Journal of Algebra 322 (2009) 2321–2334

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Degree estimate for commutators

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ARTICLE INFO

Article history:
Received 4 November 2008
Available online 3 August 2009
Communicated by J.T. Stafford

Keywords:
Degree estimate
Commutators
Jacobian
Automorphisms
Coordinates
Tame automorphisms
Wild automorphisms
Free associative algebras
Polynomial algebras
Jacobian conjecture
$K[u]$-bimodules
Equation $[u^m, s] = [u^n, r]$

ABSTRACT

Let $K\langle X\rangle$ be a finite generated free associative algebra over a field K of characteristic zero and let $f, g \in K\langle X\rangle$ generate their centralizer respectively. Assume that the f and g are algebraically independent, the leading homogeneous components of f and g are algebraically dependent and $\deg(f) \nmid \deg(g)$, $\deg(g) \nmid \deg(f)$. In this article, we construct a counterexample to a conjecture of Jie-Tai Yu that $\deg([f, g]) > \min\{\deg(f), \deg(g)\}$, which is closely related to the study of the structure of the automorphism group of $K\langle X\rangle$. We also obtain a counterexample to another related conjecture of Makar-Limanov and Jie-Tai Yu stated in terms of Malcev–Neumann formal power series. In view of the counterexamples we formulate two open problems concerning degree estimate for commutators in view of the study of the structure of the automorphism group of $K\langle X\rangle$.

The counterexamples in this article are constructed by applying the description of the free algebra $K\langle X\rangle$ considered as a bimodule of $K[u]$ where u is a monomial which is not a power of another monomial and the solution the equation $[u^m, s] = [u^n, r]$ with unknowns $r, s \in K\langle X\rangle$. The newly discovered description and the solution of the equation in this article are closely related to the combinatorial and computational aspects of free associative algebras, hence have their own independent interests.

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1 Partially supported by Grant MI-1503/2005 of the Bulgarian National Science Fund.
2 Partially supported by an RGC-CERG grant.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.07.018

0. Introduction

Let K be a field of characteristic zero and let $X = \{x_1, \dots, x_d\}$ be a finite set of indeterminates. Let $K[X]$ and $K\langle X\rangle$ be, respectively, the polynomial algebra and the free associative algebra generated by X over K. Let f and g be two polynomials in $K[X]$ or $K\langle X\rangle$, we want to estimate the minimal degree of the nonconstant elements of the subalgebra generated by f and g. This problem is closely related to the study of structure of the automorphism group of $K[X]$ and $K\langle X\rangle$, and related topics. See [DY,GY1,GY2,MLRSY,MLY,SU1,SU2,SY,U1,U2,UY,Y1,Y2].

If f and g are algebraically dependent in $K[X]$, then the theorem of Zaks [Z] gives that the integral closure of $K[f,g]$ in $K[X]$ is a subalgebra $K[h]$ of $K[X]$ (see also Eakin [E] for a simple proof and generalizations). If f and g are algebraically dependent in $K\langle X\rangle$, then they commute, see Cohn [C], and the theorem of Bergman [B1] gives that the centralizer of f (as well as the centralizer of g) is also a subalgebra $K[h]$ of $K\langle X\rangle$. In the both cases not much is known for the minimal degree of the elements of the subalgebra generated by f and g in terms of f and g, except that the minimal degree is bounded by $\deg(h)$ from the lower side. For example, the famous Abhyankar–Moh–Suzuki theorem (see [AM,Su]) gives that if $f, g \in K[x]$ generate the whole algebra $K[x]$, then $\deg(f)$ divides $\deg(g)$ or $\deg(g)$ divides $\deg(f)$.

Let $\varphi = (f, g)$ be an automorphism of $K[x, y]$ or $K\langle x, y\rangle$ (that is, $\varphi(x) = f$, $\varphi(y) = g$, and $K[f, g] = K[x, y]$ or $K\langle f, g\rangle = K\langle x, y\rangle$ respectively). Then it is well known that f and g may be of arbitrary high degrees, and one of the degrees $\deg(f)$ and $\deg(g)$ divides the other and one of the leading homogeneous components of f and g is a power of the other up to a nonzero constant factor in K^*. See, for instance, Cohn [C].

The above examples show that there is no good estimate for the minimal degree of the nonconstant elements in the subalgebra generated by f and g if f and g are algebraically dependent; or if f and g are algebraically independent, but one of the highest homogeneous components of f and g is a power of another up to a nonzero constant factor in K^*. More over, in case f and g are algebraically independent, it is obvious the problem of degree estimate becomes trivial if the highest homogeneous components of f and g are also algebraically independent. In view of that, the natural formulation of the problem should be the following

Problem 0.1. Let f and g be algebraically independent polynomials in $K[X]$ or $K\langle X\rangle$ such that the homogeneous components of maximal degree of f and g are algebraically dependent. Suppose that $\deg(f) \nmid \deg(g)$, $\deg(g) \nmid \deg(f)$. Find a good estimate of the minimal degree of the nonconstant elements of the subalgebra generated by f and g.

Shestakov and Umirbaev [SU1] obtained such an estimate for the commutative case by means of Poisson brackets. The estimate played essential role in Shestakov and Umirbaev [SU2], where they discovered an algorithm determining whether any given automorphism of $K[x, y, z]$ is tame and proved the famous Nagata Conjecture [N] that the Nagata automorphism is wild. As a by-product, Shestakov and Umirbaev obtained a new proof of the Jung–van der Kulk theorem [J,K] that the automorphisms of $K[x, y]$ are tame. The estimate was also used by Umirbaev and J.-T. Yu [UY], to prove the Strong Nagata Conjecture that there exist wild coordinates of $K[x, y, z]$.

