Bosonization for dual quasi-bialgebras and preantipode ☆

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A R T I C L E   I N F O

A B S T R A C T

To every dual quasi-bialgebra H and every bialgebra R in the category of Yetter–Drinfeld modules over H, one can associate a dual quasi-bialgebra, called bosonization. In this paper, using the fundamental theorem, we characterize as bosonizations the dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode. As an application we investigate the structure of the graded coalgebra $\operatorname{gr} A$ associated to a dual quasi-bialgebra A with the dual Chevalley property (e.g. A is pointed).

Contents

1. Introduction

Let H be a bialgebra. Consider the functor $T := (-) \otimes H : \mathfrak{M} \to \mathfrak{M}_H^H$ from the category of vector spaces to the category of right Hopf modules. It is well-known that T determines an equivalence if and only if H has an antipode i.e. it is a Hopf algebra. The fact that T is an equivalence is the so-called fundamental (or structure) theorem for Hopf modules, which is due, in the finite-dimensional case, to Larson and Sweedler, see [14, Proposition 1, p. 82]. This result is crucial in characterizing the structure of bialgebras with a projection as Radford–Majid bosonizations (see [19]). Recall that a bialgebra A has a projection onto a Hopf algebra H if there exist bialgebra maps $\sigma : H \to A$ and $\pi : A \to H$ such that $\pi \circ \sigma = \mathrm{Id}_H$. Essentially using the fundamental theorem, one proves that A is isomorphic, as a vector space, to the tensor product $R \otimes H$ where R is some bialgebra in the category $_H^H \mathcal{YD}$ of Yetter–Drinfeld modules over H. This way $R \otimes H$ inherits, from A, a bialgebra structure which is called the *Radford–Majid bosonization of R by H* and denoted by $R \# H$. It is remarkable that the graded coalgebra gr A associated to a pointed Hopf algebra A (here "pointed" means that all simple subcoalgebras of A are one-dimensional) always admits a projection onto its coradical. This is the main ingredient in the so-called lifting method for the classification of finite-dimensional pointed Hopf algebras, see [1].

In 1989 Drinfeld introduced the concept of quasi-bialgebra in connection with the Knizhnik–Zamolodchikov system of partial differential equations. The axioms defining a quasi-bialgebra are a translation of monoidality of its representation category with respect to the diagonal tensor product. In [10], the antipode for a quasi-bialgebra (whence the concept of quasi-Hopf algebra) is introduced in order to make the category of its flat right modules rigid. If we draw our attention to the category of co-representations of H, we get the concepts of dual quasi-bialgebra and of dual quasi-Hopf algebra. These notions have been introduced in [17] in order to prove a Tannaka–Krein type theorem for quasi-Hopf algebras.

A fundamental theorem for dual quasi-Hopf algebras was proved by Schauenburg in [24]. It is remarkable that the functor T giving the fundamental theorem in the case of ordinary Hopf algebras must be substituted, in the "quasi" case, by the functor $F := (-) \otimes H$ between the category $^H \mathfrak{M}$ of left H-comodules and the category $^H \mathfrak{M}_H^H$ of right dual quasi-Hopf H-bicomodules (essentially this is due to the fact that, unlike the classical case, a dual quasi-bialgebra H is not an algebra in the category of right H-comodules but it is still an algebra in the category of H-bicomodules). In [4, Theorem 3.9], we showed that, for a dual quasi-bialgebra H, the functor F is an equivalence if and only if there exists a suitable map $S : H \to H$ that we called a preantipode for H. Moreover for any dual quasi-bialgebra with antipode (i.e. a dual quasi-Hopf algebra) we constructed a specific preantipode, see [4, Theorem 3.10].

It is worth to notice that there is a dual quasi-bialgebra H which has a preantipode but not an antipode (i.e. it is not a dual quasi-Hopf algebra). Nevertheless, for a finite-dimensional dual quasi-bialgebra, the existence of an antipode is equivalent to the existence of a preantipode (cf. Remark 2.17).

The main aim of this paper is to introduce and investigate the notion of bosonization in the setting of dual quasi-bialgebras. Explicitly, we associate a dual quasi-bialgebra $R \# H$ (that we call bosonization of R by H) to every dual quasi-bialgebra H and bialgebra R in $_H^H \mathcal{YD}$. Then, using the fundamental theorem, we characterize as bosonizations the dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode. As an application, for any dual quasi-bialgebra A with the dual

Chevalley property (i.e. such that the coradical of A is a dual quasi-subbialgebra of A), under the further hypothesis that the coradical H of A has a preantipode, we prove that there is a bialgebra R in ${}_{H}^{H}\mathcal{YD}$ such that gr A is isomorphic to $R \# H$ as a dual quasi-bialgebra. In particular, if A is a pointed dual quasi-Hopf algebra, then gr A comes out to be isomorphic to $R \# \Bbbk \mathbb{G}(A)$ as dual quasi-bialgebra where R is the diagram of A and $\mathbb{G}(A)$ is the set of grouplike elements in A. We point out that the results in this paper are obtained without assuming that the dual quasi-bialgebra considered is finite-dimensional so that most of the proofs do not follow by a mere dualization of results on quasi-bialgebras nor by the case of dual quasi-Hopf algebras.

The paper is organized as follows.

Section 2 contains preliminary results needed in the next sections. Moreover in Theorem 2.16, we investigate cocommutative dual quasi-bialgebras with a preantipode and in Corollary 2.21, we provide a Cartier–Gabriel–Kostant type theorem for dual quasi-bialgebras with a preantipode. In the connected case such a result was achieved in [12, Theorem 4.3].

Section 3, is devoted to the study of the category ${}_{H}^{H}\mathcal{YD}$ of Yetter–Drinfeld modules over a dual quasi-bialgebra H. Explicitly, we consider the pre-braided monoidal category $({}_{H}^{H}\mathcal{YD}, \otimes, \Bbbk)$ of Yetter–Drinfeld modules over a dual quasi-bialgebra H and we show that the functor F, as above, induces a functor $F : {}_{H}^{H}\mathcal{YD} \rightarrow {}_{H}^{H}\mathfrak{M}_{H}^{H}$ (that is an equivalence in case H has a preantipode, see Proposition 3.8).

