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A generalization of Weyl's denominator formulas for the classical groups ☆

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Abstract

We give a generalization of Weyl's denominator formulas for the classical groups. We consider the matrices whose constituents are the characters of the respective classical groups in the restricted variables for each column of the matrices and show that the determinants of the matrices are equal to the powers of the fundamental alternating polynomials (the original denominators of Weyl's denominator formulas).

Keywords: Vandermonde determinant; Weyl's denominator formulas

1. Introduction

In this paper we give a generalization of Weyl's denominator formulas [3,4,7] for the classical groups. We consider the matrices whose constituents are the characters of the respective classical groups in the restricted variables for each column of the matrices and show that the determinants of the matrices are equal to the powers of the fundamental alternating polynomials (the original denominators of Weyl's denominator formulas). From now on, we consider all the (Laurent) polynomials in the variables $\{x_1, x_2, \dots, x_n\}$ over the rational field $\mathbb{Q}$. These formulas play a crucial role in the paper [1]. For another generalization of Weyl's denominator formulas, which is different from the one presented here, see [2].

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2. Formulas for the general linear groups

For the general linear groups $GL(n)$ and a positive integer k less than n, let M_k be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams λ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \ldots, i_k\}$ of the integer set $[n] = \{1, 2, \ldots, n\}$, whose $\lambda \times \{i_1, i_2, \ldots, i_k\}$ entry is given by $s_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$, where $s_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ denotes the Schur function (the character of $GL(n)$) in the k variables $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. We note that the number of the Young diagrams λ's included in the rectangular Young diagram $(n-k)^k$ is $\binom{n}{k}$.

We introduce the reverse lexicographic order $>$ in the partitions, namely for partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, we define the order $\lambda > \mu$ if $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\ldots$, $\lambda_{j-1} = \mu_{j-1}$ and $\lambda_j > \mu_j$ for some j. We arrange the rows of M_k in the decreasing order. Also we introduce the reverse lexicographic order into the monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ and $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} > x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ if $p_1 = q_1$, $p_2 = q_2, \ldots, p_{j-1} = q_{j-1}$ and $p_j > q_j$ for some j.

We arrange the columns of the matrix M_k in the decreasing order of the corresponding monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$.

For example, if $n = 4$ and $k = 2$, M_2 is given by

$$M_2 = \begin{pmatrix} s_{2,2}(x_1,x_2) & s_{2,2}(x_1,x_3) & s_{2,2}(x_1,x_4) & s_{2,2}(x_2,x_3) & s_{2,2}(x_2,x_4) & s_{2,2}(x_3,x_4) \\ s_{2,1}(x_1,x_2) & s_{2,2}(x_1,x_3) & s_{2,1}(x_1,x_4) & s_{2,1}(x_2,x_3) & s_{2,1}(x_2,x_4) & s_{2,1}(x_3,x_4) \\ s_2(x_1,x_2) & s_2(x_1,x_3) & s_2(x_1,x_4) & s_2(x_2,x_3) & s_2(x_2,x_4) & s_2(x_3,x_4) \\ s_{1,1}(x_1,x_2) & s_{1,1}(x_1,x_3) & s_{1,1}(x_1,x_4) & s_{1,1}(x_2,x_3) & s_{1,1}(x_2,x_4) & s_{1,1}(x_3,x_4) \\ s_1(x_1,x_2) & s_1(x_1,x_3) & s_1(x_1,x_4) & s_1(x_2,x_3) & s_1(x_2,x_4) & s_1(x_3,x_4) \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Our theorem for the general linear groups is as follows.

Theorem 1.

$$\det(M_k) = \left(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) \right)^{\binom{n-2}{k-1}}.$$

Remark 2. For $k = 1$, since $s_t(x_i) = x_i^t$, the above formula reduces to the well-known Vandermonde formula.