Based on the Lemma on radicals for the Malcev–Neumann power series, recently Makar-Limanov and J.-T. Yu [MLY] have obtained a sharp lower degree estimate for the nonconstant elements of the subalgebra generated by f, g in $K\langle X\rangle$: If f and g are as in Problem 0.1 and $p(x, y) \in K\langle x, y\rangle$, then

$$\deg\bigl(p(f, g)\bigr) \geqslant D(f, g)\, w_{\deg(f),\deg(g)}(p),$$

where

$$D(f, g) = \frac{\deg([f, g])}{\deg(fg)}$$

and $w_{\deg(f),\deg(g)}(p)$ is the weighted degree of $p(x,y)$, defined by

$$w_{\deg(f),\deg(g)}(x)=\deg(f), \qquad w_{\deg(f),\deg(g)}(y)=\deg(g).$$

Similarly, by applying the Lemma on radicals in the commutative case they obtain the lower degree estimate for subalgebras $K[f,g]$ of polynomial algebras $K[X]$

$$\deg\bigl(p(f,g)\bigr) \geqslant D(f,g)w_{\deg(f),\deg(g)}(p),$$

where $p(x,y)\in K[x,y]$,

$$D(f,g)=\left[1-\frac{(\deg(f),\deg(g))(\deg(fg)-\deg(df\wedge dg))}{\deg(f)\deg(g)}\right],$$

(m,n) is the greatest common divisor of m,n and

$$df\wedge dg=\sum\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)(dx_i\wedge dx_j)$$

is the corresponding differential 2-form.

It is not hard to see that under the nontrivial condition that p has outer rank two, (that is, no automorphism of $K\langle x,y\rangle$ or $K[x,y]$ can take $p(x,y)$ into $K[x]$), we must have $\deg(p(f,g))\geqslant\deg([f,g])$ for free associative algebras and $\deg(p(f,g))\geqslant\deg(J(f,g))+2$ for polynomial algebras, where $\deg(J(f,g))=\max_{1\leqslant i<j\leqslant d}\{\deg(J_{x_i,x_j}(f,g))\}$. See, for instance, S.-J. Gong and J.-T. Yu [GY2].

These estimates have been used by J.-T. Yu [Y1], and S.-J. Gong and J.-T. Yu [GY1,GY2], to classify retracts, test elements and automorphic orbits of $K[x,y]$ and $K\langle x,y\rangle$, as well as to give a new proof of the theorem of Czerniakiewicz and Makar-Limanov (see [Cz,ML]) for the tameness of the automorphisms of $K\langle x,y\rangle$.

Umirbaev described the structure of the group of tame automorphisms of $K[x,y,z]$ in terms of generators and defining relations in [U1]. Based on that, he affirmatively solved the famous Anick conjecture (see, for instance, Cohn [C]) that the Anick automorphism of $K\langle x,y,z\rangle$ is wild in [U2]. Based on the methodology and results in [U1,U2,UY], Drensky and J.-T. Yu [DY] proved the strong Anick conjecture that there exist wild coordinates of $K\langle x,y,z\rangle$. But to our best knowledge, there is no algorithm determining whether any given automorphism of $K\langle x,y,z\rangle$ is tame, as obtained by Shestakov and Umirbaev [SU2] in commutative case. Here a serious and essential obstacle is that there is no good estimate for $\deg([f,g])$ for any given $f,g\in K\langle X\rangle$ in terms of $\deg(f)$ and $\deg(g)$. For details, see [Y2], where J.-T. Yu raised the following

Conjecture 0.2 *(J.-T. Yu). Let f and g be algebraically independent polynomials in $K\langle X\rangle$ such that the homogeneous components of maximal degree of f and g are algebraically dependent. Let f and g generate their own centralizers in $K\langle X\rangle$ respectively. Suppose that* $\deg(f)\nmid\deg(g)$, $\deg(g)\nmid\deg(g)$. *Then*

$$\deg\bigl([f,g]\bigr)>\min\bigl\{\deg(f),\deg(g)\bigr\}.$$

The condition that the degrees of f and g do not divide each other is essential. It does not hold when $\varphi=(f,g)$ is an automorphism of $K\langle x,y\rangle$ when the commutator test of Dicks [D] gives that $[f,g]=c[x,y]$, $c\in K^*$. The condition that f and g generate their own centralizers respectively is also necessary. For example, if

$$f=y+\bigl(x+y^k\bigr)^m, \qquad g=\bigl(x+y^k\bigr)^n, \quad m>n,\ k>2,$$

then $[f,g]=[y,(x+y^k)^n]$. The homogeneous component of maximal degree of $[f,g]$ is equal to

$$\bigl[y, xy^{k(n-1)} + y^2xy^{k(n-2)} + \cdots + y^{k(n-1)}x\bigr],$$
$$\deg\bigl([f,g]\bigr) = k(n-1)+2 < kn = \deg(g) < km = \deg(f).$$

If Conjecture 0.2 were true, it would give a nice description of the group of tame automorphisms of $K\langle x,y,z\rangle$ algorithmically, much better than the description of the group of tame automorphisms of $K[x,y,z]$ (see J.-T. Yu [Y2]). Makar-Limanov and J.-T. Yu [MLY] have been working in the Malcev–Neumann algebra $\mathcal{A}(X)$ (as a natural extension K-algebra of $K\langle X\rangle$) of formal power series with monomials from the free group generated by X, allowing infinite sums of homogeneous components of negative degree and only finite number of homogeneous components of positive degree. See Section 4 of this article. It is not hard to see that Conjecture 0.2 would follow from the next conjecture formulated by Makar-Limanov and J.-T. Yu.

Conjecture 0.3 *(Makar-Limanov and J.-T. Yu). Let $g \in K\langle X\rangle$ generate its own centralizer and let the homogeneous component of maximal degree of g is an n-th power of an element of $K\langle X\rangle$. Then, for every $m > n$ which is not divisible by n, the formal power series $g^{m/n} \in \mathcal{A}(X)$ has a monomial of positive degree containing a negative power of an indeterminate in X.*

The analogue of Conjecture 0.2 for polynomial algebras is: *if f and g are algebraically independent polynomials in $K[X]$ such that the homogeneous components of maximal degree of f and g are algebraically dependent, f and g generate their own integral closures in $K[X]$ respectively, and neither of the degrees of f and g divides the other, then*

$$\deg(df \wedge dg) > \min\bigl\{\deg(f), \deg(g)\bigr\}.$$

Note that in the case of $K[x,y]$,

$$\deg(df \wedge dg) = \deg\bigl(J(f,g)\bigr) + 2 = \deg\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + 2.$$

Recently, Makar-Limanov has constructed an example (see Example 1.1 in this article) of $f, g \in K[x,y]$ such that f and g may be of arbitrarily high degree but

$$\deg(df \wedge dg) = 3,$$

which serves as a counterexample of the analogue of Conjecture 0.2 for polynomial algebras.

