Schubert complexes and degeneracy loci

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ARTICLE INFO

ABSTRACT

Given a generic map between flagged vector bundles on a Cohen–Macaulay variety, we construct maximal Cohen–Macaulay modules with linear resolutions supported on the Schubert-type degeneracy loci. The linear resolution is provided by the Schubert complex, which is the main tool introduced and studied in this paper. These complexes extend the Schubert functors of Kraśkiewicz and Pragacz, and were motivated by the fact that Schur complexes resolve maximal Cohen–Macaulay modules supported on determinantal varieties. The resulting formula in K-theory provides a "linear approximation" of the structure sheaf of the degeneracy locus, which can be used to recover a formula due to Fulton.

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Introduction

Let X be an equidimensional Cohen–Macaulay (e.g., nonsingular) variety, and let $\varphi : E \to F$ be a map of vector bundles over X, with ranks e and f respectively. Given a number $k \leqslant \min(e, f)$, let $D_k(\varphi)$ be the degeneracy locus of points x where the rank of φ restricted to the fiber of x is at most k. Then $\operatorname{codim} D_k(\varphi) \leqslant (e - k)(f - k)$, and in the case of equality, the Thom–Porteous formula expresses the homology class of $D_k(\varphi)$ as an evaluation of a multi-Schur function at the Chern classes of E and F (see [Man, §3.5.4]). Also in the case of equality, the Schur complex associated with the rectangular partition $(f - k) \times (e - k)$ (see [ABW] or [Wey, §2.4] for more about Schur complexes) of φ is a linear locally free resolution for a Cohen–Macaulay coherent sheaf whose support is $D_k(\varphi)$. This resolution gives a formula in the K-theory of X. In the case that X is smooth, there is an isomorphism from an associated graded of the K-theory of X to the Chow ring of X (see Section 3.1 for more details). Then the image of this complex recovers the Thom–Porteous formula, and the complex provides a "linear approximation" of the syzygies of $D_k(\varphi)$.

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The situation was generalized by Fulton [F1] as follows. We provide the additional data of a flag of subbundles $E_\bullet$ for E and a flag of quotient bundles $F_\bullet$ for F, and we can define degeneracy loci for an array of numbers which specifies the ranks of the restriction maps $E_p \to F_q$. The rank functions that give rise to irreducible degeneracy loci are indexed by permutations in a natural way. Under the right codimension assumptions, one can express the homology class of a given degeneracy locus as a substitution of a double Schubert polynomial with the Chern classes of the quotients E_i/E_{i-1} and the kernels $\ker(F_j \to F_{j-1})$. The motivation for this work was to complete the analogy of this situation with the previous one by constructing "Schubert complexes" which would be acyclic whenever the degeneracy loci has the right codimension.

Building on the constructions for Schubert functors by Kraśkiewicz and Pragacz of [KP], we construct these complexes over an arbitrary (commutative) ring R from the data of two free R-modules M_0, M_1, with given flags of submodules, respectively, quotient modules, and a map $\partial : M_0 \to M_1$. We can also extend the construction to an arbitrary scheme. We show that they are acyclic when a certain ideal defined in terms of minors of ∂ has the right depth, i.e., they are "depth-sensitive." Our main result is that in the situation of Fulton's theorem, the complex is acyclic and the Euler characteristic provides the formula in the same sense as above. Our proof uses techniques from commutative algebra, algebraic geometry, and combinatorics. Again, the complexes are linear and provide a "linear approximation" to the syzygies of Fulton's degeneracy loci. As a special case of Fulton's degeneracy loci, one gets Schubert varieties inside of (type A) partial flag varieties.

Using the work of Fomin, Greene, Reiner, and Shimozono [FGRS], we construct explicit bases for the Schubert complex in the case that M_0 and M_1 are free. This basis naturally extends their notion of balanced labelings and the generating function of the basis elements gives what seems to be a new combinatorial expression for double Schubert polynomials. Furthermore, the complex naturally affords a representation of the Lie superalgebra of upper triangular matrices (with respect to the given flags) in $\mathrm{Hom}(M_0, M_1)$, and its graded character is the double Schubert polynomial.

The article is structured as follows. In Section 1 we recall some facts about double Schubert polynomials and balanced labelings. We introduce balanced super labelings (BSLs) and prove some of their properties. In Section 2 we extend the construction for Schubert functors to the $\mathbf{Z}/2$-graded setting and show that they have a basis naturally indexed by the BSLs. In Section 3 we construct the Schubert complex from this $\mathbf{Z}/2$-graded Schubert functor. Using some facts about the geometry of flag varieties, we show that the acyclicity of these complexes is controlled by the depth of a Schubert determinantal ideal. In the case of acyclicity and when the coefficient ring is Cohen–Macaulay, we show that the cokernel of the complex is a Cohen–Macaulay module which is generically a line bundle on its support. We also give some examples of Schubert complexes. Finally, in Section 4, we relate the acyclicity of the Schubert complexes to a degeneracy locus formula of Fulton. We finish with some remarks and possible future directions.

Conventions

The letter K is reserved for a field of arbitrary characteristic. If X is a scheme, then $\mathcal{O}_X$ denotes the structure sheaf of X. Throughout, all schemes are assumed to be separated. A variety means a reduced scheme which is of finite type over K. We treat the notions of locally free sheaves and vector bundles as the same, and points will always refer to closed points. The fiber of a vector bundle E at a point $x \in X$ is denoted $E(x)$ and refers to the stalk E_x tensored with the residue field $k(x)$. Given a line bundle L on X, $c_1(L)$ denotes the first Chern class of L, which we think of as a degree -1 endomorphism of the Chow groups $A_*(X)$. For an element $\alpha \in A_*(X)$, and an endomorphism c of $A_*(X)$, we will use the notation $c \cap \alpha$ to denote c applied to α.

1. Double Schubert polynomials

1.1. Preliminaries

Let Σ_n be the permutation group on the set $\{1, \ldots, n\}$. Since we are thinking of Σ_n as a group of functions, we will multiply them as functions, e.g., if s_1 and s_2 are the transpositions that switch

1 and 2, and 2 and 3, respectively, then $s_1 s_2$ is the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. We will use inline notation for permutations, so that w is written as $w(1)w(2) \cdots w(n)$. Proofs for the following statements about Σ_n can be found in [Man, §2.1]. Let s_i denote the transposition which switches i and $i+1$. Then Σ_n is generated by $\{s_1, \ldots, s_{n-1}\}$, and for $w \in \Sigma_n$, we define the *length* of w to be the least number $\ell(w)$ such that $w = s_{i_1} \cdots s_{i_{\ell(w)}}$. Such a minimal expression is a *reduced decomposition* for w. All reduced expressions can be obtained from one another using only the *braid relations*: $s_i s_j = s_j s_i$ for $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. We can also write $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$. The *long word* w_0 is the unique word with maximal length, and is defined by $w_0(i) = n + 1 - i$.

We will use two partial orders on Σ_n. The *(left) weak Bruhat order*, denoted by $u \leqslant_W w$, holds if some reduced decomposition of u is the suffix of some reduced decomposition of w.[1] We denote the *strong Bruhat order* by $u \leqslant w$, which holds if some reduced decomposition of w contains a sub-word that is a reduced decomposition of u. It follows from the definition that $u \leqslant w$ if and only if $u^{-1} \leqslant w^{-1}$. For a permutation w, let $r_w(p, q) = \#\{i \leqslant p \mid w(i) \leqslant q\}$ be its *rank function*. Then $u \leqslant w$ if and only if $r_u(p, q) \geqslant r_w(p, q)$ for all p and q (the inequality on rank functions is reversed).

Given a polynomial (with arbitrary coefficient ring) in the variables $\{x_i\}_{i \geqslant 1}$, let ∂_i be the *divided difference operator*

$$(\partial_i P)(x_1, x_2, \ldots) = \frac{P(\ldots, x_{i-1}, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i-1}, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}. \tag{1.1}$$

The operators ∂_i satisfy the braid relations: $\partial_i \partial_j = \partial_j \partial_i$ when $|i - j| > 1$ and $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$.

For the long word $w_0 \in \Sigma_n$, set $\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leqslant n}(x_i - y_j)$. In general, if $\ell(ws_i) = \ell(w) - 1$, we set $\mathfrak{S}_{ws_i}(x, y) = \partial_i \mathfrak{S}_w(x, y)$, where we interpret $\mathfrak{S}_w(x, y)$ as a polynomial in the variables $\{x_i\}_{i \geqslant 1}$ with coefficients in the ring $\mathbf{Z}[y_1, y_2, \ldots]$. These polynomials are the *double Schubert polynomials*, and are well-defined since the ∂_i satisfy the braid relations and the braid relations connect all reduced decompositions of a permutation. The definition of these polynomials is due to Lascoux and Schützenberger [LS]. They enjoy the following stability property: if we embed Σ_n into Σ_{n+m} by identifying permutations of Σ_n with permutations of Σ_{n+m} which pointwise fix $\{n+1, n+2, \ldots, n+m\}$, then the polynomial $\mathfrak{S}_w(x, y)$ is the same whether we regard w as an element of Σ_n or Σ_{n+m} [Man, Corollary 2.4.5].

Define the *single Schubert polynomials* by $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$. We will use the identity [Man, Proposition 2.4.7]

$$\mathfrak{S}_w(x, y) = \sum_{u \leqslant_W w} \mathfrak{S}_u(x)\mathfrak{S}_{uw^{-1}}(-y). \tag{1.2}$$

1.2. Balanced super labelings

For the rest of this article, we fix a totally ordered alphabet $\cdots < 3' < 2' < 1' < 1 < 2 < 3 < \cdots$. The elements i' will be referred to as *marked* and the elements i will be referred to as *unmarked*.

For a permutation w, define its *diagram* $D(w) = \{(i, w(j)) \mid i < j, \ w(i) > w(j)\}$. Note that $\#D(w) = \ell(w)$. Our convention is that the box (i, j) means row number i going from top to bottom, column number j going from left to right, just as with matrix indexing. An alternative way to get the diagram of $D(w)$ is as follows: for each i, remove all boxes to the right of $(i, w(i))$ in the same row and all boxes below $(i, w(i))$ in the same column including $(i, w(i))$. The complement is $D(w)$. See Fig. 1 for an example with $w = 35142$. Here the boxes $(i, w(i))$ are marked with $\bullet$ and the other removed boxes are marked with $\times$.

Let T be a labeling of $D(w)$. The *hook* of a box $b \in D(w)$ is the set of boxes in the same column below it, and the set of boxes in the same row to the right of it (including itself). A hook is *balanced*

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[1] In [Man], the weak Bruhat order is defined in terms of prefixes. We point out that these two definitions are distinct, but this will not cause any problems.
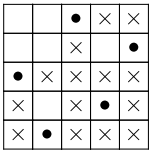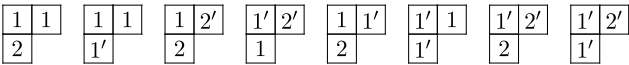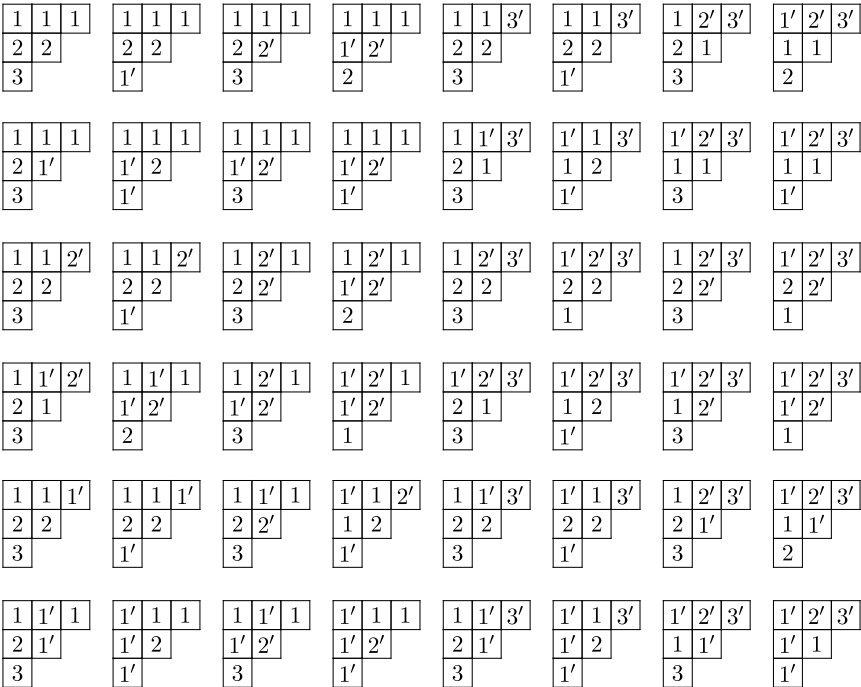
Fig. 1. $D(35\,142)$.

