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Involutions, Classical Groups, and Buildings

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In [*Invent. Math.* **58** (1980), 201–210], Curtis *et al.* construct a variation of the Tits building. The Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}, k)$ of a connected reductive k-group G has the important feature that it is a functor from the category of reductive groups defined over a field k and monomorphisms to the category of topological spaces and inclusions. An important consequence derived by Curtis *et al.* from the functorial nature of the Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}, k)$ is that if s is a semisimple element of the group $\mathsf{G}(k)$ of k-rational points, and G' is the connected component group of the centralizer of s, then the fixed point set $\mathscr{L}(\mathsf{G}, k)^s$ of s in $\mathscr{L}(\mathsf{G}, k)$ is the Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}', k)$. We generalize this result to arbitrary involutions of $\mathrm{Aut}_k(\mathsf{G})$, and we also prove an analogue in the context of affine buildings.

Key Words: nonarchimedean local field; classical group; Bruhat-Tits building; spherical building; involution.

1. INTRODUCTION

1.1.

Let k be a field and G a connected reductive group defined over k. The Tits or spherical combinatorial building $\Delta(\mathsf{G}, k)$ of G with respect to k is the simplicial complex whose simplices are the proper parabolic k-subgroups of G, reverse ordered by inclusions. A parabolic k-subgroup P is an r-simplex if and only if there are $(r+1)$ distinct $\mathsf{P}_0, \mathsf{P}_1, \ldots, \mathsf{P}_r$ maximal parabolic k-subgroups of G so that $\mathsf{P} = \mathsf{P}_0 \cap \mathsf{P}_1 \cap \cdots \cap \mathsf{P}_r$. The group $\mathscr{G} := \mathsf{G}(k)$ of k-rational points of G operates simplicially on $\Delta(\mathsf{G}, k)$. In [CLT], Curtis *et al.* construct a variation of the Tits building. The Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}, k)$ of G has the important feature that it is a functor from the category of connected reductive groups defined over k and monomorphisms to the category of topological spaces and inclusions.

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That is, given a monomorphism f: $\mathsf{G} \to \mathsf{H}$ of reductive groups defined over k, Curtis *et al.* naturally associate an embedding of topological spaces $\mathscr{L}(f)$: $\mathscr{L}(\mathsf{G}, k) \to \mathscr{L}(\mathsf{H}, k)$. An important consequence derived by Curtis *et al.* from the functorial nature of the Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}, k)$ is the following: Suppose s is a semisimple element of $\mathscr{G}$. Let G': $= (\mathsf{G}^s)^\circ$ denote the connected centralizer of s in G. Then, the fixed point set $\mathscr{L}(\mathsf{G}, k)^s$ of s in $\mathscr{L}(\mathsf{G}, k)$ is the Curtis–Lehrer–Tits building $\mathscr{L}(\mathsf{G}', k)$. One of our main goals (Theorem 3.3.1) is to establish, when $\mathrm{char}(k) \neq 2$, a generalization of the Curtis *et al.* result to arbitrary involutions in $\mathrm{Aut}_k(\mathsf{G})$.

1.2.

Suppose k is a nonarchimedean local field, and G is a connected semisimple k-group. Suppose further that τ is a k-automorphism of G so that the k-group $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple. Let $\mathscr{B}(\mathsf{G}, k)$ and $\mathscr{B}(\mathsf{G}', k)$ denote the Bruhat–Tits affine buildings of G and G', respectively. It is natural to ask what is the relation between $\mathscr{B}(\mathsf{G}, k)^\tau$ and $\mathscr{B}(\mathsf{G}', k)$. Under the assumption that the residual characteristic of k is odd, G is a special linear group and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution defining a classical group, we show (Theorem 6.7.3) that $\mathscr{B}(\mathsf{G}, k)^\tau$ can be identified with $\mathscr{B}(\mathsf{G}', k)$. This type of result is related to the question of when Bruhat–Tits buildings are functorial [L].

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2. PRELIMINARIES ON THE CURTIS–LEHRER–TITS BUILDING

2.1.

Let G be a connected reductive group defined over a field k and let $\mathscr{G}$ denote the group of k-rational points $\mathsf{G}(k)$. We review the construction of the Curtis–Lehrer–Tits (spherical) building $\mathscr{L}(\mathsf{G}, k)$ of G in [CLT]. Let S be a maximal k-split torus of G and let $\Phi(\mathsf{S}, \mathsf{G})$ be the k-roots of G with respect to S. Denote the k-cocharacters $\mathrm{Hom}_k\,(\mathrm{GL}(1), \mathsf{S})$ of S as $X_*(\mathsf{S}, k)$. The space $\mathscr{L}(\mathsf{S}, k)$ is defined as the sphere whose points represent rays in the real vector space $X_*(\mathsf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. To any point $b \in \mathscr{L}(\mathsf{S}, k)$, i.e., a ray in $X_*(\mathsf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$, we associate the parabolic k-subgroup $\mathsf{P}(b)$ defined as the largest closed subgroup of G containing S and whose Lie algebra contains the roots $\psi \in \Phi(\mathsf{S}, \mathsf{G})$ whose inner product with b is non-negative.

2.2.

Denote as $\mathscr{L}_1$ the disjoint union of the spheres $\mathscr{L}(\mathsf{S}, k)$ as S runs over all the maximal k-split tori of G. The group $\mathrm{Aut}_k(\mathsf{G})$ of k-automorphisms of G obviously acts on $\mathscr{L}_1$. In particular, $\mathscr{G}$ acts on $\mathscr{L}_1$ by inner automorphisms. We define an equivalence on points in $\mathscr{L}_1$ as follows. Given $b \in \mathscr{L}(\mathsf{S}, k)$ and $b' \in \mathscr{L}(\mathsf{S}', k)$, we define $b \sim b'$ if $\mathsf{P}(b) = \mathsf{P}(b')$ and there is a $g \in \mathsf{P}(b)$ conjugating S to S' so that $b' = Ad(g)(b)$. The *Curtis–Lehrer–Tits* (*spherical*) *building* $\mathscr{L}(\mathsf{G}, k)$ of G with respect to k is defined as the quotient of $\mathscr{L}_1$ by the equivalence relation $\sim$. It is immediate from (2.1) that $b \in \mathscr{L}(\mathsf{G}, k)$ determines unambiguously a parabolic k-subgroup $\mathsf{P}(b)$ of G. The building $\mathscr{L}(\mathsf{G}, k)$ inherits an action by $\mathrm{Aut}_k(\mathsf{G})$; in particular an action of $\mathscr{G}$.

For our purposes, the field k is fixed. Therefore, in order to simplify notation, we shall often use $X_*(\mathsf{S}), \mathscr{L}(\mathsf{S})$, and $\mathscr{L}(\mathsf{G})$ to denote $X_*(\mathsf{S}, k)$, $\mathscr{L}(\mathsf{S}, k)$, and $\mathscr{L}(\mathsf{G}, k)$, respectively.

LEMMA 2.2.1 ([CLT], Lemma 2.2). (i) *The stabilizer of a point* $b \in \mathscr{L}(\mathsf{G})$ *in* $\mathscr{G}$ *is* $\mathsf{P}(b)(k)$.

(ii) *Let* S *be a maximal k-split torus in* G. *The canonical projection* $\mathscr{L}_1 \to \mathscr{L}(\mathsf{G})$ *restricts to an injection of* $\mathscr{L}(\mathsf{S})$ *into* $\mathscr{L}(\mathsf{G})$.

The image of $\mathscr{L}(\mathsf{S})$ in $\mathscr{L}(\mathsf{G})$ is called the (*spherical*) *apartment* associated with S in $\mathscr{L}(\mathsf{G})$. For convenience, we identify $\mathscr{L}(\mathsf{S})$ with its corresponding apartment. A point b lies in an apartment $\mathscr{L}(\mathsf{S})$ precisely when $\mathsf{S} \subset \mathsf{P}(b)$.

LEMMA 2.2.2 ([CLT], Statements (2.3) and (2.4)). (i) *Any two points of* $\mathscr{L}(\mathsf{G})$ *belong to an apartment*.

(ii) *If* S *and* S' *are two maximal k-split tori*, *there is an element* $g \in \mathscr{G}$ *conjugating* S *to* S' (*hence g takes* $\mathscr{L}(\mathsf{S})$ *to* $\mathscr{L}(\mathsf{S}')$), *which fixes all points in* $\mathscr{L}(\mathsf{S}) \cap \mathscr{L}(\mathsf{S}')$.

2.3.

Let S' be any k-split torus and let S be a maximal k-split torus containing S'. The canonical injection $X_*(\mathsf{S}') \to X_*(\mathsf{S})$ induces an injection $\mathscr{L}(\mathsf{S}') \to \mathscr{L}(\mathsf{S})$ and hence an injection of $\mathscr{L}(\mathsf{S}')$ into $\mathscr{L}(\mathsf{G})$. As a map into $\mathscr{L}(\mathsf{G})$, this injection is independent of the torus S. In particular, any point $b \in \mathscr{L}(\mathsf{S})$ determines a parabolic k-subgroup $\mathsf{P}(b) \subset \mathsf{G}$.

LEMMA 2.3.1 ([CLT], Lemma 1.2). *Let* S *be a k-split torus in* G.

(i) *For* $b \in \mathscr{L}(\mathsf{S})$, *we have* $\mathsf{P}(b) \supset \mathsf{C}_{\mathsf{G}}(\mathsf{S})$.

