Journal of Algebra 320 (2008) 3199–3205

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Structure of prime finitely presented monomial algebras

Jan Okniński

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 18 December 2007
Available online 26 August 2008
Communicated by J.T. Stafford

Keywords:
Monomial algebra
Finitely presented
Primitive algebra
Semigroup algebra

ABSTRACT

The structure of a finitely presented monomial algebra $K[X]/K[I]$ over a field K is described. Here X is a finitely generated free monoid and I is a prime ideal of X that is finitely generated. As an application, a new structural proof of the recent result of Bell and Pekcagliyan [J. Bell, P. Pekcagliyan, Primitivity of finitely presented monomial algebras, preprint, arXiv: 0712.0815v1] on the primitivity of such algebras is presented, which yields a positive solution to the trichotomy problem, raised by Bell and Smoktunowicz [J. Bell, A. Smoktunowicz, The prime spectrum of algebras of quadratic growth, J. Algebra 319 (2008) 414–431], in the finitely presented case. Our approach is based on a new result on the form of prime Rees factors of semigroups satisfying the ascending chain condition on one-sided annihilators and on its refinement in the case of finitely presented factors of the form X/I.

The main motivation for this paper is to study primitivity of algebras of the form $R = K[X]/K[I]$ for an ideal I of a finitely generated free monoid X. It is well known that the free algebra $K[X]$ is left and right primitive, see [10], Corollary 11.26. We know also that R is a prime ring if and only if the Rees factor semigroup X/I is a prime semigroup, [11], Proposition 24.2. Surprisingly, one can prove that such an algebra is either left and right primitive or it satisfies a polynomial identity, provided that the ideal I is finitely generated, [2]. We present a new proof of this theorem. Our approach is different than the one presented in [2], which is based on automaton algebras. We first describe the structure of the underlying monoid X/I, see Theorem 2. It is based on a 'matrix pattern' associated to X/I. The key observation that makes such a description possible is that there are only finitely many right (and left) annihilators of elements of X/I, viewed as a submonoid of $K[X]/K[I]$. Secondly, there are certain free submonoids of X canonically associated to X/I. It turns out that, from this structural point of view, the only difference between the case where $K[X]/K[I]$ is primitive and the case where $K[X]/K[I]$ satisfies a polynomial identity is that these free monoids are of rank

E-mail address: okninski@mimuw.edu.pl.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.08.003

one in the latter case. In order to prove the main theorem we start with a result on arbitrary prime Rees factors of semigroups that satisfy certain finiteness conditions on annihilator ideals (Theorem 1), much weaker than the finiteness of the set of all such ideals. This is an extension of certain very useful results on prime Rees factors of cancellative semigroups satisfying some stronger finiteness conditions, [8], Section 4.5, and seems to be of independent interest. It is worth mentioning that the recent interest in monomial algebras, especially prime and primitive, stems from a study of various problems concerning the Gelfand–Kirillov dimension, [3,14,15]. Notice also that an explicit characterization of the radical of arbitrary monomial algebras was given in [4] and [1].

For the basic results on semigroups used in this paper we refer to [9]. Recall that, for given nonempty sets Z, Y, a group G and a $Y \times Z$ matrix $Q = (q_{yz})$ with entries in $G \cup \{\theta\}$ (group G with a zero element adjoined), the semigroup of matrix type $\mathcal{M}(G, Z, Y; Q)$ consists of all triples (g, z, y) with $g \in G, z \in Z, y \in Y$, and the zero element 0, subject to the operation $(g, z, y)(g', z', y') = (gq_{yz'}g', z, y')$ if $q_{yz'} \in G$ and $(g, z, y)(g', z', y') = 0$ if $q_{yz'} = \theta$. We will use the same notation for θ and 0, if unambiguous. If every row and every column of Q contains a nonzero element then $\mathcal{M}(G, Z, Y; Q)$ is called a completely 0-simple semigroup, [9]. Such semigroups yield a fundamental building block of semigroup theory. These are exactly semigroups (with zero) that have no nontrivial ideals and have a primitive idempotent.

