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Parabolic Kazhdan–Lusztig R-polynomials for Hermitian symmetric pairs

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Abstract

We give explicit combinatorial product formulas for the parabolic Kazhdan–Lusztig R-polynomials of Hermitian symmetric pairs. Our results imply that all the roots of these polynomials are (either zero or) roots of unity, and complete those in [F. Brenti, Kazhdan–Lusztig and R-polynomials, Young's lattice, and Dyck partitions, Pacific J. Math. 207 (2002) 257–286] on Hermitian symmetric pairs of type A. As an application of our results, we derive explicit combinatorial product formulas for certain sums and alternating sums of ordinary Kazhdan–Lusztig R-polynomials.

Keywords: Coxeter groups; Kazhdan–Lusztig theory; Hermitian symmetric pairs

1. Introduction

In their fundamental paper [13] Kazhdan and Lusztig defined, for any Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan–Lusztig polynomials of W (see, e.g., [11, Chapter 7] or [2, Chapter 5]). These polynomials play an important role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [2, Chapter 5], and the

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references cited there). In order to prove the existence of these polynomials Kazhdan and Lusztig introduced another family of polynomials, usually called the R-polynomials, whose knowledge is equivalent to that of the Kazhdan–Lusztig polynomials.

In 1987 Deodhar [7] introduced parabolic analogues of all these polynomials. These parabolic Kazhdan–Lusztig and R-polynomials reduce to the ordinary ones for the trivial parabolic subgroup of W and are also related to them in other ways (see, e.g., Proposition 2.4 below). Besides these connections the parabolic polynomials also play a direct role in several areas including the theories of generalized Verma modules [6], tilting modules [15,16], quantized Schur algebras [19], Macdonald polynomials [9,10], Schubert varieties in partial flag manifolds [12], and in the representation theory of the Lie algebra gl_n [14].

The purpose of this work is to study the parabolic Kazhdan–Lusztig R-polynomials for Hermitian symmetric pairs. More precisely, we give explicit combinatorial product formulas for these polynomials. These imply, in particular, that all the roots of these polynomials are (either zero or) roots of unity. Some of these results are used in [5] to obtain explicit combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials.

The organization of the paper is as follows. In the next section we recall definitions, notation and results that are used in the rest of this work. In Section 3 we study the parabolic R-polynomials of Hermitian symmetric pairs. We obtain two main combinatorial formulas for them, one in terms of signed permutations and one in terms of lattice paths. As an application of our results, we derive combinatorial closed product formulas for certain sums and alternating sums of ordinary Kazhdan–Lusztig R-polynomials.

2. Preliminaries

In this section we collect some definitions, notation and results that are used in the rest of this work. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$ and $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$. For $m, n \in \mathbf{N}$, $m \geqslant n$, we let $[n, m] \stackrel{\text{def}}{=} \{n, n+1, \ldots, m-1, m\}$ and $[n] \stackrel{\text{def}}{=} [1, n]$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S_n$ then we write σ in *disjoint cycle form* (see, e.g., [17, p. 17]) and we usually omit writing the 1-cycles of σ. So, for example, if $\sigma = (9, 7, 1, 3, 5)(2, 6)$ then $\sigma(1) = 3$, $\sigma(2) = 6$, $\sigma(3) = 5$, $\sigma(4) = 4$, etc. Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We follow Chapter 3 of [17] for poset notation and terminology. In particular, given a poset $(P, \leqslant)$ and $u, v \in P$ we let $[u, v] \stackrel{\text{def}}{=} \{z \in P\colon u \leqslant z \leqslant v\}$ and call this an *interval* of P. We say that v *covers* u, denoted $u \lhd v$ (or, equivalently, that u is *covered by* v) if $|[u, v]| = 2$. We say that $u, v \in P$ are *comparable* if either $u \leqslant v$ or $v \leqslant u$. If P has a minimum element, denoted $\hat{0}$, then we call a subset of the form $[\hat{0}, u]$, for $u \in P$, a *lower interval* of P. Given any $Q \subseteq P$ we will always consider Q as a poset with the partial ordering induced by P.

We follow §7.2 of [18] for any undefined notation and terminology concerning partitions. Let $\mathcal{H} \stackrel{\text{def}}{=} \{(i, j) \in \mathbf{P}^2\colon i \leqslant j\}$, with the ordering induced by the product ordering on $\mathbf{P}^2$. We call the finite order ideals of $\mathcal{H}$ *shifted partitions*. We denote by $\mathcal{S}$ the set of all finite order ideals of $\mathcal{H}$. We will always assume that $\mathcal{S}$ is partially ordered by set inclusion. It is well known that this

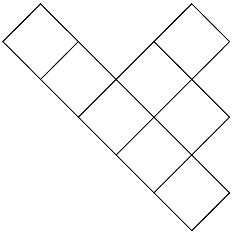

Fig. 1.

makes $\mathcal{S}$ into a distributive lattice. Note that there is an obvious bijection between partitions into distinct parts and shifted partitions, given by

$$(\lambda_1,\ldots,\lambda_k) \leftrightarrow \bigl\{(i,j)\in \mathbf{P}^2\colon\ 1\leqslant i\leqslant k,\ i\leqslant j\leqslant \lambda_i+i-1\bigr\}.$$

For this reason we will freely identify these objects, and use expressions such as "the shifted partition $(5,2,1)$." We also identify each shifted partition with its diagram, which we draw rotated counterclockwise by $\frac{3}{4}\pi$ radians with respect to the usual (Anglophone) convention. So, for example, the diagram of the shifted partition $(5,2,1)$ is shown in Fig. 1. We let $\delta_n \stackrel{\text{def}}{=} (n,n-1,\ldots,2,1)$.

We follow [11] for general Coxeter groups notation and terminology. In particular, given a Coxeter system (W,S) and $u\in W$ we denote by $l(u)$ the length of u in W, with respect to S, and we let $D(u)\stackrel{\text{def}}{=}\{s\in S\colon\ l(us)<l(u)\}$. For $u,v\in W$ we let $l(u,v)\stackrel{\text{def}}{=} l(v)-l(u)$. We denote by e the identity of W, and we let $T\stackrel{\text{def}}{=}\{usu^{-1}\colon\ u\in W,\ s\in S\}$ be the set of reflections of W. Given $J\subseteq S$ we let W_J be the parabolic subgroup generated by J and

$$W^J \stackrel{\text{def}}{=} \bigl\{u\in W\colon\ l(su)>l(u) \text{ for all } s\in J\bigr\}. \tag{1}$$

Note that $W^{\emptyset}=W$. If W_J is finite then we denote by $w_0(J)$ its longest element. We will always assume that W^J is partially ordered by *Bruhat order*. Recall (see, e.g., [11, §5.9]) that this means that $x\leqslant y$ if and only if there exist $r\in\mathbf{N}$ and $t_1,\ldots,t_r\in T$ such that $t_r\cdots t_1 x=y$ and $l(t_i\cdots t_1 x)>l(t_{i-1}\cdots t_1 x)$ for $i=1,\ldots,r$. Given $u,v\in W^J$, $u\leqslant v$, we let

$$[u,v]^J \stackrel{\text{def}}{=} \bigl\{z\in W^J\colon\ u\leqslant z\leqslant v\bigr\},$$

and $[u,v]\stackrel{\text{def}}{=}[u,v]^{\emptyset}$.

The following result is due to Deodhar, and we refer the reader to [7, §§2–3] for its proof.

