathfrak{m}$ be the graded maximal ideal $R_1 \oplus R_2 \oplus \cdots$ of $R$. Let $\mathcal{M}$ be the set of monomials in $R$. Moreover, let $M = M_0 \oplus M_1 \oplus \cdots$ be a graded $R$-module. We will always assume that $M$ is finitely generated. For any $k$-subspace $V \subseteq M_c$ and every integer $d$ we write

$$(V : R)_d = \left\{ m \in M_d \;\middle|\; am \in V \quad \text{for all } a \in R_{c-d} \right\}. \tag{1}$$

We then have that $(V : R)_d$ is a sub-vector space of $M_d$ and $\bigoplus_{d=0}^{\infty}(V : R)_d$

is a graded submodule of $M$. Moreover, we have that $(V : R)_c = V$ and $(V : R)_d = M_d$ for $d > c$.

Definition 2.1. Let $M$ be a graded $R$-module. The *socle* of $M$ is defined by

$$\operatorname{Soc} M = \{m \in M \mid am = 0 \quad \text{for all } a \in \mathfrak{m}\} = \operatorname{Hom}_R(k, M). \tag{2}$$

Definition 2.2. The graded Artin $R$-module $M$ is said to be *level* if it is generated by $M_0$ as an $R$-module and $\operatorname{Soc} M = M_c$ for some integer $c$.

If $M$ is a Cohen–Macaulay $R$-module we say that $M$ is a level module if its artinian reductions are level.

*Remark 2.1.* Since there is a natural isomorphism $\operatorname{Soc}(M) \to \operatorname{Tor}_r^R(M, k)$ of degree $r$, we note that $\operatorname{Soc}(M) = M_c$ means that $\operatorname{Tor}_r^R(M, k)$ is concentrated in degree $r + c$.

Furthermore, a level algebra $A$ is a level module with one generator of degree 0, i.e., with $A_0 = k$.

As in the case of level algebras, we have that the complete presentation of an Artin level module is determined by its forms in the degree of the socle. This fact turns out to be very useful in the study of level algebras and level modules.

Proposition 2.1. *Let $M$ be a graded $R$-module generated by $M_0$ and let $N \subseteq M$ be a homogeneous submodule such that $M/N$ is Artin. Then $M/N$ is level if and only if $N = \bigoplus_{d=0}^{\infty}(V : R)_d$ for some linear subspace $V \subseteq M_c$.*

*In particular, any graded Artin level module with socle in degree $c$ and with $t$ generators in degree 0 is a quotient of a free module $R^t = \bigoplus_{i=1}^{t} R$ by a submodule $\bigoplus_{d=0}^{\infty}(V : R)_d$ for some subspace $V$ in $R_c^t$.*

*Proof.* Suppose that $M/N$ is an Artin level module with socle in degree $c$ and let $V = N_c$. Then we have that $N_d \subseteq (V : R)_d$ for all integers $d$. Moreover we have that $N_c = (V : R)_c$. We shall show that $N_d = (V : R)_d$ by descending induction on $d$ starting in $d = c$. Assume, by induction, that $N_{d+1} = (V : R)_{d+1}$ for some $d < c$. Let $m \in (V : R)_d$. Then $am \in V$ for all $a \in R_{c-d}$ and consequently $aa'm \in V$ for all $a \in R_1$ and $a' \in R_{c-d-1}$. Hence we have that $am \in (V : R)_{d+1} = N_{d+1}$ for all $a \in R_1$. It follows that the class of $m$ is in $\operatorname{Soc}(M/N)$ and since $\operatorname{Soc}(M/N) = (M/N)_c$ we must have that $m \in N_d$. We have proved that $N_d \supseteq (V : R)_d$ and hence that $N = \bigoplus_{d=0}^{\infty}(V : R)_d$.

Conversely, let $N = \bigoplus_{d=0}^{\infty}(V : R)_d$ for some linear subspace $V \subseteq M_c$. Suppose that the class of $m \in M_d$ is in $\operatorname{Soc}(M/N)$ for some $d < c$. Then $am \in N$ for all $a \in \mathfrak{m}$ and in particular $am \in N$ for all $a \in R_{c-d}$. Hence $m \in N$ and $\operatorname{Soc}(M/N) = (M/N)_c$. ∎

### 2.1. *The Dual Module of a Level Module*

In the study of level modules we will see that the dual module plays an important role.

DEFINITION 2.3. Let $M$ be an Artin graded $R$-module. Then the dual module $M^\vee$ is defined by $M^\vee = \mathrm{Hom}_k(M, k)$, with the natural grading given by $M_d^\vee = \mathrm{Hom}_k(M_{-d}, k)$, for all $d \in \mathbf{Z}$. The module structure if $M^\vee$ is given by $x\phi(y) = \phi(xy)$, for all $x \in R$, $\phi \in M^\vee$ and $y \in M$.