First, consider a semigroup T such that there exist sets Y, Z such that T can be presented as a disjoint union $T = \bigcup_{y \in Y, z \in Z} T_{zy} \cup \{0\}$ with all T_{zy} nonempty and for every $z, z' \in Z$, $y, y' \in Y$ either $T_{zy}T_{z'y'} \subseteq T_{zy'}$ or $T_{zy}T_{z'y'} = 0$. Assume also that T is a prime semigroup. Then we claim that $T_{zy}T_{z'y'} = 0$ if and only if $T_{xy}T_{z'y'} = 0 = T_{zy}T_{z'u}$ for every $x \in Z, u \in Y$. Indeed, suppose for example that $T_{zy}T_{z'y'} = 0$ but $T_{xy}T_{z'y'} \neq 0$. Then $T_{zv}T_{xy}T_{z'y'} \subseteq (T_{zy} \cup \{0\})T_{z'y'} = 0$ for every $v \in Y$. So $T_{xy}T_{z'y'}$ annihilates on the right the right ideal $\bigcup_{v \in Y} T_{zv} \cup \{0\}$ of T. Since T is prime, we must have $T_{xy}T_{z'y'} = 0$. The second part of the claim is proved in a similar way.

Therefore, in this case we get a homomorphism $f : T \to \mathcal{M}(\{1\}, Z, Y; Q)$ onto the completely 0-simple semigroup over the trivial group with the sandwich matrix Q defined by $q_{yz} = 1$ if $T_{xy}T_{zu} \subseteq T_{xu}$ and $q_{yz} = 0$ if $T_{xy}T_{zu} = 0$ (for all $x \in Z, u \in Y$). In this case, we say that T is a semigroup with a completely 0-simple pattern and the sets $T_{zy}, z \in Z, y \in Y$, are called the components of T.

For a semigroup S with an ideal P we often identify the nonzero elements of the Rees factor S/P with the elements of S not contained in P. If $a \in S/P$ is nonzero, then we define $\mathrm{r}_{S/P}(a) = \{x \in S \mid ax \in P\}$. This is a right ideal of S containing P and can be called the right annihilator of a in S/P. Similarly, $\mathrm{l}_{S/P}(a)$ stands for the left annihilator of a in S/P.

Our first result reads as follows.

Theorem 1. *Let P be a prime ideal of a semigroup S. Assume that S satisfies the ascending chain condition on right ideals of the form $\mathrm{r}_{S/P}(x), x \in S \setminus P$. Then S has an ideal J properly containing P such that J/P has a completely 0-simple pattern.*

Proof. Let

$$J = \{a \in S \mid xay \in P,\ x, y \in S,\ \text{implies } xa \in P \text{ or } ay \in P\}.$$

Clearly $P \subseteq J$. Let $a \in J$ and $x, y, z \in S$ and assume that $xzay \in P$. Then $xza \in P$ or $ay \in P$, because $a \in J$. Hence $xza \in P$ or $zay \in P$. So $za \in J$. This shows that J is a left ideal in S. A symmetric argument allows us to show that J is an ideal of S.

If $S \setminus P$ is a subsemigroup of S, then $J = S$ and the assertion trivially follows. So assume that S/P has nonzero zero divisors. Now $J \neq P$ because elements $a \in S \setminus P$ with maximal $\mathrm{r}(a) = \mathrm{r}_{S/P}(a)$ are in J. It is clear that $J \setminus P$ coincides with the set of all elements $a \in S \setminus P$ such that $\mathrm{r}(a) = \mathrm{r}(xa)$ if $xa \notin P, x \in S$. Similarly, if we define $\mathrm{l}(b) = \mathrm{l}_{S/P}(b)$ for $b \in S \setminus P$, then $J \setminus P$ consists of all $b \in S \setminus P$ such that $\mathrm{l}(b) = \mathrm{l}(by)$ whenever $by \notin P, y \in S$.