Theorem 2.1. *Let (W,S) be a Coxeter system, and $J\subseteq S$. Then, for each $x\in\{-1,q\}$, there is a unique family of polynomials $\{R^{J,x}_{u,v}(q)\}_{u,v\in W^J}\subseteq \mathbf{Z}[q]$ such that, for all $u,v\in W^J$:*

(i) $R^{J,x}_{u,v}(q)=0$ *if* $u\not\leqslant v$;
(ii) $R^{J,x}_{u,u}(q)=1$;

(iii) *if $u < v$ and $s \in D(v)$ then*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,vs}(q), & \text{if } us < u, \\ (q-1)R^{J,x}_{u,vs}(q) + qR^{J,x}_{us,vs}(q), & \text{if } u < us \in W^J, \\ (q-1-x)R^{J,x}_{u,vs}(q), & \text{if } u < us \notin W^J. \end{cases}$$

The polynomials $R^{J,x}_{u,v}(q)$, whose existence is guaranteed by the previous theorem, are called the *parabolic R-polynomials* of W^J of type x. It follows immediately from Theorem 2.1 and from well-known facts (see, e.g., [11, §7.5]) that $R^{\emptyset,-1}_{u,v}(q)$ $(= R^{\emptyset,q}_{u,v}(q))$ are the (ordinary) *R-polynomials* of W which we will denote simply by $R_{u,v}(q)$, as customary. The parabolic R-polynomials can then be used to define and compute the parabolic Kazhdan–Lusztig polynomials of W^J of type x (see [7, Proposition 3.1]).

The parabolic R-polynomials are related to their ordinary counterparts also in the following way.

Proposition 2.2. *Let (W, S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J$. Then we have that*

$$R^{J,x}_{u,v}(q) = \sum_{w \in W_J} (-x)^{l(w)} R_{wu,v}(q),$$

for all $x \in \{-1, q\}$.

A proof of this result can be found in [7, Proposition 2.12].

There is one more property of the parabolic R-polynomials that we will use and that we recall for the reader's convenience. A proof of it can be found in [8, Corollary 2.2].

Proposition 2.3. *Let (W, S) be a Coxeter system, and $J \subseteq S$. Then*

$$q^{l(u,v)} R^{J,x}_{u,v}\left(\frac{1}{q}\right) = (-1)^{l(u,v)} R^{J,q-1-x}_{u,v}(q)$$

for all $u, v \in W^J$, and $x \in \{-1, q\}$.

The purpose of this work is to study the parabolic R-polynomials for quotients W^J such that (W, W_J) is a Hermitian symmetric pair. These quotients have been classified (see, e.g., [3]) and there are five infinite families and two exceptional ones. Using (and abusing slightly) the standard notation for the classification of the finite Coxeter systems, the Hermitian symmetric pairs are: $(A_n, A_{i-1} \times A_{n-i})$, (B_n, A_{n-1}), (B_n, B_{n-1}), (D_n, A_{n-1}), (D_n, D_{n-1}), (E_6, D_5), (E_7, E_6) $(n \geqslant 3,\ 1 \leqslant i \leqslant n)$. The parabolic R-polynomials for the pairs $(A_n, A_{i-1} \times A_{n-i})$ $(1 \leqslant i \leqslant n)$ have been computed in [4]. In this work we deal with the other ones.

We follow [2, Chapter 8] for combinatorial descriptions of the Coxeter systems of type B_n and D_n as permutation groups. In particular, we let S^B_n be the group of all bijections w of $\{-n, \ldots, -1, 1, \ldots, n\}$ in itself such that $w(-i) = -w(i)$ for all $i \in [n]$, $s_j \stackrel{\text{def}}{=} (j, j+1)(-j, -j-1)$ for $j = 1, \ldots, n-1$, $s_0 \stackrel{\text{def}}{=} (1, -1)$, and $\mathcal{B}_n \stackrel{\text{def}}{=} \{s_0, \ldots, s_{n-1}\}$. If $v \in S^B_n$ then we write $v = [a_1, \dots, a_n]$ to mean that $v(i) = a_i$, for $i = 1, \dots, n$. It is well known that $(S_n^B, \mathcal{B}_n)$ is a Coxeter system of type B_n and that the following holds. Given $v \in S_n^B$ we let

$$\mathrm{inv}(v) \stackrel{\text{def}}{=} \left|\left\{(i,j) \in [n]^2 \colon i < j,\ v(i) > v(j)\right\}\right|,$$

$N_1(v) \stackrel{\text{def}}{=} |\{i \in [n] \colon v(i) < 0\}|$ and

$$N_2(v) \stackrel{\text{def}}{=} \left|\left\{(i,j) \in [n]^2 \colon i < j,\ v(i) + v(j) < 0\right\}\right|.$$

Proposition 2.4. *Let $v \in S_n^B$. Then $l(v) = \mathrm{inv}(v) + N_1(v) + N_2(v)$, and $D(v) = \{s_i \in \mathcal{B}_n \colon v(i) > v(i+1)\}$, where $v(0) \stackrel{\text{def}}{=} 0$.*

Let $\mathcal{A}_{n-1} \stackrel{\text{def}}{=} \{s_1, \dots, s_{n-1}\}$. Then $(S_n^B)_{\mathcal{A}_{n-1}} \cong S_n$ and it is clear from Proposition 2.4 that $v \in (S_n^B)^{\mathcal{A}_{n-1}}$ if and only if $v^{-1}(1) < v^{-1}(2) < \cdots < v^{-1}(n)$. In this case

$$v^{-1}(1) < \cdots < v^{-1}(k) < 0 < v^{-1}(k+1) < \cdots < v^{-1}(n)$$

where $k = N_1(v)$, and we associate to $v \in (S_n^B)^{\mathcal{A}_{n-1}}$ the shifted partition

$$\Lambda_B(v) \stackrel{\text{def}}{=} \left(v^{-1}(-1), v^{-1}(-2), \dots, v^{-1}(-k)\right). \tag{2}$$

The next result is known, and not hard to prove.

Proposition 2.5. *The map Λ_B defined by* (2) *is a bijection between $(S_n^B)^{\mathcal{A}_{n-1}}$ and $\{\lambda \in \mathcal{S} \colon \lambda \subseteq \delta_n\}$. Furthermore $u \leqslant v$ in $(S_n^B)^{\mathcal{A}_{n-1}}$ if and only if $\Lambda_B(u) \subseteq \Lambda_B(v)$, and $l(v) = |\Lambda_B(v)|$, for all $u, v \in (S_n^B)^{\mathcal{A}_{n-1}}$.*

We find it sometimes convenient to identify a shifted partition in $\{\lambda \in \mathcal{S} \colon \lambda \subseteq \delta_n\}$ with a lattice path with $(1,1)$ (*up*) and $(1,-1)$ (*down*) steps starting at $(0,0)$ and having n steps. So, for example, the shifted partition $(5,2,1) \subseteq \delta_7$ corresponds to the lattice path illustrated in Fig. 2.

Proposition 2.6. *Let $v \in (S_n^B)^{\mathcal{A}_{n-1}}$ and $i \in [n]$. Then the ith step* (*from the left*) *of $\Lambda_B(v)$* (*seen as a lattice path*) *is an up-step if and only if $v(n+1-i) < 0$.*

Proof. From the definition of $\Lambda_B(v)$ we have that the ith step of $\Lambda_B(v)$ is an up step if and only if $n+1-i \in \{v^{-1}(-1), \dots, v^{-1}(-k)\}$. But this, by the definition of k, happens if and only if $v(n+1-i) < 0$, as desired. $\square$

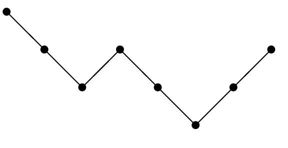

Fig. 2.