(ii) *For* $b \in \mathscr{L}(\mathsf{S})$, *let* S' *be the intersection of* S *with the radical of* $\mathsf{P}(b)$. *Then* $b \in \mathscr{L}(\mathsf{S}')$.

2.4.

Suppose G' is a connected reductive k-group and $f : \mathsf{G}' \to \mathsf{G}$ is a k-monomorphism. Let $\mathscr{L}_1'$ be the disjoint union of all $\mathscr{L}(\mathsf{S}')$ with S' a maximal k-split torus of G'. The map $f : \mathsf{G}' \to \mathsf{G}$ induces a mapping $\mathscr{L}_1' \to \mathscr{L}(\mathsf{G})$. It is shown in [CLT, Sect. 4] that two points $b, b' \in \mathscr{L}_1'$ are equivalent under the equivalence relation $\sim_{\mathsf{G}'}$ described by G' if and only if they have the same image in $\mathscr{L}(\mathsf{G})$. Thus, the map $f : \mathsf{G}' \to \mathsf{G}$ induces an injective map $\mathscr{L}(f) : \mathscr{L}(\mathsf{G}') \to \mathscr{L}(\mathsf{G})$; or equivalently the Curtis–Lehrer–Tits building is a functor from the category of connected reductive k-groups with k-monomorphisms into the category of sets with injections.

2.5.

As mentioned in the Introduction, an important consequence of the functorial nature of the Curtis–Lehrer–Tits building is the following proposition.

PROPOSITION 2.5.1 ([CLT], Proposition 5.1). *Let s be a k-rational semisimple element of a connected reductive k-group G and denote the connected component group of the centralizer of s as $\mathsf{G}' = \mathsf{C}_{\mathsf{G}}(s)^{\circ}$. Then, $\mathscr{L}(\mathsf{G}', k)$ is the fixed point set $\mathscr{L}(\mathsf{G}, k)^s$ of s acting on $\mathscr{L}(\mathsf{G}, k)$.*

3. INVOLUTIONS AND SPHERICAL BUILDINGS

3.1.

Suppose $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution, i.e., of order 2. From now on, we assume $\mathrm{char}(k) \neq 2$. Under this assumption, the induced action of τ on $\mathrm{Lie}(\mathsf{G})$ is semisimple, and therefore $\mathrm{Lie}(\mathsf{G}^{\tau})$ and G^{τ} are reductive. For notational ease, we let G' denote the connected component group $(\mathsf{G}^{\tau})^{\circ}$. Our main goal in this section is to prove

$$\mathscr{L}(\mathsf{G}, k)^{\tau} = \mathscr{L}(\mathsf{G}', k).$$

3.2.

We begin with a preliminary lemma.

LEMMA 3.2.1. (i) *If S is a maximal k-split torus in a linear algebraic k-group Q, then the intersection $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{Q})$ of S with the radical $R(\mathsf{Q})$ of Q is a maximal k-split torus in $R(\mathsf{Q})$.*

(ii) *In the situation of (i), any two maximal k-split tori in $R(\mathsf{Q})$ are conjugate by an element $u \in R_u(\mathsf{Q})(k)$, where $R_u(\mathsf{Q})$ is the unipotent radical of Q.*

(iii) *Suppose* ℓ *is a positive integer relatively prime to* $\mathrm{char}(k)$. *If* U *is a unipotent* k*-group and* $u \in \mathsf{U}(k)$, *then there is a unique* $v \in \mathsf{U}(k)$ *satisfying* $u = v^\ell$.

Proof. To prove (i), let T' be a maximal split k-torus in $R(\mathsf{Q})$ and $\mathsf{T}' \subset \mathsf{T}$ a maximal split k-torus in Q. There exists $g \in \mathsf{Q}(k)$ such that $g\mathsf{T}g^{-1} = \mathsf{S}$. Thus, $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{Q}) = g\mathsf{T}g^{-1} \cap R(\mathsf{Q}) = g(\mathsf{T} \cap R(\mathsf{Q}))g^{-1} = g\mathsf{T}'g^{-1}$ must be a maximal split k-torus in $R(\mathsf{Q})$. Statement (ii) is [B1, 11.23]. To prove statement (iii), we recall the descending central series of U is defined as $\mathsf{U}_0 := \mathsf{U} \supsetneq \mathsf{U}_1 \supsetneq \cdots \supsetneq \mathsf{U}_{r-1} \supsetneq \mathsf{U}_r = \{0\}$, where for $i \geq 1$, the group $\mathsf{U}_i := [\mathsf{U}, \mathsf{U}_{i-1}]$ is generated by the commutators of elements in U with those in U_{i-1}. We note that $\mathsf{U}(k)/\mathsf{U}_i(k) = (\mathsf{U}/\mathsf{U}_i)(k)$. Set $\mathcal{U}_i := \mathsf{U}_i(k)$. We perform an induction on the length r of the central series. If $r = 1$, then $\mathcal{U}$ is a vector space over k and assertion (iii) is obvious. For $r > 1$, the group $\mathsf{U}/\mathsf{U}_{r-1}$ has length $r - 1$. By induction, there is a unique $\bar{v} \in \mathcal{U}/\mathcal{U}_{r-1}$ satisfying $u\mathcal{U}_{r-1} = \bar{v}^\ell$ in $\mathcal{U}/\mathcal{U}_{r-1}$. Pick a representative w of $\bar{v}$. Then $w^{-\ell}u \in \mathcal{U}_{r-1}$. Since $\mathcal{U}_{r-1}$ is a vector space over k, there is a unique $y \in \mathcal{U}_{r-1}$ such that $w^{-\ell}u = y^\ell$. Set $v = wy$. But, $\mathcal{U}_{r-1}$ is contained in the center of $\mathcal{U}$; so $v^\ell = (wy)^\ell = w^\ell y^\ell = u$. This establishes existence. A trivial modification of the argument proves uniqueness as well. ∎

3.3.

We now prove this section's main result.

THEOREM 3.3.1. *Suppose* $\mathrm{char}(k) \neq 2, \mathsf{G}$ *is a connected reductive* k*-group,* $\tau \in \mathrm{Aut}_k(\mathsf{G})$ *is an involution and* $\mathsf{G}' = (\mathsf{G}^\tau)^\circ$. *Then*

$$\mathcal{L}(\mathsf{G}, k)^\tau = \mathcal{L}(\mathsf{G}', k).$$

Proof. Let S be a maximal k-split torus. Suppose $b \in \mathcal{L}(\mathsf{S}) \subset \mathcal{L}(\mathsf{G})$. Let $\mathsf{P}(b)(k)$ be the parabolic subgroup which fixes b. By Lemma 3.2.1 (i), the split k-torus $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{P}(b))$ is a maximal split k-torus in $R(\mathsf{P}(b))$, and by Lemma 2.3.1 (ii), we have $b \in \mathcal{L}(\mathsf{S}')$. Of course, $\tau(\mathsf{S}')$ is a maximal split k-torus in $R(\tau(\mathsf{P}(b)))$ and $\tau(b) \in \mathcal{L}(\tau(\mathsf{S}'))$. If b is fixed by τ, then $\mathsf{P}(b)$ is τ invariant and therefore S' and $\tau(\mathsf{S}')$ are maximal k-split tori in $R(\mathsf{P}(b))$. By Lemma 3.2.1 (ii), there is a $u \in R_u(\mathsf{P}(b))(k)$ so that $\tau(\mathsf{S}') = u\mathsf{S}'u^{-1}$. If we apply τ to this last equality, we obtain $\mathsf{S}' = \tau(u)\tau(\mathsf{S}')\tau(u^{-1}) = \tau(u)u\mathsf{S}'u^{-1}\tau(u^{-1})$. Clearly $R_u(\mathsf{P}(b))$ is τ-invariant, so $\tau(u)u \in R_u(\mathsf{P}(b))$ and $\tau(u)u$ normalizes S'. Since S' is a maximal split k-torus in $R(\mathsf{P}(b))$, we conclude $\tau(u)u = 1$. By Lemma 3.2.1 (iii), there is a unique element $v \in R_u(\mathsf{P}(b))(k)$ so that $u = v^2$. Therefore $v^2 = u = \tau(u^{-1}) = \tau(v^{-1})^2$; i.e., v and $\tau(v^{-1})$ are square roots of u in $R_u(\mathsf{P}(b))(k)$, and therefore by uniqueness of square roots, $\tau(v^{-1}) = v$. Consider the maximal split k-torus in $R(\mathsf{P}(b))$ which is $\mathsf{T} = v\mathsf{S}'v^{-1}$. Then

$$\tau(\mathsf{T}) = \tau(v)\tau(\mathsf{S}')\tau(v^{-1}) = v^{-1}u\mathsf{S}'u^{-1}v = v\mathsf{S}'v^{-1} = \mathsf{T}.$$

That is, T is a τ-stable torus. Clearly $b \in \mathscr{L}(\mathsf{T})$. So, b is a ray in $X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is τ invariant. This means b is a ray in $(X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R})^\tau = X_*(\mathsf{T}^\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. A ray in $X_*(\mathsf{T}^\tau) \otimes_{\mathbb{Z}} \mathbb{R}$ is precisely a point in $\mathscr{L}(\mathsf{T}^\tau) \subset \mathscr{L}(\mathsf{G}', k)$. Thus, $\mathscr{L}(\mathsf{G}, k)^\tau = \mathscr{L}(\mathsf{G}', k)$. ∎

3.4.