It is also not hard to see that the analogue of Conjecture 0.3 is not true in the commutative case.

In this article we construct counterexamples to Conjectures 0.2 and 0.3. The degrees of the polynomials f and g in the counter example to Conjecture 0.2 are $3(2k+1)$ and $2(2k+1)$ respectively, where $k \geqslant 2$, and $\deg([f,g]) = 2k+5 < \deg(g) < \deg(f)$. The same polynomial g is also a counterexample to Conjecture 0.3. Comparing with the example of Makar-Limanov, we see that in the commutative case the fraction

$$\frac{\deg(df \wedge dg)}{\min\{\deg(f), \deg(g)\}}$$

can be made as close to 0 as possible. In our example, the fraction

$$\frac{\deg([f,g])}{\min\{\deg(f), \deg(g)\}} > \frac{1}{2}$$

and can be made as close to 1/2 as possible.

Problem 0.4. Let f and g be algebraically independent polynomials in $K\langle X\rangle$ such that the homogeneous components of maximal degree of f and g are algebraically dependent. Let f and g generate their own centralizers in $K\langle X\rangle$ respectively. Suppose that $\deg(f)\nmid\deg(g)$ and $\deg(g)\nmid\deg(f)$. Does then there exist a fixed positive number $c\leqslant 1/2$ such that $\deg([f,g]) > c\min\{\deg(f),\deg(g)\}$ for all such pair (f,g)?

If the answer to this problem is affirmative, it would differentiate the situation in the noncommutative case from the commutative case, in view of the example of Makar-Limanov (Example 1.1). Moreover, it would still give a nice description of the group of tame automorphisms of $K\langle x,y,z\rangle$ algorithmically (see Yu [Y2]). If the answer turns to be negative, it would mean that the 'Asymptotic properties' of the commutator in the noncommutative case are similar to that of the Jacobian in the commutative case, which could be viewed as another evidence for the truth of the famous Jacobian conjecture (for polynomial algebras of rank two). See, Dicks [D]. See also Cohn [C], Mikhalev, Shpilrain and J.-T. Yu [MSY].

In particular, to our best knowledge, there are no examples that $\deg(f) > \deg(g)$, $\deg(g)\nmid\deg(f)$, and $\deg([f,g]) < \frac{1}{2}\deg(g)$ even without a condition that f and g generate their own centralizers respectively (refer to the example in the Introduction). In view of that and the fact we are not able to make the constant $\frac{1}{2}$ smaller in the counterexample to Conjecture 0.2 (see Section 3 of this article), we formulate the following

Problem 0.5. Let f and g be algebraically independent polynomials in $K\langle X\rangle$ such that the homogeneous components of maximal degree of f and g are algebraically dependent. Let $\deg(f) > \deg(g)$ and $\deg(g)\nmid\deg(f)$. Can then we conclude that $\deg([f,g]) > \frac{1}{2}\deg(g)$?

In order to construct the counterexamples mentioned above, we first study the structure of the free algebra $K\langle X\rangle$ as a bimodule of $K[u]$, where u is a monomial which is not a proper power. It turns out that $K\langle X\rangle$ is a direct sum of three types of bimodules: the polynomial algebra $K[u]$, free bimodules generated by a single monomial, and two-generated bimodules with a nontrivial defining relation. By means of the bimodule structure of $K\langle X\rangle$, we solve the equation $[u^m,s]=[u^n,r]$ with unknowns $r,s\in K\langle X\rangle$. Then using the existence of the $K[u]$-bimodules of the third kind in $K\langle X\rangle$ and by means of Malcev–Neumann power series, we successfully construct the counterexamples.

An essential part of the combinatorial theory of free associative algebras A over a field is based on the fact that A is a free ideal ring (FIR), that is, every ideal of A is a free left (right respectively) A-module generated by a free basis in A, and based on the weak (Euclidean) algorithm in A (see Cohn [C]). The theory of equations in $K\langle X\rangle$ may be considered in the framework of the recently developed universal algebraic geometry, see the survey by Plotkin [P], as in the spirit of algebraic geometry over groups, see [BMR,MR]. Another aspect for equations in $K\langle X\rangle$ is from algorithmic point of view. For example, Gupta and Umirbaev [GU] proved that, the compatibility problem for any given system of linear equations over $K\langle X\rangle$ (as well as over some other types of relatively free algebras and Lie algebras), is algorithmically recognizable.

But very little is known about the actual solutions of an explicitly given equation. Recently Remeslennikov and Stöhr [RS] have studied the equation $[x,a]+[y,b]=0$ with unknowns x,y in the free Lie algebra $L(X)$ where a, b are free generators of $L(X)$. Therefore the description of $K\langle X\rangle$ as a $K[u]$-bimodule and the solution of the equation $[u^m,s]=[u^n,r]$ are closely related to the combinatorial and algorithmic aspects of free associative algebras, hence have their own independent interests, besides as the tool to construct the counterexamples.

1. An example of Makar-Limanov

In this section we present an example constructed by Makar-Limanov of two polynomials $f,g\in K[x,y]$ such that the degrees of f and g can be arbitrarily high and do not divide each other, and when the degree of the Jacobian of f and g is equal to 1, that answered the commutative version of Conjecture 0.3 negatively. It shows that it is unlikely that one may solve the famous Jacobian conjecture (for polynomial algebras of rank two) by degree estimate for the Jacobian, as suggested in [SU1]. The example was communicated by Makar-Limanov to Jie-Tai Yu in August 2007 when they were trying to attack Conjectures 0.2 and 0.3 in MPIM Bonn.

We are grateful to Leonid Makar-Limanov for kind permission to include the example in this article, and for helpful discussion.