Proof. The weight decomposition formula tells us that $s_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = \sum_{\lambda \geqslant \mu} K_{\lambda\mu} \times m_\mu(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. Here $K_{\lambda\mu}$ denotes the Kostka number (the weight multiplicity) of the weight μ in the irreducible representation of $GL(k)$ with the highest weight λ and $m_\mu(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ denotes the monomial symmetric function. We note that $K_{\lambda\lambda} = 1$. (For the exact definitions, see [6].) Therefore applying row elementary transformations to $\det(M_k)$ successively, we can replace the $s_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$'s in the determinant by the $m_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$'s simultaneously. Namely we have the equality $\det([s_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})]_{\lambda,\{i_1,i_2,\ldots,i_k\}}) = \det([m_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})]_{\lambda,\{i_1,i_2,\ldots,i_k\}})$. So we consider the determinant of $\bar{M}_k = [m_\lambda(x_{i_1}, x_{i_2}, \ldots, x_{i_k})]_{\lambda,\{i_1,i_2,\ldots,i_k\}}$ instead of M_k.

Lemma 3. *For any pair of integers (i, j) with $1 \leqslant i < j \leqslant n$, $(x_i - x_j)^{\binom{n-2}{k-1}}$ divides* $\det(\bar{M}_k)$.

Proof. For any subset of integers $\{\ell_1, \ell_2, \ldots, \ell_{k-1}\}$ of $[n]$ such that $i, j \notin \{\ell_1, \ell_2, \ldots, \ell_{k-1}\}$, we consider the columns indexed by $\{i, \ell_1, \ell_2, \ldots, \ell_{k-1}\}$ and $\{j, \ell_1, \ell_2, \ldots, \ell_{k-1}\}$. If we take difference of the above columns (elementary transformation of columns), the difference of $\lambda \times \{i, \ell_1, \ell_2, \ldots, \ell_{k-1}\}$th entry and $\lambda \times \{j, \ell_1, \ell_2, \ldots, \ell_{k-1}\}$th entry is given by

$$m_\lambda(x_i, x_{\ell_1}, \ldots, x_{\ell_{k-1}}) - m_\lambda(x_j, x_{\ell_1}, \ldots, x_{\ell_{k-1}}) = \sum_{p_s \geqslant 1} \left(x_i^s - x_j^s\right) m_{\lambda - s}(x_{\ell_1}, \ldots, x_{\ell_{k-1}}).$$

Here p_s denotes the multiplicity of the integer s in the partition λ, namely $\lambda = 1^{p_1} 2^{p_2} \cdots u^{p_u}$ and $\lambda - s = 1^{p_1} 2^{p_2} \cdots s^{p_s - 1} \cdots u^{p_u}$ in terms of the exponential description of the partitions.

Hence the factor $(x_i - x_j)$ occurs once for each subset of integers $\{\ell_1, \ell_2, \ldots, \ell_{k-1}\}$ of $[n]$ such that $i, j \notin \{\ell_1, \ell_2, \ldots, \ell_{k-1}\}$. The number of choices of such sets is $\binom{n-2}{k-1}$, so we have the lemma. □

Lemma 4.

$$\det(\bar{M}_k) = c \left(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) \right)^{\binom{n-2}{k-1}}.$$

Here c is a constant.

From Lemma 3, $\left(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)\right)^{\binom{n-2}{k-1}}$ divides the determinant $\det(\bar{M}_k)$. So in order to prove the lemma, it is only enough to show that the degrees of both sides coincide. The degree of the right-hand side is $\binom{n}{2} \times \binom{n-2}{k-1}$. So we calculate the degree of the left-hand side. Since all the entries in the row of $\det(\bar{M}_k)$ corresponding to Young diagram λ have homogeneous degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (the size of λ), the proof of the lemma reduces to the following.

Lemma 5.

$$\sum_{\lambda \subseteq (n-k)^k} |\lambda| = \binom{n}{2} \times \binom{n-2}{k-1}.$$

Proof. We start with the following formula:

$$\frac{1}{(1 - qt)(1 - q^2 t) \cdots (1 - q^{n-k} t)} = \sum_{\lambda_1 \leqslant n-k} q^{|\lambda|} t^{\ell(\lambda)}. \tag{1}$$

Here in the right-hand side, the sum runs over all the Young diagrams with their parts less than or equal to $n - k$ and $\ell(\lambda)$ denotes the length of λ, i.e., the number of non-zero parts in λ.