Recall the following fact (cf. Bruns and Herzog [4, Theorem 3.6.19]).

PROPOSITION 2.2. *Let $M$ be a graded artinian $R$-module. Then we have that $M^\vee \cong \mathrm{Ext}_R^r(M, \omega_R)$, and for any minimal free resolution $F_\bullet$ of $M$, $F_\bullet^*$ is a minimal free resolution of $M^\vee$, which implies that we have isomorphisms*

$$\mathrm{Tor}_i^R(M, k)_d \to \mathrm{Tor}_{r-i}^R(M^\vee, k)_{r-d}, \tag{3}$$

*for $i = 0, 1, \ldots, r$, and $d \in \mathbf{Z}$.*

PROPOSITION 2.3. *Assume that $M$ is a graded Artin level $R$-module with socle in degree $c$. Then the graded $R$-module $M^\vee(-c)$ is Artin level with socle in degree $c$.*

*Proof.* Combining the fact that $\mathrm{Soc}(M)(-r) = \mathrm{Tor}_r^R(M, k)$, for any module $M$, with Proposition 2.2 yields that $M$ is generated by $M_0$ if and only if $\mathrm{Soc}(M^\vee) = M_0^\vee$. Conversely, $\mathrm{Soc}(M) = M_c$ if and only if $M^\vee$ is generated by $M_{-c}^\vee$. ∎

### 2.2. *Constructions of Artin Level Modules*

One of the interesting features of the concept of level modules is that we can do a lot of transformations of a level module and still have a level module. In the previous section we have seen that the dual of an Artin level module is an Artin level module. Now we will give two other ways to obtain level modules from level modules.

PROPOSITION 2.4. *Let $M = M_0 \oplus M_1 \oplus \cdots \oplus M_c$ be an Artin level module with socle in degree $c$ and let $i, j$ be integers with $0 \le i \le j \le c$. Then $M' = M_i \oplus M_{i+1} \oplus \cdots \oplus M_j$ is an Artin level module.*

*Proof.* It is clear that the location of the socle does not change if we go from $M_0 \oplus M_1 \oplus \cdots \oplus M_c$ to $M_i \oplus M_{i+1} \oplus \cdots \oplus M_c$. Hence $M_i \oplus M_{i+1} \oplus \cdots \oplus M_c$ is a level module.

On the other hand, we can apply the same argument to the dual module $M^\vee(-c)$ to see that $M_0 \oplus M_1 \oplus \cdots M_j$ is a level module.

If we apply the first argument on the module $M_0 \oplus M_1 \oplus \cdots M_j$ the assertion of the proposition follows. ∎

Moreover, the class of Artin level modules with socle in degree $c$ is closed under extension.

PROPOSITION 2.5. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of Artin modules. If* $M'$ *and* $M''$ *are level modules with socle in degree* $c$ *then* $M$ *is level with socle in degree* $c$.

*Proof.* It is immediate that $M$ is generated in degree 0 if $M'$ and $M''$ are. We then use the exactness of $M \mapsto M^\vee$ (cf. [4, Proposition 3.6.16]) to see that $M^\vee$ is generated in degree $-c$ if $(M')^\vee$ and $(M'')^\vee$ are. ∎

## 3. COMPRESSED LEVEL MODULES

In this section we will generalize to level modules the theory of compressed level algebras (cf. Iarrobino [6], Fröberg and Laksov [5]). The generalization reveals the beautiful symmetries in the theory. Especially the symmetry between a level module and its dual module will be examined. The existence of compressed modules of course implies the existence of compressed algebras and all previous proofs of this existence have assumed that the field $k$ is infinite. Here we will give a proof valid also in the case of a finite field. This is interesting also in the computational aspect, since computer algebra systems, such as Macaulay, work over finite fields.

*Setup.* Let $R^s$ be a free graded $R$-module with basis elements $e_1, e_2, \ldots, e_s$ of degree 0.

PROPOSITION 3.1. *Let* $V$ *be a codimension* $s'$ *subspace of* $R^s_c$. *Given elements* $\lambda_m^{(j',j)}$ *in* $k$ *for* $j' = 1, 2, \ldots, s'$, $j = 1, 2, \ldots, s$ *and* $m \in \mathcal{M}_c$ *such that*

$$V = \left\{ \sum_{\substack{j=1 \\ m\in\mathcal{M}_c}}^{s} \xi_m^{(j)} m e_j \;\middle|\; \sum_{\substack{j=1 \\ m\in\mathcal{M}_c}}^{s} \lambda_m^{(j',j)} \xi_m^{(j)} = 0, \quad \textit{for } j' = 1, 2, \ldots, s' \right\}. \tag{4}$$

*Then we have that* $(V : R)_d$ *is given by*

$$\left\{ \sum_{\substack{j=1 \\ m\in\mathcal{M}_d}}^{s} \xi_m^{(j)} m e_j \;\middle|\; \sum_{\substack{j=1 \\ m\in\mathcal{M}_d}}^{s} \lambda_{m'm}^{(j',j)} \xi_m^{(j)} = 0, \quad \begin{matrix} \textit{for } 1 \le j' \le s' \\ \textit{and } m' \in \mathcal{M}_{c-d} \end{matrix} \right\}, \tag{5}$$

*for* $d = 0, 1, \ldots, c$.