In Section 4, we prove that the equivalence between the categories ${}_{H}^{H}\mathfrak{M}_{H}^{H}$ and ${}_{H}^{H}\mathcal{YD}$ becomes monoidal if we equip ${}_{H}^{H}\mathfrak{M}_{H}^{H}$ with the tensor product $\otimes_{H}$ (or $\square_{H}$) and unit H (see Lemma 4.4 and Lemma 4.8). As a by-product, in Lemma 4.11, we produce a monoidal equivalence between $({}_{H}^{H}\mathfrak{M}_{H}^{H}, \otimes_{H}, H)$ and $({}_{H}^{H}\mathfrak{M}_{H}^{H}, \square_{H}, H)$.

Section 5 contains the main results of the paper. In Theorem 5.2, to every dual quasi-bialgebra H and bialgebra R in ${}_{H}^{H}\mathcal{YD}$ we associate a dual quasi-bialgebra structure on the tensor product $R \otimes H$ that we call the *bosonization of R by H* and denote by $R \# H$. Now, let (A, H, σ, π) be a dual quasi-bialgebra with projection and assume that H has a preantipode S. In Lemma 5.7, we prove that such an A is an object in the category ${}_{H}^{H}\mathfrak{M}_{H}^{H}$. Therefore the fundamental theorem describes A as the tensor product $R \otimes H$ of some vector space R by H. Indeed, in Theorem 5.8, we prove that the dual quasi-bialgebra structure inherited by $R \otimes H$ through the claimed isomorphism is exactly the bosonization of R by H. The analogous of this result for quasi-Hopf algebras, anything but trivial, has been established by Bulacu and Nauwelaerts in [9], but their proof cannot be adapted to dual quasi-bialgebras with a preantipode.

In Section 6 we collect some applications of our results. Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Since A is an ordinary coalgebra, we can consider the associated graded coalgebra gr A. In Proposition 6.3, we prove that gr A fits into a dual quasi-bialgebra with projection onto H. As a consequence, in Corollary 6.4, under the further assumption that H has a preantipode, we show that there is a bialgebra R in ${}_{H}^{H}\mathcal{YD}$ such that gr A is isomorphic to $R \# H$ as a dual quasi-bialgebra. When A is a pointed dual quasi-Hopf algebra it is in particular a dual quasi-bialgebra with the dual Chevalley property and its coradical has a preantipode. Using this fact, in Theorem 6.10 we obtain that gr A is of the form $R \# \Bbbk \mathbb{G}(A)$ as dual quasi-bialgebra, where R is the so-called diagram of A.

2. Preliminaries

In this section we recall the definitions and results that will be needed in the paper.

Notation 2.1. Throughout this paper $\Bbbk$ will denote a field. All vector spaces will be defined over $\Bbbk$. The unadorned tensor product $\otimes$ will stand for the tensor product over $\Bbbk$ if not stated otherwise.

2.2. Monoidal categories. Recall that (see [13, Chapter XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \rightarrow X$, $r_X : X \otimes \mathbf{1} \rightarrow X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*, that is the equality

$$(U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W \otimes X} \circ a_{U \otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom*, that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra A in $\mathcal{M}$ one can define the categories $_A\mathcal{M}$, $\mathcal{M}_A$ and $_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in $\mathcal{M}$, one can define the categories of C-comodules $^C\mathcal{M}$, $\mathcal{M}^C$, $^C\mathcal{M}^C$. For more details, the reader is refereed to [2].

Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ is abelian and both the functors $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and right exact, for any $X \in \mathcal{M}$. Given an algebra A in $\mathcal{M}$, there exist a suitable functor $\otimes_A : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A$ and constraints that make the category $({}_A\mathcal{M}_A, \otimes_A, A)$ monoidal, see [2, 1.11]. The tensor product over A in $\mathcal{M}$ of a right A-module (V, μ_V^r) and a left A-module (W, μ_W^l) is defined to be the coequalizer:

Note that, since $\otimes$ preserves coequalizers, then $V \otimes_A W$ is also an A-bimodule, whenever V and W are A-bimodules.

Dually, given a coalgebra (C, Δ, ε) in a monoidal category $\mathcal{M}$, abelian and with additive and left exact tensor functors, there exist a suitable functor $\square_C : {}^C\mathcal{M}^C \times {}^C\mathcal{M}^C \to {}^C\mathcal{M}^C$ and constraints that make the category $({}^C\mathcal{M}^C, \square_C, C)$ monoidal. The cotensor product over C in $\mathcal{M}$ of a right C-comodule (V, ρ_V^r) and a left C-comodule (W, ρ_W^l) is defined to be the equalizer:

Note that, since $\otimes$ preserves equalizers, then $V \square_C W$ is also a C-bicomodule, whenever V and W are C-bicomodules.

Definition 2.3. A dual quasi-bialgebra is a datum $(H, m, u, \Delta, \varepsilon, \omega)$ where

- (H, Δ, ε) is a coassociative coalgebra;
- $m : H \otimes H \to H$ and $u : \Bbbk \to H$ are coalgebra maps called multiplication and unit respectively; we set $1_H := u(1_\Bbbk)$;
- $\omega : H \otimes H \otimes H \to \Bbbk$ is a unital 3-cocycle i.e. it is convolution invertible and satisfies

$$\omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) = m_\Bbbk(\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) * m_\Bbbk(\omega \otimes \varepsilon) \qquad (1)$$

and

$$\omega(h \otimes k \otimes l) = \varepsilon(h)\varepsilon(k)\varepsilon(l) \quad \text{whenever } 1_H \in \{h, k, l\}; \qquad (2)$$

- m is quasi-associative and unitary i.e. it satisfies

$$m(H \otimes m) * \omega = \omega * m(m \otimes H), \tag{3}$$

$$m(1_H \otimes h) = h \quad \text{for all } h \in H, \tag{4}$$

$$m(h \otimes 1_H) = h \quad \text{for all } h \in H. \tag{5}$$

ω is called *the reassociator* of the dual quasi-bialgebra.

A morphism of dual quasi-bialgebras (see e.g. [20, Section 2])

$$\alpha : (H, m, u, \Delta, \varepsilon, \omega) \to (H', m', u', \Delta', \varepsilon', \omega')$$

is a coalgebra homomorphism $\alpha : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ such that

$$m'(\alpha \otimes \alpha) = \alpha m, \qquad \alpha u = u', \qquad \omega'(\alpha \otimes \alpha \otimes \alpha) = \omega.$$

It is an isomorphism of quasi-bialgebras if, in addition, it is invertible.