(with respect to T) if it satisfies the following property: when the entries are rearranged so that they are weakly increasing going from the top right end to the bottom left end, the label in the corner stays the same. A labeling is *balanced* if all of the hooks are balanced. Call a labeling T of $D(w)$ with entries in our alphabet a *balanced super labeling (BSL)* if it is balanced, column-strict (no repetitions in any column) with respect to the unmarked alphabet, row-strict with respect to the marked alphabet, and satisfies $j' \leqslant T(i,j) \leqslant i$ for all i and j (this last condition will be referred to as the *flag conditions*). To be consistent with the identity permutation, we say that an empty diagram has exactly one labeling.

Example 1.3. We list the BSL for some long words.

$n = 3$, $\mathfrak{S}_{321}(x, y) = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$.

$n = 4$, $\mathfrak{S}_{4321}(x, y) = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_1 - y_3)(x_2 - y_2)(x_3 - y_1)$.

Row of diagrams (first row):

| $\begin{smallmatrix}1&1'&2'\\2&2\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&1&2'\\2&2\\1'\end{smallmatrix}$ | $\begin{smallmatrix}1&1'&2'\\2&2'\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&1\\1'&2'\\2\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2\\1'\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2'\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\1'&2'\\2\end{smallmatrix}$ |

Row of diagrams (second row):

| $\begin{smallmatrix}1&1'&2'\\2&2\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&1&2'\\2&2\\1'\end{smallmatrix}$ | $\begin{smallmatrix}1&1'&2'\\2&2'\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&1\\1'&2'\\2\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2\\1'\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\2&2'\\3\end{smallmatrix}$ | $\begin{smallmatrix}1'&2'&3'\\1'&2'\\2\end{smallmatrix}$ |

Let $A = (a_{i,j})$ be an $n \times n$ array. We define left and right actions of Σ_n on A as follows. For $w \in \Sigma_n$, set $(wA)_{i,j} = A_{i,w(j)}$, and $(Aw)_{i,j} = A_{w^{-1}(i),j}$. Equivalently, $Aw = (w^{-1}A^t)^t$ where t denotes transpose. In particular, if $A = D(w)$ is the diagram of a permutation, and $\ell(wu) = \ell(w) + \ell(u)$, then $D(w)u \subseteq D(wu)$. It is enough to check this when $u = s_i$ is a transposition. In this case, the condition $\ell(ws_i) = \ell(w) + 1$ means that $w(i) < w(i+1)$, and then $D(ws_i) = D(w)s_i \cup \{(i, w(i))\}$. Similarly, $wD(u) \subseteq D(wu)$.

If w is a permutation, then $(i, j) \in D(w)$ is a *border cell* if $w(i+1) = j$. In particular, if (i, j) is a border cell, then $w(i) > w(i+1)$, so $(D(w) \setminus (i, j))s_i = D(ws_i)$.

Lemma 1.4. *Let T be a labeling of $D(w)$ with largest label M.*

(a) *Suppose (i, j) is a border cell which contains M. Then T is balanced if and only if $(T \setminus (i, j))s_i$ is balanced.*
(b) *Suppose T is a BSL and M is unmarked. Then every row which contains M must contain an M in a border cell.*

Proof. See [FGRS, Theorem 4.8] for (a). Part (b) follows from [FGRS, Lemma 4.7]. $\square$

By convention, a BSL of $D(w)$ is an $n \times n$ array which is 0 outside of $D(w)$ and takes values in our alphabet otherwise. We use the convention that $0 + i = i + 0 = i$ and $0 + i' = i' + 0 = i'$ whenever i, i' is in our alphabet, and also that $1' < 0 < 1$.

Lemma 1.5. *Let u and v be two permutations such that $\ell(uv) = \ell(u) + \ell(v)$. Let T_u be a BSL of $D(u)$ using only marked letters, and let T_v be a BSL of $D(v)$ using only unmarked letters. Then $T = T_u v + u T_v$ is a BSL for $D(uv)$, and all BSLs of $w = uv$ come from such a "factorization" in a unique way.*

Proof. The condition $j' \leqslant T(i, j) \leqslant i$ is automatic since we assumed that T_u contains only marked letters and T_v contains only unmarked letters. Similarly, the respective column-strict and row-strict conditions are automatic. So it is enough to check that T is balanced.

By Lemma 1.4, we can factor $v = s_{i_1} s_{i_2} \cdots s_{i_{\ell(v)}}$ into simple transpositions such that if we write $v_j = s_{i_1} \cdots s_{i_{j-1}} s_{i_j}$, then $T_{v_{\ell(v)}} = T_v$, and for $j < \ell(v)$, T_{v_j} is the result of removing a border cell with the largest label L_j from $T_{v_{j+1}}$ and hence is a balanced labeling. In particular, $L_1 \leqslant L_2 \leqslant \cdots \leqslant L_{\ell(v)}$. Set $T_0 = T_u$ and $T_j = T_u v_j + u T_{v_j}$ for $1 \leqslant j \leqslant \ell(v)$. Then for $1 \leqslant j \leqslant \ell(v)$, T_j is the result of switching rows i_j and i_{j+1} in T_{j-1} and replacing the newly made 0 with L_j. Since all letters in T_0 are marked, and $L_1 \leqslant L_2 \leqslant \cdots \leqslant L_{\ell(v)}$, we conclude from Lemma 1.4(a) that each T_j is balanced, and hence $T = T_{\ell(v)}$ is balanced.

The last statement also follows from Lemma 1.4: given a BSL of $D(w)$, we can successively remove border cells containing the largest labels (which are unmarked), and the result will be a BSL of a diagram $D(u)$ for some permutation u which contains only unmarked letters. The removals give the desired permutation $v = u^{-1}w$.

For uniqueness, note that if at any point we have two choices of border cells to remove in rows i and j, then $|i - j| > 1$. Otherwise, if $j = i + 1$, for example, then by the balanced condition at the hook of box $(i, w(i+2))$, $T(i, w(i+1)) = T(i, w(i+2)) = T(i+1, w(i+2))$, which contradicts our strictness conditions. Since s_i and s_j commute for $|i - j| > 1$, it does not matter which one we do first. $\square$

Given a BSL T of $D(w)$, let $f_T(i)$, respectively $f_T(i')$, be the number of occurrences of i, respectively i'. Define a monomial

$$m(T) = x_1^{f_T(1)} \cdots x_{n-1}^{f_T(n-1)} (-y_1)^{f_T(1')} \cdots (-y_{n-1})^{f_T((n-1)')}. \tag{1.6}$$

One more bit of notation: given a labeling T of $D(w)$, let T^* denote the labeling of $D(w^{-1})$ obtained by transposing T and performing the swap $i \leftrightarrow i'$.

Theorem 1.7. *For every permutation w,*

$$\mathfrak{S}_w(x, y) = \sum_T m(T),$$

where the sum is over all BSL T of $D(w)$.

Proof. Suppose we are given a BSL T of $D(w)$. By Lemma 1.5, there exists a unique pair of permutations v^{-1} and u such that $v^{-1}u = w$, $\ell(w) = \ell(v^{-1}) + \ell(u)$, a BSL $T_{v^{-1}}$ of $D(v^{-1})$ which only uses marked letters, and a BSL T_u of $D(u)$ which only uses unmarked letters, such that $T = T_u v^{-1} + u T_{v^{-1}}$. The labeling $T_v = T_{v^{-1}}^*$ gives a BSL of $D(v)$ which only uses unmarked letters.

Finally, using (1.2) coupled with the fact that $\mathfrak{S}_u(x) = \sum_T m(T)$, where the sum is over all BSL of $D(u)$ using only unmarked letters [FGRS, Theorem 6.2], we get the desired result. $\quad\square$

Remark 1.8. The operation $T \mapsto T^*$ gives a concrete realization of the symmetry $\mathfrak{S}_w(-y, -x) = \mathfrak{S}_{w^{-1}}(x, y)$ [Man, Corollary 2.4.2].

2. Double Schubert functors

2.1. Super linear algebra preliminaries

Let $V = V_0 \oplus V_1$ be a free $\mathbf{Z}/2$-graded module over a (commutative) ring R with $V_0 = \langle e_1, \ldots, e_n \rangle$ and $V_1 = \langle e'_1, \ldots, e'_m \rangle$, and let $\mathfrak{gl}(m|n) = \mathfrak{gl}(V)$ be the Lie superalgebra of endomorphisms of V. Let $\mathfrak{b}(m|n) \subset \mathfrak{gl}(m|n)$ be the standard Borel subalgebra of upper triangular matrices with respect to the ordered basis $\langle e'_m, \ldots, e'_1, e_1, \ldots, e_n \rangle$. We will mainly deal with the case $m = n$, in which case we write $\mathfrak{b}(n) = \mathfrak{b}(n|n)$, and if it is clear from context, we will drop the n and simply write $\mathfrak{b}$. Also, let $\mathfrak{b}(n)_0 = \mathfrak{gl}(V)_0 \cap \mathfrak{b}(n)$ be the even degree elements in $\mathfrak{b}(n)$, and again, we will usually denote this by $\mathfrak{b}_0$. We also write $\mathfrak{h}(n) \subset \mathfrak{b}(n)$ for the Cartan subalgebra of diagonal matrices (this is a Lie algebra concentrated in degree 0). Let $\varepsilon'_n, \ldots, \varepsilon'_1, \varepsilon_1, \ldots, \varepsilon_n$ be the dual basis vectors to the standard basis of $\mathfrak{h}(n)$. For notation, write $(a_n, \ldots, a_1 \mid b_1, \ldots, b_n)$ for $\sum_{i=1}^n (a_i \varepsilon'_i + b_i \varepsilon_i)$. The even and odd roots of $\mathfrak{b}(n)$ are $\Phi_0 = \{\varepsilon'_j - \varepsilon'_i, \ \varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$ and $\Phi_1 = \{\varepsilon'_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant n\}$, respectively. The even and odd simple roots are $\Delta_0 = \{\varepsilon'_{i+1} - \varepsilon'_i, \ \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}$ and $\Delta_1 = \{\varepsilon'_1 - \varepsilon_1\}$.

Given a highest weight representation W of $\mathfrak{b}(n)$, we have a weight decomposition $W = \bigoplus_\lambda W_\lambda$ as a representation of $\mathfrak{h}(n)$. Let Λ be the highest weight of W. Then every weight λ appearing in the weight decomposition can be written in the form $\Lambda - \sum n_\alpha \alpha$ where α ranges over the simple roots of $\mathfrak{b}(n)$ and $n_\alpha \in \mathbf{Z}_{\geqslant 0}$. For such a λ, set $\omega(\lambda) = (-1)^{\sum n_\alpha \deg \alpha}$. Then we define the *character* and *supercharacter* of W as

$$\operatorname{ch} W = \sum_\lambda (\dim W_\lambda) e^\lambda, \qquad \operatorname{sch} W = \sum_\lambda \omega(\lambda)(\dim W_\lambda) e^\lambda. \tag{2.1}$$

Here the e^λ are formal symbols with the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$.