*Proof.* Since $\mathcal{M}_d$ is a basis for $R_d$, we have that

$$\{me_j \mid m \in \mathcal{M}_d, \quad \text{and } j = 1, 2, \ldots, s\} \tag{6}$$

form a basis for $R_d^s$. Moreover, $\mathcal{M}_{c-d}$ is a basis for $R_{c-d}$. Hence we have that (5) is a reformulation of (1), which proves the proposition. ∎

DEFINITION 3.1. An Artin level module $M$ is said to be *determined* by the elements $\lambda_m^{(j',j)}$ if we have the situation described in Proposition 3.1.

DEFINITION 3.2. Let $\eta_m^{(j',j)}$ be independent variables over $k$, for $m \in \mathcal{M}_c$, $j' = 1, 2, \ldots, s'$, and $j = 1, 2, \ldots, s$. We define $s' \dim_k R_{c-d} \times s \dim_k R_d$-matrices $\Xi_d$, for $d = 0, 1, \ldots, c$ by

$$\Xi_d = \begin{pmatrix} \Xi_d^{(1,1)} & \cdots & \Xi_d^{(1,s)} \\ \vdots & \ddots & \vdots \\ \Xi_d^{(s',1)} & \cdots & \Xi_d^{(s',s)} \end{pmatrix}, \tag{7}$$

where $(\Xi_d^{(j',j)})_{m',m} = \eta_{m'm}^{(j',j)}$, for $j' = 1, 2, \ldots, s'$, $j = 1, 2, \ldots, s$, $m' \in \mathcal{M}_{c-d}$, and $m \in \mathcal{M}_d$. That is, $\Xi_d$ is a matrix with entries $\eta_{m'm}^{(j',j)}$ where $m'$ and $j'$ are row indices and $m$ and $j$ are column indices.

Let $\lambda_m^{(j',j)}$ be elements of $k$ and let $\Lambda_d$ be the matrix obtained from $\Xi_d$ by specializing the coordinates $\eta_m^{(j',j)}$ to $\lambda_m^{(j',j)}$, for $d = 0, 1, \ldots, c$.

PROPOSITION 3.2. *Let $M$ be an Artin level module determined by the elements $\lambda_m^{(j',j)}$, for $j' = 1, 2, \ldots, s'$, $j = 1, 2, \ldots, s$, and $m \in \mathcal{M}_c$. Then we have that*

$$\dim_k M_d = \operatorname{rank} \Lambda_d, \qquad \textit{for } d = 0, 1, \ldots, c. \tag{8}$$

*Proof.* We have that $\dim_k M_d = \operatorname{codim}_k((V : R)_d, R_d^s) = \dim_k R_d^s - (\dim_k R_d^s - \operatorname{rank}\Lambda_d) = \operatorname{rank}\Lambda_d$, since by Proposition 3.1 we have that $(V : R)_d$ is given by the null space of the matrix $\Lambda_d$. ∎

From Proposition 3.2 it follows that for any Artin level module with socle in degree $c$, we have inequalities $H_M(d) \leq \min\{s \dim_k R_d, s' \dim_k R_{c-d}\}$, for $d = 0, 1, \ldots, c$ . As in the case of level algebras it is natural to consider the level modules of maximal Hilbert function (cf. Miri [9]).

DEFINITION 3.3. An Artin level module $M$ with socle in degree $c$ is *compressed* if there are integers $s$ and $s'$ such that

$$H_M(d) = \dim_k M_d = \min\{s \dim_k R_d, s' \dim_k R_{c-d}\}, \tag{9}$$

for $d = 0, 1, \ldots, c$.

Hence the Hilbert function of a compressed Artin level module is increasing up to a certain point after which it is decreasing. The initial degree of a compressed level module with Hilbert function given by (9) is given by the least integer $t$ such that $s' \dim_k R_t < s \dim_k R_{c-t}$.

PROPOSITION 3.3. *Let $M$ be an Artin level module. Then $M$ is compressed if and only if $M^\vee(-c)$ is compressed.*

*Proof.* The result follows immediately from Proposition 2.3 and the fact that the Hilbert function of $M^\vee(-c)$ is equal to the Hilbert function of $M$ read backwards (cf. Definition 3.3). ∎

To prove that compressed level algebras exist over any field $k$, we need the following result.