Example 1.1. Let $a > b$ be positive integers such that $a - b > 1$ and $a - b$ divides $a + 1$. Let

$$c = a - b, \qquad k = \frac{a+1}{c},$$
$$f(x, y) = yp\big(x^a y^b\big), \qquad g(x, y) = xy\big(1 + x^a y^b\big),$$

where $p(z) \in K[z]$ is a polynomial of degree k. Then

$$\deg(f) = (a+b)k + 1 = (a+b)\frac{a+1}{a-b} + 1 = \frac{(a+b+2)a}{a-b} = \frac{(a+b+2)a}{c},$$
$$\deg(g) = a + b + 2 < (a+b+2)\frac{a}{a-b} = \deg(f).$$

Clearly, $a - b$ does not divide a because it divides $a + 1$ and $a - b > 1$. Also, the degree of f and g can be made as large as we want. Since f cannot be expressed as $p(h)$ for $p(t) \in K[t]$ with $\deg(p(t)) > 1$ and $h \in K[x, y]$, f generates its own integral closure in $K[x, y]$, so does g.

Direct computations show that

$$\begin{aligned} J(f, g) &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \\ &= y\big[-\big(1 + (a+1)x^a y^b\big)p\big(x^a y^b\big) + (a-b)x^a y^b\big(1 + x^a y^b\big)p'\big(x^a y^b\big)\big], \end{aligned}$$

where $p'(z)$ is the derivative of $p(z)$. We want to choose $p(z)$ in such a way that $J(f, g) = y$. This is equivalent to the condition

$$-(1 + kcz)p(z) + cz(1+z)p'(z) = 1.$$

Let $p(z) = -1 + p_1 z + p_2 z^2 + \cdots + p_{k-1} z^{k-1} + p_k z^k$, $p_i \in K$.

Then

$$\begin{aligned} 1 &= -(1 + kcz)p(z) + cz(1+z)p'(z) \\ &= -(1 + kcz)\big(-1 + p_1 z + p_2 z^2 + \cdots + p_k z^k\big) + cz(1+z)\big(p_1 + 2p_2 z + \cdots + kp_k z^{k-1}\big) \\ &= 1 - (p_1 - kc)z - (p_2 + kcp_1)z^2 - \cdots - (p_k + kcp_{k-1})z^k - kcp_k z^{k+1} + cp_1 z \\ &\quad + c(2p_2 + p_1)z^2 + \cdots + c\big(kp_k + (k-1)p_{k-1}\big)z^k + kcp_k z^{k+1} \\ &= 1 + \big((c-1)p_1 + kc\big)z + \big((2c-1)p_2 + c(1-k)p_1\big)z^2 \\ &\quad + \big((3c-1)p_3 + c(2-k)p_2\big)z^3 + \cdots + \big((kc-1)p_k - cp_{k-1}\big)z^k. \end{aligned}$$

Hence

$$p_1 = -\frac{kc}{c-1}, \quad p_2 = \frac{(k-1)c}{2c-1}p_1, \quad p_3 = \frac{(k-2)c}{3c-1}p_2, \quad \ldots, \quad p_k = \frac{c}{kc-1}p_{k-1}.$$

Since $p_1 \neq 0$, we get $p_i \neq 0$ for all i. Hence the degree of f is equal to the prescribed $\deg(f) = [(a+b+2)a]/c$ and $\deg(J(f,g)) = \deg(y) = 1$.

Finally note that f can be obtained as the *polynomial part* of the Malcev–Neumann power series $g^{a/c}$, see Section 4.

2. The free associative algebra $K\langle X\rangle$ as a $K[u]$-bimodule

Let $\langle X\rangle$ be the free semigroup generated by X. In this section we fix a monomial $u \in \langle X\rangle$ of positive degree which is not a proper power of another monomial. We consider the algebra $K\langle X\rangle$ as a $K[u]$-bimodule. Equivalently, $K\langle X\rangle$ is a $K[u_1,u_2]$-module with the actions of u_1 and u_2 defined by

$$u_1 w = uw, \qquad u_2 w = wu, \quad w \in \langle X\rangle.$$

Clearly, $K\langle X\rangle$ decomposes as a $K[u_1,u_2]$-module as

$$K[u] \oplus \Big(\sum K[u_1,u_2]v\Big),$$

where the inner sum runs on all $v \in \langle X\rangle$ which do not commute with u. We want to find the complete description of the $K[u_1,u_2]$-module $K\langle X\rangle$. If $v_1, v_2 \in \langle X\rangle$, we call v_1 a *head*, respectively a *tail* of v_2 if there exists $w \in \langle X\rangle$ such that $v_2 = v_1 w$, respectively $v_2 = wv_1$.

Theorem 2.1. *As a $K[u_1,u_2]$-module, $K\langle X\rangle$ is a direct sum of three types of submodules*: (i) $K[u]$; (ii) $K[u_1,u_2]t$; (iii) $K[u_1,u_2]t_1 + K[u_1,u_2]t_2$, *where*:

(i) *$K[u]$ is generated as a $K[u_1,u_2]$-module by 1, and $u^p = u_1^p \cdot 1$. The defining relation for this submodule is $u_1 \cdot 1 = u_2 \cdot 1$.*

(ii) *$K[u_1,u_2]t$ is a free $K[u_1,u_2]$-module and u is neither a head nor a tail of t. If t is a head, respectively a tail of u, and t' is the tail, respectively the head of u of the same degree as t, then $tu \neq ut'$, respectively $ut \neq t'u$.*