A dual quasi-subbialgebra of a dual quasi-bialgebra $(H', m', u', \Delta', \varepsilon', \omega')$ is a quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ such that H is a vector subspace of H' and the canonical inclusion $\alpha : H \to H'$ yields a morphism of dual quasi-bialgebras.

2.1. The category of bicomodules for a dual quasi-bialgebras

Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. It is well-known that the category $\mathfrak{M}^H$ of right H-comodules becomes a monoidal category as follows. Given a right H-comodule V, we denote by $\rho = \rho_V^r : V \to V \otimes H$, $\rho(v) = v_0 \otimes v_1$, its right H-coaction. The tensor product of two right H-comodules V and W is a comodule via diagonal coaction i.e. $\rho(v \otimes w) = (v_0 \otimes w_0) \otimes v_1 w_1$. The unit is $\Bbbk$, which is regarded as a right H-comodule via the trivial coaction i.e. $\rho(k) = k \otimes 1_H$. The associativity and unit constraints are defined, for all $U, V, W \in \mathfrak{M}^H$ and $u \in U$, $v \in V$, $w \in W$, $k \in \Bbbk$, by

$$a_{U,V,W}^H \big((u \otimes v) \otimes w \big) := u_0 \otimes (v_0 \otimes w_0) \omega(u_1 \otimes v_1 \otimes w_1),$$

$$l_U(k \otimes u) := ku \quad \text{and} \quad r_U(u \otimes k) := uk.$$

The monoidal category we have just described will be denoted by $(\mathfrak{M}^H, \otimes, \Bbbk, a^H, l, r)$.

Similarly, the monoidal categories $({}^H\mathfrak{M}, \otimes, \Bbbk, {}^H a, l, r)$ and $({}^H\mathfrak{M}^H, \otimes, \Bbbk, {}^H a^H, l, r)$ are introduced. We just point out that

$${}^H a_{U,V,W} \big((u \otimes v) \otimes w \big) := \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1}) u_0 \otimes (v_0 \otimes w_0),$$

$${}^H a_{U,V,W}^H \big((u \otimes v) \otimes w \big) := \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1}) u_0 \otimes (v_0 \otimes w_0) \omega(u_1 \otimes v_1 \otimes w_1).$$

Remark 2.4. We know that, if $(H, m, u, \Delta, \varepsilon, \omega)$ is a dual quasi-bialgebra, we cannot construct the category $\mathfrak{M}_H$, because H is not an algebra. Moreover H is not an algebra in $\mathfrak{M}^H$ or in ${}^H\mathfrak{M}$. On the other hand $((H, \rho_H^l, \rho_H^r), m, u)$ is an algebra in the monoidal category $({}^H\mathfrak{M}^H, \otimes, \Bbbk, {}^H a^H, l, r)$ with $\rho_H^l = \rho_H^r = \Delta$. Thus, the only way to construct the category ${}^H\mathfrak{M}_H^H$ is to consider the right H-modules in ${}^H\mathfrak{M}^H$. Hence, we can set

$${}^H\mathfrak{M}_H^H := \big({}^H\mathfrak{M}^H \big)_H.$$

The category $^H\mathfrak{M}^H_H$ is the so-called category of *right dual quasi-Hopf H-bicomodules* [6, Remark 2.3]. In a similar way one can introduce the category $^H_H\mathfrak{M}^H_H := {}_H({}^H\mathfrak{M}^H)_H$.

Remark 2.5. (See [2, Example 1.5(a)].) Let (A, m, u) be an algebra in a given monoidal category $(\mathcal{M}, \otimes, 1, a, l, r)$. Then the assignments $M \mapsto (M \otimes A, (M \otimes m) \circ a_{A,A,A})$ and $f \mapsto f \otimes A$ define a functor $T : \mathcal{M} \to \mathcal{M}_A$. Moreover the forgetful functor $U : \mathcal{M}_A \to \mathcal{M}$ is a right adjoint of T.

2.2. An adjunction between $^H\mathfrak{M}^H_H$ and $^H\mathfrak{M}$

We are going to recall the construction of an adjunction between $^H\mathfrak{M}^H_H$ and $^H\mathfrak{M}$ that will be crucial afterwards.

2.6. Consider the functor $L : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H$ defined on objects by $L({}^\bullet V) := {}^\bullet V^\circ$ where the upper empty dot denotes the trivial right coaction while the upper full dot denotes the given left H-coaction of V. The functor L has a right adjoint $R : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}$ defined on objects by $R({}^\bullet M^\bullet) := {}^\bullet M^{coH}$, where $M^{coH} := \{m \in M \mid m_0 \otimes m_1 = m \otimes 1_H\}$ is the space of right H-coinvariant elements in M.

By Remark 2.5, the forgetful functor $U : {}^H\mathfrak{M}^H_H \to {}^H\mathfrak{M}^H$, $U({}^\bullet M^\bullet_\bullet) := {}^\bullet M^\bullet$ has a left adjoint, namely the functor $T : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}^H_H$, $T({}^\bullet M^\bullet) := {}^\bullet M^\bullet \otimes {}^\bullet H^\bullet_\bullet$. Here the upper dots indicate on which tensor factors we have a codiagonal coaction and the lower dot indicates where the action takes place. Explicitly, the structure of $T({}^\bullet M^\bullet)$ is given as follows:

$$\rho^l_{M \otimes H}(m \otimes h) := m_{-1}h_1 \otimes (m_0 \otimes h_2),$$

$$\rho^r_{M \otimes H}(m \otimes h) := (m_0 \otimes h_1) \otimes m_1 h_2,$$

$$\mu^r_{M \otimes H}\big[(m \otimes h) \otimes l\big] = (m \otimes h)l := \omega^{-1}(m_{-1} \otimes h_1 \otimes l_1)m_0 \otimes h_2 l_2 \omega(m_1 \otimes h_3 \otimes l_3).$$

Define the functors $F := TL : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$ and $G := RU : {}^H\mathfrak{M}^H_H \to {}^H\mathfrak{M}$. Explicitly $G({}^\bullet M^\bullet_\bullet) = {}^\bullet M^{coH}$ and $F({}^\bullet V) := {}^\bullet V^\circ \otimes {}^\bullet H^\bullet_\bullet$ so that, for every $v \in V$, $h, l \in H$,

$$\rho^l_{V \otimes H}(v \otimes h) = v_{-1}h_1 \otimes (v_0 \otimes h_2),$$

$$\rho^r_{V \otimes H}(v \otimes h) = (v \otimes h_1) \otimes h_2,$$

$$\mu^r_{V \otimes H}\big[(v \otimes h) \otimes l\big] = (v \otimes h)l = \omega^{-1}(v_{-1} \otimes h_1 \otimes l_1)v_0 \otimes h_2 l_2.$$

Remark 2.7. By the right-hand version of [24, Lemma 2.1], the functor $F : {}^H\mathfrak{M} \to {}^H\mathfrak{M}^H_H$ is a left adjoint of the functor G, where the counit and the unit of the adjunction are given respectively by $\epsilon_M : FG(M) \to M$, $\epsilon_M(x \otimes h) := xh$ and by $\eta_N : N \to GF(N)$, $\eta_N(n) := n \otimes 1_H$, for every $M \in {}^H\mathfrak{M}^H_H$, $N \in {}^H\mathfrak{M}$. Moreover η_N is an isomorphism for any $N \in {}^H\mathfrak{M}$. In particular the functor F is fully faithful.