PROPOSITION 3.4. *There exist specializations $\lambda_m^{(j',j)}$ of the variables $\eta_m^{(j',j)}$ such that the matrices $\Lambda_d$ have maximal rank for all $d = 0, 1, \ldots, c$.*

*Proof.* We order the rows and the columns of the matrices $\Xi_d$ in the following way. The monomials are ordered lexicographically, and we order the rows and columns after the monomial index. If the monomial indices are equal we order after the superscripts.

These matrices $\Xi_d$ all have the following property: in every submatrix the variable in the lower right corner does not appear elsewhere in the submatrix, since its monomial index is the product of the greatest row and column indices.

Let **A** be any matrix with variable entries having this property. We will now prove by induction on the size of **A** that we can specialize the variables so that all upper left minors of **A** are non-zero. Without loss of generality we can assume that **A** is an $N \times N$-matrix. We can obviously choose the element of a $1 \times 1$ matrix so that the determinant is non-zero. Assume that we have specialized all variables in the upper left $(N-1) \times (N-1)$-submatrix of **A** in the proper way. Let $x$ be the variable in the lower right corner of **A**, i.e., at position $(N, N)$. We now specialize all variables of **A** not yet specialized, except $x$. Then the determinant of **A** is a linear function in $x$. The coefficient of $x$ is the upper left $(N-1) \times (N-1)$-minor of **A** which is non-zero by the assumption. Hence we can specialize $x$ so that the determinant is non-zero. The claim now follows by induction.

Note that we can specialize all the variables not on the main diagonal of **A** to zero, and that the rest of the variables can be specialized to one or zero.

We now apply the above argument to see that we can obtain maximal rank for all matrices $\Lambda_d$ separately. It remains to show that we can do it simultaneously. To do this we use the way we have ordered the rows and columns of the matrices. In fact, we will now show that we have ordered the rows and columns in such a way that the upper left submatrices of the matrices $\Lambda_d$ are all the same.

The lexicographical order of the monomials has the property that the smallest monomials in $\mathcal{M}_\lceil$ are $x_1^{d-d'}\mathcal{M}_{d'}$, for all $d' \leq d$. Let $t$ be the integer maximizing $\min\{s \dim_k R_t, s' \dim_k R_{c-t}\}$. Then we will show that any upper left submatrix of any matrix $\Lambda_d$ is a submatrix of $\Lambda_t$. By symmetry we can assume that $d \leq t$. The number of columns of $\Lambda_d$ is $s \dim_k R_d$ which is smaller than $s \dim_k R_t$ and $s' \dim_k R_{c-t}$, by the choice of $t$. Therefore any maximal upper left submatrix of $\Lambda_d$ will be a submatrix of the matrix with elements $\lambda^{(j',j)}_{x_1^{t-d} m' m}$, where $m' \in \mathcal{M}_{c-t}$ and $m \in \mathcal{M}_d$, which is a submatrix of $\Lambda_t$.

The conclusion is that since we can specialize the variables so that all upper left minors of $\Lambda_t$ are non-zero, we have that all upper left minors of all matrices $\Lambda_d$ are non-zero. ∎

*Remark 3.1.* In fact, the proof of Proposition 3.4 gives an algorithm for determining specializations yielding maximal rank and we need only to use the values 0 and 1.

We can now prove that there is a non-empty open set of level modules which are compressed, corresponding to the results on compressed algebras by Iarrobino [6] and Fröberg and Laksov [5].

PROPOSITION 3.5. *Let $V$ be a codimension $s'$ subspace in general position in $R_c^s$, and let $N = \bigoplus_{d \geq 0}(V : R)_d$. Then $M = R^s/N$ is a compressed level module.*

*Proof.* It follows from Proposition 2.1 that $M$ is level. Moreover, it follows from Proposition 3.1 that $M$ is compressed if and only if the matrices $\Lambda_d$ defined by (7) all have maximal rank. This shows that the condition that $M$ is compressed is an open condition on the coefficients in $\Xi_1, \Xi_2, \ldots, \Xi_c$, since it depends on the non-vanishing of the minors of $\Lambda_1, \Lambda_2, \ldots, \Lambda_c$. Observe that the open set in the affine space with coordinate functions $\eta_m^{(i,j)}$ where all the minors are non-zero is mapped onto an open set in the Grassmannian parametrizing all codimension $s'$ subspaces of $R_c^s$.

It remains to show that this open set is non-empty, but this follows immediately from Proposition 3.4. ∎

### 3.1. *Presentation of the Dual Module*

In this section, we show how to find a presentation of the dual module $M^\vee$, given a presentation of $M$, for any Artin level module $M$.

As a consequence of Proposition 3.3 we get that if $A$ is an Artin level algebra with socle in degree $c$ its dual module $A^\vee$ is a level module with socle in degree $c$, at least after a suitable twist. By Proposition 2.2 we have that $A^\vee$ is isomorphic to the canonical module $\omega_A = \operatorname{Ext}^r_R(A, R)$. Now it follows from Proposition 2.3 that it suffices to know a presentation of $\omega_A$ in degree $c$ to recover the complete presentation of $\omega_A$. Proposition 3.6, below, tells us how to find this presentation of $\omega_A$ in terms of a presentation of $A$. We will apply this approach in Section 5 to compute the resolution of a set of random points in $\mathbf{P}^r$ from the back.