We will need $\mathbf{Z}/2$-graded analogues of the divided and exterior powers (see [Wey, §2.4] for the dual versions of our definitions). Let $F = F_0 \oplus F_1$ be a free R-supermodule. Let D denote the divided

power functor, let $\bigwedge$ denote the exterior power functor, and let Sym denote the symmetric power functor. Then $\bigwedge^i F$ and $\mathrm{D}^i F$ are $\mathbf{Z}$-graded modules with terms given by

$$\left(\overset{i}{\bigwedge} F\right)_d = \overset{i-d}{\bigwedge} F_0 \otimes \mathrm{Sym}^d F_1, \qquad \left(\mathrm{D}^i F\right)_d = \mathrm{D}^{i-d} F_0 \otimes \overset{d}{\bigwedge} F_1. \tag{2.2}$$

We can define a coassociative $\mathbf{Z}$-graded comultiplication $\Delta : \mathrm{D}^{i+j} F \to \mathrm{D}^i F \otimes \mathrm{D}^j F$ as follows. On degree d, pick $0 \leqslant a \leqslant i$ and $0 \leqslant b \leqslant j$ such that $a + b = d$. Then we have the composition $\Delta_{a,b}$,

$$\left(\mathrm{D}^{i+j} F\right)_d = \mathrm{D}^{i+j-a-b} F_0 \otimes \overset{a+b}{\bigwedge} F_1$$

$$\xrightarrow{\Delta' \otimes \Delta'} \mathrm{D}^{i-a} F_0 \otimes \mathrm{D}^{j-b} F_0 \otimes \overset{a}{\bigwedge} F_1 \otimes \overset{b}{\bigwedge} F_1$$

$$\cong \mathrm{D}^{i-a} F_0 \otimes \overset{a}{\bigwedge} F_1 \otimes \mathrm{D}^{j-b} F_0 \otimes \overset{b}{\bigwedge} F_1$$

$$= \left(\mathrm{D}^i F\right)_a \otimes \left(\mathrm{D}^j F\right)_b, \tag{2.3}$$

where Δ' is the usual comultiplication, and we define Δ on the degree d part to be $\sum_{a+b=d} \Delta_{a,b}$.

Similarly, we can define an associative $\mathbf{Z}$-graded multiplication $m : \bigwedge^i F \otimes \bigwedge^j F \to \bigwedge^{i+j} F$ as follows. For degrees a and b, we have

$$\left(\overset{i}{\bigwedge} F\right)_a \otimes \left(\overset{j}{\bigwedge} F\right)_b = \overset{i-a}{\bigwedge} F_0 \otimes \mathrm{Sym}^a F_1 \otimes \overset{j-b}{\bigwedge} F_0 \otimes \mathrm{Sym}^b F_1$$

$$\cong \overset{i-a}{\bigwedge} F_0 \otimes \overset{j-b}{\bigwedge} F_0 \otimes \mathrm{Sym}^a F_1 \otimes \mathrm{Sym}^b F_1$$

$$\xrightarrow{m' \otimes m'} \overset{i+j-a-b}{\bigwedge} F_0 \otimes \mathrm{Sym}^{a+b} F_1 = \left(\overset{i+j}{\bigwedge} F\right)_{a+b}, \tag{2.4}$$

where m' is the usual multiplication.

2.2. Constructions

Define a flag of $\mathbf{Z}/2$-graded submodules

$$V^\bullet : V^{-n} \subset \cdots \subset V^{-1} \subset V^1 \subset \cdots \subset V^n \tag{2.5}$$

such that V^{-1} consists of all of the odd elements of V^n. We will say that the flag is *split* if each term and each quotient is a free module. Fix a permutation $w \in \Sigma_n$. Let $r_k = r_k(w)$, respectively $c_j = c_j(w)$, be the number of boxes in the kth row, respectively jth column, of $D(w)$. Define $\chi_{k,j}$ to be 1 if $(k, j) \in D(w)$ and 0 otherwise. Consider the map

$$\bigotimes_{k=1}^{n-1} \mathrm{D}^{r_k} V^k \xrightarrow{\otimes \Delta} \bigotimes_{k=1}^{n-1} \bigotimes_{j=1}^{n-1} \mathrm{D}^{\chi_{k,j}} V^k \cong \bigotimes_{j=1}^{n-1} \bigotimes_{k=1}^{n-1} \mathrm{D}^{\chi_{k,j}} V^k$$

$$\xrightarrow{\otimes m} \bigotimes_{j=1}^{n-1} \bigwedge^{c_j} V^{w^{-1}(j)}$$

$$\xrightarrow{\otimes \pi} \bigotimes_{j=1}^{n-1} \bigwedge^{c_j} \left(V^{w^{-1}(j)} / V^{-j-1} \right), \tag{2.6}$$

where $\otimes \pi$ denotes the product of projection maps. Note that $D^1 V^k = V^k$ and $D^0 V^k = R$, so that the multiplication above makes sense. Then its image $\mathscr{S}_w(V^\bullet)$ is the $\mathbf{Z}/2$-*graded Schubert functor*, or *double Schubert functor*. By convention, the empty tensor product is R, so that if w is the identity permutation, then $\mathscr{S}_w(V^\bullet) = R$.

This definition is clearly functorial: given an even map of flags $f : V^\bullet \to W^\bullet$, i.e., $f(V^k) \subset W^k$ for $-n \leqslant k \leqslant n$, we have an induced map $f : \mathscr{S}_w(V^\bullet) \to \mathscr{S}_w(W^\bullet)$.

We will focus on the case when $V^{-i} = \langle e'_n, e'_{n-1}, \ldots, e'_i \rangle$ and $V^i = V^{-1} + \langle e_1, e_2, \ldots, e_i \rangle$, so that $\mathscr{S}_w = \mathscr{S}_w(V^\bullet)$ is a $\mathfrak{b}(n)$-module.

Remark 2.7. One could dually define the double Schubert functor as the image of (dual) exterior powers mapping to symmetric powers. One has to be careful, because the $\mathbf{Z}/2$-graded version of exterior powers are not self-dual. For the dual of our definition, one uses $(\bigwedge^i F)_d = \bigwedge^{i-d} F_0 \otimes D^d F_1$. We have chosen our definitions to be consistent with [KP]. This will be especially convenient for Theorem 2.13.

Remark 2.8. We could also define $\mathscr{S}_D(V^\bullet)$ for an arbitrary diagram D which does not necessarily come from a permutation. This is relevant in [KP, §4], whose proof we use in Theorem 2.13. However, since the details will go through without significant changes, we will have no need to elaborate on this point.

Lemma 2.9. *Let* $w \in \Sigma_n$ *and* $v \in \Sigma_m$ *be two permutations. Define a new permutation* $u \in \Sigma_{n+m}$ *by* $u(i) = w(i)$ *for* $i = 1, \ldots, n$ *and* $u(n+j) = v(j)$ *for* $j = 1, \ldots, m$. *Also, define a permutation* $v' \in \Sigma_{n+m}$ *by* $v'(i) = i$ *for* $i = 1, \ldots, n$ *and* $v'(n+j) = v(j)$ *for* $j = 1, \ldots, m$. *Then*

$$\mathscr{S}_u(V^\bullet) \cong \mathscr{S}_w(V^\bullet) \otimes \mathscr{S}_{v'}(V^\bullet).$$

Proof. This follows from the definition of double Schubert functors, the fact that $D(u) = D(w) \cup D(v')$, and the fact that no two cells of $D(w)$ and $D(v')$ lie in the same row or column. □

Example 2.10. Consider $n = 3$ and $w = 321$. Then $r_1 = 2$, $r_2 = 1$, $c_1 = 2$, and $c_2 = 1$. We need to calculate the image of the map

$$D^2 V^1 \otimes V^2 \xrightarrow{\Delta \otimes 1} (V^1 \otimes V^1) \otimes V^2 \xrightarrow{t_{2,3}} (V^1 \otimes V^2) \otimes V^1 \xrightarrow{m \otimes 1} \bigwedge^2 (V^3 / V^{-2}) \otimes V^2 / V^{-3},$$

where $t_{2,3}$ is the map that switches the second and third parts of the tensor product. Write x, y for e_1, e_2, and x', y' for e'_1, e'_2. We can ignore e_3 and e'_3 since they will not appear in the image. We can write $D^2 V^1 = \langle x^2, x \otimes x', x \otimes y', x' \wedge y' \rangle$ and $V^2 = \langle x, y, x', y' \rangle$. Then we have

$$m\left(t_{2,3}\left(\Delta\left(x^2 \otimes x\right)\right)\right) = m\left(t_{2,3}(x \otimes x \otimes x)\right) = 0,$$

$$m\left(t_{2,3}\left(\Delta\left(x^2 \otimes y\right)\right)\right) = m\left(t_{2,3}(x \otimes x \otimes y)\right) = (x \wedge y) \otimes x,$$

$$m\left(t_{2,3}\left(\Delta\left(x^2 \otimes x'\right)\right)\right) = m\left(t_{2,3}(x \otimes x \otimes x')\right) = (x \otimes x') \otimes x,$$

$$m\left(t_{2,3}\left(\Delta\left(x^2 \otimes y'\right)\right)\right) = m\left(t_{2,3}(x \otimes x \otimes y')\right) = 0,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes x'\otimes x\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes x'+x'\otimes x\big)\otimes x\big)\big)=\big(x\otimes x'\big)\otimes x,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes x'\otimes y\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes x'+x'\otimes x\big)\otimes y\big)\big)=(x\wedge y)\otimes x'+\big(y\otimes x'\big)\otimes x,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes x'\otimes x'\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes x'+x'\otimes x\big)\otimes x'\big)\big)=\big(x\otimes x'\big)\otimes x'+x'^2\otimes x,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes x'\otimes y'\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes x'+x'\otimes x\big)\otimes y'\big)\big)=0,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes y'\otimes x\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes y'+y'\otimes x\big)\otimes x\big)\big)=0,$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes y'\otimes y\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes y'+y'\otimes x\big)\otimes y\big)\big)=(x\wedge y)\otimes y',$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes y'\otimes x'\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes y'+y'\otimes x\big)\otimes x'\big)\big)=\big(x\otimes x'\big)\otimes y',$$

$$m\big(t_{2,3}\big(\Delta\big(x\otimes y'\otimes y'\big)\big)\big)=m\big(t_{2,3}\big(\big(x\otimes y'+y'\otimes x\big)\otimes y'\big)\big)=0,$$

$$m\big(t_{2,3}\big(\Delta\big(x'\wedge y'\otimes x\big)\big)\big)=m\big(t_{2,3}\big(\big(x'\otimes y'-y'\otimes x'\big)\otimes x\big)\big)=\big(x\otimes x'\big)\otimes y',$$

$$m\big(t_{2,3}\big(\Delta\big(x'\wedge y'\otimes y\big)\big)\big)=m\big(t_{2,3}\big(\big(x'\otimes y'-y'\otimes x'\big)\otimes y\big)\big)=\big(y\otimes x'\big)\otimes y',$$

$$m\big(t_{2,3}\big(\Delta\big(x'\wedge y'\otimes x'\big)\big)\big)=m\big(t_{2,3}\big(\big(x'\otimes y'-y'\otimes x'\big)\otimes x'\big)\big)=x'^2\otimes y',$$

$$m\big(t_{2,3}\big(\Delta\big(x'\wedge y'\otimes y'\big)\big)\big)=m\big(t_{2,3}\big(\big(x'\otimes y'-y'\otimes x'\big)\otimes y'\big)\big)=0.$$

Here is a combinatorial description of the map (2.6). The elements of $\bigotimes_{k=1}^{n-1}\mathrm{D}^{r_k}V^k$ can be thought of as labelings of $D=D(w)$ such that in row k, only the labels $n',(n-1)',\ldots,1',1,\ldots,k$ are used, such that there is at most one use of i' in a given row, and such that the entries in each row are ordered in the usual way (i.e., $n'<(n-1)'<\cdots<1'<1<\cdots<k$). Let Σ_D be the permutation group of D. We say that $\sigma\in\Sigma_D$ is *row-preserving* if each box and its image under σ are in the same row. Denote the set of row-preserving permutations as $\mathrm{Row}(D)$. Let T be a labeling of D that is row-strict with respect to the marked letters. Let $\mathrm{Row}(D)_T$ be the subgroup of $\mathrm{Row}(D)$ that leaves T fixed, and let $\mathrm{Row}(D)^T$ be the set of cosets $\mathrm{Row}(D)/\mathrm{Row}(D)_T$. Given $\sigma\in\mathrm{Row}(D)^T$, and considering the boxes as ordered from left to right, let $\alpha(T,\sigma)_k$ be the number of inversions of σ among the *marked letters* in the kth row, and define $\alpha(T,\sigma)=\sum_{k=1}^{n-1}\alpha(T,\sigma)_k$. Note that this number is independent of the representative chosen since T is row strict with respect to the marked letters. Then the comultiplication sends T to $\sum_{\sigma\in\mathrm{Row}(D)^T}(-1)^{\alpha(T,\sigma)}\sigma T$ where σT is the result of permuting the labels of T according to σ.