(iii) *t_1 and t_2 are of the same degree and are, respectively, a proper head and a proper tail of u such that $t_1 u = ut_2$. The defining relation of this submodule is $u_2 t_1 = u_1 t_2$. There exist $v_1, v_2 \in \langle X\rangle$ with $v_1 v_2 \neq v_2 v_1$ and a positive integer k such that*

$$u = (v_1 v_2)^k v_1, \qquad t_1 = v_1 v_2, \qquad t_2 = v_2 v_1.$$

Proof. The statement (i) is obvious so we only concentrate on (ii) and (iii). Each $v \in \langle X\rangle$ has the form $v = u^a v'$, where u is not a head of v'. Similarly, $v' = tu^b$, where u is not a tail of t. Hence, by the property that u is not a proper power of another monomial, we conclude that $K\langle X\rangle$ is generated as a $K[u_1,u_2]$-module by 1 and monomials t which do not commute with u and u is neither a head nor a tail of t. Let

$$\sum_{i=1}^{p} \gamma_i u^{a_i} t_i u^{b_i} = 0, \quad 0 \neq \gamma_i \in K, \tag{1}$$

be a relation between such t_i, where the triples (a_i, b_i, t_i) are pairwise different, with possible $t_i = t_j$ for some $i \neq j$. We may assume that this relation is homogeneous, i.e., $(a_i + b_i)\deg(u) + \deg(t_i)$ is the same for all monomials. For each t_i there exists a t_j such that $u^{a_i} t_i u^{b_i} = u^{a_j} t_j u^{b_j}$. Let $u^{a_1} t_1 u^{b_1} = u^{a_2} t_2 u^{b_2}$. We may assume that $a_1 \leqslant a_2$. Cancel u^{a_1} and obtain that $t_1 u^{b_1} = u^a t_2 u^{b_2}$, $a = a_2 - a_1$. Similarly, if $b_1 \leqslant b_2$, then $t_1 = u^a t_2 u^b$, $b = b_2 - b_1$. By the choice of t_1, t_2, we get $t_1 = t_2$, $a = b = 0$, which contradicts with $(a_1, b_1, t_1) \neq (a_2, b_2, t_2)$. If $b_1 > b_2$, then for $b = b_1 - b_2$

$$t_1 u^b = u^a t_2, \quad a, b > 0.$$

If $\deg(t_1) \geqslant \deg(u)$, then u is a head of t_1 which is impossible. Hence t_1 is a head of u. Similarly if $\deg(t_2) \geqslant \deg(u)$, then u is a tail of t_2 which is also impossible. Hence t_2 is a tail of u. In this way, in the relation (1) all t_i are heads or tails of u. Since (1) is homogeneous, the degree of t_i is equal to the residue of the division of the degree of the relation by the degree of u. Hence all t_i are of the same degree smaller than the degree of u. Since the heads and the tails of u are determined by their degrees, we obtain that in (1) $p = 2$ and t_1 is a head and t_2 is a tail of u. Let $u = t_1 w_1 = w_2 t_2$, $w_1, w_2 \in \langle X\rangle$. Since $\deg(t_1) = \deg(t_2) < \deg(u)$,

$$\deg(w_1) = \deg(u) - \deg(t_1) = \deg(u) - \deg(t_2) = \deg(w_2).$$

Substituting $u = t_1 w_1 = w_2 t_2$ in $t_1 u^b = u^a t_2$, we obtain

$$t_1 u^b = t_1(w_2 t_2)\cdots(w_2 t_2) = (t_1 w_1)\cdots(t_1 w_1)t_2 = u^a t_2.$$

Both sides of this equality start with $t_1 w_2$ and $t_1 w_1$, respectively. Since w_1 and w_2 are of the same degree, this implies that $w_1 = w_2 = w$ and $u = t_1 w = w t_2$. Hence

$$t_1 u = t_1(w t_2) = (t_1 w)t_2 = u t_2.$$

If $t_1 = t_2$, then $t_1 u = u t_1$ which is impossible because u is not a proper power, and u generates its own centralizer. Hence $t_1 \neq t_2$. Using the relation $t_1 u = u t_2$, we get the elements of $K[u_1, u_2]t_1 + K[u_1, u_2]t_2$ as linear combinations of $u_1^a t_1 = u^a t_1$ and $u_1^b u_2^c t_2 = u^b t_2 u^c$. It is easy to see that $u^{b_1} t_2 u^{c_1} \neq u^{b_2} t_2 u^{c_2}$, because $t_2 u \neq u t_2$ and $(b_1, c_1) \neq (b_2, c_2)$. Similarly $u^a t_1 = u^b t_2 u^c$ is also impossible, because t_1 is not a tail of u (hence $c = 0$) and $\deg(t_1) = \deg(t_2)$, $t_1 \neq t_2$. Hence all relations in the $K[u_1, u_2]$-module generated by t_1 and t_2 follow from $t_1 u = u t_2$. Let $u = t_1^k v_1$, where k is the maximum with this property. Then $t_1 u = u t_2$ implies that

$$t_1^{k+1} v_1 = t_1^k v_1 t_2, \qquad t_1 v_1 = v_1 t_2.$$

Since t_1 is not a head of v_1 (otherwise $u = t_1^{k+1} v_1'$), we obtain that v_1 is a head of t_1 and $t_1 = v_1 v_2$ for some $v_2 \in \langle X\rangle$. Now $t_1 v_1 = v_1 t_2$ gives $v_1 v_2 v_1 = v_1 t_2$ and $t_2 = v_2 v_1$. Hence

$$u = (v_1 v_2)^k v_1, \qquad t_1 = v_1 v_2, \qquad t_2 = v_2 v_1$$

and $v_1 v_2 \neq v_2 v_1$ because u is not a proper power. □

Remark 2.2. For a fixed $u \in \langle X\rangle$ there may be more than one pair (t_1, t_2) satisfying the condition (iii) of Theorem 2.1 but different pairs must have different degrees. For example, if $u = (xy)^k x$, $k > 1$, then for any positive $\ell \leqslant k$ the monomials $t_{1\ell} = (xy)^\ell$, $t_{2\ell} = (yx)^\ell$ satisfy $t_{1\ell} u = u t_{2\ell}$.

Now we are going to solve the equation $[u^m, s] = [u^n, r]$. It is more convenient to replace m and n by ℓm and ℓn, respectively, where m and n are relatively prime.