2.3. The notion of preantipode

Next result characterizes when the adjunction (F, G) is an equivalence of categories in terms of the existence of a suitable map τ.

Proposition 2.8. (See [4, Proposition 3.3].) Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The following assertions are equivalent.

(i) *The adjunction (F, G) is an equivalence.*

(ii) *For each $M \in {}^H\mathfrak{M}_H^H$, there exists a $\Bbbk$-linear map $\tau : M \to M^{coH}$ such that*

$$\tau(mh) = \omega^{-1}\big[\tau(m_0)_{-1} \otimes m_1 \otimes h\big]\tau(m_0)_0 \quad \text{for all } h \in H, \ m \in M, \tag{6}$$

$$m_{-1} \otimes \tau(m_0) = \tau(m_0)_{-1}m_1 \otimes \tau(m_0)_0 \quad \text{for all } m \in M, \tag{7}$$

$$\tau(m_0)m_1 = m, \quad \forall m \in M. \tag{8}$$

(iii) *For each $M \in {}^H\mathfrak{M}_H^H$, there exists a $\Bbbk$-linear map $\tau : M \to M^{coH}$ such that (8) holds and*

$$\tau(mh) = m\varepsilon(h) \quad \text{for all } h \in H, \ m \in M^{coH}. \tag{9}$$

Remark 2.9. Let $\tau : M \to M^{coH}$ be a $\Bbbk$-linear map such that (8) holds. By [4, Remark 3.4], the map τ fulfills (9) if and only if it fulfills (6) and (7).

Definition 2.10. Following [4, Definition 3.6] we will say that a preantipode for a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ is a $\Bbbk$-linear map $S : H \to H$ such that, for all $h \in H$,

$$S(h_1)_1h_2 \otimes S(h_1)_2 = 1_H \otimes S(h), \tag{10}$$

$$S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H, \tag{11}$$

$$\omega\big(h_1 \otimes S(h_2) \otimes h_3\big) = \varepsilon(h). \tag{12}$$

Remark 2.11. (See [4, Remark 3.7].) Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. Then the following equalities hold

$$h_1 S(h_2) = \varepsilon S(h)1_H = S(h_1)h_2 \quad \text{for all } h \in H. \tag{13}$$

Lemma 2.12. *(See [4, Lemma 3.8].) Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. For any $M \in {}^H\mathfrak{M}_H^H$ and $m \in M$, set*

$$\tau(m) := \omega\big[m_{-1} \otimes S(m_1)_1 \otimes m_2\big]m_0 S(m_1)_2. \tag{14}$$

Then (14) defines a map $\tau : M \to M^{coH}$ which fulfills (6), (7) and (8).

Theorem 2.13. *(See [4, Theorem 3.9].) For a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ the following are equivalent.*

(i) *The adjunction (F, G) of Remark 2.7 is an equivalence of categories.*
(ii) *There exists a preantipode.*

We include here some new results that will be needed later on in the paper.

Lemma 2.14. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. Then*

$$\omega^{-1}\big[S(h_1) \otimes h_2 \otimes S(h_3)\big] = \varepsilon S(h) \quad \text{for all } h \in H. \tag{15}$$

Proof. Set $\alpha := \omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) * m_{\Bbbk}(\omega^{-1} \otimes \varepsilon)$ and $\beta = m_{\Bbbk}(\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H)$. Fix $h \in H$. We have

$$\alpha\big(S(h_1) \otimes h_2 \otimes S(h_3) \otimes h_4\big)$$

$$= \omega\big[S(h_1)_1 \otimes h_2 \otimes S(h_5)_{(1)}h_6\big]\omega\big[S(h_1)_2 h_3 \otimes S(h_5)_{(2)} \otimes h_7\big]\omega^{-1}\big[S(h_1)_3 \otimes h_4 \otimes S(h_5)_{(3)}\big]$$

$$\overset{(10)}{=} \omega\big[S(h_1)_1 \otimes h_2 \otimes 1_H\big]\omega\big[S(h_1)_2 h_3 \otimes S(h_5)_{(1)} \otimes h_6\big]\omega^{-1}\big[S(h_1)_3 \otimes h_4 \otimes S(h_5)_{(2)}\big]$$

$$= \omega\big[S(h_1)_1 h_2 \otimes S(h_4)_{(1)} \otimes h_5\big]\omega^{-1}\big[S(h_1)_2 \otimes h_3 \otimes S(h_4)_{(2)}\big]$$

$$\overset{(10)}{=} \omega\big[1_H \otimes S(h_4)_{(1)} \otimes h_5\big]\omega^{-1}\big[S(h_1) \otimes h_3 \otimes S(h_4)_{(2)}\big]$$

$$= \omega^{-1}\big[S(h_1) \otimes h_2 \otimes S(h_3)\big],$$

and

$$\beta\big(S(h_1) \otimes h_2 \otimes S(h_3) \otimes h_4\big) = \omega\big[h_2 \otimes S(h_4)_{(1)} \otimes h_5\big]\omega\big[S(h_1) \otimes h_3 S(h_4)_{(2)} \otimes h_6\big]$$

$$\overset{(11)}{=} \omega\big[h_2 \otimes S(h_3) \otimes h_4\big]\omega\big[S(h_1) \otimes 1_H \otimes h_5\big]$$

$$= \omega\big[h_2 \otimes S(h_3) \otimes h_4\big]\varepsilon S(h_1)$$

$$\overset{(12)}{=} \varepsilon S(h).$$

By the cocycle condition we have $\alpha = \beta$. $\quad\square$

Definition 2.15. (See [15, p. 66].) A *dual quasi-Hopf algebra* $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra anti-homomorphism

$$s : H \to H,$$

called *antipode*, and two maps α, β in H^*, such that, for all $h \in H$:

$$h_1 \beta(h_2) s(h_3) = \beta(h) 1_H, \tag{16}$$

$$s(h_1)\alpha(h_2)h_3 = \alpha(h)1_H, \tag{17}$$

$$\omega\big(h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5\big) = \varepsilon(h) = \omega^{-1}\big(s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)\big). \tag{18}$$

In [4, Theorem 3.10], we proved that any dual quasi-Hopf algebra has a preantipode. The following result proves that the converse holds true whenever H is also cocommutative.