Let us differentiate both sides in the variable q and put $q = 1$. Then the right-hand side of Eq. (1) is

$$\sum_{\lambda_1 \leqslant n-k} |\lambda| t^{\ell(\lambda)},$$

and what we want here is the sum of the coefficients of t^i for $1 \leqslant i \leqslant k$.

On the other hand, the left-hand side is given by

$$\frac{t+2t+\cdots+(n-k)t}{(1-t)^{n-k+1}}=\binom{n-k+1}{2}\frac{t}{(1-t)^{n-k+1}}.$$

The generalized binomial coefficient theorem tells us that

$$\begin{aligned}
&t(1-t)^{-n+k-1}\\
&=t\biggl(1-\binom{-n+k-1}{1}t+\binom{-n+k-1}{2}t^2+\cdots+(-1)^{k-1}\binom{-n+k-1}{k-1}t^{k-1}\\
&\qquad+\text{higher terms in } t\biggr).
\end{aligned}$$

So the sum of the coefficients of t^i for $1\leqslant i\leqslant k$ in the above is given by the sum

$$\binom{n+1-k}{0}+\binom{n+1-k}{1}+\binom{n+2-k}{2}+\cdots+\binom{n-1}{k-1}.$$

It is easy to see that this sum is equal to $\binom{n}{k-1}$ by the successive use of the well-known equality $\binom{n}{i-1}+\binom{n}{i}=\binom{n+1}{i}$. Since $\binom{n-k+1}{2}\times\binom{n}{k-1}=\binom{n}{2}\times\binom{n-2}{k-1}$, we have the lemma. □

Finally we show that the coefficient c in Lemma 4 is 1. Let us prepare some notations.

For any polynomial $P(x_1,x_2,\ldots,x_n)$, we define the dominant monomial of P by the maximum monomial in the order defined before among the monomials with the non-zero coefficients in P and denote it by $d[P]$ and define the dominant coefficient of P by the coefficient of the dominant monomial in P and denote it by $cd[P]$. It is easy to see that for any product of two polynomials P and Q, $d[PQ]=d[P]d[Q]$ and $cd[PQ]=cd[P]cd[Q]$.

Since the dominant monomial of $(\prod_{1\leqslant i<j\leqslant n}(x_i-x_j))^{\binom{n-2}{k-1}}$ is $(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})^{\binom{n-2}{k-1}}$ and its coefficient is one, if we prove the same for the polynomial $\det(\bar{M}_k)$, we can conclude that $c=1$.

Let us prove that the dominant monomial of $\det(\bar{M}_k)$ is $(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})^{\binom{n-2}{k-1}}$ and its coefficient is one.

Claim 6. *Only the diagonal components of $\bar{M}_k$ contribute to the dominant monomial of* $\det(\bar{M}_k)$ *and the dominant monomial of the determinant is given by* $d[\det(\bar{M}_k)]=(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})^{\binom{n-2}{k-1}}$ *and its coefficient in* $\det(\bar{M}_k)$ *is one.*

We prove this claim by induction on n (the number of the variables). The dominant monomials of the entries occurred in the first $\binom{n-1}{k-1}$ rows in $\bar{M}_k$ are of the form $x_{i_1}^{n-k}x_{i_2}^{p_2}\cdots x_{i_k}^{p_k}$, where $i_1<i_2<\cdots<i_k$ and $t\geqslant p_2\geqslant\cdots\geqslant p_k$.

From the definition of the order in the monomials, the dominant monomial in all the minors of the first $\binom{n-1}{k-1}$ rows of size $\binom{n-1}{k-1}\times\binom{n-1}{k-1}$ must be taken from the first $\binom{n-1}{k-1}$ columns, in other words, i_1 must be one and at that time the highest power of x_1 in the monomials of $\det(\bar{M}_k)$ is $(n-k)\times\binom{n-1}{k-1}$.