The next result shall be used when we discuss involutions and affine buildings. Geometrically, it implies that when k is a finite field of odd order and τ is a nontrivial involution of G, then any point $b \in \mathscr{L}(\mathsf{G}, k)^\tau$ lies in a τ-invariant apartment of $\mathscr{L}(\mathsf{G}, k)$.

PROPOSITION 3.4.1. *Suppose k is a finite field of odd order q, G is a connected reductive k-group, $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution, and $\mathsf{G}' = (\mathsf{G}^\tau)^\circ$. If T is a k-split torus in G', then there exists a maximal k-split torus S in G which contains T and is τ-invariant.*

Proof. Let $\mathsf{M} = C_{\mathsf{G}}(\mathsf{T})$, a Levi k-subgroup of G. We make three trivial observations.

(i) The group M is τ-invariant.

(ii) Any maximal k-split torus in M must contain the torus T.

(iii) The reductive k-rank of M and G are equal. In particular, a maximal split k-torus in M is also a maximal split k-torus in G. Denote the rank by r.

Let B (resp. U) be a Borel k-subgroup of M (resp. the unipotent radical of B). Then, the number of maximal split k-tori in B is equal to the order of the group $\mathsf{U}(k)$. The k-subgroup $\tau(\mathsf{B})$ is obviously also a Borel subgroup of M. Let Ω denote the set of maximal split k-tori in $\mathsf{B} \cap \tau(\mathsf{B})$. Obviously, since $\mathsf{B} \cap \tau(\mathsf{B})$ is a τ-invariant group, the set Ω is a τ-invariant set. The intersection of any two minimal parabolic k-subgroups must contain a minimal Levi k-subgroup. Thus, the intersection $\mathsf{B} \cap \tau(\mathsf{B})$ contains a minimal Levi k-subgroup and so a k-torus of dimension r; i.e., Ω is nonempty. Any two elements in Ω are conjugate in $\mathsf{B} \cap \tau(\mathsf{B})$. We conclude Ω has order a power of $q = |k|$, in particular has odd order. Thus, there must be a τ-fixed point S in Ω. The torus S is our desired torus. ∎

REMARK 3.4.2. The referee pointed out that if k is algebraically closed, then Proposition 3.4.1 follows from 7.2, 7.3, and 7.5 in [St].

4. SIMPLE ALGEBRAS, INVOLUTIONS, AND CLASSICAL GROUPS

4.1.

In the previous section, under the assumption that k is a field of odd characteristic, G is connected reductive k-group, $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution, and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$, we showed $\mathscr{L}(\mathsf{G}, k)^\tau = \mathscr{L}(\mathsf{G}', k)$. In this section, we recall the relationship between the simple classical groups and simple algebras with involutions.

Let k be a field and suppose A is a finite dimensional simple algebra over k. A fundamental result of Weddeburn asserts that A is isomorphic as a k-algebra to a matrix algebra $M_n(D)$ of a finite dimensional division algebra D over k. Furthermore, if D_1 and D_2 are two finite dimensional algebras over k and $M_{n_1}(D_1)$ and $M_{n_2}(D_2)$ are isomorphic as k-algebras, then $n_1 = n_2$ and D_1 and D_2 are isomorphic k-algebras.

4.2.

A k-involution of a k-algebra A is an invertible k-linear map $J : A \to A$ satisfying

$$J^2 = I \text{ (Identity)}, \quad \text{and} \quad J(ab) = J(b)J(a). \tag{4.2.1}$$

An element $a \in A$ is *J-symmetric* if $J(a) = a$ and *J-skew* if $J(a) = -a$. The sets of J-symmetric and J-skew elements are trivially k-linear subspaces of A. If the characteristic of k is not 2, then A is the additive direct sum of these two linear subspaces.

4.3.

Involutions fall into two distinct types. Suppose J is a k-involution of A. Since J is surjective, it must preserve the center C of the algebra A. Since we are assuming A is a simple algebra, the center C must be a finite dimensional field extension of k. The involution J is said to be of the *first* or *second kind* depending on whether C lies in the symmetrical elements of J. That is, J is of the first kind if the center C is elementwise fixed by J, and J is of the second kind if there are elements in the center C which are not J symmetric. When J is an involution of the second kind, the symmetric elements C^J in the center C is a finite extension of k so that $[C : C^J] = 2$.

Let C be the center of a simple finite dimensional k-algebra A and suppose S is a k-subfield of C. We call a k-involution J of A an *involution over S of A* if the J-symmetric elements in C are precisely S. We remark that C is either equal to S or to a quadratic extension of S.

THEOREM 4.3.1 ([A], Theorem 11, p. 154). *Let C be the center of a simple finite dimensional k-algebra A. Suppose S is a k-subfield of C and J is an involution of A over S. Then, a k-anti-isomorphism $T : A \to A$ is a k-involution over S if and only if there exists a J-symmetric or J-skew invertible element $y \in A$, i.e., $J(y) = \pm y$, such that*

$$T(a) = y^{-1}J(a)y \quad a \in A.$$

4.3.2.

Let D be a finite dimensional division algebra over k and let $A = M_n(D)$. Denote the identity matrix in A as I. We identify $d \in D$ with the matrix $dI \in A$. The center C of D becomes the center of A. For $1 \leq i, j \leq n$, let $e_{i,j}$ be the matrix whose r, s entry is $\delta_{i,r}\delta_{j,s}$ (Kronecker delta).

THEOREM 4.3.3 ([A], Theorem 12, p. 156). *Let D be a finite dimensional division algebra over k and set $A = M_n(D)$. Let C be the center of D and S a k-subfield of C. Then, A has a k-involution over S if and only if there exists a k-involution of D over S. In this case, A has a k-involution J over S such that $J(D) = D$ and $J(e_{i,j}) = e_{j,i}$.*

4.4.

Suppose D is a finite dimensional division algebra over k. For $a \in M_n(D)$, denote by a^t the transpose of a. Also, if J is a k-anti-isomorphism of D, and $a = (a_{i,j}) \in A$, set $J(a) := (J(a_{i,j}))$. As a consequence of Theorems 4.3.1 and 4.3.3, we conclude that any k-involution T of $A = M_n(D)$ over $S \subset C$ has the form

$$T(a) = y^{-1}J(a^t)y \quad a \in A, \tag{4.4.1}$$

where J is a k-involution of D over S and $y \in A^\times$ satisfies $J(y^t) = \pm y$.

Let T be an involution of $A = M_n(D)$. Write T as in (4.4.1). Set $\mathcal{G} := A^\times = GL_n(D)$ and define $\tau : \mathcal{G} \to \mathcal{G}$ by

$$\tau(a) := T(a^{-1}). \tag{4.4.2}$$

Set

$$\mathcal{G}^\tau := \{a \in \mathcal{G} \mid \tau(a) = a\}. \tag{4.4.3}$$

Let $V := D^n$ be the (A, D)-bimodule of size n column vectors with entries in D. We use the element y, which is either J-symmetric or J-skew, to define a form $\langle\, , \rangle$ on V

$$\langle v, w\rangle := J(w^t)yv. \tag{4.4.4}$$

In particular, if $d_1, d_2 \in D$, we have

$$\langle vd_1, wd_2 \rangle = J(d_2)\langle v, w \rangle d_1.$$

Define $\epsilon \in \{\pm 1\}$ by $J(y^t) = \epsilon y$. The form $\langle\ ,\ \rangle$ is ϵ-hermitian; i.e.,

$$J(\langle v, w \rangle) = \epsilon \langle w, v \rangle. \tag{4.4.5}$$

In terms of the form $\langle\ ,\ \rangle$, the group $\mathscr{G}^\tau$ has the description

$$\begin{aligned} \mathscr{G}^\tau &= \left\{ a \in \mathscr{G} \mid \tau(a) = a \right\} \\ &= \left\{ a \in \mathscr{G} \mid \langle av, aw \rangle = \langle v, w \rangle \ \forall\, v, w \in V \right\}. \end{aligned} \tag{4.4.6}$$

The two groups $\mathscr{G}$ and $\mathscr{G}^\tau$ are the k-rational points of two reductive k-groups Γ and Γ^τ. Their explicit description as functors is the following: if K is an extension field of k then

$$\Gamma(K) = (A \otimes_k K)^\times \tag{4.4.7}$$

and

$$\Gamma^\tau(K) = \left\{ g \in \Gamma(K) \mid \tau(g) = g \right\}. \tag{4.4.8}$$

The k-group Γ (resp. Γ^τ) is of course a general linear (resp. classical) group. Recall the reduced norm is a k-group homomorphism

$$N_{D/k} : \Gamma \to \mathsf{GL}(1).$$

Define $\mathsf{G} := \ker(N_{D/k})$ and $\mathsf{G}^\tau := \ker(N_{D/k}|_{\Gamma^\tau})$. Then,

$$\mathsf{G}(K) = \left\{ g \in \Gamma(K) \mid N_{D/k}(g) = 1 \right\} \tag{4.4.9}$$

and

$$\mathsf{G}^\tau(K) = \mathsf{G}(K) \cap \Gamma^\tau(K). \tag{4.4.10}$$

5. INVOLUTIONS AND AFFINE BUILDINGS I

5.1.

Our goal in this and the next section is to prove an analogue of Theorem 3.3.1 for affine buildings. Let k be a nonarchimedean local field with ring of integers $\mathfrak{o}$ and residue field $\mathfrak{f}$ of odd characteristic. Suppose G is a connected semisimple group defined over k. Set $\mathscr{G} = \mathsf{G}(k)$ and let $\mathscr{B} = \mathscr{B}(\mathsf{G}, k)$ denote the Bruhat–Tits building of G with respect to k.