Let ρ be the relation on S defined by $(a, b) \in \rho$ if either $a = b$ or $a, b \in J$ and $\mathrm{r}(a) = \mathrm{r}(b)$ and $\mathrm{l}(a) = \mathrm{l}(b)$. We claim that ρ is a congruence on S. Clearly, this is an equivalence relation on S. So suppose that $a, b \in J$ are such that $\mathrm{r}(a) = \mathrm{r}(b)$ and $\mathrm{l}(a) = \mathrm{l}(b)$. Let $x \in S$ be such that $ax \neq 0$ in S/P.

Then we also have $bx \neq 0$. Hence $\mathrm{l}(ax) = \mathrm{l}(a) = \mathrm{l}(b) = \mathrm{l}(bx)$. Moreover $axy = 0$ if and only if $bxy = 0$ for $y \in S$ because $\mathrm{r}(b) = \mathrm{r}(a)$. This means that $(ax, bx) \in \rho$. Similarly one shows that $(xa, xb) \in \rho$. It follows that ρ is a congruence on S.

Suppose that for every $a \in J/P$ we have $a^2 = 0$. If $x \in S$ then $axax = 0$ in S/P and the definition of J implies that $axa = 0$. Then $a(S/P)a = 0$ and primeness of S/P implies that $a = 0$. Since $a \in J/P$ was arbitrary, we get that $J = P$, a contradiction. So, choose an element $e \in J/P$ such that $e^2 \neq 0$. Consider the set $Y \subseteq S$ such that $\mathrm{r}(ey)$, $y \in Y$, are all distinct right annihilators of nonzero elements of the form ex for $x \in S/P$. Similarly, choose a set $Z \subseteq S$ such that $\mathrm{l}(ze)$, $z \in Z$, are all distinct left annihilators of nonzero elements of the form xe for $x \in S/P$.

Let $b \in J/P, b \neq 0$. Then $bxex'b \neq 0$ for some $x, x' \in S$ because S/P is prime. So $\mathrm{l}(b) = \mathrm{l}(bxe)$ and $\mathrm{r}(b) = \mathrm{r}(ex'b)$. Hence $\mathrm{l}(b) = \mathrm{l}(ze)$ and $\mathrm{r}(b) = \mathrm{r}(ey)$ for some $z \in Z$ and $y \in Y$. Then $zeey \neq 0$ by the definition of J and we get $\mathrm{r}(b) = \mathrm{r}(ey) = \mathrm{r}(zeey)$ and $\mathrm{l}(b) = \mathrm{l}(ze) = \mathrm{l}(zeey)$. So $(b, zeey) \in \rho$. Clearly, all elements $zeey$, where $y \in Y$ and $z \in Z$, are in different ρ-classes. Therefore we have a decomposition $J/P = \bigcup_{z \in Z, y \in Y} J_{zy} \cup \{0\}$, where

$$J_{zy} = \bigl\{ b \in J \setminus P \bigm| \mathrm{r}(b) = \mathrm{r}(ey),\ \mathrm{l}(b) = \mathrm{l}(ze) \bigr\}$$

are nonempty sets.

Suppose $a \in J_{zy}, b \in J_{z'y'}$ are such that $ab \neq 0$ in J/P. Also let $c \in J_{xy}$. Then $\mathrm{r}(a) = \mathrm{r}(c) = \mathrm{r}(ey)$. So $cb \neq 0$. Since $b, c \in J$, we get $\mathrm{r}(cb) = \mathrm{r}(b) = \mathrm{r}(ey')$ and $\mathrm{l}(cb) = \mathrm{l}(c) = \mathrm{l}(xe)$. Therefore $cb \in J_{xy'}$ and hence $J_{xy} b \subseteq J_{xy'}$. Then, a symmetric argument allows us to prove that $J_{xy} J_{z'y''} \subseteq J_{xy''}$ for every $y'' \in Y$.

It is now clear that $(J/P)/\rho \cong \mathcal{M}(\{1\}, Z, Y; Q)$, a semigroup of matrix type over a trivial group with the sandwich matrix $Q = (q_{yz})$ defined by: $q_{yz} = 1$ if $J_{xy} J_{zy'} \subseteq J_{xy'}$ and $q_{yz} = 0$ if $J_{xy} J_{zy'} = 0$ (these conditions do not depend on the choice of $x \in X$ and $y' \in Y$). It is completely 0-simple because J/P is a prime semigroup. This proves the assertion. $\square$

The above theorem has the flavor of classical results on prime rings R satisfying certain finiteness conditions (such as, for example, prime Goldie rings). Namely, a matrix structure is also associated to any such ring R. Our main results, Theorem 2 and Theorem 4, heavily depend on a significant improvement of Theorem 1 that is possible in the case of finitely presented monomial algebras.