PROPOSITION 3.6. *Let $M$ be an Artin level module with socle in degree $c$ determined by the matrices $\Lambda_d$, for $d = 0, 1, \dots, c$. Then $M^\vee(-c)$ is determined by the matrices $\Lambda_d^\vee$, for $d = 0, 1, \dots, c$, where $\Lambda_d^\vee$ is the transpose of $\Lambda_{c-d}$.*

*Proof.* Let $f_1, f_2, \dots, f_{s'}$ be free basis elements of degree 0 in $R^{s'}$. Since $M^\vee(-c)$ is level, by Proposition 2.3, it suffices to verify that the degree $c$ part of the kernel of the map $R^{s'} \to M^\vee(-c)$ is given by

$$\left\{ \sum_{\substack{j'=1\\ m\in\mathcal{M}_c}}^{s'} \xi_m^{(j')} m f_{j'} \;\middle|\; \sum_{\substack{j'=1\\ m\in\mathcal{M}_c}}^{s'} \xi^{(j')} \lambda_m^{(j',j)} = 0, \qquad \text{for } j = 1, 2, \dots, s \right\}. \tag{10}$$

For all $j' = 1, 2, \dots, s'$ we can define a homomorphism $\phi_{j'}\colon R_c^{s'} \to k$ by $\phi_{j'}(me_j) = \lambda_m^{(j',j)}$, for all $m \in \mathcal{M}_c$ and $j = 1, 2, \dots, s$. Since $V$, from (4), is defined by the vanishing of all $\phi_{j'}$, we have that $V = \bigcap_{j'} \ker(\phi_{j'})$ and there is an exact sequence

$$0 \to V \to R_c^s \to M_c \to 0. \tag{11}$$

Hence $\phi_{j'}$ defines a homomorphism $\psi_{j'}\colon M_c \to k$, for all $j' = 1, 2, \dots, s'$, and we can define a surjective map $R^{s'} \to M^\vee(-c)$ by $f_{j'} \mapsto \psi_{j'}$, for $j' = 1, 2, \dots, s'$. We are now interested in what is generated by $\psi_{j'}$ in $M^\vee(-c)_c$. But $M^\vee(-c)_c = \operatorname{Hom}_k(M_0, k)$ and we have assumed that $s = \dim_k M_0 = \dim_k R_0^s$. Hence the relations defining $M^\vee(-c)_c$ as a quotient of $R_c^{s'}$ are exactly those obtained from linear combinations $\sum \xi_m^{(j')} m f_{j'}$ that are mapped to the zero homomorphism from $M_0$ to $k$. This is the same set of relations which is described by (10) which finishes the proof. ∎

## 4. BETTI NUMBERS OF COMPRESSED LEVEL MODULES

We now give a homological criterion for an Artin level module to be compressed. It is known that the graded Betti numbers of compressed algebras are concentrated in two degrees (cf. Fröberg and Laksov [5, Proposition 16]). We now prove that the converse is also true, and we generalize this result to level modules.

PROPOSITION 4.1. *Let $M$ be an Artin level module with socle of dimension $s'$ in degree $c$ and let $t$ be the initial degree of the kernel of $R^s \to M$. Then we have that $\operatorname{Tor}_i^R(M,k)$ is concentrated in degrees $t-1+i$ and $t+i$, for $i = 1, 2, \dots, r-1$ if and only if $M$ is compressed.*

*Furthermore, if $M$ is compressed, then we have that*

$$\begin{aligned}\dim_k \operatorname{Tor}_i^R(M,k)_{i+t-1} &- \dim_k \operatorname{Tor}_{i-1}^R(M,k)_{i+t-1} \\ &= s\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} - s'\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1},\end{aligned} \tag{12}$$

*for $i = 1, 2, \dots, r$.*

*Proof.* Since $M$ is compressed the initial degree $t$ of the kernel of $R^s \to M$ is the smallest integer such that $s \dim_k R_t > s' \dim_k R_{c-t}$. We have that $\operatorname{Tor}_1^R(M,k)_d = 0$ for $d < t$. Because of the relation between the Hilbert function of $M$ and the Hilbert function of $M^\vee(-c)$ we see that the initial degree of the kernel of $R^{s'} \to M^\vee(-c)$ is at least $c - t + 1$. Hence we have that $\operatorname{Tor}_1^R(M^\vee(-c),k)_d = 0$ for $d < c-t+1$. We have that $\operatorname{Tor}_i^R(M,k)_d = 0$, for $d < t+i-1$ and $i = 1, 2, \dots, r-1$. Similarly, we have that $\operatorname{Tor}_i^R(M^\vee(-c),k)_d = 0$ for $d < c-t+i$ and $i = 1, \dots, r-1$. By Proposition 2.2 we have that $\operatorname{Tor}_i^R(M,k)_d = \operatorname{Tor}_{r-i}^R(M^\vee(-c),k)_{r+c-d}$, which is zero for $r+c-d < c-t+r-i$, i.e., for $d > t+i$. Hence we have that $\operatorname{Tor}_i^R(M,k)$ is concentrated in degrees $t+i-1$ and $t+i$, for $i = 1, 2, \dots, r-1$.