We set some notation. Let r denote the k-rank of G. A *face* E of a (closed) chamber C of $\mathscr{B}$ is the closure of a $(r-1)$-facet of C. If $x \in \mathscr{B}$, denote by G_x the reductive $\mathfrak{f}$-group which is the reduction modulo $\mathfrak{p}$ (the prime ideal of $\mathfrak{o}$) of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup fixing the point x. Let $\mathrm{Star}(x)$ denote the union of the (closed) chambers of $\mathscr{B}$ containing the point x. If S is a maximal k-split torus, let $A(\mathsf{S}, k)$ denote the apartment of S. Like our previous situation for the Curtis–Lehrer–Tits building, we will often drop k in our notation. Thus, we shorten $\mathscr{B}(\mathsf{G}, k)$ to $\mathscr{B}(\mathsf{G})$ and $A(\mathsf{S}, k)$ to $A(\mathsf{S})$. Suppose $A(\mathsf{S})$ is an apartment containing the point x. Then S determines a $\mathfrak{o}$-scheme whose reduction modulo $\mathfrak{p}$ is a maximal $\mathfrak{f}$-split torus S_x in G_x. Every maximal $\mathfrak{f}$-split torus in G_x is obtained as an S_x. If S and S' are two maximal k-split tori with $x \in A(\mathsf{S})$ and $x \in A(\mathsf{S}')$, then $\mathsf{S}_x = \mathsf{S}'_x$ precisely when $A(\mathsf{S}) \cap \mathrm{Star}(x) = A(\mathsf{S}') \cap \mathrm{Star}(x)$.

5.2.

Let $\mathscr{L}(x)$ denote the set of geodesic rays in $\mathrm{Star}(x)$ which begin at the point x. This set can be identified with the Curtis–Lehrer–Tits building of G_x as follows. Given $l \in \mathscr{L}(x)$, let S be a maximal k-torus with $l \subset A(\mathsf{S})$. The ray l determines a unique ray in $\mathrm{Hom}_k(GL(1), \mathsf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathfrak{f}}(GL(1), \mathsf{S}_x) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e., a point $\theta_{\mathsf{S}_x}(l)$ in $\mathscr{L}(\mathsf{S}_x, \mathfrak{f}) \subset \mathscr{L}(\mathsf{G}_x, \mathfrak{f})$. If S' is another k-torus with $l \subset A(\mathsf{S}')$, there exists $g \in \mathscr{G}_x$ fixing l so that $g\mathsf{S}g^{-1} = \mathsf{S}'$. That g fixes l means the image of g in $\mathsf{G}_x(\mathfrak{f})$ lies in $\mathsf{P}(l)$ and conjugates S_x to S'_x. So, $\theta_{\mathsf{S}_x}(l) = \theta_{\mathsf{S}'_x}(l)$. We drop the subscript and merely write $\theta : \mathscr{L}(x) \to \mathscr{L}(\mathsf{G}_x, \mathfrak{f})$. We leave it to the reader to show that θ is a bijection (see [T], Sect. 3.5.4).

5.3.

Suppose $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is a nontrivial involution. The associated action of τ on $\mathscr{B}$ is a polysimplicial isometry. It is elementary that $\mathscr{B}^\tau$ is nonempty. The set $\mathscr{B}^\tau$ is also obviously convex and so it has a well-defined dimension. Let m denote its dimension. Suppose $x \in \mathscr{B}^\tau$. The following are elementary:

(i) τ acts on $\mathrm{Star}(x)$ and $\mathscr{L}(x)$.

(ii) The parahoric subgroup fixing x is stable under τ.

(iii) τ induces an involution of the $\mathfrak{f}$-group G_x. For ease of notation, we also denote this induced involution by τ.

(iv) $\theta \circ \tau = \tau \circ \theta$.

Let $\mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau)$ denote the (reductive) $\mathfrak{f}$-rank of $(\mathsf{G}_x)^\tau$. The dimension of $\mathscr{L}(\mathsf{G}_x, \mathfrak{f})^\tau$ equals $\mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau) - 1$. Since we have seen that $\mathscr{L}(\mathsf{G}_x, \mathfrak{f})^\tau$ can be identified with the rays in $\mathrm{Star}(x)^\tau$ starting from x, we also have $\dim(\mathscr{L}(\mathsf{G}_x, \mathfrak{f})^\tau)$ equals $m - 1$, and thus $m = \mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau)$.

PROPOSITON 5.4. *Suppose* $x \in \mathcal{B}^{\tau}$. *Then, there exists a* τ*-invariant maximal* k*-split torus* $\mathsf{S} \subset \mathsf{G}$ *so that*

(i) $x \in A(\mathsf{S})^{\tau}$.

(ii) $\dim(A(\mathsf{S})^{\tau}) = m$.

5.5. *Preliminary Remarks on the Proof of Proposition* 5.4.

We begin by outlining our proof. By Proposition 3.4.1, there exists a τ-invariant maximal $\mathfrak{f}$-split torus T_0 in G_x, so that T_0^{τ} is a maximal $\mathfrak{f}$-split torus in $(\mathsf{G}_x)^{\tau}$. In particular, $\dim_{\mathfrak{f}}(\mathsf{T}_0^{\tau}) = m$. Let T be a maximal k-split torus so that $x \in A(\mathsf{T})$ and $\mathsf{T}_0 = \mathsf{T}_x$. The torus T is not necessarily τ-invariant. However, the τ-invariance of T_0 implies that $A(\mathsf{T}) \cap \mathrm{Star}(x)$ is τ-invariant. Furthermore, $\dim(A(\mathsf{T}) \cap \mathrm{Star}(x))^{\tau} = m$.

If $\mathscr{A}$ is an apartment containing $A(\mathsf{T}, k) \cap \mathrm{Star}(x)$, set

$$\mathscr{A}_{\tau} := \{x \in \mathscr{A} \mid \tau(x) \in \mathscr{A}\}.$$

It is elementary that $\mathscr{A}_{\tau}$ is τ-invariant, convex, and a union of (closed) chambers of $\mathscr{A}$. To prove Proposition 5.4, our goal is to begin with the apartment $\mathscr{A}_0 := A(\mathsf{T})$ and show that there is a sequence of apartments $\mathscr{A}_i$ containing $A(\mathsf{T}) \cap \mathrm{Star}(x)$ with the property that the sets $(\mathscr{A}_i)_{\tau}$ increase and converge to a τ-invariant apartment A. The k-torus S corresponding to A satisfies the conclusions (i) and (ii) of Proposition 5.4.

If A is an apartment and $E \subset A$ is a face of a chamber, let H_E denote the affine root hyperplane in A which contains E. Suppose E' is another (possibly the same) face of H_E. The intersection of all apartments containing the two faces E and E' is a convex $(r-1)$-dimensional set which we denote by $\mathscr{C}(E, E')$. Obviously, $\mathscr{C}(E, E')$ is contained in H_E. Define $\mathscr{K} := \mathscr{K}(E, E')$ to be the collection of the sets Ω satisfying the following properties:

(i) Ω contains $\mathscr{C}(E, E')$,

(ii) Ω is convex and a union of chambers,

(iii) Ω is minimal (under inclusion) among all sets satisfying (i) and (ii).

LEMMA 5.5.1. *Suppose* H_{ψ} *is an affine root hyperplane in an apartment* A *and* E, E' *are two faces in* H_{ψ}.

(i) *If* D *is a chamber with* E *as a face, then there exists* $\Omega \in \mathscr{K}(E, E')$ *such that* $D \subset \Omega$.

(ii) *If* $\Omega_1, \Omega_2 \in \mathscr{K}(E, E')$ *share a chamber* D, *then* $\Omega_1 = \Omega_2$.

Proof. We first consider the situation where $\mathscr{C}(E, E')$ and D all belong to the apartment A. The existence of an $\Omega \in \mathscr{K}$ reduces to the existence of a union of chambers in A which contains $\mathscr{C}(E, E')$, is convex, and is minimal under inclusion. In the latter situation, if we take Ω to be the intersection of all half apartments in A which contain D and $\mathscr{C}(E, E')$, then clearly Ω is minimal under inclusion. This implies existence. Uniqueness of Ω is a consequence of the property that Ω must contain the union of all chambers (in A) whose interiors meet any geodesic segment with one endpoint in the interior of the chamber D and the other endpoint in $\mathscr{C}(E, E')$. To treat the general situation, we observe that the affine root group $\mathscr{U}_\psi$ acts on the set $\mathscr{K}$. Indeed, the $\mathscr{U}_\psi$ action on the chambers containing E has two orbits (one for each of the two chambers in A containing E). Parts (i) and (ii) follow. ∎

COROLLARY 5.5.2. *Let* $p = \mathrm{char}(k) \neq 2$. *Then*, $|\mathscr{K}|$ *has the form* $1 + p^t$.

Proof. Let C_1 and C_2 denote the two chambers of A containing the face E and let $\Omega(C_1)$ and $\Omega(C_2)$ be the elements of $\mathscr{K}$ which contain C_1 and C_2, respectively. As stated in the proof of Lemma 5.5.1, the affine root group $\mathscr{U}_\psi$ acts on $\mathscr{K}$ with two orbits—the orbits of $\Omega(C_1)$ and $\Omega(C_2)$. It is clear $\Omega(C_1)$ is a singleton orbit and the size of the $\mathscr{U}_\psi$-orbit of $\Omega(C_2)$ is a power of p. The corollary follows. ∎

5.6. *Completion of the Proof of Proposition* 5.4.