Example 2.3. Let $u \in \langle X\rangle$ be a monomial of positive degree which is not a power of another polynomial. Let ℓ, m, n be positive integers such that $m > n$ and m, n are relatively prime. We consider the equation

$$\left[u^{\ell m}, s\right] = \left[u^{\ell n}, r\right]. \tag{2}$$

Applying Theorem 2.1, we write r and s in the form

$$r = r_1(u) + \sum pt + \sum (p_1t_1 + p_2t_2), \quad r_1 \in K[u],\ p, p_1, p_2 \in K[u_1, u_2],$$
$$s = s_1(u) + \sum qt + \sum (q_1t_1 + q_2t_2), \quad s_1 \in K[u],\ q, q_1, q_2 \in K[u_1, u_2],$$

where the sums run, respectively, on all monomials t and t_1, t_2 described in parts (ii) and (iii) of Theorem 2.1. Clearly, $r_1(u)$ and $s_1(u)$ may be arbitrary polynomials and we have to solve the following systems for each t and t_1, t_2:

$$\left[u^{\ell m}, q(u_1, u_2)t\right] = \left[u^{\ell n}, p(u_1, u_2)t\right], \tag{3}$$
$$\left[u^{\ell m}, q_1t_1 + q_2t_2\right] = \left[u^{\ell n}, p_1t_1 + p_2t_2\right]. \tag{4}$$

We rewrite (3) in the form

$$\left(u_1^{\ell m} - u_2^{\ell m}\right)q(u_1, u_2) = \left(u_1^{\ell n} - u_2^{\ell n}\right)p(u_1, u_2).$$

Since m and n are relatively prime, the greatest common divisor of the polynomials $u_1^{\ell m} - u_2^{\ell m}$ and $u_1^{\ell n} - u_2^{\ell n}$ is equal to $u_1^{\ell} - u_2^{\ell}$ and we obtain that

$$p(u_1, u_2) = \frac{u_1^{\ell m} - u_2^{\ell m}}{u_1^{\ell} - u_2^{\ell}} r_2(u_1, u_2), \qquad q(u_1, u_2) = \frac{u_1^{\ell n} - u_2^{\ell n}}{u_1^{\ell} - u_2^{\ell}} r_2(u_1, u_2),$$

where $r_2(u_1, u_2) \in K[u_1, u_2]$ is an arbitrary polynomial.

Now we assume that $\deg(t_1) = \deg(t_2) < \deg(u)$ and u, t_1, t_2 satisfy the condition $t_1u = ut_2$. Using this relation we present $p_1t_1 + p_2t_2$ and $q_1t_1 + q_2t_2$ in (4) in the form

$$p_1t_1 + p_2t_2 = p_1(u_1)t_1 + p_2(u_1, u_2)t_2, \qquad q_1t_1 + q_2t_2 = q_1(u_1)t_1 + q_2(u_1, u_2)t_2$$

and rewrite (4) as

$$\left(u_1^{\ell m} - u_2^{\ell m}\right)\left(q_1(u_1)t_1 + q_2(u_1, u_2)t_2\right) = \left(u_1^{\ell n} - u_2^{\ell n}\right)\left(p_1(u_1)t_1 + p_2(u_1, u_2)t_2\right).$$

Replace u_2t_1 by u_1t_2 and obtain

$$u_1^{\ell m}q_1t_1 - u_1u_2^{\ell m-1}q_1t_2 + \left(u_1^{\ell m} - u_2^{\ell m}\right)q_2t_2 = u_1^{\ell n}p_1t_1 - u_1u_2^{\ell n-1}p_1t_2 + \left(u_1^{\ell n} - u_2^{\ell n}\right)p_2t_2.$$

Comparing the coefficients of t_1 and t_2, we derive

$$u_1^{\ell m}q_1(u_1) = u_1^{\ell n}p_1(u_1),$$
$$-u_1u_2^{\ell m-1}q_1 + \left(u_1^{\ell m} - u_2^{\ell m}\right)q_2 = -u_1u_2^{\ell n-1}p_1 + \left(u_1^{\ell n} - u_2^{\ell n}\right)p_2.$$

It is sufficient to solve these equations when p_i, q_i are homogeneous. We may assume that $\deg(q_1) = \deg(q_2) = a$, $\deg(p_1) = \deg(p_2) = a + \ell(m - n)$. Hence

$$p_1(u_1) = \xi u_1^{a+\ell(m-n)}, \qquad q_1(u_1) = \xi u_1^a, \quad \xi \in K,$$
$$-\xi u_1^{a+1} u_2^{\ell m-1} + \left(u_1^{\ell m} - u_2^{\ell m}\right) q_2 = -\xi u_1^{a+\ell(m-n)+1} u_2^{\ell n-1} + \left(u_1^{\ell n} - u_2^{\ell n}\right) p_2,$$
$$\left(u_1^{\ell m} - u_2^{\ell m}\right) q_2 + \xi u_1^{a+1} u_2^{\ell n-1}\left(u_1^{\ell(m-n)} - u_2^{\ell(m-n)}\right) = \left(u_1^{\ell n} - u_2^{\ell n}\right) p_2.$$

Defining the polynomial

$$\Phi_b(u_1, u_2) = \frac{u_1^{\ell b} - u_2^{\ell b}}{u_1^{\ell} - u_2^{\ell}} = u_1^{\ell(b-1)} + u_1^{\ell(b-2)} u_2^{\ell} + \cdots + u_2^{\ell(b-1)}, \quad b \geqslant 1,$$

and using that

$$\Phi_m(u_1, u_2) = u_1^{\ell(m-n)} \Phi_n(u_1, u_2) + u_2^{\ell n} \Phi_{m-n}(u_1, u_2),$$

the equation for ξ, p_2, q_2 becomes

$$\left(u_1^{\ell(m-n)} \Phi_n + u_2^{\ell n} \Phi_{m-n}\right) q_2 + \xi u_1^{a+1} u_2^{\ell n-1} \Phi_{m-n} = \Phi_n p_2,$$
$$\left(p_2 - u_1^{\ell(m-n)} q_2\right) \Phi_n = u_2^{\ell n-1}\left(u_2 q_2 + \xi u_1^{a+1}\right) \Phi_{m-n}.$$