In the case where $t = c+1$, we have that $M$ is compressed with Hilbert function $H_M(d) = s \dim_k R_d$, for $d = 0, 1, 2 \dots, c$. We will have that $\operatorname{Tor}_i^R(M,k)$ is concentrated in degree $c+i$, for $i = 1, 2, \dots, r$. Hence we can assume that $t \le c$.

We will use the identity

$$(1-z)^r \sum_{d=0}^{l} \binom{r-1+d}{r-1} z^d = 1 + \sum_{j=1}^{r} (-1)^j \binom{l+j-1}{j-1}\binom{l+r}{r-j} z^{l+j}, \tag{13}$$

which can be easily, but tediously, checked.

Using the additivity of the $k$-dimension on a minimal free resolution of $M$ we get

$$(1-z)^r \operatorname{Hilb}_M(z) = \sum_{i=0}^{r}\sum_{j=1}^{b_i} (-1)^i z^{n_{j,i}} = \sum_{i=0}^{r}\sum_{d=0}^{\infty} (-1)^i \dim_k \operatorname{Tor}_i^R(M,k)_d z^d. \tag{14}$$

From the first assertion we have that the last sum is taken from $d = i+t-1$ to $d = i+t$.

Since $M$ is compressed we have that

$$\dim_k M_d = \min\left\{ s\binom{r-1+d}{r-1}, s'\binom{r-1+c-d}{r-1} \right\}, \tag{15}$$

which yields that

$$
\begin{aligned}
&(1-z)^r \mathrm{Hilb}_M(z) \\
&\quad = (1-z)^r \sum_{d=0}^{t-1} s \binom{r-1+d}{r-1} z^d + s'(1-z)^r \sum_{d=0}^{c-t} \binom{r-1+d}{r-1} z^{c-d}.
\end{aligned} \tag{16}
$$

Since $t \leq c$ it follows from (13) and (16) that

$$
\begin{aligned}
(1-z)^r \mathrm{Hilb}_M(z) &= s + s\sum_{i=1}^{r}(-1)^i \binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} z^{t-1+i} \\
&\quad + s'(-1)^r z^{r+c} + s'(-1)^r z^{r+c}\sum_{i=1}^{r}(-1)^i \binom{c-t+i-1}{i-1}\binom{c-t+r}{r-i} z^{t-c-i} \\
&= s + s'(-1)^r z^{r+c} + \sum_{i=1}^{r}(-1)^i \left( s\binom{t-1+i-1}{i-1}\binom{t-1+r}{r-i} \right. \\
&\quad \left. - s'\binom{c-t+r-i}{r-i}\binom{c-t+r}{i-1} \right) z^{i+t-1}.
\end{aligned} \tag{17}
$$

From (14) we have that

$$
\begin{aligned}
(1-z)^r \mathrm{Hilb}_M(z) &= s + \sum_{i=1}^{r-1}(-1)^i \dim_k \mathrm{Tor}_i^R(M,k)_{i+t-1} z^{i+t-1} \\
&\quad + \sum_{i=1}^{r-1}(-1)^i \dim_k \mathrm{Tor}_i^R(M,k)_{i+t} z^{i+t} + s'(-1)^{r+c} z^{r+c}.
\end{aligned} \tag{18}
$$

The second statement of the proposition now follows from equating the coefficients of $z^{i+t-1}$ in Eq. (18) with the same coefficient in the final expression of (17). ∎

## 5. COUNTEREXAMPLES TO THE MINIMAL RESOLUTION CONJECTURE

We will now show how to compute the Betti numbers of a generic set of points in projective space, under the hypothesis that there are some quadrics through the points. We will also in this way be able to find nine examples where the minimal resolution conjecture of Lorenzini [7] does not hold for a random set of points. It is worth remarking that these counterexamples do not give counterexamples to the corresponding conjecture for artinian algebras, not necessarily coming from points [3]. Moreover, there is a systematic behavior of the counterexamples, which is very interesting if we want to guess what the Betti numbers should be.