Theorem 2.16. *Let* $(H, m, u, \Delta, \varepsilon, \omega, S)$ *be a dual quasi-bialgebra with a preantipode. If H is cocommutative, then* $(H, m, u, \Delta, \varepsilon, s)$ *is an ordinary Hopf algebra, where, for all $h \in H$,*

$$s(h) := S(h_3)_1 \omega\big[h_1 \otimes S(h_3)_2 \otimes h_2\big].$$

Furthermore $(H, m, u, \Delta, \varepsilon, \omega, \alpha, \beta, s)$ *is a dual quasi-Hopf algebra, where* $\alpha := \varepsilon$ *and* $\beta := \varepsilon S$. *Moreover one has* $S = \beta * s$.

Proof. By (3), cocommutativity and convolution invertibility of ω, we get that $(hk)l = h(kl)$ for all $h, k, l \in H$. Therefore m is associative and hence $(H, m, u, \Delta, \varepsilon)$ is an ordinary bialgebra. Let us check that s is an antipode for H. Using cocommutativity, (10) and (12) one proves that $s(h_1)h_2 = 1_H \varepsilon(h)$ for all $h \in H$. Similarly one gets $h_1 s(h_2) = 1_H \varepsilon(h)$ for all $h \in H$. Hence $(H, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra. Note that, for all $h \in H$,

$$S(h) = S(h_1)[h_2 s(h_3)] = [S(h_1)h_2]s(h_3) \overset{(13)}{=} \varepsilon S(h_1)s(h_2) = \beta(h_1)s(h_2). \tag{19}$$

Let us check that $(H, m, u, \Delta, \varepsilon, \omega, \alpha, \beta, s)$ is a dual quasi-Hopf algebra. For all $h \in H$,

$$h_1 \beta(h_2)s(h_3) \overset{(19)}{=} h_1 S(h_2) \overset{(13)}{=} 1_H \varepsilon S(h),$$

$$s(h_1)\alpha(h_2)h_3 = s(h_1)h_2 = 1_H \varepsilon(h) = 1_H \alpha(h),$$

$$\omega[h_1 \otimes \beta(h_2)s(h_3)\alpha(h_4) \otimes h_5] \overset{(19)}{=} \omega[h_1 \otimes S(h_2) \otimes h_3] \overset{(12)}{=} 1_H \varepsilon(h).$$

Now, since $(H, m, u, \Delta, \varepsilon, s)$ is an ordinary Hopf algebra, we have that s is an anti-coalgebra map. Thus

$$S(h)_1 \otimes S(h)_2 \overset{(19)}{=} \beta(h_1)s(h_2)_1 \otimes s(h_2)_2 = \beta(h_1)s(h_3) \otimes s(h_2)$$

$$\overset{\text{cocom.}}{=} \beta(h_1)s(h_2) \otimes s(h_3) \overset{(19)}{=} S(h_1) \otimes s(h_2)$$

so that

$$\omega^{-1}[s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)] \overset{(19)}{=} \omega^{-1}[s(h_1) \otimes h_2 \otimes S(h_3)]$$

$$= \omega^{-1}[S(h_3)_1 \otimes h_4 \otimes S(h_5)]\omega[h_1 \otimes S(h_3)_2 \otimes h_2]$$

$$= \omega^{-1}[S(h_3) \otimes h_5 \otimes S(h_6)]\omega[h_1 \otimes s(h_4) \otimes h_2]$$

$$\overset{\text{cocom.}}{=} \omega^{-1}[S(h_2) \otimes h_3 \otimes S(h_4)]\omega[h_1 \otimes s(h_5) \otimes h_6]$$

$$\overset{(15)}{=} \varepsilon S(h_2)\omega[h_1 \otimes s(h_3) \otimes h_4]$$

$$\overset{(19)}{=} \omega[h_1 \otimes S(h_2) \otimes h_3] \overset{(12)}{=} 1_H \varepsilon(h). \qquad \square$$

Remark 2.17. Following the Referee's suggestions, we here give an extended version of [4, Remark 3.12].

(1) We recall an example of a dual quasi-bialgebra considered in [21, Example 4.5.1] which has a preantipode but not an antipode.

Consider the subgroup $2^{\mathbb{Q}}$ of the multiplicative group $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$. Let the infinite cyclic group $\langle \sigma \rangle$ act on $2^{\mathbb{Q}}$ by $\sigma \rightharpoonup 2^x = 2^{2x}$ and consider the semidirect group $G := 2^{\mathbb{Q}} \rtimes \langle \sigma \rangle$. The multiplication is given by $(2^x \sigma^s)(2^y \sigma^t) = 2^{x+2^s y}\sigma^{s+t}$ and the inverse of $2^x \sigma^s$ is $2^{-2^{-s}x}\sigma^{-s}$. Let $B := 2^{\mathbb{Z}} \subseteq 2^{\mathbb{Q}}$ regarded as a subgroup of G in the obvious way. Consider the group algebras $H := \Bbbk[G]$ and $K := \Bbbk[B]$. Define $\pi : H \to K$ on generators by setting $\pi(2^x \sigma^s) = 2^{\lfloor x \rfloor}$ where $\lfloor x \rfloor$ denotes the floor of x i.e. $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leqslant x\}$. One easily checks that π is a left K-linear, unitary coalgebra map. In particular it is convolution invertible. If we regard H and K as dual quasi-bialgebras with trivial reassociators, we get that H is an object in $\mathfrak{E}_l(K)$ in the sense of [21, Definition 4.1.1], where the map $i : K \to H$ is the canonical inclusion such that $i(2^z) = 2^z \sigma^0$. Since H and K are cocommutative we have that H^{cop} is cocleft as a left K^{cop}-module coalgebra (with cocleaving π). Let us check that every