For the multiplication map, we can interpret the columns as being alternating in the *unmarked letters* and symmetric in the *marked letters*. We write $m(T)$ for the image of T under this equivalence relation. Therefore, the map (2.6) can be defined as

$$T\mapsto\sum_{\sigma\in\mathrm{Row}(D)^T}(-1)^{\alpha(T,\alpha)}m(\sigma T).\tag{2.11}$$

2.3. A basis and a filtration

In order to prove properties of $\mathscr{S}_w$, we will construct a filtration by submodules, which is based on the filtration of the single Schubert functors introduced by Kraśkiewicz and Pragacz [KP].

Let $w\in\Sigma_n$ be a nonidentity permutation. Consider the set of pairs (α,β) such that $\alpha<\beta$ and $w(\alpha)>w(\beta)$. Choose (α,β) to be maximal with respect to the lexicographic ordering. Let $\gamma_1<\cdots<\gamma_k$ be the numbers such that $\gamma_t<\alpha$ and $w(\gamma_t)<w(\beta)$, and such that $\gamma_t<i<\alpha$ implies that $w(i)\notin\{w(\gamma_t),w(\gamma_t)+1,\ldots,w(\beta)\}$. Then we have the following identity of double Schubert polynomials

$$\mathfrak{S}_w=\mathfrak{S}_v\cdot(x_\alpha-y_{w(\beta)})+\sum_{t=1}^{k}\mathfrak{S}_{\psi_t},\tag{2.12}$$

where $v = wt_{\alpha,\beta}$ and $\psi_t = wt_{\alpha,\beta}t_{\gamma_t,\alpha}$. Here $t_{i,j}$ denotes the transposition which switches i and j. See, for example, [Man, Exercise 2.7.3]. The formula in (2.12) will be called a *maximal transition* for w. Define the *index* of a permutation u to be the number $\sum_k (k-1)\#\{j > k \mid u(k) > u(j)\}$. Note that the index of ψ_t is smaller than the index of w.

When $w = s_i$ is a simple transposition, $v = 1$ is the identity, $k = 1$, and $\psi_1 = s_{i-1}$. See Example 3.11 for more details regarding the filtration in this case.

Theorem 2.13. *Let $V^\bullet$ be a split flag as in (2.5). Given a nonidentity permutation $w \in \Sigma_n$, let (2.12) be the maximal transition for w. Then there exists a functorial $\mathfrak{b}$-equivariant filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset F' \subset F = \mathscr{S}_w(V^\bullet)$$

such that $F/F' \cong \mathscr{S}_v(V^\bullet) \otimes V^\alpha/V^{\alpha-1}$, $F'/F_k \cong \mathscr{S}_v(V^\bullet) \otimes V^{-w(\beta)}/V^{-w(\beta)-1}$, and $F_t/F_{t-1} \cong \mathscr{S}_{\psi_t}(V^\bullet)$ for $t = 1, \ldots, k$.

Proof. For notation, write $W^i = V^i/V^{-w(i)-1}$, and let $p : W^\beta \to V^\beta/V^{\alpha-1}$ be the projection map. Define the $\mathfrak{b}$-equivariant morphism φ' by the composition

$$\overset{c_{w(1)}}{\bigwedge} W^1 \otimes \cdots \otimes \overset{c_{w(\alpha)}}{\bigwedge} W^\alpha \otimes \cdots \otimes \overset{c_{w(\beta)}}{\bigwedge} W^\beta \otimes \cdots \xrightarrow{1\otimes\cdots\otimes 1\otimes\cdots\otimes\Delta\otimes\cdots}$$

$$\overset{c_{w(1)}}{\bigwedge} W^1 \otimes \cdots \otimes \overset{c_{w(\alpha)}}{\bigwedge} W^\alpha \otimes \cdots \otimes \overset{c_{w(\beta)}-1}{\bigwedge} W^\beta \otimes W^\beta \otimes \cdots \xrightarrow{T}$$

$$\left(\overset{c_{w(1)}}{\bigwedge} W^1 \otimes \cdots \otimes \overset{c_{w(\beta)}-1}{\bigwedge} W^\beta \otimes \cdots \otimes \overset{c_{w(\alpha)}}{\bigwedge} W^\alpha \otimes \cdots \right) \otimes W^\beta$$

where T is the map which switches the order of the tensor product in the way prescribed. Let $\varphi = (1 \otimes p) \circ \varphi'$. We set $F' = \ker \varphi$. Let C_w and C_v be the $\mathfrak{b}$-cyclic generators of $\mathscr{S}_w(V^\bullet)$ and $\mathscr{S}_v(V^\bullet)$, respectively. These are given by BSLs where the ith row only has the label i. By maximality of the pair (α, β) (with respect to the property $\alpha < \beta$ and $w(\alpha) > w(\beta)$), the lowest box in column $w(\beta)$ is in row α. Hence, restricting φ to $F = \mathscr{S}_w(V^\bullet)$, we get $\varphi(C_w) = C_v \otimes e_\alpha$. This gives an isomorphism $F/F' \to \mathscr{S}_v(V^\bullet) \otimes V^\alpha/V^{\alpha-1}$.

Let $X \in \mathfrak{b}$ be the matrix defined by $X(e_\alpha) = e'_{w(\beta)}$ and $X(e_i) = 0$ for $i \neq \alpha$ and $X(e'_j) = 0$ for all j. We claim that the rightmost box in row α has column index $w(\beta)$. If not, then there is a box $(\alpha, w(\beta')) \in D(w)$ with $w(\beta') > w(\beta)$ and $\beta' > \alpha$. If $\beta' < \beta$, then $(\beta', \beta) > (\alpha, \beta)$ which contradicts the maximality of (α, β). Otherwise, if $\beta' > \beta$, we have $(\alpha, \beta') > (\alpha, \beta)$ which also contradicts maximality. This contradiction proves the claim. The claim implies that $\varphi'(X(C_w)) = C_v \otimes e'_{w(\beta)}$, and hence $X(C_w) \in \ker \varphi$. Letting F'' be the $\mathfrak{b}$-submodule generated by $X(C_w)$, we get an isomorphism $F''/\ker \varphi' \cong \mathscr{S}_v(V^\bullet) \otimes V^{-w(\beta)}/V^{-w(\beta)-1}$.

Using the notation of [KP, §4] with the obvious changes (see also [KP, Remark 5.3]), let $F_t = \sum_{r \leqslant t} S_{\mathcal{I}_r}(V^\bullet)$.[2] The proofs from [KP, §4] of the fact that there is a surjection $F_t/F_{t-1} \to \mathscr{S}'_{\psi_t}(V^\bullet)$ in the ungraded case extend to the $\mathbf{Z}/2$-graded case. We just need to show that these surjections are actually isomorphisms and that $F' = F''$. Since this is all defined over $\mathbf{Z}$ and obtained for arbitrary R via extension of scalars, it is enough to prove the corresponding statements when R is a field of arbitrary characteristic.

We will use the proof of Step 3 in [KP, §4]. The key steps there involve a tensor product identity [KP, Lemma 1.8], using the maximal transitions, and verifying the theorem for the simple transpositions. The tensor product identity in our case is Lemma 2.9, and the maximal transitions still

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[2] There is a typo in the definition of $\mathcal{F}_t$ in [KP] regarding $\leqslant$ versus $<$.

exist. Also, the fact that the statement is valid for simple transpositions can be seen directly, or see Example 3.11. The rest of the proof goes through using the definitions $d_w = \dim_R \mathscr{S}_w(V^\bullet)$ and $z_w = \mathfrak{S}_w(1, -1)$. So we have defined the desired filtration, and the functoriality is evident from the constructions. $\quad\square$

Given any labeling T, denote its weight by $w(T) = (a_{-n}, \ldots, a_{-1} | a_1, \ldots, a_n)$, where a_i is the number of times that the label i is used, and a_{-i} is the number of times that the label i' is used. We define a *dominance order* $\geqslant$ by $(a_{-n}, \ldots, a_{-1} | a_1, \ldots, a_n) \geqslant (a'_{-n}, \ldots, a'_{-1} | a'_1, \ldots, a'_n)$ if $\sum_{i=-n}^k a_i \geqslant \sum_{i=-n}^k a'_i$ for all $-n \leqslant k \leqslant n$.

Theorem 2.14. *Assume that the flag $V^\bullet$ is split. The images of the BSLs under (2.6) form a basis over R for $\mathscr{S}_w$.*

Proof. Since the BSLs are defined when $R = \mathbf{Z}$, and are compatible with extension of scalars, it is enough to show that the statement is true when $R = K$ is an infinite field of arbitrary characteristic, so we will work in this case.

We can show linear independence of the BSLs following the proof of [FGRS, Theorem 7.2]. Combined with Theorem 2.13, this will show that they form a basis. First, we note that $\mathscr{S}_w$ has a weight decomposition since it is a highest weight module of $\mathfrak{b}(n)$, so we only need to show linear independence of the BSLs in each weight space. Second, if T is column-strict, row-strict, and satisfies the flagged conditions, then $T \neq \pm \sigma T$ in $\mathscr{S}_w$ whenever $\sigma \in \mathrm{Row}(D)_T$. This implies that if T is a BSL, then the image of T under (2.6) is nonzero.

First write $L = \ell(w)$. We assign to T a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_L}$ of w following the method in Lemma 1.5 by induction on L. If T contains unmarked letters, let M be the largest such label, and let i_L denote the smallest row index which contains M in a border cell. Let $s_{i_1} \cdots s_{i_{L-1}}$ be the reduced decomposition assigned to the labeling $T \setminus (i_L, w(i_L))$ of ws_{i_L}, so that we get the reduced decomposition $s_{i_1} \cdots s_{i_L}$ for w. If T contains no unmarked letters, let $s_{i_1^*} \cdots s_{i_L^*}$ be the reduced decomposition associated to the labeling T^* of w^{-1} and then assign the reduced decomposition $s_{i_L^*} \cdots s_{i_1^*}$ to T. So we can write this reduced decomposition as $s^*(T)s(T)$ where $s^*(T)$, respectively $s(T)$, corresponds to the transpositions coming from removing marked, respectively unmarked, letters. We will totally order reduced decompositions as follows: $s_{i_1} \cdots s_{i_L} < s_{i'_1} \cdots s_{i'_L}$ if there exists a j such that $i_j < i'_j$ and $i_k = i'_k$ for $j + 1 \leqslant k \leqslant L$. We say that $s^*(T')s(T') \leqslant s^*(T)s(T)$ if either $s(T') < s(T)$ (the ordering for reduced decompositions), or $s(T') = s(T)$ and $s^*(T')^{-1} < s(T)^{-1}$ (the inverse means write the decomposition backwards).