We define the sequence $\mathscr{A}_i$ inductively. If at any stage $(\mathscr{A}_i)_\tau = \mathscr{A}_i$, i.e., the apartment $\mathscr{A}_i$ is τ-invariant, we take A to be $\mathscr{A}_i$ and we are done. Therefore, we shall always suppose $(\mathscr{A}_i)_\tau \neq \mathscr{A}_i$. We can choose two chambers $C, D \subset \mathscr{A}_i$ so that:

(i) $F = C \cap D$ is a face of C (hence D).

(ii) $C \subset (\mathscr{A}_i)_\tau, D \not\subseteq (\mathscr{A}_i)_\tau$.

We descriptively refer to the face F as a *boundary face* of $(\mathscr{A}_i)_\tau$. We can and do assume (for the purpose of causing convergence in the compact open topology) that the distance of the boundary face F to the point x is minimal for all boundary faces of $(\mathscr{A}_i)_\tau$. Denote as H_F the affine root hyperplane in $\mathscr{A}_i$ determined by F. Let J denote the half apartment of $\mathscr{A}_i \backslash H_F$ containing C. Our proof divides into the following two cases: Case (i) $H_F \neq H_{\tau(F)}$. Case (ii) $H_F = H_{\tau(F)}$.

Case (i) $H_F \neq H_{\tau(F)}$. Let J' denote (open) half apartment of $\mathscr{A}_i \backslash H_{\tau(F)}$ containing $\tau(C)$. The set $(\mathscr{A}_i)_\tau$ is contained in $J \cap J'$. Let ψ denote affine root with the property $H_{\tau(F)} = H_\psi$ and the property that the affine root group $\mathscr{U}_\psi$ fixes $\tau(C)$. The group $\mathscr{U}_\psi$ fixes the half apartment J' and permutes the chambers in the interior of $\mathrm{Star}(\tau(F)) \backslash \tau(C)$ transitively. Choose $u \in \mathscr{U}_\psi$ so that $\tau(D) \subset u\mathscr{A}_i$. Our hypothesis that $H_F \neq H_{\tau(F)}$ implies D and C, hence D and $(\mathscr{A}_i)_\tau$, lie on the same side of $H_{\tau(F)}$. We conclude from this that u fixes D and $(\mathscr{A}_i)_\tau$. Set $\mathscr{A}_{i+1} := u\mathscr{A}_i$. Then $(\mathscr{A}_i)_\tau \cup D \subset \mathscr{A}_{i+1}$ and indeed $(\mathscr{A}_i)_\tau \cup D \subset (\mathscr{A}_{i+1})_\tau$.

Case (ii) $H_F = H_{\tau(F)}$. Since J contains $(\mathscr{A}_i)_\tau \supset C$ and $(\mathscr{A}_i)_\tau$ is τ-invariant, we have $\tau(C) \subset J$. The two sets $\mathscr{C}(F, \tau(F))$ and $\mathscr{K}(F, \tau(F))$ are τ-invariant. Since we are assuming the residual characteristic of k is odd, we conclude from Corollary 5.5.2 that $|\mathscr{K}(F, \tau(F))|$ is even. The unique element of $\mathscr{K}(F, \tau(F))$ containing C is obviously τ-invariant; hence there must also exist $\Omega \in \mathscr{K}(F, \tau(F))$ which is τ-invariant and does not contain C. Choose $u \in \mathscr{U}_\psi$ so that $uD \subset \Omega$ and set $\mathscr{A}_{i+1} = u\mathscr{A}_i$. Then, $(\mathscr{A}_{i+1})_\tau \supset (\mathscr{A}_i)_\tau \cup \Omega$. This completes our induction step.

As already mentioned, in our construction of the sequence $\mathscr{A}_i$, at each stage, if there exist boundary faces on $(\mathscr{A}_i)_\tau$ we choose a boundary face F whose distance to x is minimal among boundary faces. Then, $F \subset (\mathscr{A}_{i+1})_\tau$ and is no longer a boundary face. It follows that the sequence of apartments $\mathscr{A}_i$ converge in the compact open topology to an apartment A which is τ-invariant. Hence the maximal k-split torus S associated to A is τ-invariant, and clearly (i) $x \in A$ and (ii) $\dim(A^\tau) = m$. ∎

COROLLARY 5.7. (i) *Suppose* G *is a reductive* k*-group*, $\tau \in \mathrm{Aut}_k(\mathsf{G})$ *is an involution and* T *is a* τ*-invariant* k*-split torus*. *Then*, *there exists a* τ*-invariant maximal* k*-split torus* S *containing* T.

(ii) *Suppose* G *is a semisimple* k*-group*, $\tau \in \mathrm{Aut}_k(\mathsf{G})$ *is an involution and* $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$. *Then*, $\dim(\mathscr{B}^\tau) = \mathrm{rk}_k(\mathsf{G}')$.

Proof. To prove (i), let $\mathsf{L} := \mathsf{L}(\mathsf{T})$ denote the Levi k-subgroup which is the centralizer of T, and let $\mathsf{Z} := \mathsf{Z}(\mathsf{L})$ denote the center of L. The quotient $\mathsf{M} := \mathsf{L}/\mathsf{Z}$ is a (semisimple) k-group and τ induces a k-involution in $\mathrm{Aut}_k(\mathsf{M})$, which we shall for ease of notation also denote by τ. By Proposition 5.4, there is a τ-invariant, maximal k-split torus $\overline{\mathsf{S}}$ in M. Let C denote the preimage in L of $\overline{\mathsf{S}}$. We can take S to be the maximal k-split torus of C.

To prove (ii), suppose T is a maximal k-split torus of G^τ. By (i), there exists a τ-invariant maximal k-split torus S of G containing T. We have

$$\dim(\mathscr{B}^\tau) \geq \dim(A(\mathsf{S})^\tau) \geq \dim(\mathsf{T}) = \mathrm{rk}_k(\mathsf{G}').$$

Also, by Proposition 5.4, there exists a τ-invariant maximal k-split torus V so that

$$\dim(\mathscr{B}^\tau) = \dim A(\mathsf{V})^\tau = \dim_k(\mathsf{V}^\tau) \leq \mathrm{rk}_k(\mathsf{G}')$$

So $\dim(\mathscr{B}^\tau) = \mathrm{rk}_k(\mathsf{G}')$. ∎

6. INVOLUTIONS AND AFFINE BUILDINGS II

6.1.

We continue with the assumptions of Section 5: k is a nonarchimedean local field with ring of integers $\mathfrak{o}$ and residue field $\mathfrak{f}$ of odd characteristic, G is a connected semisimple group defined over k, and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is a nontrivial involution. Let G' denote the connected reductive group $(\mathsf{G}^\tau)^\circ$ and set $\mathscr{G}' := \mathsf{G}'(k) \subset \mathscr{G} = \mathsf{G}(k)$. If $\mathscr{H}$ is a subgroup of $\mathscr{G}$, let $\mathscr{C}_{\mathscr{G}}(\mathscr{H})$ and $\mathscr{N}_{\mathscr{G}}(\mathscr{H})$ denote the centralizer and normalizer of $\mathscr{H}$, respectively. We use similar notation for subgroups of $\mathscr{G}^\tau$ and $\mathscr{G}'$.

The main result in this section is Theorem 6.7.3. It states under the assumption that G and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ are as in Section 4 and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple that $\mathscr{B}' := \mathscr{B}(\mathsf{G}', k) = \mathscr{B}(\mathsf{G}')$ identifies naturally with $\mathscr{B}(\mathsf{G})^\tau$. We outline our plan to accomplish this.

(i) Suppose T is a maximal k-split torus of G'. Set $\mathscr{T} := \mathsf{T}(k)$. Let $A(\mathsf{T}, \mathsf{G}')$ denote the apartment in $\mathscr{B}(\mathsf{G}')$ associated to T. We find a canonical affine subspace $A(\mathsf{T}, \mathsf{G})$ of $\mathscr{B}^\tau$ which we eventually identify with the apartment $A(\mathsf{T}, \mathsf{G}')$.

(ii) We show that the restriction of the action of $\mathscr{G}$ on $\mathscr{B}$ to the group $\mathscr{N}_{\mathscr{G}'}(\mathscr{T}) \subset \mathscr{G}$ yields an action on the affine subspace $A(\mathsf{T}, \mathsf{G})$ which is equivalent to the action of $\mathscr{N}_{\mathscr{G}'}(\mathscr{T})$ on $A(\mathsf{T}, \mathsf{G}')$ coming from the action of $\mathscr{G}'$ on $\mathscr{B}'$.

(iii) Suppose $x \in A(\mathsf{T}, \mathsf{G})$ and S is a τ-stable maximal k-split torus of G containing T. Set $\mathscr{S} := \mathsf{S}(k)$. Let $\mathscr{V}_x$ (resp. $\mathscr{U}_x$) denote the subgroup generated by the affine $\mathscr{T}$-root groups (resp. affine $\mathscr{S}$-root groups) of $\mathscr{G}'$ (resp. $\mathscr{G}$) fixing the point x. We show $\mathscr{V}_x \subset \mathscr{U}_x \cap \mathscr{G}'$.

We then identify $\mathscr{B}^\tau$ with $\mathscr{B}'$ so that the restriction action of $\mathscr{G}' \subset \mathscr{G}$ on $\mathscr{B}^\tau$ is equivalent to its action on $\mathscr{B}'$.

6.2.