(B, B)-double coset in G is the union of finitely many left B-cosets. In fact such a double coset is of the form $B(2^x \sigma^s)B$. We have

$$B(2^x \sigma^s)B = \bigcup_{z \in \mathbb{Z}} B(2^x \sigma^s) 2^z = \bigcup_{z \in \mathbb{Z}} B(2^{2^s z + x} \sigma^s).$$

Now, if $s \geqslant 0$, then $B(2^x \sigma^s)B = B(2^x \sigma^s)$ i.e. it is a left coset. If $s < 0$ we can write $z = q2^{-s} + r$ with $q \in \mathbb{Z}$ and $0 \leqslant r < 2^{-s}$ and hence $B(2^x \sigma^s)B = \bigcup_{0 \leqslant r < 2^{-s}} B(2^{2^s r + x} \sigma^s)$ (then $B(2^x \sigma^s)B$ is union of 2^{-s} left cosets). By [21, p. 49], we have that $Q := H/K^+H$ is locally finite right K-module. Hence, we can apply [21, Theorem 4.4.3] to get that the category ${}^H_K \underline{\mathcal{M}}_K$ is left and right rigid. By [21, Theorem 4.3.1], the category ${}^H_K \underline{\mathcal{M}}_K$ is equivalent to the category ${}^{\widetilde{H}} \mathfrak{M}_f$ where $\widetilde{H}$ is a suitable dual quasi-bialgebra (note that, by [21, Remark 4.4.4(3)], the underlying vector space of $\widetilde{H}$ is isomorphic to $Q \otimes K^\circ$). By the left-handed version of [24, Theorem 2.6], we conclude that $\widetilde{H}$ has a preantipode (cf. Theorem 2.13).

Note also that $B(2^x \sigma^s)B$ as above can be written as union of right cosets as follows $B(2^x \sigma^s)B = \bigcup_{z \in \mathbb{Z}} (2^{z+x} \sigma^s)B$. Now, if $s \geqslant 0$, then we can write $z = q2^s + r$ with $q \in \mathbb{Z}$ and $0 \leqslant r < 2^s$ and hence $B(2^x \sigma^s)B = \bigcup_{0 \leqslant r < 2^s} (2^{r+x} \sigma^s)B$ (then $B(2^x \sigma^s)B$ is union of 2^s right cosets). If $s < 0$ then $B(2^x \sigma^s)B = (2^x \sigma^s)B$ is a right coset. Thus, for $s \neq 0$, the (B, B)-double coset $D := B(2^x \sigma^s)B$ does not contain as many left as right cosets. As in [21, p. 49], one can consider $V := \Bbbk[D]/\Bbbk[B]^+\Bbbk[D]$ and $\dim(V)$ is the number of left cosets in D while $\dim({}^\vee V)$ is the number of right cosets in D, where ${}^\vee V$ denotes the right dual of V in ${}^{\widetilde{H}} \mathfrak{M}_f$. Note that if $\widetilde{H}$ had an antipode, then ${}^\vee V$ would have been isomorphic to the $\Bbbk$-dual V^* as a vector space (see e.g. [21, p. 13]). Hence $\widetilde{H}$ has a preantipode but not an antipode.

(2) Let H be a finite-dimensional dual quasi-bialgebra. Then H^* is a quasi-bialgebra. Moreover H has an antipode if and only if H^* has an antipode. By the right-handed version of [24, Theorem 3.1] the latter is equivalent to require that the monoidal category $(\mathfrak{M}_{H^*})_f \cong {}^H \mathfrak{M}_f$ is rigid. By the left-handed version of [24, Theorem 2.6], this means that H has a preantipode. Thus, in the finite-dimensional case, a dual quasi-bialgebra H has an antipode if and only if it has a pre-antipode.

Definition 2.18. A dual quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \omega)$ is called pointed if the underlying coalgebra is pointed, i.e. all its simple subcoalgebras are one-dimensional.

Definition 2.19. Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The set

$$\mathbb{G}(A) = \big\{ a \in A \mid \Delta(a) = a \otimes a \text{ and } \varepsilon(a) = 1 \big\}$$

is called the set of the grouplike elements of A.

Remark 2.20. Let A be a pointed dual quasi-bialgebra. We know that the one-dimensional sub-coalgebras of A are exactly those of the form $\Bbbk g$ for $g \in G$ [25, p. 57]. Thus the coradical of A is $A_0 = \sum_{g \in G} \Bbbk g = \Bbbk \mathbb{G}(A)$.

The following results extend the so-called Cartier–Gabriel–Kostant theorem to dual quasi-bialgebras with a preantipode. In the connected case such a result was achieved in [12, Theorem 4.3].

Corollary 2.21. *Let H be a dual quasi-bialgebra with a preantipode over a field $\Bbbk$ of characteristic zero. If H is cocommutative and pointed, then H is an ordinary Hopf algebra isomorphic to the biproduct $U(P(H)) \# \Bbbk \mathbb{G}(H)$, where $P(H)$ denotes the Lie algebra of primitive elements in H.*

Proof. By Theorem 2.16, H is an ordinary Hopf algebra. By [25, Section 13.1, p. 279], we conclude (see also [18, p. 79]). $\quad \square$

3. Yetter–Drinfeld modules over a dual quasi-bialgebra

The main aim of this section is to restrict the equivalence between ${}^H\mathfrak{M}_H^H$ and ${}^H\mathfrak{M}$ of Theorem 2.13, to an equivalence between ${}_H^H\mathfrak{M}_H^H$ and ${}_H^H\mathcal{YD}$ (the category of Yetter–Drinfeld modules over H) for any dual quasi-bialgebra H with a preantipode.

Definition 3.1. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The category ${}_H^H\mathcal{YD}$ of Yetter–Drinfeld modules over H, is defined as follows. An object in ${}_H^H\mathcal{YD}$ is a tern $(V, \rho_V, \triangleright)$, where

- (V, ρ) is an object in ${}^H\mathfrak{M}$,
- $\triangleright : H \otimes V \to V$ is a $\Bbbk$-linear map such that, for all $h, l \in H$ and $v \in V$

$$(hl) \triangleright v = \left[\begin{array}{c} \omega^{-1}(h_1 \otimes l_1 \otimes v_{-1}) \omega(h_2 \otimes (l_2 \triangleright v_0)_{-1} \otimes l_3) \\ \omega^{-1}((h_3 \triangleright (l_2 \triangleright v_0)_0)_{-1} \otimes h_4 \otimes l_4)(h_3 \triangleright (l_2 \triangleright v_0)_0)_0 \end{array} \right], \qquad (20)$$

$$1_H \triangleright v = v \qquad (21)$$

and

$$(h_1 \triangleright v)_{-1} h_2 \otimes (h_1 \triangleright v)_0 = h_1 v_{-1} \otimes (h_2 \triangleright v_0). \qquad (22)$$

A morphism $f : (V, \rho, \triangleright) \to (V', \rho', \triangleright')$ in ${}_H^H\mathcal{YD}$ is a morphism $f : (V, \rho) \to (V', \rho')$ in ${}^H\mathfrak{M}$ such that $f(h \triangleright v) = h \triangleright' f(v)$.