*Setup.* Let $\mathbf{P}^r$ be the projective space over $k$ with homogeneous coordinate ring $S = k[x_0, x_1, \ldots, x_r]$. We consider a set $X$ of $n$ points given by coordinates $(a_0^{(i)}, a_1^{(i)}, \ldots, a_r^{(i)})$, for $i = 1, 2, \ldots, n$. By a projective transformation we can move $r+1$ points to the points $(1, 0, \ldots, 0)$, $(1, -1, 0, \ldots, 0)$, $\ldots$,$(1, 0, \ldots, 0, , -1)$. We are interested in the homogeneous ideal $I(X)$ in $R$ generated by all forms vanishing on $X$. A priori we get a one-dimensional coordinate ring $k[x_0, x_1, \ldots, x_r]/I(X)$, but if we assume that no points of $X$ lie on the hypersurface $x_0 = 0$, we have that $x_0$ is not a zero-divisor in $S/I(X)$. Hence we can look at the artinian reduction $S/(I(X) \oplus (x_0))$. We are therefore interested in finding the image of the ideal $I(X)$ in $R = k[x_1, x_2, \ldots, x_r] = S/(x_0)$. We concentrate on the case when the ideal $I(X)$ is generated by quadrics. Then we have that $r+1 < n < (r+1)(r+2)/2$.

We can write equations for the generators of the ideal $I(X)$ as

$$\left\{ \sum_{0 \le i \le j \le r} \xi_{i,j} x_i x_j \;\Big|\; \sum_{0 \le i \le j \le r} \xi_{i,j} a_i^{(l)} a_j^{(l)} = 0, \qquad \text{for } l = 1, 2, \ldots, n \right\}. \tag{19}$$

Since we have chosen the last $r+1$ points to be special points, we can write the equations for $\xi_{i,j}$ as

$$\begin{aligned} \xi_{0,0} &= 0 \\ \xi_{0,0} - \xi_{0,j} + \xi_{j,j} &= 0, \qquad \text{for } j = 1, 2, \ldots, r \\ \sum_{0 \le i \le j \le r} \xi_{i,j} a_i^{(l)} a_j^{(l)} &= 0, \qquad \text{for } l = 1, 2, \ldots, s, \end{aligned}$$

where $s = n - 1 - r$. Hence we can eliminate the variables $\xi_{0,i}$, for $i = 0, 1, \ldots, r$ and get that the image, $I$, of $I(X)$ in the ring $R = S/(x_0)$ is given by

$$\left\{ \sum_{1 \le i \le j \le r} \xi_{i,j} x_i x_j \;\Big|\; \sum_{1 \le i \le j \le r} \xi_{i,j} a_i^{(l)} a_j^{(l)} + \sum_{1 \le j \le r} \xi_{j,j} a_0^{(l)} a_j^{(l)} = 0,\ 1 \le l \le s \right\}. \tag{20}$$

We now assume that the points are in general position. Then the Hilbert series of $A = k[x_1, x_2, \ldots, x_r]/I$ is given by $\mathrm{Hilb}_A(z) = 1 + rz + sz^2$ and by a result of Trung and Valla [11], we have that the socle of $A$ is contained in $A_2$, and thus $A$ is a compressed level algebra with socle of dimension $s$ in degree 2. Hence we can use the methods of Section 3 to obtain the equations for the dual, or canonical, module $A^\vee = \omega_A$. It will be sufficient to consider the linear part of the resolution of $A^\vee$, since the module $\mathrm{Tor}_i^R(A^\vee(-2), k)$ is concentrated in degrees $i$ and $i+1$.

Let $F_0$ be a free $R$-module of rank $s$ with basis elements $e_1, e_2, \dots, e_s$ of degree 0. By Proposition 3.6 and Proposition 3.1, we can find the kernel of the quotient map $F_0 \to A^\vee(-2) \to 0$ as

$$\left\{ \sum_{i=1}^{r} \sum_{l=1}^{s} \xi_{i,l} x_i e_l \mid \sum_{i=1}^{r} \sum_{l=1}^{s} \xi_{i,l} \lambda_{x_j x_i}^{(l)} = 0, \qquad \text{for } j = 1, 2, \dots, r \right\}, \tag{21}$$

where the elements $\lambda_{x_i x_j}^{(l)}$ are given by

$$\lambda_{x_i x_j}^{(l)} = \begin{cases} a_i^{(l)} a_j^{(l)} & \text{if } i \neq j \\ a_i^{(l)} a_i^{(l)} + a_0^{(l)} a_i^{(l)} & \text{if } i = j \end{cases} \tag{22}$$

for $l = 1, 2, \dots, s$, and $1 \leq i \leq j \leq r$.

This choice of the elements $\lambda_{x_i x_j}^{(l)}$ is not very general and it is therefore not so surprising that the Betti numbers of the module defined by them are not equal to the generic Betti numbers for compressed level modules.

We have by means of Macaulay [1] calculated nine examples where the MRC does not hold. The Betti numbers of $A^\vee(-2)$ were determined for a random set of points and they are presented in Table I. Only the linear part of the resolution needed to be calculated which corresponds to the numbers $b_i' = \dim_k \operatorname{Tor}_i^R(A^\vee(-2), k)_i$.