We let S_n^D be the subgroup of S_n^B defined by

$$S_n^D \stackrel{\text{def}}{=} \{w \in S_n^B\colon N_1(w) \equiv 0 \pmod 2\},$$

$\tilde{s}_0 \stackrel{\text{def}}{=} (1, -2)(2, -1)$, and $\mathcal{D}_n \stackrel{\text{def}}{=} \{\tilde{s}_0, s_1, \ldots, s_{n-1}\}$. It is then well known that $(S_n^D, \mathcal{D}_n)$ is a Coxeter system of type D_n, and that the following holds (see, e.g., [2, §8.2]).

Proposition 2.7. *Let $v \in S_n^D$. Then $l(v) = \mathrm{inv}(v) + N_2(v)$, and $D(v) = \{s_i \in \mathcal{D}_n\colon v(i) > v(i+1)\}$, where $v(0) \stackrel{\text{def}}{=} -v(2)$.*

Let $v \in S_n^D$. It is then clear from Proposition 2.7 that $v \in (S_n^D)^{\mathcal{A}_{n-1}}$ if and only if $v^{-1}(1) < \cdots < v^{-1}(n)$. In this case

$$v^{-1}(1) < \cdots < v^{-1}(k) < 0 < v^{-1}(k+1) < \cdots < v^{-1}(n) \tag{3}$$

where $k = N_1(v)$, and we associate to $v \in (S_n^D)^{\mathcal{A}_{n-1}}$ the shifted partition

$$\Lambda_D(v) \stackrel{\text{def}}{=} \bigl(v^{-1}(-1) - 1, v^{-1}(-2) - 1, \ldots, v^{-1}(-k) - 1\bigr). \tag{4}$$

The next result is almost certainly known. However, for lack of an adequate reference, and for completeness, we include its proof here.

Proposition 2.8. *The map Λ_D defined by* (4) *is a bijection between $(S_n^D)^{\mathcal{A}_{n-1}}$ and $\{\lambda \in \mathcal{S}\colon \lambda \subseteq \delta_{n-1}\}$. Furthermore $u \leqslant v$ in $(S_n^D)^{\mathcal{A}_{n-1}}$ if and only if $\Lambda_D(u) \subseteq \Lambda_D(v)$, and $l(v) = |\Lambda_D(v)|$, for all $u, v \in (S_n^D)^{\mathcal{A}_{n-1}}$.*

Proof. Since $l(\Lambda_B(v)) = N_1(v)$ for all $v \in (S_n^B)^{\mathcal{A}_{n-1}}$ and $(S_n^D)^{\mathcal{A}_{n-1}} = \{v \in (S_n^B)^{\mathcal{A}_{n-1}}\colon N_1(v) \equiv 0 \pmod 2\}$ it follows immediately from Proposition 2.5 that Λ_B is a bijection between $(S_n^D)^{\mathcal{A}_{n-1}}$ and $\{\lambda \in \mathcal{S}\colon \lambda \subseteq \delta_n,\ l(\lambda) \equiv 0 \pmod 2\}$. But the map $(\lambda_1, \ldots, \lambda_k) \mapsto (\lambda_1 - 1, \ldots, \lambda_k - 1)$ is clearly a bijection between $\{\lambda \in \mathcal{S}\colon \lambda \subseteq \delta_n,\ l(\lambda) \equiv 0 \pmod 2\}$ and $\{\lambda \in \mathcal{S}\colon \lambda \subseteq \delta_{n-1}\}$ so the first statement follows. To prove the second statement note that, by Proposition 2.5 and the definition of Λ_B, it is equivalent to the statement that, for all $u, v \in (S_n^D)^{\mathcal{A}_{n-1}}$, $u \leqslant v$ in S_n^D if and only if $u \leqslant v$ in S_n^B. It is well known that if $u \leqslant v$ in S_n^D then $u \leqslant v$ in S_n^B (see, e.g., [2, Theorems 8.1.8 and 8.2.8]). So assume that $u \leqslant v$ in S_n^B. By [2, Theorems 8.1.8 and 8.2.8] it is enough to show that if $a, b \in [n]$ are such that $[-a, a] \times [-b, b]$ is an empty rectangle for both u and v, and $u[-a-1, b+1] = v[-a-1, b+1]$ (we refer the reader to [2, §8.2, p. 257] for the definition of this notation and terminology), then $u[-1, b+1] \equiv v[-1, b+1] \pmod 2$. Note that, for any $w \in (S_n^D)^{\mathcal{A}_{n-1}}$ and any $a, b \in [n]$, $w[-1, b+1] = \max\{0, N_1(w) - b\}$, and if $[-a, a] \times [-b, b]$ is empty for w then $N_1(w) \geqslant b$ (for if $N_1(w) < b$ then $w^{-1}(b) > 0$ and hence, since $[-a, a] \times [-b, b]$ is empty for w, $w^{-1}(b) > a$ and $|w(1)| > b$, hence $a < w^{-1}(b) < w^{-1}(|w(1)|) \leqslant 1$, which is a contradiction). Hence, under our hypotheses, $u[-1, b+1] = N_1(u) - b \equiv N_1(v) - b = v[-1, b+1] \pmod 2$, as desired. Finally, $l_D(v) = l_B(v) - N_1(v) = |\Lambda_B(v)| - l(\Lambda_B(v)) = |\Lambda_D(v)|$ for any $v \in (S_n^D)^{\mathcal{A}_{n-1}}$ by (4) and Propositions 2.4, 2.7, and 2.5. $\square$

Proposition 2.9. *Let $v \in (S_n^D)^{\mathcal{A}_{n-1}}$ and $i \in [n-1]$. Then the ith step (from the left) of $\Lambda_D(v)$ (seen as a lattice path) is an up-step if and only if $v(n+1-i) < 0$. In particular, $v(1) < 0$ if and only if $\Lambda_D(v)$ has an odd number of up-steps.*

Proof. This follows immediately from Proposition 2.6 and the fact that the ith step $(1 \leqslant i \leqslant n-1)$ of $(\lambda_1, \dots, \lambda_k) \subseteq \delta_n$ is up if and only if the ith step of $(\lambda_1 - 1, \dots, \lambda_k - 1) \subseteq \delta_{n-1}$ is, for all $(\lambda_1, \dots, \lambda_k) \in \mathcal{S}$. $\square$

Let $u \in S_n^B$, and $\mathcal{B}_{n-1} \stackrel{\text{def}}{=} \{s_0, \dots, s_{n-2}\} \subseteq \mathcal{B}_n$. Then $(S_n^B)_{\mathcal{B}_{n-1}} \cong S_{n-1}^B$ and by Proposition 2.4 we have that

$$\left(S_n^B\right)^{\mathcal{B}_{n-1}} = \left\{w \in S_n^B\colon\ 0 < w^{-1}(1) < \cdots < w^{-1}(n-1)\right\}.$$

Therefore $u \in (S_n^B)^{\mathcal{B}_{n-1}}$ if and only if there exist $i \in [n]$ and $\varepsilon \in \{1, -1\}$ such that

$$u = [1, 2, \dots, i-1, \varepsilon n, i, \dots, n-1].$$

Let, for brevity, $u_{\varepsilon i} \stackrel{\text{def}}{=} [1, 2, \dots, i-1, \varepsilon n, i, \dots, n-1]$. It then follows easily from the definition of Bruhat order that

$$u_n < u_{n-1} < \cdots < u_1 < u_{-1} < \cdots < u_{-n}.$$

So $(S_n^B)^{\mathcal{B}_{n-1}}$, partially ordered by Bruhat order, is a chain with $2n$ elements.