3.2. The category ${}_H^H\mathcal{YD}$ is isomorphic to the weak right center of ${}^H\mathfrak{M}$ (regarded as a monoidal category as in Section 2), see Theorem A.2. As a consequence ${}_H^H\mathcal{YD}$ has a pre-braided monoidal structure given as follows. The unit is $\Bbbk$ regarded as an object in ${}_H^H\mathcal{YD}$ via trivial structures i.e. $\rho_\Bbbk(k) = 1_H \otimes k$ and $h \triangleright k = \varepsilon(h)k$. The tensor product is defined by

$$(V, \rho_V, \triangleright) \otimes (W, \rho_W, \triangleright) = (V \otimes W, \rho_{V \otimes W}, \triangleright)$$

where $\rho_{V \otimes W}(v \otimes w) = v_{-1} w_{-1} \otimes v_0 \otimes w_0$ and

$$h \triangleright (v \otimes w) = \left[\begin{array}{c} \omega(h_1 \otimes v_{-1} \otimes w_{-2}) \omega^{-1}((h_2 \triangleright v_0)_{-2} \otimes h_3 \otimes w_{-1}) \\ \omega((h_2 \triangleright v_0)_{-1} \otimes (h_4 \triangleright w_0)_{-1} \otimes h_5)(h_2 \triangleright v_0)_0 \otimes (h_4 \triangleright w_0)_0 \end{array} \right]. \qquad (23)$$

The constraints are the same of ${}^H\mathfrak{M}$ viewed as morphisms in ${}_H^H\mathcal{YD}$. The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by

$$c_{V,W}(v \otimes w) = (v_{-1} \triangleright w) \otimes v_0. \qquad (24)$$

Remark 3.3. It is easily checked that condition (20) holds for all $h, l \in H$ and $v \in V$ if and only if

$$c_{H \otimes H, V} = {}^H a_{V,H,H} \circ (c_{H,V} \otimes H) \circ {}^H a_{H,V,H}^{-1} \circ (H \otimes c_{H,V}) \circ {}^H a_{H,H,V},$$

where ${}^H a$ is the associativity constraint in ${}^H\mathfrak{M}$. Now, the displayed equality above can be written as

$${}^H a_{V,H,H}^{-1} \circ c_{H \otimes H, V} \circ {}^H a_{H,H,V}^{-1} = (c_{H,V} \otimes H) \circ {}^H a_{H,V,H}^{-1} \circ (H \otimes c_{H,V}).$$

One easily checks that this is equivalent to ask that

$$\omega(h_1 \otimes l_1 \otimes v_{-1})\omega\big(((h_2 l_2) \triangleright v_0)_{-1} \otimes h_3 \otimes l_3\big)((h_2 l_2) \triangleright v_0)_0$$

$$= \omega\big(h_1 \otimes (l_1 \triangleright v)_{-1} \otimes l_2\big)h_3 \triangleright (l_1 \triangleright v)_0$$

holds for all $h, l \in H$ and $v \in V$. This equation is the left-handed version of [5, (3.3)]. In conclusion, the axioms defining the category $^H_H \mathcal{YD}$ are the left-handed version of the ones appearing in [5, Definition 3.1].

3.1. The restriction of the equivalence (F, G)

Let H be a dual quasi-bialgebra. From Theorem 2.13, we know that the adjunction (F, G) of Remark 2.7 is an equivalence of categories when H has a preantipode. Next aim is to prove that (F, G) restricts to an equivalence between the categories $^H_H \mathcal{YD}$ and $^H_H \mathfrak{M}^H_H$.

Lemma 3.4. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. For all* $U \in {}^H \mathfrak{M}$ *and* $M \in {}^H_H \mathfrak{M}^H_H$, *we have a map*

$$\xi_{U,M} : F(U) \otimes_H M \to U \otimes M,$$

$$\xi_{U,M}\big((u \otimes h) \otimes_H m\big) = \omega^{-1}(u_{-1} \otimes h_1 \otimes m_{-1})u_0 \otimes h_2 m_0$$

which is a $\Bbbk$-*linear natural isomorphism with inverse given by* $\xi^{-1}_{U,M}(u \otimes m) = (u \otimes 1_H) \otimes_H m$. *Moreover:*

(1) *the map* $\xi_{U,M}$ *is a natural isomorphism in* $^H_H \mathfrak{M}^H_H$ *where* $U \otimes M$ *has the following structures:*

$$\rho^l_{U \otimes M}(u \otimes m) = u_{-1} m_{-1} \otimes (u_0 \otimes m_0),$$

$$\rho^r_{U \otimes M}(u \otimes m) = (u \otimes m_0) \otimes m_1,$$

$$\mu^r_{U \otimes M}\big((u \otimes m) \otimes h\big) = \omega^{-1}(u_{-1} \otimes m_{-1} \otimes h_1)u_0 \otimes m_0 h_2;$$

(2) *if* $U \in {}^H_H \mathcal{YD}$, *the map* $\xi_{U,M}$ *is a natural isomorphism in* $^H_H \mathfrak{M}^H_H$ *where* $U \otimes M$ *has the structures above along with the following left module structure:*

$$\mu^l{}_{U \otimes M}\big(h \otimes (u \otimes m)\big) = \omega(h_1 \otimes u_{-1} \otimes m_{-2})\omega^{-1}\big((h_2 \triangleright u_0)_{-1} \otimes h_3 \otimes m_{-1}\big)(h_2 \triangleright u_0)_0 \otimes h_4 m_0.$$

Proof. Inspired by [23, Lemma 3.4]. See also [24, p. 541]. □

Lemma 3.5. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. For all* $U, V \in {}^H \mathfrak{M}$, *consider the map*

$$\alpha_{U,V} : U \otimes (V \otimes H) \to (U \otimes V) \otimes H,$$

$$\alpha_{U,V}\big(u \otimes (v \otimes k)\big) = \omega(u_{-1} \otimes v_{-1} \otimes k_1)(u_0 \otimes v_0) \otimes k_2.$$

(1) *The map* $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ *is a natural isomorphism in* $^H_H \mathfrak{M}^H_H$, *where* $U \otimes F(V)$ *has the structure described in Lemma 3.4 for* $M = F(V)$.
(2) *If* $U, V \in {}^H_H \mathcal{YD}$, *then* $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ *is a natural isomorphism in* $^H_H \mathfrak{M}^H_H$, *where* $U \otimes F(V)$ *has the structure described in Lemma 3.4 for* $M = F(V)$.