So let us consider the minor consisting of the first $\binom{n-1}{k-1}$ rows and the first $\binom{n-1}{k-1}$ columns of $\bar{M}_k$. From the above, in order to obtain the dominant monomial of that minor, we can simultaneously replace each of its entries by the partial sum of the monomials containing x_1^{n-k}. After we put out the powers of x_1 from each column of that minor, the induction hypothesis can apply to that minor with $n-1$ variables $\{x_2, x_3, \dots, x_n\}$ since its rows are indexed by the Young diagrams included in the rectangular $(n-k)^{k-1}$ and its columns are indexed by the $k-1$ subsets of $\{x_2, x_3, \dots, x_n\}$. The dominant monomial of that minor comes from the product of the diagonal components and is given by $(x_2^{n-2} \cdots x_{n-1})^{\binom{n-3}{k-2}}$ and its coefficient is one. We consider the minor of the remaining rows and columns of $\bar{M}_k$. This minor consists of the columns indexed only by the variables $\{x_2, \dots, x_n\}$ and of the rows indexed only by the Young diagrams included in $(n-k-1)^k$. So we can apply the induction hypothesis to that minor and the dominant monomial is $(x_2^{n-2} \cdots x_{n-1})^{\binom{n-3}{k-1}}$.

Therefore the dominant monomial of $\det(\bar{M}_k)$ is $x_1^{(n-k)\binom{n-1}{k-1}} \times (x_2^{n-2} \cdots x_{n-1})^{\binom{n-3}{k-2}} \times (x_2^{n-2} \cdots x_{n-1})^{\binom{n-3}{k-1}}$ and its coefficient is one. It is easy to see that this monomial is equal to $(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})^{\binom{n-2}{k-1}}$ and the claim, therefore the theorem is proved. $\square$

Remark 7. We can skip the degree argument (Lemma 5) since the above claim holds without the degree argument and we already know that $(\prod_{1 \leqslant i<j \leqslant n}(x_i - x_j))^{\binom{n-2}{k-1}}$ divides $\det(\bar{M}_k)$.

3. Formulas for the symplectic groups

Let $\lambda_{Sp(2n)}(x_1, x_2, \dots, x_n)$ denote the irreducible character of type C_n with the dominant integral weight λ. $\lambda_{Sp(2n)}(x_1, x_2, \dots, x_n)$ is by definition, the Laurent polynomial (i.e., the element in $\mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$) and invariant under the action of the Weyl group $W(C_n)$. From now on, we assume $n \geqslant 2$.

For a positive integer k less than n, let M_k^{Sp} be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams λ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \dots, i_k\}$ of the integer set $[n]$, whose $\lambda \times \{i_1, i_2, \dots, i_k\}$ entry is given by $\lambda_{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$.

The rows and columns of M_k^{Sp} are arranged in the same way as in the case of A_n.

Our theorem for the type C_n is as follows.

Theorem 8.

$$\det\bigl(M_k^{Sp}\bigr) = \Biggl(\prod_{1 \leqslant i<j \leqslant n}(x_i - x_j)\bigl(1 - x_i^{-1}x_j^{-1}\bigr)\Biggr)^{\binom{n-2}{k-1}}.$$

Remark 9. In the above, if $k=1$, we have $(t)_{Sp(2)}(x_i) = x_i^t + x_i^{t-2} + \cdots + x_i^{-t+2} + x_i^{-t}$. So up to the factor $\prod_{i=1}^n (x_i - x_i^{-1})$, the above formula gives us the original Weyl's denominator formula for the type C_n.

Proof. Actually the proof goes in the same way as in the case of A_n literally. Since $\lambda_{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k}) = \sum_{\lambda \geqslant \mu} K_{\lambda\mu}^{Sp(2k)} m_\mu^{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ and $K_{\lambda\lambda}^{Sp(2k)} = 1$, where $K_{\lambda\mu}^{Sp(2k)}$ denotes the weight multiplicity of the weight μ in the irreducible representation $\lambda_{Sp(2k)}$ and $m_\mu^{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ denotes the orbit sum of the monomial $x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \cdots x_{i_k}^{\mu_k}$ under the action of the Weyl group $W(C_k)$. Therefore applying row elementary transformations to $\det(M_k^{Sp})$ successively, we can replace the $\lambda_{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$'s in the determinant by the $m_\lambda^{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})$'s simultaneously. So if we put $\bar{M}_k^{Sp(2k)} = [m_\lambda^{Sp(2k)}(x_{i_1}, x_{i_2}, \dots, x_{i_k})]_{\lambda, \{i_1, i_2, \dots, i_k\}}$, we have $\det(M_k^{Sp}) = \det(\bar{M}_k^{Sp})$.