Theorem 1 is an extension of the fact that certain prime homomorphic images of cancellative semigroups are monomial semigroups over some groups, see [8], Section 4.5, or [13]. However, in that case, a stronger finiteness condition has to be assumed on S, and as a result one shows that the ideal J/P has a pattern determined by a Brandt semigroup and the components J_{zy} of J/P such that $q_{yz} \neq 0$ are cancellative subsemigroups of J/P. (A Brandt semigroup is a semigroup of the form $\mathcal{M}(\{1\}, Z, Z; Q)$, where Z is a set and Q is the identity $Z \times Z$-matrix.) Actually, J/P is an order in a completely 0-simple inverse semigroup in the sense of [7].

One of our main motivating classes for the above result consists of a broad class of Rees factors of free monoids. In this case, one can prove a stronger assertion on the factor J/P arising from Theorem 1. By X we denote a finitely generated free monoid. If $w \in X$ then $|w|$ stands for the length of w.

Theorem 2. *Let I be a finitely generated ideal of X. Then X satisfies the ascending chain condition on right and left ideals of the form $\mathrm{r}_{X/I}(x), \mathrm{l}_{X/I}(x)$ for $x \in X \setminus I$. Moreover, assume that X/I is prime and J, ρ are the ideal of $S = X$ and the congruence on J/P constructed for $P = I$ as in Theorem* 1. *Then*

(1) *J contains the ideal J' of X consisting of all elements of length at least $N = \max\{|w_i| - 1 \mid i = 1, \ldots, m\}$, where $w_1, \ldots, w_m$ is a set of generators of the ideal I,*
(2) *if $J_0 = (J')^2 \cup I$ then J_0/I embeds into a completely* 0*-simple semigroup with the same pattern as J/I,*
(3) *the completely* 0*-simple semigroup $(J/I)/\rho$ is finite.*

Proof. We may assume that $w_1, \ldots, w_m$ form a minimal set of generators of the ideal I. Let $s \in X \setminus I$. Then s is a left zero divisor in X/I if and only if there exist $w, t \in X$ such that $s = wt$, $w \neq 1$ and

$w_i = uw$ for some i and some $u \in X, u \neq 1$. So $\mathrm{l}_{X/I}(s)$ contains Xu, and actually $\mathrm{l}_{X/I}(s)$ must be then a union of principal left ideals of the form Xu, where u is a proper initial segment of some w_i. So, there are only finitely many left annihilator ideals in X/I. A symmetric argument applies to the right annihilators. Therefore the first assertion follows.

Now, assume that I is a prime ideal of X. Then we get the ideal $J = \{w \in X \mid xwy \in I,\ x, y \in X$, implies $xw \in I$ or $wy \in I\}$ of X constructed in Theorem 1. Clearly, if the length of w is at least $|w_i| - 1$ for $i = 1, \dots, m$, then $w \in J$. Hence, assertion (1) follows.

We will look more closely at the annihilators of elements in X/I. Let A be the set of all elements $a \in X$ such that $w_i \in aX$ and $w_i \neq a$ for some $i \in \{1, \dots, m\}$. Similarly, let B be the set of all elements $b \in X$ such that $w_i \in Xb$ and $w_i \neq b$ for some $i \in \{1, \dots, m\}$. Notice that $A \cap I = \emptyset = B \cap I$ by the minimality of the chosen generating set of I. For every $a \in A$ define the set

$$R_a = \bigl\{w \in J \setminus I \bigm| w \in Xa,\ \ w \notin Xa' \text{ for every } a' \in A \text{ with } |a'| > |a|\bigr\}.$$