Although we shall eventually assume that G and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ are as in Section 4, our preliminary results (Proposition 6.2.1 and 6.3.1) hold in the generality of G an arbitrary semisimple k-group and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ an arbitrary involution.

PROPOSITION 6.2.1. *Suppose* T *is a maximal* k*-split torus of* G'.

(i) *There exists a* τ*-invariant maximal* k*-split torus* S *of* G *containing* T.

(ii) *If* S_1 *and* S_2 *are two* τ*-invariant maximal* k*-split tori of* G *containing* T, *then*

$$A(\mathsf{S}_1)^\tau = A(\mathsf{S}_2)^\tau.$$

Proof. Assertion (i) is Corollary 5.7 (i). To prove assertion (ii), we have by Corollary 5.7 (ii) that $\dim(\mathscr{B}(\mathsf{G})^\tau) = \mathrm{rk}_k(\mathsf{G}') = \dim(A(\mathsf{S}_i)^\tau)$. Let L denote the Levi k-subgroup $\mathsf{C}_{\mathsf{G}}(\mathsf{T})$. Set

$$\mathscr{B}_{\mathsf{L}} := \text{union of the apartments } A(\mathsf{S}) \subset \mathscr{B}(\mathsf{G}) \text{ with } \mathsf{S} \text{ containing } \mathsf{T}. \tag{6.2.2}$$

It is obvious that $\mathscr{B}_{\mathsf{L}}$ is convex, τ-stable. The dimension of the convex set $(\mathscr{B}_{\mathsf{L}})^\tau$ satisfies

$$\dim(\mathscr{B}(\mathsf{G})^\tau) \geq \dim((\mathscr{B}_{\mathsf{L}})^\tau) \geq \mathrm{rk}_k(\mathsf{G}') = \dim(\mathscr{B}(\mathsf{G})^\tau),$$

so $\dim((\mathscr{B}_{\mathsf{L}})^\tau) = \mathrm{rk}_k(\mathsf{G}')$. The vector space $V(\mathsf{T}) := X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ acts naturally on $(\mathscr{B}_{\mathsf{L}})^\tau$. Let $(\mathscr{B}_{\mathsf{L}})^\tau/V(\mathsf{T})$ denote the quotient space. We have

$$\dim((\mathscr{B}_{\mathsf{L}})^\tau) = \mathrm{rk}_k(\mathsf{T}) + \dim((\mathscr{B}_{\mathsf{L}})^\tau/V(\mathsf{T})).$$

So, $\dim((\mathscr{B}_{\mathsf{L}})^\tau/V(\mathsf{T})) = 0$ and therefore the convex set $(\mathscr{B}_{\mathsf{L}})^\tau/V(\mathsf{T})$ must be a singleton point. This in particular means $(A(\mathsf{S}_1))^\tau = (A(\mathsf{S}_2))^\tau$. ∎

As a consequence of this proposition, any maximal k-split torus T of G' determines a canonical affine subspace of $\mathscr{B}(\mathsf{G})^\tau$ of dimension $\mathrm{rk}_k(\mathsf{G}')$. We denote this set as $A(\mathsf{T}, \mathsf{G})$.

6.3.

The maximal k-split tori of G^τ and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ obviously coincide. Suppose T is such a maximal k-split torus of G' and $\mathscr{T} := \mathsf{T}(k)$. Let $\mathscr{N}_{\mathscr{G}}(\mathscr{T})$ (resp. $\mathscr{C}_{\mathscr{G}}(\mathscr{T})$) denote the normalizer (resp. centralizer) of $\mathscr{T}$ in $\mathscr{G}$. All these groups are τ-stable. The group $\mathscr{N}_{\mathscr{G}}(\mathscr{T})$ acts on the space $\mathscr{B}_{\mathsf{L}}$ of (6.2.2); thus $\mathscr{N}_{\mathscr{G}}(\mathscr{T})^\tau = \mathscr{N}_{\mathscr{G}^\tau}(\mathscr{T})$ acts on $A(\mathsf{T}, \mathsf{G})$. The groups $\mathscr{C}_{\mathscr{G}^\tau}(\mathscr{T}) = \mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau$ and $\mathscr{C}_{\mathscr{G}'}(\mathscr{T}) \subset \mathscr{C}_{\mathscr{G}^\tau}(\mathscr{T})$ are compact modulo $\mathscr{T}$ and so they have unique maximal bounded subgroups $(\mathscr{C}_{\mathscr{G}^\tau}(\mathscr{T}))^b$ and $(\mathscr{C}_{\mathscr{G}'}(\mathscr{T}))^b$, respectively.

PROPOSITION 6.3.1. *Suppose* T *is a maximal k-split torus of* G'. *The maximal bounded subgroup* $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ *acts trivially on* $A(\mathsf{T}, \mathsf{G})$.

Proof. The compact group $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ acts isometrically on the Euclidean space $A(\mathsf{T}, \mathsf{G})$; hence there is a fixed point $x \in A(\mathsf{T}, \mathsf{G})$. Any $s \in \mathscr{N} = \mathscr{N}_{\mathscr{G}}(\mathscr{T})^\tau$ normalizes $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ and hence $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ must fix sx. This means $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ must fix all the points $\mathscr{N}x$ and in particular all the points $\mathscr{T}x$. But $\mathscr{T}$ acts on $A(\mathsf{T}, \mathsf{G})$ cocompactly, so we conclude $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ must act trivially on the points of a cocompact lattice of $A(\mathsf{T}, \mathsf{G})$ and thus $(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^\tau)^b$ must fix $A(\mathsf{T}, \mathsf{G})$. ∎

6.4.

It follows immediately from Proposition 6.3.1 that the action of $\mathcal{N}_{\mathscr{G}}(\mathscr{T})^{\tau}$ on $A(\mathsf{T},\mathsf{G})$ factors to an action of $\mathcal{N}_{\mathscr{G}}(\mathscr{T})^{\tau}/(\mathscr{C}_{\mathscr{G}}(\mathscr{T})^{\tau})^{b}$ on $A(\mathsf{T},\mathsf{G})$. Our next step is to establish the compatibility of the action of $\mathcal{N}_{\mathscr{G}}(\mathscr{T})^{\tau}$ on $A(\mathsf{T},\mathsf{G})$ with the action of $\mathcal{N}_{\mathscr{G}'}(\mathscr{T})$ on $A(\mathsf{T},\mathsf{G}')$. Due to present limitations in our proof, we shall only be able to accomplish this when G is a special linear group and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution as in Section 4.

As a first step in this direction, we consider the action of $\mathscr{C}_{\mathscr{G}'}(\mathscr{T})$ on the affine spaces $A(\mathsf{T},\mathsf{G}')$ and $A(\mathsf{T},\mathsf{G})$. Define the homomorphism $\nu : \mathscr{C}_{\mathscr{G}'}(\mathscr{T}) \to X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R} \subset X_*(\mathsf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ as in ([SS], Sect. 1.1). Then, the action of an element $v \in \mathscr{T}$ on both $A(\mathsf{T},\mathsf{G}')$ and $A(\mathsf{T},\mathsf{G}) \subset A(\mathsf{S},\mathsf{G})$ is a translation

$$tx = x + \nu(v).$$

6.4.1.

Suppose now that Γ, G and $\tau \in \mathrm{Aut}_k(\Gamma)$ are as defined in Section 4, more specifically by (4.4.8) and (4.4.2). Recall also $\mathcal{g} = \Gamma(k) = \mathrm{GL}_D(V)$, where $V = D^n$. Let $\langle\ ,\ \rangle$ be the form (4.4.4). Set $\mathsf{G}' := (\mathsf{G}^{\tau})^{\circ}$.

LEMMA 6.4.2. *In the notation of* (6.4.1), *suppose* T *is a maximal* k*-split torus in* G' *and suppose* $\widetilde{\mathsf{S}}$ *is a* τ*-stable maximal* k*-split torus in* Γ *containing* T. *Let* $\widetilde{\mathcal{S}}$ *be* $\widetilde{\mathsf{S}}(k)$. *Then*

$$\big(\mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T}) \cap \mathcal{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})\big)\mathscr{C}_{\mathcal{G}^{\tau}}(\mathscr{T})^{b} = \mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T}). \tag{6.4.3}$$