Taking into account the description (2.11), we show that if $m(T') = \pm m(\sigma T)$ where T' and T are BSLs and $\sigma \in \mathrm{Row}(D)$, then $s^*(T')s(T') \leqslant s^*(T)s(T)$. Note that since we assume that T and T' have the same weights, we have $s^*(T')s(T') = s^*(T)s(T)$ if and only if $T = T'$. The BSLs are linearly independent in $\bigotimes_k \mathrm{D}^{r_k} V^k$, so by induction on $\leqslant$, we see that the coefficients of any linear dependence of their images in $\mathscr{S}_w$ must all be zero.

So suppose that $m(T') = \pm m(\sigma T)$ holds and choose representatives T' and σT that realize this equality. First suppose that T contains an unmarked letter. Then so does T', and let M be the largest such one. Write $s(T) = s_{i_r} \cdots s_{i_L}$ and $s(T') = s_{i'_r} \cdots s_{i'_L}$. Since m only affects entries within the same column, the M in row i'_L is moved to some row with index $\leqslant i'_L$ because M occupies a border cell. By definition of i_L, all instances of M in σT lie in rows with index $\geqslant i_L$ since σ is row-preserving. Hence the equality $m(T') = m(\sigma T)$ implies that $i'_L \leqslant i_L$.

If $i'_L < i_L$, there is nothing left to do. So suppose that $i'_L = i_L$. Then M lies in the same border cell b in both T and T'. Hence $T' \setminus b = \sigma(T \setminus b)$, and we conclude by induction. So we only need to handle the case that T (and hence T') do not contain any unmarked letters. In this case, we pass to T^* and T'^*, and the above shows that $s^*(T')^{-1} \leqslant s^*(T)^{-1}$, so we are done. $\quad\square$

The above proof does not establish how one can write the image of an arbitrary labeling as a linear combination of the images of the BSLs. Such a straightening algorithm is preferred, but we have not been successful in finding one, so we leave this task as an open problem.

Problem 2.15. Find an algorithm for writing the image of an arbitrary labeling of $D(w)$ as a linear combination of the images of the BSLs of $D(w)$.

Corollary 2.16. *Identify $x_i = -e^{\varepsilon_i}$ and $y_i = -e^{\varepsilon'_i}$ for $1 \leqslant i \leqslant n$. Then*

$$\mathrm{ch}\, \mathscr{S}_w = \mathfrak{S}_w(-x, y), \qquad \mathrm{sch}\, \mathscr{S}_w = \mathfrak{S}_w(x, y).$$

Corollary 2.17. *Choose an ordering of the set of permutations below w in the weak Bruhat order: $1 = v_1 \prec v_2 \prec \cdots \prec v_N = w$ such that $v_i \prec v_{i+1}$ implies that $\ell(v_i) \leqslant \ell(v_{i+1})$. Then there exists a $\mathfrak{b}$-equivariant filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathscr{S}_w$$

such that

$$F_i/F_{i-1} \cong \mathscr{S}'_{v_i} \otimes \mathscr{S}''_{wv_i^{-1}}$$

as $\mathfrak{b}_0$-modules.

Proof. Let $\mathscr{S}'$, respectively $\mathscr{S}''$, denote the usual Schubert functor which uses only unmarked, respectively marked, letters. Let $W_v = \mathscr{S}'_v \otimes \mathscr{S}''_{wv^{-1}}$. Theorem 2.14 implies that we have a $\mathfrak{b}_0$-equivariant decomposition $\mathscr{S}_w = \bigoplus_{v \leqslant_w w} W_v$. Let $F_i = \bigoplus_{j \leqslant i} W_{v_j}$. Then F_i is a $\mathfrak{b}_0$-submodule, and applying an element of $\mathfrak{b} \setminus \mathfrak{b}_0$ to W_{v_j} can only give elements in W_{v_k} where $\ell(v_k) < \ell(v_j)$. So F_i is in fact a $\mathfrak{b}$-submodule, and we have the desired filtration. $\square$

3. Schubert complexes

Now we can use the above machinery to define Schubert complexes. We start with the data of two split flags $F_0^\bullet : 0 = F_0^0 \subset F_0^1 \subset \cdots \subset F_0^n = F_0$ and $F_1^\bullet : F_1^{-n} \subset F_1^{-n+1} \subset \cdots \subset F_1^{-1} = F_1$, and a map $\partial : F_0 \to F_1$ between them. Given the flag for F_0, we pick an ordered basis $\{e_1, \ldots, e_n\}$ for it such that $e_i \in F_0^i \setminus F_0^{i-1}$. Similarly, we pick an ordered basis $\{e'_1, \ldots, e'_n\}$ for F_1 such that $e'_i \in F_1^{-i} \setminus F_1^{-i-1}$. Given these bases, we can represent ∂ as a matrix. This matrix representation will be relevant for the definition of certain ideals later.

Equivalently, we can give $F_1^\bullet$ as a quotient flag $F_1 = G^n \twoheadrightarrow G^{n-1} \twoheadrightarrow \cdots \twoheadrightarrow G^1 \twoheadrightarrow G^0 = 0$, so that the correspondence is given by $F_1^{-i} = \ker(G^n \twoheadrightarrow G^{i-1})$. Note that $F_1^{-i}/F_1^{-i-1} = \ker(G^i \twoheadrightarrow G^{i-1})$. We assume that each quotient has rank 1. Then we form a flagged $\mathbf{Z}/2$-graded module F with even part F_0 and odd part F_1. The formation of divided and exterior products commutes with the differential ∂ by functoriality, so we can form the *Schubert complex* $\mathscr{S}_w(F)$ for a permutation $w \in \Sigma_n$.

Proposition 3.1. *The ith term of $\mathscr{S}_w(F)$ has a natural filtration whose associated graded is*

$$\bigoplus_{\substack{v \leqslant_w w \\ \ell(v) = i}} \mathscr{S}_v(F_0) \otimes \mathscr{S}_{wv^{-1}}(F_1).$$

Proof. This is a consequence of Corollary 2.17. $\square$

Proposition 3.2. *Let $\partial : F_0 \to F_1$ be a map. With the notation as in Theorem 2.13, there is a functorial $\mathfrak{b}$-equivariant filtration of complexes*

$$0 = C_0 \subset C_1 \subset \cdots \subset C_k \subset C' \subset C = \mathscr{S}_w(\partial)$$

such that $C/C' \cong \mathscr{S}_v(\partial)[-1] \otimes F_0^\alpha/F_0^{\alpha-1}$, $C'/C_k \cong \mathscr{S}_v(\partial) \otimes F_1^{-w(\beta)}/F_1^{-w(\beta)+1}$, and $C_t/C_{t-1} \cong \mathscr{S}_{\psi_t}(\partial)$ for $t = 1, \ldots, k$.

Proof. The filtration of Theorem 2.13 respects the differentials since everything is defined in terms of multilinear operations. The grading shift of C/C' follows from the fact that the F_0 terms have homological degree 1. □

Corollary 3.3. *Let $\partial : F_0 \to F_1$ be a flagged isomorphism. Then $\mathscr{S}_w(\partial)$ is an exact complex whenever $w \neq 1$.*

Proof. This is an immediate consequence of Proposition 3.2 using induction on length and index, and the long exact sequence on homology: when $w = s_i$, exactness is obvious. □

3.1. Flag varieties and K-theory

Throughout this section, we use [F2] as a reference. The reader may wish to see [F2, Appendix B.1, B.2] for the conventions used there.

We will need some facts about the geometry of flag varieties. Let V be a vector space with ordered basis $\{e_1, \ldots, e_n\}$. Then the complete flag variety **Flag**(V) can be identified with **GL**$(V)/B$ where B is the subgroup of upper triangular matrices with respect to the given basis. For a permutation $w \in \Sigma_n$, we define the *Schubert cell* Ω_w to be the B-orbit of the flag

$$\langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots, e_{w(n-1)} \rangle \subset V.$$

Then Ω_w is an affine space of dimension $\ell(w)$ (see [Man, §3.6]), and the flag variety is a disjoint union of the Ω_w. The *Schubert variety* X_w is the closure of Ω_w. Alternatively,

$$X_w = \left\{ W_\bullet \in \textbf{Flag}(V) \,\middle|\, \dim\bigl(W_p \cap \langle e_1, \ldots, e_q \rangle\bigr) \geqslant r_w(p, q) \right\}.$$

Recall from Section 1.1 that $r_w(p, q) = \#\{i \leqslant p \mid w(i) \leqslant q\}$. Given a matrix ∂ and a permutation w, let $I_w(\partial)$ be the ideal generated by the $(r_w(p, q) + 1) \times (r_w(p, q) + 1)$ minors of the upper left $p \times q$ submatrix of ∂. It is clear that $I_v \subseteq I_w$ if and only if $v \leqslant w$. In the case that ∂ is a generic matrix of variables over some coefficient ring R, let $X(w)$ be the variety defined by $I_w(\partial) \subset R[\partial_{i,j}]$. We refer to the ideals $I_w(\partial)$ as *Schubert determinantal ideals*, and the varieties $X(w)$ as *matrix Schubert varieties*. Given a permutation w, we say that a cell α in the diagram $D(w)$ is a *southeast corner* if the cells to the immediate right of α and immediately below α do not belong to $D(w)$.

Theorem 3.4. *Let ∂ be a generic matrix defined over a field, and let w be a permutation.*

(a) *$I_w(\partial)$ is generated by the minors coming from the submatrices whose lower right corner is a southeast corner of $D(w)$.*
(b) *$I_w(\partial)$ is a prime ideal of codimension $\ell(w)$.*
(c) *$X(w)$ is a normal variety.*

Proof. See [MS, Chapter 15] for (a) and (b). For (c), we can realize $X(w)$ as a product of an affine space with an open subset of a Schubert variety in the complete flag variety (see Step 2 of the proof of Theorem 3.8 for more details), so it is enough to know that Schubert varieties are normal. This is proven in [RR, Theorem 3].

See also [KM, Theorem 2.4.3] for more about the relationship of local properties for Schubert varieties and local properties of a product of matrix Schubert varieties with affine spaces. □

Given any scheme X, we let $K(X)$ denote the K-theory of coherent sheaves on X. This is the free Abelian group generated by the symbols $[\mathcal{F}]$ for each coherent sheaf $\mathcal{F}$ modulo the relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for each short exact sequence of the form

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

Given a finite complex $\mathcal{C}_\bullet$ of coherent sheaves, we set $[\mathcal{C}_\bullet] = \sum_i (-1)^i [\mathcal{C}_i] = \sum_i (-1)^i [H_i(\mathcal{C}_\bullet)]$. If X is nonsingular and finite-dimensional, then $K(X)$ has a ring structure given by

$$[\mathcal{F}][\mathcal{F}'] = \sum_{i=0}^{\dim X} (-1)^i \big[\mathcal{T}\mathrm{or}_i^{\mathcal{O}_X} (\mathcal{F}, \mathcal{F}') \big].$$

Now suppose that X is an equidimensional smooth quasi-projective variety over K. For $k \geqslant 0$, let $F^k K(X)$ be the subgroup of $K(X)$ generated by coherent sheaves whose support has codimension at least k, and set $\mathrm{gr}^k K(X) = F^k K(X)/F^{k+1} K(X)$. This filtration is compatible with the ring structure on $K(X)$ [Gro, Théorème 2.12, Corollaire 1], and we set $\mathrm{gr}\, K(X) = \bigoplus_{k \geqslant 0} \mathrm{gr}^k K(X)$ to be the associated graded ring.