In particular, $R_1 = \{w \in J \setminus I \mid wx \notin I$ for every $x \in X \setminus I\}$, the set of all elements in J/I that are not left zero divisors in X/I. Define also

$$L_b = \bigl\{w \in J \setminus I \bigm| w \in bX,\ \ w \notin b'X \text{ for every } b' \in B \text{ with } |b'| > |b|\bigr\}.$$

It is easy to see that elements of a given nonempty set R_a have the same right annihilators and elements from different $R_a, a \in A$, have different right annihilators in X/I. Similarly, the nonempty sets $L_b, b \in B$, correspond to all different left annihilators of nonzero elements of J/I. Let $A_0 = \{a \in A \mid R_a \neq \emptyset\}$ and $B_0 = \{b \in B \mid L_b \neq \emptyset\}$. Define $T_{ba} = R_a \cap L_b$. Since X/I is prime, it follows that $L_b X R_a \not\subseteq I$ for $a \in A_0, b \in B_0$. Clearly $L_b X R_a \subseteq T_{ba}$ and so $T_{ba} \neq \emptyset$. Hence, we get a disjoint union decomposition

$$J/I = \bigcup_{a \in A_0, b \in B_0} T_{ba} \cup \{0\}.$$

From the proof of Theorem 1 we know that this determines a completely 0-simple pattern on the factor J/I. Consider the semigroup of matrix type $M = \mathcal{M}(X, B_0, A_0; Q)$ with the sandwich matrix $Q = (q_{ab})$ defined by $q_{ab} = ab$ if $ab \notin I$ and $q_{ab} = 0$ otherwise. For every $w \in T_{ba} \cap J_0$ we put $\phi(w) = (u, b, a)$, where $w = bua$ for some $u \in X$. If $w' = b'u'a' \in T_{b'a'} \cap J_0$ is such that $ww' \notin I$ then

$$\phi(ww') = \phi(buab'u'a') = \phi\bigl(b(uab'u')a'\bigr) = (uab'u', b, a') = \phi(w)\phi(w').$$

On the other hand, if $ww' \in I$ then we define $\phi(ww') = 0$. In this case, some w_i is a subword of $buab'u'a'$ (but not of bua or $b'u'a'$). Then, by the definition of R_a, this occurrence of w_i is contained in the subword $ab'u'a'$ of $buab'u'a'$. Similarly, by the definition of $L_{b'}$, every occurrence of w_i is contained in the subword $buab'$ of $buab'u'a'$. Therefore w_i is a subword of ab'. Hence $ab' \in I$. Therefore we also get $\phi(w)\phi(w') = 0 = \phi(ww')$ in this case. So $\phi : J_0/I \to M$ is a homomorphism. It is clear that ϕ defines an embedding of J_0/I into M.

Now, $M \subseteq M' = \mathcal{M}(G, B_0, A_0; Q)$ where G is the free group generated by X. Since X/I is prime, J_0/I also is prime. Therefore the sandwich matrix Q has no zero rows or columns. Hence M' is a completely 0-simple semigroup. Since J_0/I intersects every $\mathcal{H}$-class of M', we get that J_0/I has the same completely 0-simple pattern as J/I. Finally, (3) follows because A_0, B_0 are finite sets. □

Notice that the semigroup J_0/I constructed above is a uniform subsemigroup of $M' = \mathcal{M}(G, B_0, A_0; Q)$ in the sense of [12], Section 3.1. This means that J_0/I intersects nontrivially every $\mathcal{H}$-class $\{(g, b, a) \mid g \in G\}$ of M'.

Example 3. We determine the completely 0-simple pattern on an ideal of the prime semigroup X/P, where $X = \langle x, y, z\rangle$ is a free monoid of rank 3 and $P = XxyzX \cup XyzxX \cup XzxyX$.

Proof. Suppose that $w, v \in X \setminus P$ are such that $wXv \subseteq P$. We may assume that $w, v \neq 1$. Let a be the terminal letter in w and let b be the initial letter in v. Then $wabv$ does not have any of xyz, yzx, zxy as a subword, whence $wabv \notin P$. This contradiction shows that P is indeed a prime ideal of X.