Let $v \in S_n^D$, and $\mathcal{D}_{n-1} \stackrel{\text{def}}{=} \{\tilde{s}_0, s_1, \dots, s_{n-2}\} \subseteq \mathcal{D}_n$. Then $(S_n^D)_{\mathcal{D}_{n-1}} \cong S_{n-1}^D$ and by Proposition 2.7 we have that

$$\left(S_n^D\right)^{\mathcal{D}_{n-1}} = \left\{w \in S_n^D\colon\ w^{-1}(-2) < w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n-1)\right\}.$$

Hence, if $w \in (S_n^D)^{\mathcal{D}_{n-1}}$, then $0 < w^{-1}(2)$ so

$$\left(S_n^D\right)^{\mathcal{D}_{n-1}} = \left\{v_{\varepsilon i}\colon\ i \in [n],\ \varepsilon \in \{-1, 1\}\right\},$$

where

$$v_{\varepsilon i} \stackrel{\text{def}}{=} \begin{cases} [\varepsilon 1, 2, \dots, i-1, \varepsilon n, i, \dots, n-1], & \text{if } i \geqslant 2, \\ [\varepsilon n, \varepsilon 1, 2, \dots, n-1], & \text{if } i = 1, \end{cases}$$

for $i \in [n]$, $\varepsilon \in \{-1, 1\}$. Furthermore, it follows easily from the definition of Bruhat order (and from Proposition 2.7) that

$$v_n < v_{n-1} < \cdots < v_2 < v_1 < v_{-2} < \cdots < v_{-n},$$

$v_2 < v_{-1} < v_{-2}$, and v_{-1} and v_1 are incomparable.

3. Parabolic R-polynomials

In this section we obtain explicit combinatorial product formulas for the parabolic R-polynomials of Hermitian symmetric pairs. These show, in particular, that all the roots of these polynomials are (either zero or) roots of unity. As an application of our results, we derive explicit combinatorial product formulas for certain sums and alternating sums of ordinary Kazhdan–Lusztig R-polynomials.

Let $u, v \in S_n^B$. For $j \in [n]$ let

$$b_j(u,v) \stackrel{\text{def}}{=} \left|\{r \geqslant j\colon\ v(r) < 0\}\right| - \left|\{r \geqslant j\colon\ u(r) < 0\}\right|.$$

For example, if $u = [3,4,5,6,-2,7,-1]$ and $v = [-4,5,-3,6,7,-2,-1]$ then $(b_1(u,v), \dots, b_7(u,v)) = (2,1,1,0,0,1,0)$. Note that it follows easily from Propositions 2.5 and 2.8 that if $u, v \in (S_n^B)^{\mathcal{A}_{n-1}}$ (respectively, $(S_n^D)^{\mathcal{A}_{n-1}}$) then $b_j(u,v) \geqslant 0$ for $j = 1, \dots, n$ if and only if $u \leqslant v$ in $(S_n^B)^{\mathcal{A}_{n-1}}$ (respectively, $(S_n^D)^{\mathcal{A}_{n-1}}$). We let

$$N(u,v) \stackrel{\text{def}}{=} \{r \in [n]\colon\ u(r)\, v(r) < 0\}$$

and

$$D(u,v) \stackrel{\text{def}}{=} \{r \in N(u,v)\colon\ (-1)^{b_r(u,v)} < 0\}. \tag{5}$$

Theorem 3.1. *Let $W^J \in \{(S_n^B)^{\mathcal{A}_{n-1}}, (S_n^D)^{\mathcal{A}_{n-1}}\}$ and $u, v \in W^J$, $u \leqslant v$. Then*

$$R_{u,v}^{J,-1}(q) = q^{l(u,v)} \prod_{j \in D(u,v)} \left(1 - q^{-\tilde{b}_j(u,v)}\right),$$

where

$$\tilde{b}_j(u,v) \stackrel{\text{def}}{=} \begin{cases} b_j(u,v), & \text{if } u(j) > 0, \\ b_j(u,v) + 1, & \text{if } u(j) < 0. \end{cases} \tag{6}$$

Proof. Let, for brevity, $R_{u,v}(q) \stackrel{\text{def}}{=} R_{u,v}^{J,-1}(q)$ for $u, v \in W^J$. We proceed by induction on $l(v)$ the result being trivially true if $v = e$. So suppose that $l(v) \geqslant 1$. Let $s \in D(v)$.

Suppose first that $s = (i, i+1)(-i-1, -i)$ for some $i \in [n-1]$. Then, by Propositions 2.4 and 2.7, $v(i) > v(i+1)$. Since $v \in W^J$, this implies that $v(i) > 0 > v(i+1)$. We have three cases to consider.

(a) $us < u$. Then, since $u \in W^J$, $u(i) > 0 > u(i+1)$. Therefore $N(u,v) = N(us,vs)$ and $b_j(u,v) = b_j(us,vs)$ for all $j \in [n]$. So $D(u,v) = D(us,vs)$, $\tilde{b}_j(u,v) = \tilde{b}_j(us,vs)$ for all $j \in D(u,v)$, and by Theorem 2.1 and our induction hypothesis we have that

$$\begin{aligned} R_{u,v}(q) &= R_{us,vs}(q) \\ &= q^{l(us,vs)} \prod_{j \in D(us,vs)} \left(1 - q^{-\tilde{b}_j(us,vs)}\right) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \left(1 - q^{-\tilde{b}_j(u,v)}\right), \end{aligned}$$

as desired.

(b) $u < us \notin W^J$. Then, by Propositions 2.4 and 2.7, either $u(i) < u(i+1) < 0$ or $0 < u(i) < u(i+1)$. In the first case

$$N(u,vs) = \left(N(u,v) \setminus \{i\}\right) \cup \{i+1\}$$

and $b_j(u,v) = b_j(u,vs)$ for $j \in [n] \setminus \{i+1\}$, $b_i(u,v) = b_{i+1}(u,vs)$. Therefore

$$D(u,vs) \setminus \{i+1\} = D(u,v) \setminus \{i\}$$

and $i \in D(u,v)$ if and only if $i+1 \in D(u,vs)$. Hence, by Theorem 2.1 and our induction hypothesis

$$\begin{aligned} R_{u,v}(q) &= q R_{u,vs}(q) \\ &= q\, q^{l(u,vs)} \prod_{j \in D(u,vs)} \left(1 - q^{-\tilde{b}_j(u,vs)}\right) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \left(1 - q^{-\tilde{b}_j(u,v)}\right), \end{aligned}$$

as desired.

In the second case $N(u,vs) = (N(u,v) \setminus \{i+1\}) \cup \{i\}$, and $b_j(u,vs) = b_j(u,v)$ for $j \in [n] \setminus \{i+1\}$, $b_i(u,vs) = b_{i+1}(u,v)$. Therefore $D(u,vs) \setminus \{i\} = D(u,v) \setminus \{i+1\}$, and $i \in D(u,vs)$ if and only if $i+1 \in D(u,v)$, and the result follows as before.