Let $A^*(X)$ be the Chow ring of X. We identify this with the direct sum of Chow groups $A_*(X)$ of X via the isomorphism $c \mapsto c \cap [X]$. Let $\varphi : A^*(X) \to \mathrm{gr}\, K(X)$ be the functorial morphism of graded rings which for a closed subvariety $V \subseteq X$ sends $[V]$ to $[\mathcal{O}_V]$. If $\mathcal{F}$ is a coherent sheaf whose support has codimension at least k, then we have $\varphi(Z^k(\mathcal{F})) = [\mathcal{F}]$ as elements of $\mathrm{gr}^k K(X)$ where

$$Z^k(\mathcal{F}) = \sum_{\mathrm{codim}\, V = k} m_V(\mathcal{F})[V], \tag{3.5}$$

and $m_V(\mathcal{F})$ is the length of the stalk of $\mathcal{F}$ at the generic point of V. We will need to know later that φ becomes an isomorphism after tensoring with $\mathbf{Q}$. See [F2, Example 15.1.5, 15.2.16] for more details. For a finite complex of vector bundles $C_\bullet$ such that $[C_\bullet] \in F^k K(X)$, we use $[C_\bullet]_k$ to denote the corresponding element of $\mathrm{gr}^k K(X)$.

Lemma 3.6. *The identity $\varphi(\mathfrak{S}_w(x, y)) = [C_\bullet]_{\ell(w)}$ holds.*

Proof. For a line bundle L of the form $\mathcal{O}(D)$ where D is an irreducible divisor, we have $c_1(L) \cap [X] = [D]$ [F2, Theorem 3.2(f)]. Hence

$$\varphi\big(c_1(L) \cap [X]\big) = \big(1 - [L^\vee]\big)_1 \in \mathrm{gr}^1 K(X)$$

by the short exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

So the same formula holds for all L by linearity, and

$$\varphi(x_i) = 1 - [E_i/E_{i-1}], \qquad \varphi(y_j) = 1 - \big[\ker(F_j \twoheadrightarrow F_{j-1})\big].$$

Let a and b be a new set of variables. We have $\mathfrak{S}_w(a, b) = \sum_{u \leqslant_W w} \mathfrak{S}_u(a) \mathfrak{S}_{uw^{-1}}(-b)$ by (1.2). Doing the transformation $a_i \mapsto x_i - 1$ and $b_j \mapsto y_j - 1$, we get

$$\varphi\big(\mathfrak{S}_w(a, b)\big) = \sum_{u \leqslant_W w} (-1)^{\ell(u)} \mathfrak{S}_u(E) \mathfrak{S}_{uw^{-1}}(F).$$

By Proposition 3.1, this sum is $[\mathscr{S}_w(\partial)]$ (the change from uw^{-1} to wu^{-1} is a consequence of the fact that F_1 in Proposition 3.1 contains only odd elements). So it is enough to show that the substitution $a_i \mapsto a_i + 1$, $b_j \mapsto b_j + 1$ leaves the expression $\mathfrak{S}_w(a, b)$ invariant. This is clearly true for $\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leqslant n}(x_i - y_j)$, and holds for an arbitrary permutation because the divided difference operators (see (1.1)) applied to a substitution invariant function yield a substitution invariant function. $\quad\square$

The flag variety **Flag**(V) is smooth, and its K-theory is freely generated as a group by the structure sheaves $[\mathcal{O}_{X_w}]$ (see [F2, Examples 1.9.1, 15.2.16]). Also, the irreducible components of any B-equivariant subvariety of **Flag**(V) must be Schubert varieties. There is a tautological flag of subbundles

$$0 = \mathcal{R}_0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_{n-1} \subset \mathcal{R}_n = V \times \textbf{Flag}(V)$$

on **Flag**(V), where the fiber of $\mathcal{R}_i$ over a flag $W_\bullet$ is the space W_i. Setting $x_i = -c_1(\mathcal{R}_i/\mathcal{R}_{i-1})$, the Schubert polynomial $\mathfrak{S}_w(x_1, \ldots, x_n)$ represents the Poincaré dual of the Schubert variety $X_{w_0 w}$ (see, for example, [Man, Theorem 3.6.18]).

Corollary 3.7. *Let V be an n-dimensional vector space and let $\mathcal{C}$ be the Schubert complex associated with the permutation w and the identity map of $V \times \textbf{Flag}(V)$ to itself, where the subspace flag consists of the tautological subbundles and the quotient flag consists of trivial vector bundles. Then $[\mathcal{C}]_{\ell(w)} = [\mathcal{O}_{X_{w_0 w}}]$ in $\mathrm{gr}^{\ell(w)} K(\textbf{Flag}(V))$.*

Proof. Both quantities agree with $\varphi(\mathfrak{S}_w(x, 0))$ where $x_i = -c_1(\mathcal{R}_i/\mathcal{R}_{i-1})$ for $i = 1, \ldots, n$. $\quad\square$

3.2. Generic acyclicity of Schubert complexes

Theorem 3.8. *Let $A = K[\partial_{i,j}]$ be a polynomial ring over a field K, and let $\partial : F_0 \to F_1$ be a generic map of variables between two free A-modules.*

(a) *The Schubert complex $\mathscr{S}_w(\partial)$ is acyclic, and resolves a Cohen–Macaulay module M of codimension $\ell(w)$ supported in $I_{w^{-1}}(\partial) \subseteq A$.*
(b) *The restriction of M to $X(w^{-1})$ is a line bundle outside of a certain codimension 2 subset.*
(c) *The Schubert complex defined over the integers is acyclic.*

Before we begin the proof, let us outline the strategy. The main idea is to use the filtration given by Proposition 3.2 and work by induction. The main difficulty is the fact that there is a homological shift in the filtration, which only allows one to conclude that $H_i(\mathscr{S}_w(\partial)) = 0$ for $i > 1$ (see (3.9)). Hence the class of $C = \mathscr{S}_w(\partial)$ in an appropriate Grothendieck group is $[H_0(C)] - [H_1(C)]$. To make this expression more useful, we work with a sheaf version $\mathcal{C}$ of C over a flag variety, where the K-theory possesses a nice basis. To get a handle on $[\mathcal{C}]$, we work with an associated graded of K-theory and show that the top degree terms of $[H_0(\mathcal{C})]$ and $[\mathcal{C}]$ agree. Finally, we show that the support of $H_1(\mathcal{C})$ must be a proper closed subset of the support of $\mathcal{C}$, and we use this to show that $H_1(\mathcal{C})$ must be 0.

Proof. We will prove the statement by induction first on $\ell(w)$ and second on the index of w (see Section 2.3 for definitions). The case $w = 1$ is immediate. Using the notation of Proposition 3.2, it is immediate that C' is acyclic by induction and the long exact sequence on homology. Hence we only need to analyze the short exact sequence

$$0 \to C' \to C \to \mathscr{S}_v(\partial)[-1] \otimes F_0^\alpha/F_0^{\alpha-1} \to 0.$$

The induced long exact sequence is

$$0 \to H_1(C) \to H_0(\mathscr{S}_v(\partial)) \otimes F_0^\alpha/F_0^{\alpha-1} \to H_0(C') \to H_0(C) \to 0, \qquad (3.9)$$

so we have to show that $H_1(C) = 0$, and that the support of $H_0(C) = M$ is $P = I_{w^{-1}}(\partial)$. We proceed in steps.

Step 1. We first show that the length of $H_0(C)_P$ restricted to $X(w^{-1})$ is at most 1.

The short exact sequence

$$0 \to C_k \to C' \to \mathscr{S}_v(\partial) \otimes \langle e'_{w(\beta)} \rangle \to 0$$

induces the sequence

$$0 \to H_0(C_k) \to H_0(C') \to H_0(\mathscr{S}_v(\partial)) \otimes \langle e'_{w(\beta)} \rangle \to 0.$$

By induction on the filtration in Proposition 3.2, the support of $H_0(C_k)$ is in the union of the $X(\psi_t^{-1})$, and hence does not contain $X(w^{-1})$. So localizing at P, we get an isomorphism

$$H_0(C')_P \cong H_0(\mathscr{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle.$$

So we can restrict this isomorphism to $X(w^{-1})$. Localizing (3.9) at P and then restricting to $X(w^{-1})$, we get a surjection

$$H_0(\mathscr{S}_v(\partial))_P \otimes \langle e'_{w(\beta)} \rangle \to H_0(C)_P \to 0.$$

By induction, (b) gives that the first term has length 1 over the generic point of $X(w^{-1})$, so $\text{length}(H_0(C)_P) \leqslant 1$.

Step 2. We show that the length of $H_0(C)_P$ restricted to $X(w^{-1})$ is exactly 1.

Without loss of generality, we may extend ∂ to a generic $2n \times 2n$ matrix by embedding it in the upper left corner. Since $w \in \Sigma_n$, these new variables do not affect the Schubert complex when we interpret w as a permutation of Σ_{2n} by having it fix $\{n + 1, \ldots, 2n\}$, so we will refer to them as *irrelevant variables*. Now consider the Schubert complex $\mathcal{C}$ on the complete flag variety Z of a vector space of dimension $2n$, where the even flag is given by the tautological flag of vector bundles $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}_n$, and the odd flag is given by the trivial vector bundles $V_i = \langle e_1, \ldots, e_i \rangle$. We identify Z with a quotient $\mathbf{GL}(V_{2n})/B$. Restricting to the unique open B-orbit $\Omega = \Omega_{w_0}$ of Z (which is an affine space), we return to the current situation with some of the irrelevant variables of ∂ specialized to 1 and some specialized to 0. So to finish this step, we only need to show that $\text{length}(H_0(\mathcal{C})_P) = 1$.

Let $w_0 \in \Sigma_{2n}$ be the long word. Identify V_i over Ω with $F_1^{-2n-1+i}$. Then the intersection $X_{w_0 w} \cap \Omega$ is defined by the ideal $I_{w^{-1}}(\partial)$:

$$\dim(W_p \cap V_q) \geqslant r_{w_0 w}(p, q) \iff \dim\left(W_p \cap F_1^{-2n-1+q}\right) \geqslant r_{w_0 w}(p, q)$$

$$\iff \text{rank}\left(W_p \to F_1/F_1^{-2n-1+q}\right) \leqslant p - r_{w_0 w}(p, q)$$

$$\iff \text{rank}\left(W_p \to F_1/F_1^{-1-q}\right) \leqslant p - r_{w_0 w}(p, 2n - q) = r_w(p, q),$$

and the map $W_p \to F_1/F_1^{-1-q}$ is given by the upper left $q \times p$ submatrix of ∂.

From the earlier discussion, $[\mathcal{C}] = [H_0(\mathcal{C})] - [H_1(\mathcal{C})]$. By Corollary 3.7, the top dimension term of $[\mathcal{C}]$ is $[\mathcal{O}_{X_{w_0 w}}]$. So $\text{length}(H_0(\mathcal{C})_P) - \text{length}(H_1(\mathcal{C})_P) = 1$. We showed above that the first length is at

most 1, which means that it must be 1, and the stalk of $H_1(\mathcal{C})$ at the generic point of X_{w_0w} must be 0.

Step 3. The annihilator of $H_0(C)$ properly contains $I_{v^{-1}}(\partial)$.

We have that $D(w) = D(v) \cup \{(\alpha, w(\beta))\}$, and $(\alpha, w(\beta))$ is a southeast corner of $D(w)$: no boxes of $D(w)$ lie directly below or to the right of it. This means in particular that $I_{w^{-1}}$ is generated by $I_{v^{-1}}$ and the $(r+1) \times (r+1)$ minors of the upper $w(\beta) \times \alpha$ submatrix of ∂, where $r = r_w(\alpha, w(\beta))$. We will show that a minor in $I_{w^{-1}}$ which is not in $I_{v^{-1}}$ annihilates $H_0(C)$.