Proof. Let T and $\widetilde{\mathsf{S}}$ be as in the lemma. Let m be the Witt index of $\langle\ ,\ \rangle$ and $m_0 = n - 2m$. By the Witt basis theorem, we have a basis $\{e_i, f_j \mid i = \pm 1, \ldots, \pm m, j = 1, \ldots, m_0\}$ which consists of eigenspaces of $\mathscr{T}$ with $\langle e_i, e_{-j}\rangle = \delta_{ij} = \epsilon\langle e_{-j}, e_i\rangle$ for $i > 0$, $\langle e_i, f_j\rangle = 0$, and $\langle f_i, f_j\rangle = 0$ for $i \neq j$. Let V_s be a subspace of V generated by $\{e_i \mid i = \pm 1, \ldots, \pm m\}$ over D and let V_0 be generated by $\{f_j \mid j = 1, \ldots, m_0\}$. Then $V = V_s \oplus V_0$ is an orthogonal decomposition with respect to $\langle\ ,\ \rangle$. We can naturally identify $\mathrm{GL}_D(V_s)$, $\mathrm{GL}_D(V_0)$, and $\mathrm{GL}_D(V_s) \times \mathrm{GL}(V_0)$ as subgroups of $\mathcal{G} = \mathrm{GL}_D(V)$. Let $\mathcal{G}(V_s) = \mathcal{G}^{\tau} \cap \mathrm{GL}_D(V_s)$ and $\mathcal{G}(V_0) = \mathcal{G}^{\tau} \cap \mathrm{GL}_D(V_0)$. Then we have a natural embedding of $\mathcal{G}(V_s) \times \mathcal{G}(V_0)$ into $\mathcal{G}^{\tau}$ and of $\mathrm{GL}_D(V_s) \times \mathrm{GL}_D(V_0)$ into $\mathcal{G}$. Via these embeddings, we have $\mathscr{T} \subset \mathcal{G}(V_s) \times \mathcal{G}(V_0)$ and $\widetilde{\mathcal{S}} \subset \mathrm{GL}_D(V_s) \times \mathrm{GL}_D(V_0)$. Moreover, $\mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T})$ can be decomposed into a product $\mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T}) \cap \mathcal{G}(V_s)$ and $\mathcal{G}(V_0)$. In (6.4.3), $\subset$ is obvious. Since $(\mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T}) \cap \mathcal{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})) \supset (\mathcal{N}_{\mathcal{G}^{\tau}}(\mathscr{T}) \cap \mathcal{G}(V_s))$ and $\mathscr{C}_{\mathcal{G}^{\tau}}(\mathscr{T})^{b} \supset \mathcal{G}(V_0)$, $\supset$ also follows. ∎

Since $\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b$ is not contained in $\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$ in general, it is not obvious that there is a well-defined monomorphism $\mathscr{N}_{\mathscr{G}'}(\mathscr{T})/\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b \longrightarrow \mathscr{N}_{\mathscr{G}}(\mathscr{S})/\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$. In the following corollary, we define such a map by Lemma 6.4.2. Note that there are four different groups $\mathscr{G}', \mathscr{G}, \mathcal{G}^\tau$, and $\mathcal{G}$ involved in the proof.

COROLLARY 6.4.4. *We keep the notation from* (6.4.1) *and* (6.4.2). *Let* $\mathsf{S} = \widetilde{\mathsf{S}} \cap \mathsf{G}$. *If* $n\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b$ *is a coset of* $\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b$ *in* $\mathscr{N}_{\mathscr{G}'}(\mathscr{T})$, *let* $n' \in n\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b \cap \mathscr{N}_{\mathscr{G}}(\mathscr{S})$. *The element* n' *is unique modulo the subgroup* $\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$ *and the map*

$$\phi : \mathscr{N}_{\mathscr{G}'}(\mathscr{T})/\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b \longrightarrow \mathscr{N}_{\mathscr{G}}(\mathscr{S})/\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$$
$$\phi\big(nC_{\mathscr{G}'}(\mathscr{T})^b\big) = n'C_{\mathscr{G}}(\mathscr{S})^b$$

is a monomorphism of $\mathscr{N}_{\mathscr{G}'}(\mathscr{T})/\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b$ *into* $\mathscr{N}_{\mathscr{G}}(\mathscr{S})/\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$.

Proof. Observe that $\mathscr{N}_{\mathcal{G}}(\mathscr{S}) = \mathscr{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})$ and that there are natural monomorphisms

$$\phi_1 : \mathscr{N}_{\mathscr{G}'}(\mathscr{T})/\mathscr{C}_{\mathscr{G}'}(\mathscr{T})^b \longrightarrow \mathscr{N}_{\mathcal{G}^\tau}(\mathscr{T})/\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b,$$
$$\phi_2 : \mathscr{N}_{\mathscr{G}}(\mathscr{S})/\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b \longrightarrow \mathscr{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})/\mathscr{C}_{\mathcal{G}}(\widetilde{\mathcal{S}})^b.$$

By the above lemma, we can also define a monomorphism

$$\phi_3 : \mathscr{N}_{\mathcal{G}^\tau}(\mathscr{T})/\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b \longrightarrow \mathscr{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})/\mathscr{C}_{\mathcal{G}}(\widetilde{\mathcal{S}})^b$$

such that for $m\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b$, a coset of $\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b$ in $\mathscr{N}_{\mathcal{G}^\tau}(\mathscr{T})$ and $m' \in m\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b \cap \mathscr{N}_{\mathcal{G}}(\widetilde{\mathcal{S}})$, $\phi_3(m\mathscr{C}_{\mathcal{G}^\tau}(\mathscr{T})^b) = m'\mathscr{C}_{\mathcal{G}}(\widetilde{\mathcal{S}})^b$. Then the image of $\phi_3 \circ \phi_1$ factors through $\mathscr{N}_{\mathscr{G}}(\mathscr{S})/\mathscr{C}_{\mathscr{G}}(\mathscr{S})^b$; that is, we have $\phi_3 \circ \phi_1 = \phi_2 \circ \phi$ and the corollary is proved. ∎

6.5.

We introduce some notation. Suppose T is a maximal k-split torus in G'. If Y is a subset of the apartment $A(\mathsf{T}, \mathsf{G}')$, and $w \in X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, set

$$Y + w := \{x + w \mid x \in Y\}$$
$$Y(w) := \{x + tw \mid x \in Y \text{ and } t \geq 0\}.$$

If ψ is a nonconstant affine root on $A(\mathsf{T}, \mathsf{G}')$ with gradient $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$ and H_ψ is the vanishing hyperplane of ψ, then $H_\psi(\alpha)$ is a half space in $A(\mathsf{T}, \mathsf{G}')$. We use similar notation for G.

Suppose $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$ and u is a nonidentity element of the α root group $\mathscr{V}_\alpha \subset \mathscr{G}'$. The unique element in $\mathscr{V}_{-\alpha}u\mathscr{V}_{-\alpha} \cap \mathscr{N}_{\mathscr{G}'}(\mathscr{T})$ is expressible ([BT], Sect. 4.1.5.2) as $m(u) = vuv$, where $v \in \mathscr{V}_{-\alpha}$. The element $m(u)$ acts on $A(\mathsf{T}, \mathsf{G}')$ as a reflection across a hyperplane $H = H_u$. The elements v and u, as elements in $\mathscr{G}'$, fix the half-spaces $H(-\alpha)$ and $H(\alpha)$

of $A(\mathsf{T}, \mathsf{G}')$, respectively. We compare this to the situation where we view v and u as elements of $\mathscr{G}$ acting on $\mathscr{B}(\mathsf{G})$. It follows from the Witt basis in Lemma 6.4.2 and Corollary 6.4.4, that $m(u)$ is reflection across a hyperplane h_u in $A(\mathsf{T}, \mathsf{G})$. We intend to identify $A(\mathsf{T}, \mathsf{G})$ and $A(\mathsf{T}, \mathsf{G}')$ so that the hyperplane h_u is identified with the hyperplane $H = H_u$. As a step toward this goal, we now show that in $A(\mathsf{T}, \mathsf{G})$ the elements v and u fix the affine half-space $h_u(-\alpha)$ and $h_u(\alpha)$, respectively.

Choose a τ-invariant maximal k-split torus S in G containing T. When viewed as an element in $\mathscr{G}$, u is a product of elements in affine root groups $\mathscr{U}_\phi$ (of G with respect to S) satisfying

$$\operatorname{grad}(\phi|_{A(\mathsf{T},\mathsf{G})}) \text{ is a multiple of } \alpha. \tag{6.5.1}$$

Suppose ϕ satisfies (6.5.1). Then, the affine root group $\mathscr{U}_\phi$ fixes the half-space $H_\phi(\alpha)$ and so fixes the affine half-subspace $h_u(\alpha) + R\alpha$ provided $R \geq 0$ is sufficiently large. Since u lies in a product of affine root groups satisfying (6.5.1), it follows that there exists an $R_u \geq 0$ so that u fixes the affine half-subspace $h_u(\alpha) + R_u\alpha$. This in combination with convexity of the u-fixed points $A(\mathsf{T}, \mathsf{G})^u$ implies either $A(\mathsf{T}, \mathsf{G})^u$ equals $A(\mathsf{T}, \mathsf{G})$ or is a half-space. We shall see soon that $A(\mathsf{T}, \mathsf{G})^u$ is a half-space.

In a similar fashion, there is an $R_v \geq 0$ so that v fixes the affine half-subspaces $h_u(-\alpha) - R_v\alpha$, and the fixed point set is either a half-space or all of $A(\mathsf{T}, \mathsf{G})$. Suppose either $A(\mathsf{T}, \mathsf{G})^u$ or $A(\mathsf{T}, \mathsf{G})^v$ is all of $A(\mathsf{T}, \mathsf{G})$. From $m(u) = vuv$, we conclude that $m(u)$ must fix a half-space of $A(\mathsf{T}, \mathsf{G})$. This is clearly a contradiction to the action of $m(u)$ being a reflection across the hyperplane h_u. Hence u and v fix half-spaces in $A(\mathsf{T}, \mathsf{G})$. Let $b(u)$ and $b(v)$ denote the boundary hyperplanes of the two half-spaces $A(\mathsf{T}, \mathsf{G})^u$ and $A(\mathsf{T}, \mathsf{G})^v$. Heuristically, the action of u on $A(\mathsf{T}, \mathsf{G})$ is that it fixes pointwise the half-space $A(\mathsf{T}, \mathsf{G})^u$ and "folds along the hyperplane $b(u)$" the closure of the complementary half-space "away" from $A(\mathsf{T}, \mathsf{G})$. The situation for v is similar.