(c) $u < us \in W^J$. Then, by Propositions 2.4 and 2.7, $u(i) < 0 < u(i+1)$. Therefore $N(u,vs) = N(u,v) \setminus \{i, i+1\}$, $N(us,vs) = N(u,v)$, $b_j(us,vs) = b_j(u,vs) = b_j(u,v)$ for $j \in [n] \setminus \{i+1\}$, and

$$b_{i+1}(us,vs) + 1 = b_i(u,v) = b_{i+1}(u,v) - 1. \tag{7}$$

So

$$D(u,vs) = D(u,v) \setminus \{i, i+1\} = D(us,vs) \setminus \{i, i+1\}$$

and $i+1 \in D(u,v)$ if and only if $i \notin D(u,v)$ if and only if $i \notin D(us,vs)$ if and only if $i+1 \in D(us,vs)$. Hence, we have from our induction hypothesis that

$$\begin{aligned} R_{u,vs}(q) &= q^{l(u,vs)} \prod_{j \in D(u,vs)} \left(1 - q^{-\tilde{b}_j(u,vs)}\right) \\ &= q^{l(u,v)-1} \prod_{j \in D(u,v) \setminus \{i,i+1\}} \left(1 - q^{-\tilde{b}_j(u,v)}\right), \end{aligned} \tag{8}$$

and, if $us \leqslant vs$,

$$\begin{aligned} R_{us,vs}(q) &= q^{l(us,vs)} \prod_{j \in D(us,vs)} \left(1 - q^{-\tilde{b}_j(us,vs)}\right) \\ &= q^{l(u,v)-2} \prod_{j \in D(u,v) \setminus \{i,i+1\}} \left(1 - q^{-\tilde{b}_j(u,v)}\right) A(u,v), \end{aligned}$$

where

$$A(u,v) \stackrel{\text{def}}{=} \begin{cases} (1-q^{-\tilde{b}_i(us,vs)}), & \text{if } i \in D(u,v), \\ (1-q^{-\tilde{b}_{i+1}(us,vs)}), & \text{if } i \notin D(u,v). \end{cases}$$

Therefore, by Theorem 2.1,

$$R_{u,v}(q) = q^{l(u,v)-1} \prod_{j \in D(u,v)\setminus\{i,i+1\}} \bigl(1-q^{-\tilde{b}_j(u,v)}\bigr)\bigl(q-1+A(u,v)\bigr)$$

and the result follows since $\tilde{b}_i(u,v)-1=\tilde{b}_i(us,vs)$ if $i \in D(u,v)$ and $\tilde{b}_{i+1}(us,vs)=\tilde{b}_{i+1}(u,v)-1$ if $i \notin D(u,v)$.

If $us \nleqslant vs$ then from (7) and the comments at the beginning of this section we conclude that $b_i(u,v)=0$, $b_{i+1}(u,v)=1$. Hence $i+1 \in D(u,v)$ and by Theorem 2.1 and (8) we have that

$$\begin{aligned} R_{u,v}(q) &= (q-1)R_{u,vs}(q) \\ &= (q-1)q^{l(u,v)-1} \prod_{j \in D(u,v)\setminus\{i,i+1\}} \bigl(1-q^{-\tilde{b}_j(u,v)}\bigr) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \bigl(1-q^{-\tilde{b}_j(u,v)}\bigr), \end{aligned}$$

since $\tilde{b}_{i+1}(u,v)=1$, and the result again follows.

Assume now that $W^J=(S_n^B)^{\mathcal{A}_{n-1}}$ and that $s=(-1,1)$. Then, by Proposition 2.4, $v(1)<0$. We have two cases to consider.

(a) $us<u$. Then $u(1)<0$. Therefore $N(u,v)=N(us,vs)$, $b_j(u,v)=b_j(us,vs)$ for all $j \in [n]$, and hence $D(u,v)=D(us,vs)$, and $1 \notin D(u,v)$. So, by Theorem 2.1 and our induction hypothesis,

$$\begin{aligned} R_{u,v}(q) &= R_{us,vs}(q) \\ &= q^{l(us,vs)} \prod_{j \in D(us,vs)} \bigl(1-q^{-\tilde{b}_j(us,vs)}\bigr) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \bigl(1-q^{-\tilde{b}_j(u,v)}\bigr), \end{aligned}$$

as desired.

(b) $u<us$. Then $u(1)>0$ and necessarily $us \in (S_n^B)^{\mathcal{A}_{n-1}}$. Hence $N(u,v)=N(us,vs)$, $N(u,vs)=N(u,v)\setminus\{1\}$, and $b_j(u,v)=b_j(us,vs)=b_j(u,vs)$ for all $j \in [n]\setminus\{1\}$,

$$b_1(u,vs)=b_1(u,v)-1=b_1(us,vs)+1. \tag{9}$$

So

$$D(u,vs)=D(u,v)\setminus\{1\}=D(us,vs)\setminus\{1\}$$

and $1 \in D(u,v)$ if and only if $1 \in D(us,vs)$. Hence we have from our induction hypothesis that

$$\begin{aligned} R_{u,vs}(q) &= q^{l(u,vs)} \prod_{j\in D(u,vs)} \bigl(1-q^{-\tilde b_j(u,vs)}\bigr) \\ &= q^{l(u,v)-1} \prod_{j\in D(u,v)\setminus\{1\}} \bigl(1-q^{-\tilde b_j(u,v)}\bigr), \end{aligned} \tag{10}$$

and, if $us \leqslant vs$,

$$\begin{aligned} R_{us,vs}(q) &= q^{l(us,vs)} \prod_{j\in D(us,vs)} \bigl(1-q^{-\tilde b_j(us,vs)}\bigr) \\ &= q^{l(u,v)-2} \prod_{j\in D(u,v)\setminus\{1\}} \bigl(1-q^{-\tilde b_j(u,v)}\bigr) A(u,v), \end{aligned}$$

where

$$A(u,v) \stackrel{\text{def}}{=} \begin{cases} 1-q^{-\tilde b_1(u,v)+1}, & \text{if } 1\in D(u,v),\\ 1, & \text{if } 1\notin D(u,v). \end{cases} \tag{11}$$

Therefore, by Theorem 2.1,

$$R_{u,v}(q) = q^{l(u,v)-1} \prod_{j\in D(u,v)\setminus\{1\}} \bigl(1-q^{-\tilde b_j(u,v)}\bigr)\bigl(q-1+A(u,v)\bigr)$$

and the result follows from (11).

If $us \nleqslant vs$ then from (9), and the comments at the beginning of this section, we conclude that $b_1(u,v)=1$. Hence $1\in D(u,v)$ and by Theorem 2.1 and (10) we have that

$$\begin{aligned} R_{u,v}(q) &= (q-1)R_{u,vs}(q) \\ &= (q-1)q^{l(u,v)-1} \prod_{j\in D(u,v)\setminus\{1\}} \bigl(1-q^{-\tilde b_j(u,v)}\bigr) \\ &= q^{l(u,v)} \prod_{j\in D(u,v)} \bigl(1-q^{-\tilde b_j(u,v)}\bigr), \end{aligned}$$

since $\tilde b_1(u,v)=1$, and the result again follows.

Assume now that $W^J=(S_n^D)^{\mathcal{A}_{n-1}}$, and that $s=(-1,2)(-2,1)$. Then, by Proposition 2.7, $v(1)+v(2)<0$. This, since $v\in(S_n^D)^{\mathcal{A}_{n-1}}$, implies that $v(1), v(2)<0$ (for if $v(1)<0<v(2)$ then $v(-1)>0$ so, by (3), $v(2)=v(-1)+1$ and hence $v(1)+v(2)=1>0$, and similarly if $v(1)>0>v(2)$). We have two cases to consider.