Since the polynomials $\Phi_n(u_1, u_2)$ and $u_2^{\ell n-1} \Phi_{m-n}(u_1, u_2)$ are relatively prime, we obtain

$$p_2 - u_1^{\ell(m-n)} q_2 = u_2^{\ell n-1} \Phi_{m-n} r_3, \qquad u_2 q_2 + \xi u_1^{a+1} = \Phi_n r_3,$$

where $r_3(u_1, u_2) \in K[u_1, u_2]$. Hence it is sufficient to solve the equation

$$u_2 q_2(u_1, u_2) + \xi u_1^{a+1} = \Phi_n(u_1, u_2) r_3(u_1, u_2)$$

for $\xi \in K$ and for homogeneous $q_2, r_3 \in K[u_1, u_2]$. Comparing the coefficients of u_1^{a+1} and using that $\Phi_n = u_1^{\ell(n-1)} + u_2^{\ell} \Phi_{n-1}$, we obtain

$$a + 1 \geqslant \deg(\Phi_n) = \ell(n-1),$$
$$r_3(u_1, u_2) = \xi u_1^{a-\ell(n-1)+1} + u_2 s_3(u_1, u_2),$$
$$q_2(u_1, u_2) = \left(u_1^{\ell(n-1)} + u_2^{\ell} \Phi_{n-1}\right) s_3(u_1, u_2) + \xi u_1^{a-\ell(n-1)+1} u_2^{\ell-1} \Phi_{n-1},$$

for any $\xi \in K$, and arbitrary homogeneous polynomial $s_3(u_1, u_2) \in K[u_1, u_2]$ of degree $a - \ell(n-1)$ or $s_3(u_1, u_2) = 0$.

It is natural to ask whether the structure of $K\langle X\rangle$ considered as a bimodule of $K[f]$, when $f \in K\langle X\rangle$ is an arbitrary polynomial, is similar to that in Theorem 2.1. The following example shows that in this case some strange phenomenon appear similar to the case in the Buchberger algorithm of the Gröbner bases for ideals. We do not expect that there exists a nice bimodule structure of $K\langle X\rangle$ in general.

Example 2.4. Let us order the monomials of $\langle x, y\rangle$ first by degree and then lexicographically, assuming that $x > y$. Let

$$f = xyx + yxx, \qquad u = xyx, \qquad t_1 = xy, \qquad t_2 = yx.$$

The leading monomial of f is u and we have $t_1u = ut_2$. Direct computation gives that

$$t_1 f - f t_2 + t_2 f = (xy + yx)yxx$$

belongs to the $K[f]$-bimodule generated by t_1 and t_2 but its leading monomial $xyyxx$ neither starts or ends with u.

3. A counterexample to Conjecture 0.2

Theorem 3.1. *Let $X = \{x, y\}$, $k \geqslant 2$, and let*

$$u = (xy)^k x, \qquad v = xy, \qquad w = yx,$$
$$f = u^3 + r, \qquad r = uv + uw + wu,$$
$$g = u^2 + s, \qquad s = v + w.$$

Then f and g are algebraically independent polynomials which generate their own centralizers in $K\langle x, y\rangle$ respectively. The homogeneous components of maximal degree of f and g are algebraically dependent and neither of the degrees of f and g divides the other. Then

$$\deg\bigl([f, g]\bigr) < \deg(g) < \deg(f).$$

The fraction $\deg([f, g])/\deg(g) = (2k + 5)/(4k + 2)$ *is strictly larger than* $1/2$ *and can be made as close to* $1/2$ *as possible by increasing k.*

Proof. First note neither f nor g can be expressed as $p(h)$, where $p(t) \in K[t]$ with $\deg(p) > 1$, and $h \in K\langle x, y\rangle$. Hence f and g generalize their own centralizers in $K\langle x, y\rangle$ respectively.

Let $u = (xy)^k x$, $v = t_1 = xy$, $w = t_2 = yx$. Hence $vu = uw$. As a special case of Example 2.3 with $l = 1$, $m = 3$, $\xi = 1$, $s_3 = 0$, $r_2 = 0$, $r_1 = s_1 = 0$, hence $p_1 = u_1$, $p_2 = u_1 + u_2$, $q_1 = q_2 = 1$, $p = q = 0$, we obtain a solution $r = ut_1 + ut_2 + t_2u = uv + uw + wu$, $s = t_1 + t_2 = v + w$ for the equation $[u^3, s] = [u^2, r]$. Now $[f, g] = [u^3 + r, u^2 + s] = [r, s]$, $\deg([f, g]) = \deg(r) + \deg(s) = 2k + 5 < 4k + 2 = \deg(g) < \deg(f)$. $\square$

By the above approach, we were not able to obtain a pair (f, g) such that $\deg([f, g]) \leqslant \deg(g)/2$ as indicated below.

In order to decrease the degree of $[f, g]$ further, as in the example of Makar-Limanov, we may try to add new homogeneous summands to f, that is,

$$f = u^m + r + r_1, \qquad \deg(r_1) < \deg(r),$$

such that $[r, s] + [r_1, u^n] = 0$. But then we face some essential difficulties: the monomials of $[r, s]$ are of the form $u^a t_i u^b t_j u^c$, $t_i, t_j = v, w$. Using the relation $vu = uw$, we may assume that $b = 0$ if $t_i = v$ or $t_j = w$. Hence

$$[r, s] = \sum h_b w u^b v + h_{11} vv + h_{12} vw + h_{22} ww, \quad h_b, h_{ij} \in K[u_1, u_2].$$

Since the monomials $wu^b v, vv, vw, ww$ are neither heads nor tails of u, we have to work in a free $K[u]$-bimodule and it seems impossible to find r, s, r_1 of sufficiently small degree such that

$[f, g] = [r_1, s]$ and $\deg([f, g]) \leqslant \deg(g)/2$. The computations become even worse if we add one more component to g as we have tried unsuccessfully:

$$f = u^m + r + r_1, \qquad \deg(r_1) < \deg(r),$$
$$g = u^n + s + s_1, \qquad \deg(s_1) < \deg(s).$$