The module $H_0(C)$ is generated by the BSLs of $D(w)$ that only contain marked letters. We have that $w(\beta) - r$ is the number of boxes in $D(w)$ in the αth row. Let $J = \{w(\beta) - r, \ldots, w(\beta)\}$ and let I be an $(r+1)$-subset of $\{1, \ldots, \alpha\}$. Set $M_{J,I}$ to be the minor of ∂ consisting of the rows indexed by J and the columns indexed by I. We will show that $M_{J,I}$ annihilates $H_0(C)$.

Given a label j, and a labeling T of the first $\alpha - 1$ rows of $D(w)$, let $T(j)$ be the labeling of $D(w)$ that agrees with T for the first $\alpha - 1$ rows, and in which the ith box in the αth row (going from left to right) has the label i' for $i = 1, \ldots, w(\beta) - r - 1$, and the box $(\alpha, w(\beta))$ has the label j. Let $d: C_1 \to C_0$ denote the differential. Then $d(T(j)) = \sum_{k=1}^{w(\beta)} \partial_{k,j} T(k')$. Note that $T(k') = 0$ whenever $1 \leqslant k < w(\beta) - r$ since in this case the label k' appears in the bottom row twice. Since $\alpha > r$, the α equations

$$\sum_{k=w(\beta)-r}^{w(\beta)} \partial_{k,j} T(k') = 0 \quad \text{for } j = 1, \ldots, \alpha$$

in $H_0(C)$ show that $M_{J,I}$ annihilates $T(k')$ for $1 \leqslant k \leqslant w(\beta)$.

It remains to show that $M_{J,I}$ annihilates the elements T where the labels in the first $w(\beta) - r - 1$ boxes of the αth row of T are allowed to take values in $\{(w(\beta) - r)', \ldots, w(\beta)'\}$. It is enough to show how to vary the entries one box at a time by decreasing their values (remembering that $i' < j'$ if $i > j$). So fix a column index c which contains the ith box in row α and choose $j > i$. Let T_j denote the labeling obtained from T by changing the label in (α, c) from i' to j'. Let $X \in \mathfrak{b}$ be the matrix which sends the basis vector e'_i to e'_j and kills all other basis vectors. Then $X \cdot T$ is equal to T_j plus other terms whose labels in the αth row are the same as those of T, and hence are annihilated by $M_{J,I}$. Since the actions of $\mathfrak{b}$ and A commute with one another, we conclude that $M_{J,I}$ annihilates T_j.

Step 4. We show that $H_1(C) = 0$.

By examining different open affine charts of Z, Step 3 shows that the support of $H_0(\mathcal{C})$ is a proper subset of X_{w_0v}. The argument in Step 2 implies that the same is true for $H_1(\mathcal{C})$ since the structure sheaves of the Schubert varieties form a basis for $K(Z)$. So the codimension of the support of $H_1(\mathcal{C})$ is at least $\ell(w)$. Name the differentials in the complex $d_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$. Restrict to an open affine set. Let r_i be the rank of d_i, and set $I(d_i)$ to be the ideal generated by the $r_i \times r_i$ minors of d_i. Let Q be a prime ideal which does not contain $\sqrt{I(d_1)} = \text{ann}(H_0(\mathcal{C}))$. Then $H_0(\mathcal{C})_Q = 0$, which makes the localization $(d_1)_Q : (\mathcal{C}_1)_Q \to (\mathcal{C}_0)_Q$ a split surjection. Let $\mathcal{C}'_1$ be the quotient of a splitting of $(d_1)_Q$, so that we have a free resolution

$$0 \to (\mathcal{C}_{\ell(w)})_Q \to \cdots \to (\mathcal{C}_2)_Q \to \mathcal{C}'_1$$

of $H_1(\mathcal{C})_Q$. Hence the projective dimension of $H_1(\mathcal{C})_Q$ is at most $\ell(w) - 1$. In general, localizing can only increase the codimension of a module (if we interpret the codimension of 0 to be ∞), so $\text{codim} \, H_1(\mathcal{C})_Q \geqslant \ell(w)$. This is also equal to the depth of its annihilator since Z is nonsingular. So the inequality

$$\text{proj.dim} \, H_1(\mathcal{C})_Q < \text{depth ann} \, H_1(\mathcal{C})_Q$$

contradicts [Eis, Corollary 18.5] unless $H_1(\mathcal{C})_Q = 0$. This implies that $\sqrt{I(d_2)_Q}$ is the unit ideal, which means that $\sqrt{I(d_2)} \nsubseteq Q$. Hence we conclude that any prime ideal which contains $\sqrt{I(d_2)}$ also contains $\sqrt{I(d_1)}$. Since a radical ideal is the intersection of the prime ideals containing it, we conclude that $\sqrt{I(d_1)} \subseteq \sqrt{I(d_2)}$. We also get the inclusions

$$\sqrt{I(d_2)} \subseteq \sqrt{I(d_3)} \subseteq \cdots \subseteq \sqrt{I\big(d_{\ell(w)}\big)}$$

since the rest of the homology of $\mathcal{C}$ vanishes [Eis, Corollary 20.12], so depth $I(d_i) \geqslant$ depth $I(d_1) \geqslant \ell(w)$ for all i. We conclude the acyclicity of $\mathcal{C}$ using the Buchsbaum–Eisenbud criterion [Eis, Theorem 20.9] (the complex is acyclic at the generic point, and the rank of a map over an integral domain stays the same upon passing to its field of fractions, so the rank conditions of this criterion are satisfied).

Step 5. We show that the restriction of $M = H_0(C)$ to $X(w^{-1})$ is a line bundle, and that its support is exactly $X(w^{-1})$.

Since the projective dimension of M is 1 more than the projective dimension of $H_0(\mathcal{S}_v(\partial))$, the codimension of its support can increase by at most 1 by the Auslander–Buchsbaum formula [Eis, Theorem 19.9]. Thus if we can show that P is contained in the annihilator of M, then it must be equal to its annihilator. We have already done this by showing that the stalk of M at the generic point of $X(w^{-1})$ is nonzero. Thus the codimension and projective dimension of M coincide, which means that it is Cohen–Macaulay by the Auslander–Buchsbaum formula. So (a) is proven.

Let Q be the prime ideal associated with a codimension 1 subvariety of $X(w^{-1})$. To prove (b), we only need to show that M_Q is generated by 1 element. Since $X(w^{-1})$ is normal (Theorem 3.4(c)), the local ring $R = \mathcal{O}_{X(v^{-1}),Q}$ is a discrete valuation ring, and hence regular. Furthermore, we have established already that M is Cohen–Macaulay, so M_Q is a free R-module by the Auslander–Buchsbaum formula. So M is free in some open neighborhood around Q. Since M is generated by a single element generically (after further localization), we conclude that M_Q must also be generated by 1 element.

Now (c) follows since we have shown acyclicity over an arbitrary field. $\square$

Corollary 3.10. *Let X be an equidimensional Cohen–Macaulay scheme, and let $\partial : E \to F$ be a map of vector bundles on X. Let $E_1 \subset \cdots \subset E_n = E$ and $F^{-n} \subset \cdots \subset F^{-1} = F$ be split flags of subbundles. Let $w \in \Sigma_n$ be a permutation, and define the degeneracy locus*

$$D_w(\partial) = \big\{ x \in X \mid \mathrm{rank}\big(\partial_x \colon E_p(x) \to F/F^{-q-1}(x)\big) \leqslant r_w(p,q) \big\},$$

where the ideal sheaf of $D_w(\partial)$ is locally generated by the minors given by the rank conditions. Suppose that $D_w(\partial)$ has codimension $\ell(w)$.

(a) *The Schubert complex $\mathcal{S}_w(\partial)$ is acyclic, and the support of its cokernel $\mathcal{L}$ is $D_w(\partial)$.*
(b) *The degeneracy locus $D_w(\partial)$ is Cohen–Macaulay.*
(c) *The restriction of $\mathcal{L}$ to $D_w(\partial)$ is a line bundle outside of a certain codimension 2 subset.*

Proof. The statement is local, so we can replace X by $\mathrm{Spec}\, R$ where R is a local Cohen–Macaulay ring. In this case, $D_w(\partial)$ is defined by the ideal $I_{w^{-1}}(\partial)$. Let ∂^g denote the generic matrix, and let $(C_\bullet, d_\bullet)$ be the complex over $\mathbf{Z}[\partial^g_{i,j}]$ as in Theorem 3.8. We get $(C'_\bullet, d'_\bullet) = \mathcal{S}_w(\partial)$ by specializing the variables $\partial^g_{i,j}$ to elements of R and base changing to R. Let r_i be the rank of d_i, and let $I(d_i)$ be the ideal generated by the $r_i \times r_i$ minors of d_i. By [Eis, Corollary 20.12], $\sqrt{I(d_1)} = \sqrt{I(d_2)} = \cdots = \sqrt{I(d_{\ell(w)})}$ since C is acyclic and since depth $I_{w^{-1}}(\partial^g) = \ell(w)$.

Specializing ∂ to elements of R, the same equalities hold when replacing d_i with d'_i. Noting that $I(d'_1) = \mathrm{ann}\,\mathrm{coker}\,d'_1 \supseteq I_{w^{-1}}(\partial)$, we get that

$$\mathrm{depth}\, I\big(d'_1\big) \geqslant \mathrm{depth}\, I_{w^{-1}}(\partial) = \mathrm{codim}\, I_{w^{-1}}(\partial) = \ell(w)$$

by the assumptions that R is Cohen–Macaulay and that $D_w(\partial)$ has codimension $\ell(w)$. Hence depth $I(d_i') \geqslant \ell(w)$ for $i = 1, \ldots, \ell(w)$, which means that C' is acyclic by the Buchsbaum–Eisenbud acyclicity criterion [Eis, Theorem 20.9]. Finally, since the length of the Schubert complex is $\ell(w)$, we conclude that the depth of the cokernel must be $\ell(w)$ by the Auslander–Buchsbaum formula. So in fact ann coker $d_1' = I_{w^{-1}}(\partial)$, which implies that the support of the cokernel is $D_w(\partial)$. This establishes (a) and (b).

Now (c) follows from Theorem 3.8(b). $\quad\square$

3.3. Examples

Example 3.11. Let s_i denote the simple transposition that switches i and $i+1$. For $w = s_i$, the maximal transition (2.12) simplifies to $(\alpha, \beta) = (i, i+1)$, $v = 1$, $k = 1$, and $\psi_1 = s_{i-1}$. This is also evident from the fact that $\mathfrak{S}_{s_i}(x, y) = x_1 + \cdots + x_i - y_1 - \cdots - y_i$.

Let F_0 and F_1 be vector spaces of dimension n with $n \geqslant i$. Given a map $\partial : F_0 \to F_1$ with distinguished bases $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ (coming from a flag of F_0 and a quotient flag of F_1), respectively, the associated Schubert complex $\mathscr{S}_{s_i}(\partial)$ is obtained from ∂ by taking the upper left $i \times i$ submatrix of ∂.

The filtration of Proposition 3.2 can be described as follows. First, it should look like

$$0 = C_0 \subset C_1 \subset C' \subset C = \mathscr{S}_{s_i}(\partial)$$

where $C_1 \cong \mathscr{S}_{s_{i-1}}(\partial)$, $C'/C_1 \cong F_1^{-i}/F_1^{-i+1}$ and $C/C' \cong (F_0^i/F_0^{i-1})[-1]$.

Then C' is the subcomplex $\langle e_1, \ldots, e_{i-1} \rangle \to \langle e_1', \ldots, e_i' \rangle$ of C, so the quotient is then F_0^i/F_0^{i-1} concentrated in degree 1. Finally, C_1 is the subcomplex $\langle e_1, \ldots, e_{i-1} \rangle \to \langle e_1', \ldots, e_{i-1}' \rangle$ which is isomorphic to $\mathscr{S}_{s_{i-1}}(\partial)$ and the quotient C'/C_1 is F_1^{-i}/F_1^{-i+1} as required.