(a) $us<u$. Then $u(1), u(2)<0$. Therefore $N(u,v)=N(us,vs)$, $b_j(u,v)=b_j(us,vs)$ for all $j\in[n]$, and hence $D(u,v)=D(us,vs)$, and $1,2\notin D(u,v)$, and we conclude as in the corresponding case for $(S_n^B)^{\mathcal{A}_{n-1}}$.

(b) $u<us$. Then, by Proposition 2.7, $u(1)+u(2)>0$. There are then three subcases to consider.

(i) $u(1)<0<u(2)$. Then, since $u\in(S_n^D)^{\mathcal{A}_{n-1}}$, $u(2)=1+u(-1)$ and hence $us\notin(S_n^D)^{\mathcal{A}_{n-1}}$. Therefore $N(u,vs)=(N(u,v)\setminus\{2\})\cup\{1\}$, $b_j(u,v)=b_j(u,vs)$ for all $j\in[n]\setminus\{1,2\}$, and

$b_2(u,v) = b_1(u,v) \equiv 0 \pmod 2$. So $1, 2 \notin D(u,v)$ and hence $D(u,vs) = D(u,v)$. Therefore we have from our induction hypothesis and Theorem 2.1 that

$$\begin{aligned} R_{u,v}(q) &= q R_{u,vs}(q) \\ &= q q^{l(u,vs)} \prod_{j \in D(u,vs)} \bigl(1 - q^{-\tilde{b}_j(u,vs)}\bigr) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \bigl(1 - q^{-\tilde{b}_j(u,v)}\bigr), \end{aligned}$$

as desired.

(ii) $u(1) > 0 > u(2)$. Then $u(2) = u(-1) + 1$ and hence $us \notin (S_n^D)^{\mathcal{A}_{n-1}}$. Therefore $N(u,vs) = (N(u,v) \setminus \{1\}) \cup \{2\}$, $b_j(u,v) = b_j(u,vs)$ for all $j \in [n] \setminus \{1,2\}$, and $b_2(u,vs) + 1 = b_1(u,v) - 1 \equiv 1 \pmod 2$. So $1, 2 \notin D(u,v)$ and hence $D(u,vs) = D(u,v)$, and we conclude as above.

(iii) $0 < u(1), u(2)$. Then $u(2) = u(1) + 1$ and hence $us \in (S_n^D)^{\mathcal{A}_{n-1}}$. Therefore $N(us,vs) = N(u,v)$, $N(u,vs) = N(u,v) \setminus \{1,2\}$, $b_j(u,v) = b_j(u,vs) = b_j(us,vs)$ for all $j \in [n] \setminus \{1,2\}$, and

$$b_2(u,vs) = b_2(u,v) - 1 = b_2(us,vs) + 1 = b_1(u,vs) \equiv 0 \pmod 2, \tag{12}$$

$$b_2(u,v) - 1 = b_1(u,v) - 2 = b_1(us,vs) + 2 \equiv 0 \pmod 2. \tag{13}$$

So $2 \in D(u,v)$, $D(us,vs) = D(u,v)$ and $D(u,vs) = D(u,v) \setminus \{1,2\}$. Hence, by induction

$$\begin{aligned} R_{u,vs}(q) &= q^{l(u,vs)} \prod_{j \in D(u,vs)} \bigl(1 - q^{-\tilde{b}_j(u,vs)}\bigr) \\ &= q^{l(u,v)-1} \prod_{j \in D(u,v)\setminus\{2\}} \bigl(1 - q^{-\tilde{b}_j(u,v)}\bigr), \end{aligned}$$

and, if $us \leqslant vs$,

$$\begin{aligned} R_{us,vs}(q) &= q^{l(us,vs)} \prod_{j \in D(us,vs)} \bigl(1 - q^{-\tilde{b}_j(us,vs)}\bigr) \\ &= q^{l(u,v)-2} \bigl(1 - q^{-\tilde{b}_2(u,v)+1}\bigr) \prod_{j \in D(u,v)\setminus\{2\}} \bigl(1 - q^{-\tilde{b}_j(u,v)}\bigr) \end{aligned}$$

so

$$\begin{aligned} R_{u,v}(q) &= (q-1) R_{u,vs}(q) + q R_{us,vs}(q) \\ &= q^{l(u,v)-1} \bigl(q - q^{-\tilde{b}_2(u,v)+1}\bigr) \prod_{j \in D(u,v)\setminus\{2\}} \bigl(1 - q^{-\tilde{b}_j(u,v)}\bigr) \\ &= q^{l(u,v)} \prod_{j \in D(u,v)} \bigl(1 - q^{-\tilde{b}_j(u,v)}\bigr) \end{aligned}$$

as desired. If $us \nleqslant vs$ then, by (12), (13) and the comments at the beginning of this section, $b_2(u,v)=1$, hence

$$\begin{aligned} R_{u,v}(q) &= (q-1)R_{u,vs}(q) \\ &= q^{l(u,v)-1}(q-1)\prod_{j\in D(u,v)\setminus\{2\}}\bigl(1-q^{-\tilde{b}_j(u,v)}\bigr) \\ &= q^{l(u,v)}\prod_{j\in D(u,v)}\bigl(1-q^{-\tilde{b}_j(u,v)}\bigr), \end{aligned}$$

and the result again follows.

This concludes the induction step and hence the proof. □

We illustrate the preceding theorem with an example. Suppose $W^J=(S_7^B)^{\mathcal{A}_6}$, $u=[3,4,5,6,-2,7,-1]$ and $v=[-4,5,-3,6,7,-2,-1]$. Then $(\tilde{b}_1(u,v),\dots,\tilde{b}_7(u,v))=(2,1,1,0,1,1,1)$ so by Proposition 2.4 and Theorem 3.1 we have that $R_{u,v}^{J,-1}(q)=q^{17-12}(1-q^{-1})^2$. On the other hand, if $W^J=(S_7^D)^{\mathcal{A}_6}$ then $R_{u,v}^{J,-1}(q)=q^{13-10}(1-q^{-1})^2$.

We now consider the Hermitian symmetric pairs $(S_n^B,\mathcal{B}_{n-1})$ and $(S_n^D,\mathcal{D}_{n-1})$.

Theorem 3.2. *Let $u,v\in(S_n^B)^{\mathcal{B}_{n-1}}$, $u<v$. Then*

$$R_{u,v}^{J,-1}(q)=q^{l(u,v)}\bigl(1-q^{-1}\bigr). \tag{14}$$

Proof. We proceed by induction on $l(v)$, the result being clear if $l(v)=1$. So assume that $l(v)\geqslant 2$ and let $u<v$. If $l(u,v)=1$ then the result is easy to check, so assume $l(u,v)\geqslant 2$. Let $s\in D(v)$. Then from Theorem 2.1 and our induction hypothesis we conclude that

$$R_{u,v}^{J,-1}(q)=\begin{cases} q^{l(us,vs)}(1-q^{-1}), & \text{if } us<u, \\ (q-1)q^{l(u,vs)}(1-q^{-1})+qq^{l(us,vs)}(1-q^{-1}), & \text{if } u<us\in(S_n^B)^{\mathcal{B}_{n-1}}, \\ qq^{l(u,vs)}(1-q^{-1}), & \text{if } u<us\notin(S_n^B)^{\mathcal{B}_{n-1}} \end{cases}$$

(note that, if $us\in(S_n^B)^{\mathcal{B}_{n-1}}$, then $us<vs$ since $(S_n^B)^{\mathcal{B}_{n-1}}$ is a chain), and (14) follows. □

Recall the notation $v_{\varepsilon i}$ ($\varepsilon\in\{-1,1\}$, $i\in[n]$) introduced in Section 2 for the elements of $(S_n^D)^{\mathcal{D}_{n-1}}$.