Let $J = \{w \in X \mid swt \in X, s, t \in X,$ implies $sw \in P$ or $wt \in P\}$, the ideal of X defined as in the proof of Theorem 1. Then $X \setminus J = \{x, y, z, 1\}$. Moreover $e = x^2$ is a non-nilpotent in J/P. The right ideals of the form $\mathrm{r}(ea), a \in X$, are:

$$xyX = \mathrm{r}(ez), \quad yzX = \mathrm{r}(ex), \quad zxX = \mathrm{r}(ey),$$

$$xX = \mathrm{r}(eyyz), \quad yX = \mathrm{r}(exz), \quad zX = \mathrm{r}(exy).$$

The left ideals of the form $\mathrm{l}(be), b \in X$, are:

$$Xxy = \mathrm{l}(ze), \quad Xyz = \mathrm{l}(xe), \quad Xzx = \mathrm{l}(ye),$$

$$Xx = \mathrm{l}(yzze), \quad Xy = \mathrm{l}(zxe), \quad Xz = \mathrm{l}(xye).$$

For $u, v \in V = \{x, y, z, xy, yz, zx\}$ we define the sets

$$S_{u,v} = \bigl\{w \in J \setminus P \bigm| \mathrm{l}(w) = \mathrm{l}(u),\ \mathrm{r}(w) = \mathrm{r}(v)\bigr\}.$$

Then it is easy to see that $J/P = \bigcup_{u,v \in V} S_{u,v} \cup \{0\}$, a disjoint union of nonempty sets. Let $\mathcal{M}(\{1\}, 6, 6; Q)$ be the completely 0-simple semigroup over the trivial group, where the sandwich matrix Q is defined by

$$Q = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that there is a natural homomorphism $J/P \to M$, where the matrix pattern on J/P is determined by the following ordering of the set V indexing the rows and the columns: x, y, z, xy, yz, zx. $\square$

We note that a faithful irreducible left $K[X]/K[P]$-module for X and P as in the above example (actually, in a slightly more general class of examples) has been explicitly constructed in [6].

The following trichotomy conjecture was stated in [3]: if A is a prime finitely generated monomial algebra, then A is either primitive or it satisfies a polynomial identity or it has a nonzero Jacobson radical. In a recent paper, Bell and Pekcagliyan [2] show that the answer is positive for finitely presented monomial algebras and moreover that the third possibility can be dropped. As an application of Theorem 2 we are able to give another, more structural, proof of this result.

Using the notation of Theorem 2, let $T = J/I$ and let C be a cancellative component of T, so in other words it is one of the sets $T_{ba}, b \in B_0, a \in A_0$, such that $T_{ba}T_{ba} \subseteq T_{ba}$. Say, $C = T_{dc}$. Then define

$$F = \bigl\{w \in T_{da} \bigm| a \in A_0,\ w^2 \notin I\bigr\}.$$

In other words, F is the union of all $T_{da}, a \in A_0$, that are subsemigroups of J. Let

$$F' = \{w \in X \mid wF \subseteq F,\ Fw \subseteq F\}.$$

Then F' is a submonoid of X and F is an ideal of F'. Moreover $C \subseteq F$. Let

$$F'' = \{w \in X \mid wF' \cap F' \neq \emptyset,\ \ F'w \cap F' \neq \emptyset\}.$$

We claim that $F'' = F'$. Clearly, $F' \subseteq F''$. So, assume that $w \in X$ is such that $wF' \cap F' \neq \emptyset$ and $F'w \cap F' \neq \emptyset$. Then $\emptyset \neq wF'F \cap F'F \subseteq wF \cap F$. Clearly, $wF \cap F$ is a right ideal in F. Since $T_{da}T_{da'} \subseteq T_{da'}$ for every $T_{da}, T_{da'}$ contained in F, it follows that $wF \cap F$ intersects nontrivially every component T_{da} of F.

Let $y \in F$. Say, $y \in T_{da}$. Then choose $x \in T_{da}$ such that $wx \in F$. As shown in the proof of Theorem 1, the relation ρ is a congruence on X. Therefore the element wy is in the same ρ-class as wx. This means that $wy \in F$ and hence $wF \subseteq F$. Similarly, one shows that $Fw \subseteq F$. Therefore $w \in F'$ and the claim $F'' = F'$ follows.