Lemma 10. *For any pair of integers (i, j) with $1 \leqslant i < j \leqslant n$, $((x_i - x_j)(1 - x_i^{-1} x_j^{-1}))^{\binom{n-2}{k-1}}$ divides $\det(\bar{M}_k^{Sp})$ in the algebra $\mathbb{Q}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$.*

Proof. For any subset of integers $\{\ell_1, \ell_2, \dots, \ell_{k-1}\}$ of $[n]$ such that $i, j \notin \{\ell_1, \ell_2, \dots, \ell_{k-1}\}$, we consider the columns indexed by $\{i, \ell_1, \ell_2, \dots, \ell_{k-1}\}$ and $\{j, \ell_1, \ell_2, \dots, \ell_{k-1}\}$. If we take difference of the above columns (elementary transformation of columns), the difference $\lambda \times \{i, \ell_1, \ell_2, \dots, \ell_{k-1}\}$th entry and $\lambda \times \{j, \ell_1, \ell_2, \dots, \ell_{k-1}\}$th entry is given by

$$\begin{aligned} &m_\lambda^{Sp(2k)}(x_i, x_{\ell_1}, \dots, x_{\ell_{k-1}}) - m_\lambda^{Sp(2k)}(x_j, x_{\ell_1}, \dots, x_{\ell_{k-1}}) \\ &\quad = \sum_{p_s \geqslant 1} \left(x_i^s + x_i^{-s} - x_j^s - x_j^{-s}\right) m_{\lambda - s}^{Sp(2k)}(x_{\ell_1}, \dots, x_{\ell_{k-1}}). \end{aligned}$$

Since $(x_i - x_j)(1 - x_i^{-1} x_j^{-1})$ divides $(x_i^s + x_i^{-s} - x_j^s - x_j^{-s})$, we have the lemma. □

From the lemma, if we put the quotient

$$Q(x_1, x_2, \dots, x_n) = \frac{\det(\bar{M}_k^{Sp})}{(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(1 - x_i^{-1} x_j^{-1}))^{\binom{n-2}{k-1}}},$$

Q is a Laurent polynomial. For any integer i with $1 \leqslant i \leqslant n$, Q is invariant under the action $x_i \to x_i^{-1}$ and $x_j \to x_j$ for $j \neq i$ since $(x_i - x_j)(1 - x_i^{-1} x_j^{-1})$ and the characters are invariant under this action.

Since the dominant monomial argument goes well in this case, we have $d[\det(\bar{M}_k^{Sp})] = d[(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(1 - x_i^{-1} x_j^{-1}))^{\binom{n-2}{k-1}}] = (x_1^{n-1} x_2^{n-2} \cdots x_{n-1})^{\binom{n-2}{k-1}}$ and its coefficients in both polynomials coincide and are one.

If Q is not a constant, Q contains some monomials $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ with non-negative exponents and some $p_i > 0$, since Q is invariant under the action $x_i \to x_i^{-1}$. This contradicts the fact that $d[Q] d[(\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(1 - x_i^{-1} x_j^{-1}))^{\binom{n-2}{k-1}}]$ must be equal to $d[\det(\bar{M}_k^{Sp})]$. So Q must be constant and must be one. □

4. Formulas for the orthogonal groups and the pin groups

Finally the formula for the type B_n is almost the same for the type C_n. We also assume $n \geqslant 2$. For the type B_n, the dominant integral weights are given by

$$P^+ = \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n;\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

Here $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{R}^n$ and all the λ_i's are integers or half-integers (namely $1/2 + \mathbb{Z}$) simultaneously.

So the formulas in this case are as follows.