Theorem 3.3. *Let $u,v\in(S_n^D)^{\mathcal{D}_{n-1}}$, $u<v$. Then*

$$R_{u,v}^{J,-1}(q)=\begin{cases} q^{l(u,v)}(1-q^{-1})(1-q^{-\frac{l(u,v)}{2}}), & \textit{if } l(u)+l(v)=2n-2, \\ q^{l(u,v)}(1-q^{-1}), & \textit{otherwise.} \end{cases}$$

Proof. We proceed by induction on $l(v)$, the result being clear if $l(v)=1$. So assume that $l(v)\geqslant 2$. If $[u,v]^{\mathcal{D}_{n-1}}$ is a chain then the result follows similarly as in the proof of Theorem 3.2.

If $[u,v]^{\mathcal{D}_{n-1}}$ is not a chain then $u = v_i$ and $v = v_{-j}$ for some $i, j \in [2,n]$. Since $s_{j-1} \in D(v_{-j})$ and $D(v_i) = \{s_i\}$ we conclude from Theorem 2.1 that

$$R_{v_i,v_{-j}}(q) = \begin{cases} R_{v_{i+1},v_{-j+1}}(q), & \text{if } j = i+1, \\ (q-1)R_{v_i,v_{-j+1}}(q) + qR_{v_{i-1},v_{-j+1}}(q), & \text{if } j = i, \\ qR_{v_i,v_{-j+1}}(q), & \text{otherwise,} \end{cases}$$

and the result follows from our induction hypothesis (note that $v_{i-1} \leqslant v_{-i+1}$ if and only if $i \neq 2$). $\square$

The results in this section, together with those in [4, §3] and Proposition 2.3, complete the computation of the parabolic R-polynomials of Hermitian symmetric pairs.

A consequence of these results is the following one, which implies that all the roots of these polynomials are (either zero or) roots of unity.

Corollary 3.4. *Let (W, W_J) be a Hermitian symmetric pair, and $u, v \in W^J$, $u < v$. Then there exist $\tilde{b}_1(u,v), \ldots, \tilde{b}_r(u,v) \in \mathbf{P}$ ($r = r(u,v) \in \mathbf{P}$) such that*

$$R_{u,v}^{J,q}(q) = (-1)^{l(u,v)} \prod_{i=1}^{r} \bigl(1 - q^{\tilde{b}_i(u,v)}\bigr) \tag{15}$$

and

$$R_{u,v}^{J,-1}(q) = q^{l(u,v)} \prod_{i=1}^{r} \bigl(1 - q^{-\tilde{b}_i(u,v)}\bigr). \tag{16}$$

Proof. This follows immediately from Theorems 3.1–3.3, Proposition 2.3, the main result of [4, §3], and computer calculations. $\square$

The parabolic Kazhdan–Lusztig R-polynomials for the exceptional Hermitian symmetric pairs have been computed by implementing in Maple 6 the recursion given by Theorem 2.1. These implementations used two Maple packages for handling finite Coxeter groups (Version 2.3) and posets (Version 2.2) developed by John Stembridge.

It would be interesting to have a unified proof of this result, and a unified interpretation of the integers $\tilde{b}_1(u,v), \ldots, \tilde{b}_r(u,v)$ appearing in (15) and (16).

In the case of a lower interval, more can be said. For a finite Coxeter group W let $E(W) \subset \mathbf{P}$ be its set of exponents.

Corollary 3.5. *Let (W, W_J) be a Hermitian symmetric pair and $v \in W^J$. Then there exists $T(v) \subseteq E(W)$ such that*

$$R_{e,v}^{J,q}(q) = (-1)^{l(v)} \prod_{j \in T(v)} \bigl(1 - q^j\bigr)$$

and

$$R_{e,v}^{J,-1}(q) = q^{l(v)} \prod_{j \in T(v)} \bigl(1 - q^{-j}\bigr).$$

Proof. The result follows immediately from Corollary 3.5 of [4] for the Hermitian symmetric pairs of type A, from Theorems 3.2 and 3.3 if $W^J \in \{(S_n^B)^{\mathcal{B}_{n-1}}, (S_n^D)^{\mathcal{D}_{n-1}}\}$, from Theorem 3.1 if $W^J \in \{(S_n^B)^{\mathcal{A}_{n-1}}, (S_n^D)^{\mathcal{A}_{n-1}}\}$, and from computer calculations for the exceptional Hermitian symmetric pairs. □

It is an open problem, in the theory of the (ordinary) R-polynomials, to know if given $u, v \in W$ there exists $w \in W$ such that $R_{u,v}(q) = R_{e,w}(q)$ [1]. The last two results (and simple examples) show that, in general, this is false for the parabolic R-polynomials of Hermitian symmetric pairs.

As a further consequence of our main results we obtain combinatorial closed product formulas for certain sums and alternating sums of ordinary R-polynomials.

Corollary 3.6. *Let (W, W_J) be a Hermitian symmetric pair, and $u, v \in W^J$, $u < v$. Then there exist $\tilde{b}_1(u, v), \ldots, \tilde{b}_r(u, v) \in \mathbf{P}$ ($r = r(u, v) \in \mathbf{P}$) such that*

$$\sum_{w \in W_J} (-x)^{l(w)} R_{wu,v}(q) = (q - x - 1)^{l(u,v)} \prod_{i=1}^{r} \left(1 - \left(\frac{x^2}{q}\right)^{\tilde{b}_i(u,v)}\right),$$

for all $x \in \{-1, q\}$.

Proof. This follows immediately from Corollary 3.4 and Proposition 2.2. □

Note that the integers $\tilde{b}_1(u, v), \ldots, \tilde{b}_r(u, v)$ in Corollary 3.6 are explicitly determined in Theorems 3.1–3.3, and Theorem 3.1 of [4] (see also Corollary 3.8 below).

We conclude by showing that, for the Hermitian symmetric pairs $(S_n^B, \mathcal{A}_{n-1})$, $(S_n^D, \mathcal{A}_{n-1})$ and $(S_n, \mathcal{A}_{n-1} \setminus \{s_i\})$ ($n \geqslant 3$, $1 \leqslant i \leqslant n-1$) (the "interesting" Hermitian symmetric pairs, according also to [3, §1, p. 279]) there is a unified interpretation of the exponents appearing in Corollary 3.4.

Let $W^J \in \{(S_n^B)^{\mathcal{A}_{n-1}}, (S_{n+1}^D)^{\mathcal{A}_n}, (S_n)^{\mathcal{A}_{n-1}\setminus\{s_i\}}\}$ ($1 \leqslant i \leqslant n-1$) and $u \in W^J$. By (2), (4), and (2) of [4], and the comments following Propositions 2.5 and 2.8 of [4] we may associate to u a lattice path, which we will denote, for simplicity, by $\Lambda(u)$, with $(1, 1)$ and $(1, -1)$ steps, starting at $(0, 0)$ and having n steps. Furthermore, by Propositions 2.5, 2.8 and 2.8 of [4], we have that, for all $u, v \in W^J$, $u \leqslant v$ if and only if $\Lambda(u)$ lies (weakly) below $\Lambda(v)$ (write $\Lambda(u) \leqslant \Lambda(v)$, if this is the case).