If the intersection of the two half-spaces $A(\mathsf{T}, \mathsf{G})^u$ and $A(\mathsf{T}, \mathsf{G})^v$ contains an open neighborhood U, then U must be fixed by the reflection $m(u)$, an impossibility. Hence, either the intersection of the two half-spaces is empty or a hyperplane, which subsequently must be h_u. To rule out the intersection being empty, we argue by contradiction. Assume the intersection is empty. Choose a point x in the hyperplane $b(v)$. Then, the empty intersection hypothesis implies

(i) $A(\mathsf{T}, \mathsf{G})^u$ is contained in $A(\mathsf{T}, \mathsf{G}) - A(\mathsf{T}, \mathsf{G})^v$,

(ii) the geodesic segment $[x, ux]$ meets $A(\mathsf{T}, \mathsf{G})^v$ precisely at the point x,

(iii) there exists y in the open interval (x, ux) so that $(x, y]$ lies in $A(\mathsf{T}, \mathsf{G}) - A(\mathsf{T}, \mathsf{G})^v$.

Now $v[x, ux] = [vx, m(u)x]$ lies in $A(\mathsf{T}, \mathsf{G})$ since its two endpoints lie in $A(\mathsf{T}, \mathsf{G})$. We conclude that the geodesic interval $[x, y]$ is sent by v to another geodesic interval inside $A(\mathsf{T}, \mathsf{G})$. But, since $[x, y]$ meets $A(\mathsf{T}, \mathsf{G})^v$ in the point x, and v folds the half-space which is the closure of $A(\mathsf{T}, \mathsf{G}) - A(\mathsf{T}, \mathsf{G})^u$ off $A(\mathsf{T}, \mathsf{G})$ along the hyperplane $b(u)$, it is impossible for $v[x, y]$ to be in $A(\mathsf{T}, \mathsf{G})$. This is a contradiction. We conclude that v and u must fix a common point in $A(\mathsf{T}, \mathsf{G})$, from which we deduce that $b(u) = b(v) = h_u$ is the common fixed set of u and v in $A(\mathsf{T}, \mathsf{G})$. Thus, v fixes the affine half-subspace $h_u(-\alpha)$, and u fixes $h_u(\alpha)$.

6.6.

Let ι_α: $A(\mathsf{T}, \mathsf{G}') \to A(\mathsf{T}, \mathsf{G})$ be any isometry, i.e., a $\mathscr{C}_{\mathscr{G}'}(\mathscr{T})$-equivariant map, which takes h_u to H_u. The map ι_α is determined up to a translation by a vector parallel to H_u. Our discussion implies

(i) ι_α commutes with the actions of $m(u)$ on $A(\mathsf{T}, \mathsf{G}')$ and $A(\mathsf{T}, \mathsf{G})$.

(ii) The element v fixes the affine half-subspace $H_u(-\alpha) = \iota_\alpha(h_u(-\alpha))$, and u fixes $H_u(\alpha) = \iota_\alpha(h_u(\alpha))$.

The above discussion is for a single root $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$. If we consider all roots, the assumption that G' is semisimple, i.e., $X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ is spanned by the simple roots, implies there is a unique $\mathscr{C}_{\mathscr{G}'}(\mathscr{T})$-equivariant map $\iota :$ $A(\mathsf{T}, \mathsf{G}') \to A(\mathsf{T}, \mathsf{G})$ so that

(i) For any root $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$ and $u \in \mathscr{V}_\alpha \backslash \{1\}$, the map ι takes h_u to H_u and ι commutes with the actions of $m(u) = vuv$ on $A(\mathsf{T}, \mathsf{G})$ and $A(\mathsf{T}, \mathsf{G}')$.

(ii) The element v fixes the affine half-subspace $H_u(-\alpha) = \iota(h_u(-\alpha))$, while u fixes $H_u(\alpha) = \iota(h_u(\alpha))$.

For $x \in A(\mathsf{T}, \mathsf{G}')$, set

$$\begin{aligned} \mathscr{V}_x := {} & \text{subgroup of } \mathscr{G}' \text{ generated by all the affine root groups} \\ & \mathscr{V}_\psi \text{ with respect to } \mathscr{T} \text{ which fix the point } x \end{aligned}$$

and

$$\begin{aligned} \mathscr{U}_{\iota(x)} := {} & \text{subgroup of } \mathscr{G} \text{ generated by all the affine root groups} \\ & \mathscr{U}_\phi \text{ with respect to } \mathscr{S} \text{ which fix the point } \iota(x). \end{aligned}$$

From the above, we see

$$\mathscr{V}_x \subset \mathscr{U}_{\iota(x)}.$$

6.7.

We are now ready to establish the existence of an injective map $\pi : \mathcal{B}(\mathsf{G}') \to \mathcal{B}(\mathsf{G})^\tau$. To do this, we choose a maximal k-split torus T of G' and a τ-stable maximal k-split torus S of G which contains T. Recall that any point $z \in \mathcal{B}(\mathsf{G}')$ is expressible as $z = gx$, where $x \in \mathcal{G}'$ and $x \in A(\mathsf{T}, \mathsf{G}')$. This expression is not unique, but if $z = hy$ is a second representation of z, with $h \in \mathsf{G}'$ and $y \in A(\mathsf{T}, \mathsf{G}')$, then there exists an $n \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ so that (i) $y = nx$, and (ii) $g^{-1}hn \in \mathcal{V}_x$. Define $\pi : \mathcal{B}(\mathsf{G}') \to \mathcal{B}(\mathsf{G})^\tau$ as follows: If z is expressible as $z = gx$, set

$$\pi(z) = g\iota(x). \tag{6.7.1}$$

Here, $\iota(x) \in A(\mathsf{T}, \mathsf{G})$ and the action of g on $\iota(x)$ is by viewing g as being in $\mathcal{G}$.

Proposition 6.7.2. *Let k be a nonarchimedean local field with odd residual characteristic. Let G be as in* (4.4.9) *and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ be as in* (4.4.2). *Suppose $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple. Let π be defined as in* (6.7.1).

(i) *π is a well-defined map.*

(ii) *π is an injection.*

Proof. To prove (i), suppose $z = gx$ and $z = hy$ are two expressions for the point z; i.e., $g, h \in \mathcal{G}'$ and $x, y \in A(\mathsf{T}, \mathsf{G}')$. Choose $n' \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ so that $y = n'x$, and $g^{-1}hn \in \mathcal{V}_x$. According to Lemma 6.4.2 and Corollary 6.4.4, we can choose $n \in n'\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ so that $\iota(y) = n\iota(x)$. Since $\mathcal{V}_x \subset \mathcal{U}_{\iota(x)}$, we have $g^{-1}hn \in \mathcal{U}_{\iota(x)}$. So, $\iota(x) = g^{-1}hn\iota(x)$; i.e., $g\iota(x) = hn\iota(x) = h\iota(y)$. Hence, π is well defined.

To prove (ii), suppose $z_1 = gx$ and $z_2 = hy$ map to the same point under π; i.e., $g\iota(x) = h\iota(y)$. The Bruhat decomposition, $\mathcal{G}' = \mathcal{V}_x\mathcal{N}_{\mathcal{G}'}(\mathcal{T})\mathcal{V}_y$, allows us to express the element $g^{-1}h$ as

$$g^{-1}h = u_x n^{-1} u_y \text{ with } u_x \in \mathcal{V}_x \subset \mathcal{U}_{\iota(x)},\ u_y \in \mathcal{V}_y \subset \mathcal{U}_{\iota(y)}, \text{ and } n \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T}).$$

Upon substitution, the equality $\iota(x) = g^{-1}h\iota(y)$ becomes $\iota(x) = u_x n^{-1} \times u_y\iota(y)$ from which we conclude that $y = nx$ and $g^{-1}hn = u_x n^{-1} u_y n \in \mathcal{V}_x$. So $z_1 = gx = hy = z_2$, and therefore π is injective. ∎

Theorem 6.7.3. *Let k be a nonarchimedean local field with odd residual characteristic. Let G be as in* (4.4.9) *and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ be as in* (4.4.2). *Suppose $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple. Then the map π defined by* (6.7.1) *is a $\mathsf{G}'(k)$-equivariant isometry of $\mathcal{B}(\mathsf{G}')$ onto $\mathcal{B}(\mathsf{G})^\tau$.*

Proof. Normalize distance on $A(\mathsf{T}, \mathsf{G})$ so that the map $\pi|_{A(\mathsf{T}, \mathsf{G}')} : A(\mathsf{T}, \mathsf{G}') \to A(\mathsf{T}, \mathsf{G})$ is an isometry. For $g \in \mathcal{G}'$, and $z \in \mathcal{B}(\mathcal{G}')$, we have $g\pi(z) = \pi(gz)$. Given any two points $z_1, z_2 \in \mathcal{B}(\mathcal{G}')$, there exist points $x_1, x_2 \in A(\mathsf{T}, \mathsf{G}')$ and a $g \in \mathscr{G}'$ so that $z_i = gx_i$; thus $\pi(z_i) = \pi(gx_i) = g\pi(x_i)$. Then, $\mathrm{dist}(\pi(z_1), \pi(z_2)) = \mathrm{dist}(\pi(x_1), \pi(x_2)) = \mathrm{dist}(x_1, x_2) = \mathrm{dist}(z_1, z_2)$. So π is an isometry. The final assertion that π is onto $\mathscr{B}(\mathsf{G})^\tau$ is Proposition 5.4 and Corollary 5.7. ∎

Added in notes. After the submission of this manuscript, we received a preprint by G. Prasad and J.-K. Yu, whose main results specialize to the main results in this manuscript.

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