When all λ_i's are integers, namely λ is a partition, then the irreducible representation with the highest weight λ comes from the group $SO(2n+1)$, so we denote the irreducible character of type B_n by $\lambda_{SO(2n+1)}(x_1, x_2, \ldots, x_n)$. Also for the dominant integral weight $(1/2+\delta_1, 1/2+\delta_2, \ldots, 1/2+\delta_n)$, where $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ is a partition, we denote this irreducible character by $[\Delta, \delta]_{Spin(2n+1)}$.

$\lambda_{Sp(2n)}(x_1, x_2, \ldots, x_n)$ is an element in $\mathbb{Q}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and $[\Delta, \delta]_{Spin(2n+1)}$ is an element in $\mathbb{Q}[x_1^{1/2}, x_1^{-1/2}, \ldots, x_n^{1/2}, x_n^{-1/2}]$. Both polynomials are invariant under the action of the Weyl group $W(B_n) = W(C_n)$.

For a positive integer k less than n, let $M_k^{SO,odd}$ be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams λ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \ldots, i_k\}$ of the integer set $[n]$, whose $\lambda \times \{i_1, i_2, \ldots, i_k\}$ entry is given by $\lambda_{SO(2k+1)}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$.

The rows and columns of $M_k^{SO,odd}$ are arranged in the same way as in the case of A_n. The proof of the next theorem is the same as in the type C_n.

Theorem 11.

$$\det\bigl(M_k^{SO,odd}\bigr) = \left(\prod_{1\leqslant i<j\leqslant n} (x_i - x_j)\bigl(1 - x_i^{-1}x_j^{-1}\bigr)\right)^{\binom{n-2}{k-1}}.$$

Remark 12. In the above, if $k = 1$, we have $(t)_{SO(3)}(x_i) = x_i^t + x_i^{t-1} + \cdots + x_i^{-t+1} + x_i^{-t}$. So up to the factor $\prod_{i=1}^n (x_i{}^{1/2} - x_i^{-1/2})$, the above formula gives us the original Weyl's denominator formula for the type B_n.

For a positive integer k less than n, let $M_k^{Spin,odd}$ be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams δ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \ldots, i_k\}$ of the integer set $[n]$, whose $\delta \times \{i_1, i_2, \ldots, i_k\}$ entry is given by $[\Delta, \delta]_{Spin(2k+1)}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$.

The rows and columns of $M_k^{Spin,odd}$ are arranged in the same way as in the case of A_n. Then our theorem is as follows.

Theorem 13.

$$\det\bigl(M_k^{Spin,odd}\bigr) = \left(\prod_{i=1}^n \bigl(x_i{}^{1/2} + x_i{}^{-1/2}\bigr)\right)^{\binom{n-1}{k-1}} \left(\prod_{1\leqslant i<j\leqslant n} (x_i - x_j)\bigl(1 - x_i^{-1}x_j^{-1}\bigr)\right)^{\binom{n-2}{k-1}}.$$

This theorem follows from the formula (see Theorem 10.1(ii) in [5])

$$[\Delta, \delta]_{Spin(2n+1)}(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^n \bigl(x_i{}^{1/2} + x_i{}^{-1/2}\bigr)\right)\delta_{Sp(2n)}(x_1, x_2, \ldots, x_n)$$

and Theorem 8.

Remark 14. In the above, if $k=1$, the matrix components in the left-hand side are

$$\big[\Delta, (t)\big]_{Spin(3)}(x_i) = \big({x_i}^{1/2} + x_i^{-1/2}\big)\big(x_i^t + x_i^{t-2} + \cdots + x_i^{-t+2} + x_i^{-t}\big).$$

For the type D_n, the dominant integral weights are given by

$$P^+ = \big\{\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n;\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant |\lambda_n| \geqslant 0\big\}.$$

Here all the λ_i's are integers or half-integers (namely $1/2+\mathbb{Z}$) simultaneously. We denote the corresponding character by $\lambda_{Spin(2n)}(x_1, x_2, \ldots, x_n)$. If all the λ_i's are integers, the corresponding irreducible representation comes from that of $SO(2n)$, we also write $\lambda_{SO(2n)}(x_1, x_2, \ldots, x_n)$ instead of $\lambda_{Spin(2n)}(x_1, x_2, \ldots, x_n)$.