The second and the third examples in Table I were the first known examples where the Betti numbers of generic points do not satisfy the MRC. They were found by F. O. Schreyer. The four first examples were known by Beck and Kreuzer [2] but the last five examples are new.

The binomial coefficients $1, 4, 10$ and $1, 5, 15, 35, 70, 126$ appearing as the failing Betti numbers in Table I suggest some kind of structure. This was not so easily seen from only the first four examples.

TABLE I
Nine Cases of Betti Numbers of $A^\vee$ for Random Points
Not Satisfying the Minimal Resolution Conjecture

| | Betti numbers | | | | | MRC Betti numbers | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $b_0'$ | $b_1'$ | $b_2'$ | $b_3'$ | $b_4'$ | $b_0'$ | $b_1'$ | $b_2'$ | $b_3'$ | $b_4'$ |
| 11 points in $\mathbf{P}^6$ | 4 | 18 | 25 | 1 | 0 | 4 | 18 | 25 | 0 | 0 |
| 12 points in $\mathbf{P}^7$ | 4 | 21 | 36 | 4 | 0 | 4 | 21 | 36 | 0 | 0 |
| 13 points in $\mathbf{P}^8$ | 4 | 24 | 49 | 10 | 0 | 4 | 24 | 49 | 8 | 0 |
| 16 points in $\mathbf{P}^{10}$ | 5 | 40 | 126 | 160 | 1 | 5 | 40 | 126 | 160 | 0 |
| 17 points in $\mathbf{P}^{11}$ | 5 | 44 | 155 | 231 | 5 | 5 | 44 | 155 | 231 | 0 |
| 18 points in $\mathbf{P}^{12}$ | 5 | 48 | 187 | 320 | 15 | 5 | 48 | 187 | 320 | 0 |
| 19 points in $\mathbf{P}^{13}$ | 5 | 52 | 222 | 429 | 35 | 5 | 52 | 222 | 429 | 0 |
| 20 points in $\mathbf{P}^{14}$ | 5 | 56 | 260 | 560 | 70 | 5 | 56 | 260 | 560 | 0 |
| 21 points in $\mathbf{P}^{15}$ | 5 | 60 | 301 | 715 | 126 | 5 | 60 | 301 | 715 | 105 |

To calculate the example of 20 points in $\mathbf{P}^{14}$ Macaulay used 142 Mb of memory and the complete set of Betti numbers for $A$ is

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 100 | 840 | 3640 | 10192 | 20020 | 28600 | 30030 | 22880 | 12012 | 3640 | 70 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 70 | 560 | 260 | 56 | 5 |

To to calculate the last example of 21 points in $\mathbf{P}^{15}$ Macaulay needed 253 Mb of memory. Hence it would be practically impossible to calculate the resolution from the right side.

Beck and Kreuzer [2] have found a way to compute a presentation of the canonical module $\omega_A$ of the coordinate ring $A$ of a set of points using Gröbner basis techniques. By this method they have been able to find the four first counterexamples in Table I.

## REFERENCES


1. D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from `zariski.harvard.edu` via anonymous ftp.
2. S. Beck and M. Kreuzer, How to compute the canonical module of a set of points, *in* “Algorithms in algebraic geometry and applications (Santander, 1994),” Progr. Math., Vol. 143, pp. 51–78, Birkhäuser, Basel, 1996.
3. M. Boij, Betti numbers of compressed level algebras, *J. Pure Appl. Algebra* **134**, No. 2 (1999), 111–131.
4. W. Bruns and J. Herzog, “Cohen–Macaulay Rings,” Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge Univ. Press, Cambridge, UK, 1993.
5. R. Fröberg and D. Laksov, Compressed algebras, *in* “Lecture Notes in Math.,” Vol. 1092, pp. 121–151, Springer-Verlag, Berlin, 1984.
6. A. Iarrobino, Compressed algebras: Artin algebras having given socle degrees and maximal length, *Trans. Amer. Math. Soc.* **285** (1984), 337–378.
7. A. Lorenzini, The minimal resolution conjecture, *J. Algebra* **156** (1993), 5–35.
8. A. Miri, “Artin Modules Having Extremal Hilbert Series: Compressed Modules,” Ph.D. thesis, Northeastern University, 1985.
9. A. Miri, Compressed Gorenstein modules: Artin modules of type one having extremal Hilbert functions, *Comm. Algebra* **21**, No. 8 (1993), 2837–2857.
10. R. Stanley, Cohen–Macaulay complexes, *in* “Higher Combinatorics” (M. Aigner, Ed.), pp. 51–62, Reidel, Dordrecht/Boston, 1977.
11. N. V. Trung and G. Valla, The Cohen–Macaulay type of points in generic position, *J. Algebra* **125** (1989), 110–119.