Let λ, μ be two such lattice paths, with $\mu \leqslant \lambda$. Let $j \in [n]$ and consider the jth step of λ (from the left). We say that such a step is *shifted-allowable* (or, *s-allowable*, for short) with respect to μ if the jth step of μ is not parallel to it, and $\tilde{a}_j(\mu, \lambda)$ is odd, where $\tilde{a}_j(\mu, \lambda)$ is the vertical distance (divided by two, since it is always even) between the (right end of the) jth step of λ and the (right end of the) jth step of μ. Let $|\lambda \setminus \mu| \stackrel{\text{def}}{=} \sum_{j=1}^{n} \tilde{a}_j(\mu, \lambda)$.

Proposition 3.7. *Let $u, v \in (S_n^B)^{\mathcal{A}_{n-1}}$, $u \leqslant v$. Then*

$$b_i(u, v) = \tilde{a}_{n+1-i}\big(\Lambda_B(u), \Lambda_B(v)\big), \tag{17}$$

for $i = 1, \ldots, n$. Furthermore $n + 1 - i \in D(u, v)$ if and only if the ith step of $\Lambda_B(v)$ is s-allowable with respect to $\Lambda_B(u)$.

Proof. Let $i \in [n]$. Clearly, the vertical height of the path $\Lambda_B(v)$ after i steps, $h_i(\Lambda_B(v))$, equals the difference between the number of up-steps and that of down-steps among the first i steps of $\Lambda_B(v)$. But, by Proposition 2.6, the jth step of $\Lambda_B(v)$ is an up-step if and only if $v(n+1-j) < 0$. Therefore

$$\begin{aligned} h_i\big(\Lambda_B(v)\big) &= \big|\{j \in [i]\colon v(n+1-j) < 0\}\big| - \big|\{j \in [i]\colon v(n+1-j) > 0\}\big| \\ &= 2\big|\{j \in [i]\colon v(n+1-j) < 0\}\big| - i. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{a}_i\big(\Lambda_B(u), \Lambda_B(v)\big) &= \frac{1}{2}\big(h_i\big(\Lambda_B(v)\big) - h_i\big(\Lambda_B(u)\big)\big) \\ &= \big|\{j \in [i]\colon v(n+1-j) < 0\}\big| - \big|\{j \in [i]\colon u(n+1-j) < 0\}\big| \\ &= b_{n+1-i}(u, v). \end{aligned}$$

Furthermore, again by Proposition 2.6, $n+1-i \in N(u,v)$ if and only if the ith steps of $\Lambda_B(v)$ and $\Lambda_B(u)$ are not parallel, and the result follows from the definition of $D(u,v)$ and (17). $\square$

We can now give the following unified interpretation of the exponents appearing in Corollary 3.4.

Corollary 3.8. *Let $W^J \in \{(S_n^B)^{\mathcal{A}_{n-1}}, (S_{n+1}^D)^{\mathcal{A}_n}, (S_n)^{\mathcal{A}_{n-1}\setminus\{s_i\}}\}$ for some $1 \leqslant i \leqslant n-1$ and $u, v \in W^J$, $u \leqslant v$. Then*

$$R_{u,v}^{J,q}(q) = (-1)^{|\lambda\setminus\mu|} \prod_j \big(1 - q^{\bar{a}_j(\mu,\lambda)}\big)$$

where $\mu \stackrel{\text{def}}{=} \Lambda(u)$, $\lambda \stackrel{\text{def}}{=} \Lambda(v)$, j runs over all the s-allowable steps of λ with respect to μ, and

$$\bar{a}_j(\mu,\lambda) \stackrel{\text{def}}{=} \begin{cases} \tilde{a}_j(\mu,\lambda), & \text{if the } j\text{th step of } \mu \text{ is down,} \\ \tilde{a}_j(\mu,\lambda) + 1, & \text{if the } j\text{th step of } \mu \text{ is up.} \end{cases} \tag{18}$$

Proof. The result follows immediately from Theorem 3.1, and Propositions 2.3 and 3.7 if $W^J = (S_n^B)^{\mathcal{A}_{n-1}}$. If $W^J = (S_{n+1}^D)^{\mathcal{A}_n}$ then it follows similarly using the facts that if $v \in (S_{n+1}^D)^{\mathcal{A}_n}$ then $\Lambda_D(v)$ (seen as a lattice path) consists of the first n steps of $\Lambda_B(v)$, and $1 \notin D(u,v)$ if $u, v \in (S_{n+1}^D)^{\mathcal{A}_n}$.

Finally, suppose $W^J = (S_n)^{\mathcal{A}_{n-1}\setminus\{s_i\}}$ for some $i \in [n-1]$. Let $[n]_-^+(\mu,\lambda)$ be the set of all $j \in [n]$ such that the jth step of λ is up and the jth step of μ is down, and define $[n]_+^-(\mu,\lambda)$ similarly. Then, by Corollary 3.4 of [4],

$$R_{u,v}^{J,q}(q) = (-1)^{|\lambda\setminus\mu|} \prod_{j \in [n]_-^+(\mu,\lambda)} \big(1 - q^{\tilde{a}_j(\mu,\lambda)}\big). \tag{19}$$

On the other hand, by (18)

$$\prod_j \left(1 - q^{\bar{a}_j(\mu,\lambda)}\right) = \prod_j \left(1 - q^{\tilde{a}_j(\mu,\lambda)}\right) \prod_j \left(1 - q^{\tilde{a}_j(\mu,\lambda)+1}\right) \tag{20}$$

where the first (respectively, second, third) product is over all $j \in [n]_-^+(\mu,\lambda) \cup [n]_+^-(\mu,\lambda)$ (respectively, $[n]_-^+(\mu,\lambda)$, $[n]_+^-(\mu,\lambda)$) such that $\tilde{a}_j(\mu,\lambda) \equiv 1 \pmod 2$. Because μ and λ end at the same point (see the comments following Proposition 2.8 in [4]) $\tilde{a}_n(\mu,\lambda) = 0$ so there is a bijection $j \mapsto j'$ from $[n]_-^+(\mu,\lambda)$ to $[n]_+^-(\mu,\lambda)$ such that $\tilde{a}_j(\mu,\lambda) = \tilde{a}_{j'-1}(\mu,\lambda)$ for all $j \in [n]_-^+(\mu,\lambda)$ (note that $1 \notin [n]_+^-(\mu,\lambda)$ since $\mu \leqslant \lambda$). Hence

$$\prod_j \left(1 - q^{\tilde{a}_j(\mu,\lambda)+1}\right) = \prod_j \left(1 - q^{\tilde{a}_{j-1}(\mu,\lambda)}\right) = \prod_j \left(1 - q^{\tilde{a}_j(\mu,\lambda)}\right) \tag{21}$$

where the first two products are over all $j \in [n]_+^-(\mu,\lambda)$ such that $\tilde{a}_j(\mu,\lambda) \equiv 1 \pmod 2$ while the third product is over all $j \in [n]_-^+(\mu,\lambda)$ such that $\tilde{a}_j(\mu,\lambda) \equiv 0 \pmod 2$. Therefore, by (20) and (21)

$$\prod_j \left(1 - q^{\bar{a}_j(\mu,\lambda)}\right) = \prod_{j \in [n]_-^+(\mu,\lambda)} \left(1 - q^{\tilde{a}_j(\mu,\lambda)}\right)$$

where the first product is over all $j \in [n]_-^+(\mu,\lambda) \cup [n]_+^-(\mu,\lambda)$ such that $\tilde{a}_j(\mu,\lambda) \equiv 1 \pmod 2$, and the result follows from (19). □

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