Definition 15 *(Definition of $\lambda_{Pin(2n)}$ and $\lambda_{O(2n)}$).*

(1) If $\lambda_n = 0$ for the type D_n, we denote $\lambda_{O(2n)} = \lambda_{SO(2n)}$.
(2) If $\lambda_n \neq 0$ for the type D_n and we denote

$$\lambda_{Pin(2n)}(x_1, x_2, \ldots, x_n) = (\lambda_1, \lambda_2, \ldots, \lambda_n)_{Spin(2n)} + (\lambda_1, \lambda_2, \ldots, -\lambda_n)_{Spin(2n)}$$

and furthermore if all the λ_i's are integers in the above, we denote

$$\lambda_{O(2n)}(x_1, x_2, \ldots, x_n) = (\lambda_1, \lambda_2, \ldots, \lambda_n)_{SO(2n)} + (\lambda_1, \lambda_2, \ldots, -\lambda_n)_{SO(2n)}.$$

These characters are obtained by restricting the irreducible characters of $Pin(2n,\mathbb{C})$ to $Spin(2n,\mathbb{C})$ and $O(2n,\mathbb{C})$ to $SO(2n,\mathbb{C})$, respectively.

For a positive integer k less than n, let $M_k^{O,even}$ be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams λ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \ldots, i_k\}$ of the integer set $[n]$, whose $\lambda \times \{i_1, i_2, \ldots, i_k\}$ entry is given by $\lambda_{O(2k)}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$.

The rows and columns of $M_k^{O,even}$ are arranged in the same way as in the case of A_n. The proof of the next theorem is the same as in the type C_n.

Theorem 16.

$$\det\big(M_k^{O,even}\big) = \Bigg(\prod_{1\leqslant i<j\leqslant n}(x_i - x_j)\big(1 - x_i^{-1}x_j^{-1}\big)\Bigg)^{\binom{n-2}{k-1}}.$$

Remark 17. In the above, if $k=1$, the matrix components in the left-hand side are $(t)_{O(2)}(x_i) = x_i^t + x_i^{-t}$. So this is the original Weyl's denominator formula for type D_{2n}.

For a positive integer k less than n, let $M_k^{Pin,even}$ be a matrix of size $\binom{n}{k}$ with rows indexed by all the Young diagrams δ's included in the rectangular Young diagram $(n-k)^k$ and with columns indexed by all the size k subsets $\{i_1, i_2, \ldots, i_k\}$ of the integer set $[n]$, whose $\delta \times \{i_1, i_2, \ldots, i_k\}$ entry is given by $(\delta + 1/2)_{Pin(2k)}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$, where $(\delta + 1/2)$ denotes the dominant weight $(\delta_1 + 1/2, \delta_2 + 1/2, \ldots, \delta_n + 1/2)$.

The rows and columns of $M_k^{Pin,even}$ are arranged in the same way as in the case of A_n. Then our theorem is as follows.

Theorem 18.

$$\det\left(M_k^{Pin,even}\right) = \left(\prod_{i=1}^{n}\left({x_i}^{1/2} + {x_i}^{-1/2}\right)\right)^{\binom{n-1}{k-1}} \left(\prod_{1\leqslant i<j\leqslant n}(x_i - x_j)\left(1 - x_i^{-1}x_j^{-1}\right)\right)^{\binom{n-2}{k-1}}.$$

This theorem follows from Theorem 11 and the formula

$$\begin{aligned}&(\delta + 1/2)_{Pin(2k)}(x_1, x_2, \dots, x_n)\\ &\quad = \left(\prod_{i=1}^{n}\left({x_i}^{1/2} + {x_i}^{-1/2}\right)\right)(-1)^{|\delta|}\delta_{SO(2n+1)}(-x_1, -x_2, \dots, -x_n).\end{aligned}$$

As for the above formula, see Theorem 10.1(iii) in [5] and there $(\delta + 1/2)_{Pin(2k)}(x_1, x_2, \dots, x_n)$ is denoted by the different notation $\lambda^{(+)}_{Spin(2n)}(x_1, x_2, \dots, x_n)$.

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