Here is a combinatorial description of the differentials in the Schubert complex for a flagged isomorphism. We will work with just the tensor product complex $\bigotimes_{k=1}^{n-1} D^{r_k(w)}(F)$. Then the basis elements of its terms are row-strict labelings. The differential sends such a labeling to the signed sum of all possible ways to change a single unmarked letter to the corresponding marked letter. If T' is obtained from T by marking a letter in the ith row, then the sign on T' is $(-1)^n$, where n is the number of unmarked letters of T in the first $i-1$ rows.

Example 3.12. Consider the permutation $w = 1423$. Then $D(w) = \{(2, 2), (2, 3)\}$, and if we use the identity matrix I, $\mathscr{S}_w(I)$ looks like

$$\boxed{2}\,\boxed{2} \mapsto \boxed{2}\,\boxed{2'}, \qquad \boxed{2}\,\boxed{2'} \mapsto 0, \qquad \boxed{1'}\,\boxed{2'} \mapsto 0,$$

$$\boxed{2}\,\boxed{1} \mapsto \boxed{1}\,\boxed{2'} + \boxed{2}\,\boxed{1'}, \quad \boxed{1}\,\boxed{2'} \mapsto \boxed{1'}\,\boxed{2'}, \qquad \boxed{1'}\,\boxed{3'} \mapsto 0,$$

$$\boxed{1}\,\boxed{1} \mapsto \boxed{1}\,\boxed{1'}, \qquad\qquad \boxed{2}\,\boxed{1'} \mapsto -\boxed{1'}\,\boxed{2'}, \quad \boxed{2'}\,\boxed{3'} \mapsto 0.$$

$$\boxed{1}\,\boxed{1'} \mapsto 0,$$

$$\boxed{1}\,\boxed{3'} \mapsto \boxed{1'}\,\boxed{3'},$$

$$\boxed{2}\,\boxed{3'} \mapsto \boxed{2'}\,\boxed{3'},$$

If we use a generic map $e_1 \mapsto a e_1' + b e_2' + c e_3'$ and $e_2 \mapsto d e_1' + e e_2' + f e_3'$ (the images of e_3 and e_4 are irrelevant, and it is also irrelevant to map to e_4') instead, then the complex looks like

$$0 \to A^3 \xrightarrow{\begin{pmatrix} e & b & 0 \\ 0 & e & b \\ d & a & 0 \\ 0 & d & a \\ 0 & f & c \\ f & c & 0 \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} d & a & -e & -b & 0 & 0 \\ 0 & 0 & -f & -c & a & d \\ -f & -c & 0 & 0 & b & e \end{pmatrix}} A^3 \to M \to 0.$$

The cokernel M is Cohen–Macaulay of codimension 2 over $A = K[a, b, c, d, e, f]$.

Example 3.13. Consider the permutation $w = 2413$. Then $D(w) = \{(1, 1), (2, 1), (2, 3)\}$, and if we use the identity matrix I, $\mathscr{S}_w(I)$ looks like (note the negative signs which come from the fact that we are working with an image of a tensor product of two divided power complexes)

Using a generic matrix defined by $e_1 \mapsto ae'_1 + be'_2 + ce'_3$ and $e_2 \mapsto de'_1 + ee'_2 + fe'_3$ (the other coefficients are irrelevant) instead of the identity matrix gives the following complex

$$0 \to A^2 \xrightarrow{\begin{pmatrix} -d & -a \\ -e & -b \\ -f & -c \\ 0 & -d \\ a & 0 \\ 0 & a \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & a & 0 & d \\ e & -d & 0 & b & 0 & e \\ f & 0 & -d & c & 0 & f \\ a & 0 & 0 & 0 & d & a \\ 0 & a & 0 & 0 & e & b \\ 0 & 0 & a & 0 & f & c \end{pmatrix}} A^6 \xrightarrow{\begin{pmatrix} -b & a & 0 & -e & d & 0 \\ -c & 0 & a & -f & 0 & d \end{pmatrix}} A^2 \to M \to 0.$$

Its cokernel M is Cohen–Macaulay of codimension 3 over $A = K[a, b, c, d, e, f]$.

4. Degeneracy loci

4.1. A formula of Fulton

Suppose we are given a map $\partial : E \to F$ of vector bundles of rank n on a scheme X, together with a flag of subbundles $E_1 \subset E_2 \subset \cdots \subset E_n = E$ and a flag of quotient bundles $F = F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$ such that rank $E_i = $ rank $F_i = i$. We assume that the quotients E_i/E_{i+1} are locally free. For a permutation w, define

$$D_w(\partial) = \big\{ x \in X \mid \mathrm{rank}\big(\partial_x \colon E_p(x) \to F_q(x)\big) \leqslant r_w(p, q) \big\}.$$

Then codim $D_w(\partial) \leqslant \ell(w)$. Define Chern classes $x_i = -c_1(E_i/E_{i-1})$ and $y_i = -c_1(\ker(F_i \twoheadrightarrow F_{i-1}))$.

Theorem 4.1 *(Fulton). Suppose that X is an equidimensional Cohen–Macaulay scheme of finite type over a field K and $D_w(\partial)$ has codimension $\ell(w)$. Then the identity*

$$\bigl[D_w(\partial) \bigr] = \mathfrak{S}_w(x, y) \cap [X]$$

holds in the Chow group $\mathrm{A}_{\dim(D_w(\partial))}(X)$.

See [F1, §8] for a more general statement which does not enforce a codimension requirement on $D_w(\partial)$ or assume that X is Cohen–Macaulay. In order to state the connection between the Schubert complex and Fulton's formula, we will need the following lemma which was observed in [Pra, Appendix 6].

Lemma 4.2. *Let X be an equidimensional smooth scheme of finite type over a field K, and let D be an irreducible closed subscheme of X of codimension k. Let $C_\bullet$ be a finite complex of vector bundles on X and let $\alpha \in \mathrm{A}^k(X)$. If*

$$\operatorname{supp} C_\bullet = X \setminus \bigl\{ x \in X \mid (C_\bullet)|_x \text{ is an exact complex} \bigr\}$$

is contained in D, and $\varphi(\alpha) = [C_\bullet]_k$, then $c[D] = \alpha$ for some $c \in \mathbf{Q}$.

For completeness (and since we have changed notation from [Pra]), we will reproduce the proof.

Proof. Let $i : D \to X$ and $j : X \setminus D \to X$ be the inclusions. Let the subscript $(-)_\mathbf{Q}$ denote tensoring with $\mathbf{Q}$. Then the diagram (of Abelian groups)

$$
\begin{array}{ccccccc}
\mathrm{A}^*(D)_\mathbf{Q} & \xrightarrow{\; i_* \;} & \mathrm{A}^*(X)_\mathbf{Q} & \xrightarrow{\; j_\mathrm{A}^* \;} & \mathrm{A}^*(X \setminus D)_\mathbf{Q} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \varphi_\mathbf{Q}} & & \Big\downarrow {\scriptstyle \varphi_\mathbf{Q}} & & \\
 & & \operatorname{gr} \mathrm{K}(X)_\mathbf{Q} & \xrightarrow{\; j_\mathrm{K}^* \;} & \operatorname{gr} \mathrm{K}(X \setminus D)_\mathbf{Q} & &
\end{array}
$$

commutes by functoriality of φ, and the first row is exact [F2, Proposition 1.8]. Since $\operatorname{supp}(C_\bullet) \subseteq D$, we have $j_\mathrm{K}^*([C_\bullet]) = 0$, and since $\varphi_\mathbf{Q}(\alpha) = [C_\bullet]$, we conclude that $j_\mathrm{A}^*(\alpha) = 0$ because $\varphi_\mathbf{Q}$ is an isomorphism [F2, Example 15.2.16(b)]. Since we assumed that $\alpha \in \mathrm{A}^k(X)$ we have $\alpha = i_*(\beta)$ for some $\beta \in \mathrm{A}^0(D)_\mathbf{Q}$. But D is irreducible, and hence β is some rational multiple of $[D]$. $\quad\square$

We will verify Theorem 4.1 in the case that X is a smooth quasi-projective variety. The general case can be reduced to this case using a "universal construction" and Chow's lemma (see [F1, §8]).

Proof of Theorem 4.1. We will use Lemma 4.2 with $D = D_w(\partial)$, $C_\bullet = \mathscr{S}_w(\partial)$, and $\alpha = \mathfrak{S}_w(x, y)$ using the notation from the beginning of this section. We know that $\operatorname{supp} C_\bullet \subseteq D$ and that the codimension of D is $\ell(w) = \deg \alpha$ by Corollary 3.10. So in order to conclude Theorem 4.1, we need to check that $\varphi(\alpha) = [C_\bullet]$, which is the content of Lemma 3.6. Finally, it remains to show that the constant given by Lemma 4.2 is 1. This follows from (3.5) and Corollary 3.10(c). $\quad\square$

4.2. Some remarks

Remark 4.3. The previous constructions do not require that the flags be complete, so that one can omit certain subbundles or quotient bundles as desired. The appropriate generalization would be to use partial flag varieties, but we have omitted such generality to keep the notation simpler.

Remark 4.4. A permutation $w \in \Sigma_n$ is *Grassmannian* if it has at most one descent, i.e., there exists r such that $w(1) < w(2) < \cdots < w(r) > w(r+1) < \cdots < w(n)$. Suppose that w is *bigrassmannian*, which means that w and w^{-1} are Grassmannian permutations. This is equivalent to saying that $D(w)$ is a rectangle. In this case, the double Schubert polynomial $\mathfrak{S}_w(x, y)$ is a multi-Schur function for the partition $D(w)$ (one can use [Man, Proposition 2.6.8] combined with (1.2)). The degeneracy locus $D_w(\partial)$ can then be described by a single rank condition of the map $\partial : E \to F$, so the degeneracy locus formula of Fulton specializes to the Thom–Porteous formula mentioned in the introduction. So in principle, the action of $\mathfrak{b}$ on $\mathscr{S}_w(\partial)$ should extend to an action of a general linear superalgebra, but it is not clear why this should be true without appealing to Schur polynomials.

Remark 4.5. The Schubert complex only gives a formula for the structure sheaf of the given degeneracy locus in the associated graded of K-theory. A formula for the structure sheaf in the actual K-theory is given in [FL, Theorem 3] using the so-called Grothendieck polynomials, but the formula is not obtained by constructing a complex, so it would be interesting to try to construct these complexes. The degeneracy loci for bigrassmannian permutations are determinantal varieties, and the resolutions in characteristic 0 are explained in [Wey, §6.1]. We should point out that the terms of the resolutions may change with the characteristic, see [Wey, §6.2].

Remark 4.6. We have seen that the modules which are the cokernels of generic Schubert complexes have linear minimal free resolutions. These modules can be thought of as a "linear approximation" to the ideal which defines the matrix Schubert varieties, which in general have rich and complicated minimal free resolutions. More precisely, we have shown that matrix Schubert varieties possess maximal Cohen–Macaulay modules with linear resolutions. In general, the question of whether or not every graded ring possesses such a module is open (see [ES, p. 543] for further information).

Furthermore, such modules can be obtained geometrically, as outlined in [Wey, Chapter 6, Exercises 34–36] for the case of generic determinantal varieties and their symmetric and skew-symmetric analogues, which we will denote by $D \subset \mathbf{A}^N$. The idea is to find a projective variety V and a subbundle $Z \subset V \times \mathbf{A}^N$ such that the projection $Z \to D$ is a desingularization. In each case, one can find a vector bundle on Z whose pushforward to $\mathbf{A}^N$ provides the desired module supported on D. The proof that its minimal free resolution is linear involves some sheaf cohomology calculations. It would be interesting to try to do this for matrix Schubert varieties, which are our affine models of Fulton's degeneracy loci. The desingularizations of matrix Schubert varieties one might try to use could be given by some analogue of Bott–Samelson varieties. The problem would then be to find the appropriate vector bundle and do the relevant sheaf cohomology calculations. It is the latter part that seems to be complicated.

Acknowledgments

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