In view of a well known characterization of free submonoids of X, see [9], Proposition 5.2.2, this implies that F' is a free submonoid of X.

Theorem 4. *Let $R = K[X]/K[I]$ for a prime ideal I of a finitely generated free monoid X. Assume that I is a finitely generated ideal of X. Then R is left primitive and it is right primitive or R satisfies a polynomial identity. Moreover, X has an ideal J_0 such that X/J_0 is finite, $I \subseteq J_0$ and J_0/I embeds into a completely 0-simple semigroup $\mathcal{M}(G, Z, Y; Q)$ over a free group G, for some finite sets Z and Y. If R satisfies a polynomial identity then G is abelian and R is semiprimitive.*

Proof. Let T_{dc}, F, F' be chosen as in the comment preceding the theorem. We know that F' is a free monoid. Suppose first that it is not commutative. Then $K[F']$ is a left (and right) primitive ring, see [10], Corollary 11.26. Therefore also its ideal $K[F]$ is a left primitive ring.

Since the opposite algebra $(K[X]/K[I])^{op}$ also is monomial and finitely presented, it is enough to prove that $K[X]/K[I]$ is left primitive. Since I is a prime ideal of X, it is easy to see that $K[X]/K[I]$ is a prime ring, see [11], Proposition 24.2. We use the notation of the proof of Theorem 2. Then it is enough to prove that the ideal $K[J_0]/K[I]$ is a left primitive ring.

We know that $J_0/I = \bigcup_{b\in B_0, a\in A_0}(T_{ba} \cap J_0) \cup \{0\}$, is a uniform subsemigroup in a completely 0-simple semigroup $\mathcal{M}(G, B_0, A_0; Q)$. Write $S = J_0/I$ and $S_{ba} = T_{ba} \cap J_0$ for $a \in A_0, b \in B_0$. So $K[J_0]/K[I] = K_0[S]$, the contracted semigroup algebra of J_0/I.

Let $S_d = \bigcup_{a\in A_0} S_{da} \cup \{0\}$. So S_d is a right ideal of S and $E \subseteq S_d$ consists of non-nilpotents in S_d. Then $E = F \cap J_0$ is an ideal of F, whence $K[E]$ is left primitive. Thus, there exists a left ideal L of $K[E]$ such that $K[E]/L$ is a faithful irreducible $K[E]$-module. Moreover $(S_d \setminus E)S_d = 0$. Therefore $N = E \cup SE$ is a left ideal of S and $V = K_0[N] = K[E] + K_0[SE]$ is a left ideal of $K_0[S]$ such that $K_0[S]L \cap K[S_d] \subseteq L$. Also $V_0 = L + K_0[S]L$ is a left ideal of $K_0[S]$. Define $W = V/V_0$ and let $W_0 = \{v' \in W \mid Nv' = 0\}$. This is a submodule of W. We claim that W/W_0 is a faithful irreducible left $K_0[S]$-module.

Since $K[E] \cap V_0 = L$, it follows that $K[E]/L$ embeds into W. So we identify it with a submodule of W. Suppose that $v \in K[E]$ is such that its image $v' \in K[E]/L$ satisfies $Nv' = 0$. Then $Ev \subseteq L$. As $K[E]/L$ is irreducible as a $K[E]$-module, this implies that $v \in L$. Therefore $v' = 0$, which shows that $(K[E]/L) \cap W_0 = 0$ and $K[E]/L$ embeds into W/W_0.

Suppose that the annihilator D of W/W_0 in $K_0[S]$ is nonzero. Then $D \cap K[E]K_0[S]K[E] \neq 0$ because $K_0[S]$ is a prime ring. Since we also know that $(K[E]K_0[S])K[E] \subseteq K_0[S_d]K[E] \subseteq K[E]$, this contradicts the fact that $K[E]/L$ is a faithful $K[E]$-module. Therefore $D = 0$ and W/W_0 is a faithful $K_0[S]$-module.