4. Working in the Malcev–Neumann algebra

Let $F(X)$ be the free group generated by X. We define the total degree of $u = x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1} \in F(X)$ in the usual way, assuming that $\deg(x_i^{\pm 1}) = \pm 1$. By the theorem of Neumann and Shimbireva [N1,S], the group $F(X)$ can be ordered linearly in many ways. In particular, see Theorem 2.3 in [N1], if H is a linearly ordered factor group of $F(X)$, then the order of H can be lifted to a linear order of $F(X)$. Defining a partial order on the free abelian group generated by X by total degree and then refining it lexicographically, we obtain a linear order on $F(X)$ such that if $\deg(u_1) < \deg(u_2)$, then $u_1 < u_2$. Since $\langle X\rangle \subset F(X)$, we assume that the elements of $\langle X\rangle$ are linearly ordered in the same way. If $g = g(X) = \sum_{i=1}^{p} \alpha_i u_i$, $0 \neq \alpha_i \in K$, $u_i \in \langle X\rangle$, $u_1 > u_2 > \cdots > u_p$, we denote by $\nu(g)$ the leading monomial $\alpha_1 u_1$ of g. We denote by $\mathcal{A}(X)$ the Malcev–Neumann algebra of formal power series used by Malcev and Neumann [M,N2] to show that the group algebra of an ordered group can be embedded into a division ring. The algebra $\mathcal{A}(X)$ consists of all formal sums $\tau = \sum_{u\in\Delta} \alpha_u u$, $\alpha_u \in K$, where Δ is a well-ordered subset of $F(X)$. (In the commutative case when $F(X)$ itself is the free abelian group generated by X, the similar construction was discovered and used by Hahn [H].)

Use $\mathcal{A}(X)$ as in Makar-Limanov and Yu [MLY] and assume that Δ is well ordered relative to the opposite ordering, that is, any nonempty subset of Δ has a largest element. Again, if $0 \neq \tau \in \mathcal{A}(X)$, denote by $\nu(\tau)$ its leading monomial $\alpha_1 u_1$, $\alpha_1 \in K$, $u_1 \in F(X)$. The following lemma on radicals of Bergman [B2,B3] plays a crucial role in [MLY].

Lemma 4.1. *Let $0 \neq \tau \in \mathcal{A}(X)$ such that $\nu(\tau) = (\beta u)^n$ for a positive integer n, where $\beta \in K$, $u \in F(X)$. Then there exists a $\rho \in \mathcal{A}(X)$ such that $\tau = \rho^n$.*

Now we are going to prove that the polynomial g from the counterexample to Conjecture 0.2 serves also as a counterexample to Conjecture 0.3.

Theorem 4.2. *Let $X = \{x, y\}$, $k \geqslant 2$, and let*

$$u = (xy)^k x, \qquad v = xy, \qquad w = yx,$$
$$g = u^2 + s, \qquad s = v + w.$$

Then g generates its own centralizer in $K\langle x, y\rangle$ and $g^{3/2} \in \mathcal{A}(x, y)$ contains no monomial of positive degree containing a negative power of x or y.

Proof. First by Theorem 3.1, g generates its own centralizer in $K\langle x, y\rangle$. Now let $\rho := g^{1/2} = u + a_1 + a_2 + \cdots$, where a_i are homogeneous polynomials such that

$$2k + 1 = \deg(u) > \deg(a_1) > \deg(a_2) > \cdots.$$

These polynomials are determined step-by-step in a unique way from the condition

$$g = u^2 + s = \rho^2 = u^2 + (ua_1 + a_1 u) + \left(a_1^2 + ua_2 + a_2 u\right) + \cdots.$$

Comparing the homogeneous components of g and ρ^2 and their degrees, we obtain

$$ua_1 + a_1u = s, \qquad \deg(a_1) = \deg(s) - \deg(u) = 1 - 2k,$$
$$a_1^2 + ua_2 + a_2u = 0, \qquad \deg(a_2) = 2\deg(a_1) - \deg(u) = 1 - 6k,$$
$$\deg(a_i) = \deg(a_1) + \deg(a_{i-1}) - \deg(u) = 1 - 2(2i-1)k, \quad i \geqslant 1.$$

As in the proof of Theorem 3.1, we have $vu = uw$, hence $wu^{-1} = u^{-1}v$ and $u(wu^{-1}) + (wu^{-1})u = v + w = s$, therefore $a_1 = wu^{-1}$. Now

$$\rho^3 = u^3 + \sum\bigl(u^2a_i + ua_iu + a_iu^2\bigr) + \sum(ua_ia_j + a_iua_j + a_ia_ju) + \sum a_ia_ja_l,$$
$$\deg(ua_1u) = \deg\bigl(a_1u^2\bigr) = \deg\bigl(u^2a_1\bigr) = 2(2k+1) + (1-2k) = 2k+3,$$
$$\begin{aligned}\deg(ua_iu) = \deg\bigl(a_iu^2\bigr) = \deg\bigl(u^2a_i\bigr) &\leqslant \deg\bigl(u^2a_2\bigr)\\ &= 2(2k+1) + (1-6k) = 3 - 2k < 0, \quad i \geqslant 2,\end{aligned}$$
$$\begin{aligned}\deg(a_iua_j) = \deg(a_ia_ju) = \deg(ua_ia_j) &\leqslant \deg\bigl(ua_1^2\bigr)\\ &= (2k+1) + 2(1-2k) = 3 - 2k < 0, \quad i,\ j \geqslant 1,\end{aligned}$$

and $\deg(a_ia_ja_k) \leqslant \deg(a_1^3) = 3(1-2k) < 0$, $i, j, k \geqslant 1$, hence the component of positive degree of ρ^3 is

$$\begin{aligned}u^3 + \bigl(u^2a_1 + ua_1u + a_1u^2\bigr) &= u^3 + \bigl[u^2\bigl(wu^{-1}\bigr) + u\bigl(wu^{-1}\bigr)u + \bigl(wu^{-1}\bigr)u^2\bigr]\\ &= u^3 + (uv + uw + wu) = u^3 + r = f,\end{aligned}$$

as $wu^{-1} = u^{-1}v$, where $f = u^3 + r$ is the other polynomial from Theorem 3.1. Therefore ρ^3 does not contain monomials of positive degree with negative powers of x or y. $\square$

Acknowledgments

We are grateful to Leonid Makar-Limanov for kind permission to include Example 1.1, and for helpful discussion. We also thank the referee for insightful comments and suggestions. In particular, the formulation of Problem 0.5 is due to the referee.

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