Let $w' \in W \setminus W_0$ be the image of some $w \in V$. Then $Nw' \neq 0$ in W, whence $Ew' \neq 0$ by the definition of N. This means that $Ew \not\subseteq V_0 = L + K_0[S]L$. However $Ew \subseteq EV \subseteq K[E]$. Since $K[E]/L$ is an irreducible $K[E]$-module, we get that $K[E]/L \subseteq K[E]w$ in $W = V/V_0$. Hence $K_0[S]w = W$. It follows that W/W_0 is an irreducible $K_0[S]$-module. This implies that $K_0[S]$ is left primitive.

It remains to consider the case where the free monoid F' is commutative. From Theorem 2 we know that $K[J_0]/K[I]$ has finite codimension in R. Also, as above, $K[J_0]/K[I]$ embeds into $K_0[M]$, where $M = \mathcal{M}(G, B_0, A_0; Q)$ for a group G and for finite sets B_0 and A_0. However, in this case, G can be assumed to be an infinite cyclic group because the cancellative component S_{dc} satisfies

$S_{dc} \subseteq E \subseteq F'$, see [12], Proposition 3.1. Hence, it is well known and easy to check that $K_0[M]$ satisfies a polynomial identity, [11], Lemma 5.3. Therefore R also is a PI-algebra. Then the Jacobson radical of R is nilpotent, [5], whence R must be semiprimitive. This completes the proof of the theorem. $\square$

The assertion of the theorem is no longer valid if the ideal I is not finitely generated. For example, a finitely generated prime monomial algebra that is not semiprimitive, and has quadratic growth, was constructed in [14].

Finally, we note that, in view of Theorem 1, some of the methods used in this paper can be applied to a much wider class of finitely presented prime semigroups S than those of the form X/I, considered in Theorem 2. However, in this case, the non-null components of the matrix pattern on the appropriate ideal J of S need not be even cancellative. For example, S could be a prime semigroup (with zero) such that $S \setminus \{0\}$ is a non-cancellative semigroup.

Acknowledgment

This research was supported by the MNiSW research grant N201 004 32/0088 (Poland).

References

[1] K.I. Beidar, Y. Fong, On radicals of monomial algebras, Comm. Algebra 26 (1998) 3913–3919.
[2] J. Bell, P. Pekcagliyan, Primitivity of finitely presented monomial algebras, preprint, arXiv: 0712.0815v1.
[3] J. Bell, A. Smoktunowicz, The prime spectrum of algebras of quadratic growth, J. Algebra 319 (2008) 414–431.
[4] A. Belov, T. Gateva-Ivanova, Radicals of monomial algebras, in: First International Tainan—Moscow Algebra Workshop. Proceedings of the International Conference, Tainan, 1994, de Gruyter, Berlin, 1996, pp. 159–169.
[5] A. Braun, The nilpotency of the radical in a finitely generated P.I. ring, J. Algebra 89 (1984) 375–396.
[6] F. Cedo, E. Jespers, J. Okniński, Finitely presented algebras with homogeneous relations, preprint.
[7] J. Fountain, M. Petrich, Completely 0-simple semigroups of quotients III, Math. Proc. Cambridge Philos. Soc. 105 (1989) 263–275.
[8] E. Jespers, J. Okniński, Noetherian Semigroup Algebras, Springer-Verlag, Dordrecht, 2007.
[9] G. Lallement, Semigroups and Combinatorial Applications, Wiley, New York, 1979.
[10] T.Y. Lam, A First Course in Noncommutative Rings, second edition, Springer-Verlag, New York, 2001.
[11] J. Okniński, Semigroup Algebras, Marcel Dekker, New York, 1991.
[12] J. Okniński, Semigroups of Matrices, World Sci., Singapore, 1998.
[13] J. Okniński, Prime ideals of cancellative semigroups, Comm. Algebra 32 (2004) 2733–2742.
[14] A. Smoktunowicz, U. Vishne, An affine prime non-semiprimitive monomial algebra with quadratic growth, Adv. in Appl. Math. 37 (4) (2006) 511–513.
[15] U. Vishne, Primitive algebras with arbitrary Gelfand–Kirillov dimension, J. Algebra 211 (1998) 150–158.