Proof. It is straightforward. □

Lemma 3.6. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. The functor* $F : (-) \otimes H : {}^H\mathfrak{M} \to {}^H\mathfrak{M}_H^H$ *of* 2.6 *induces a functor* $F : {}^H_H\mathcal{YD} \to {}^H_H\mathfrak{M}_H^H$*. Explicitly* $F(M) \in {}^H_H\mathfrak{M}_H^H$ *with the following structures, for all* $m \in M$, $h, l \in H$,

$$\mu^l_{M \otimes H}\big[l \otimes (m \otimes h)\big] := l \cdot (m \otimes h) := \omega(l_1 \otimes m_{-1} \otimes h_1)(l_2 \triangleright m_0 \otimes l_3) \cdot h_2$$

$$= \omega(l_1 \otimes m_{-1} \otimes h_1)\omega^{-1}\big((l_2 \triangleright m_0)_{-1} \otimes l_3 \otimes h_2\big)(l_2 \triangleright m_0)_0$$

$$\otimes l_4 h_3, \tag{25}$$

$$\mu^r_{M \otimes H}\big[(m \otimes h) \otimes l\big] := (m \otimes h) \cdot l := \omega^{-1}(m_{-1} \otimes h_1 \otimes l_1)m_0 \otimes h_2 l_2,$$

$$\rho^l_{M \otimes H}(m \otimes h) := m_{-1}h_1 \otimes (m_0 \otimes h_2),$$

$$\rho^r_{M \otimes H}(m \otimes h) := (m \otimes h_1) \otimes h_2. \tag{26}$$

Proof. It is analogue to [23, Corollary 8.3]. $\quad\square$

Lemma 3.7. *Let* $(H, m, u, \Delta, \varepsilon, \omega, S)$ *be a dual quasi-bialgebra with a preantipode. The functor* $G : (-)^{coH} :$ ${}^H_H\mathfrak{M}_H^H \to {}^H\mathfrak{M}$ *of* 2.6 *induces a functor* $G : {}^H_H\mathfrak{M}_H^H \to {}^H_H\mathcal{YD}$*. Explicitly* $G(M) \in {}^H_H\mathcal{YD}$ *with the following structures, for all* $m \in M^{coH}$, $h \in H$,

$$\rho^l_{M^{coH}}(m) := \rho^l_M(m),$$

$$\mu^l_{M^{coH}}(h \otimes m) := h \triangleright m := \tau(hm) = \omega\big[h_1 m_{-1} \otimes S(h_3)_1 \otimes h_4\big](h_2 m_0)S(h_3)_2.$$

Proof. Let $M \in {}^H_H\mathfrak{M}_H^H$. We already know that $G(M) \in {}^H\mathfrak{M}$. In order to prove that $G(M)$ is in ${}^H_H\mathcal{YD}$, we consider the canonical isomorphism $\epsilon_M : FG(M) \to M$ of Remark 2.7. A priori, this is a morphism in ${}^H\mathfrak{M}_H^H$. Since M is in ${}^H_H\mathfrak{M}_H^H$, we can endow $FG(M)$ with a left H-module structure as follows

$$l \cdot (m \otimes h) := \epsilon_M^{-1}\big(l\epsilon_M(m \otimes h)\big) = \epsilon_M^{-1}\big(l(mh)\big) = \tau\big[l_1(m_0 h_1)\big] \otimes l_2(m_1 h_2)$$

$$= \tau\big[l_1(mh_1)\big] \otimes l_2 h_2 = l_1 \triangleright (mh_1) \otimes l_2 h_2$$

so that

$$l \cdot (m \otimes h) = l_1 \triangleright (mh_1) \otimes l_2 h_2 \quad \text{for all } m \in M^{coH}, \ h \in H. \tag{27}$$

By associativity we have

$$(lk) \cdot (m \otimes h) = \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1}h_1)l_2\big(k_2(m_0 \otimes h_2)\big)\omega(l_3 \otimes k_3 \otimes h_3)$$

i.e., for $h = 1_H$,

$$(lk) \cdot (m \otimes 1_H) = \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})l_2\big(k_2(m_0 \otimes 1_H)\big).$$

The first term is

$$(lk) \cdot (m \otimes 1_H) \stackrel{(27)}{=} (l_1 k_1) \triangleright m \otimes l_2 k_2.$$

The second term is

$$\omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})l_2\big(k_2(m_0 \otimes 1_H)\big)$$

$$\stackrel{(27)}{=} \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})l_2(k_2 \triangleright m_0 \otimes k_3) \stackrel{(27)}{=} \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})l_2 \triangleright \big((k_2 \triangleright m_0)k_3\big) \otimes l_3 k_4$$

$$= \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})\tau\big[l_2\big((k_2 \triangleright m_0)k_3\big)\big] \otimes l_3 k_4$$

$$= \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})\omega\big(l_2 \otimes (k_2 \triangleright m_0)_{-1} \otimes k_3\big)\tau\big[\big(l_3(k_2 \triangleright m_0)_0\big)k_4\big] \otimes l_4 k_5$$

$$= \left[\begin{array}{c} \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})\omega(l_2 \otimes (k_2 \triangleright m_0)_{-1} \otimes k_3) \\ \omega^{-1}((l_3 \triangleright (k_2 \triangleright m_0)_0)_{-1} \otimes l_4 \otimes k_4)(l_3 \triangleright (k_2 \triangleright m_0)_0)_0 \otimes l_5 k_5 \end{array}\right].$$

Hence, we obtain

$$(l_1 k_1) \triangleright m \otimes l_2 k_2 = \left[\begin{array}{c} \omega^{-1}(l_1 \otimes k_1 \otimes m_{-1})\omega(l_2 \otimes (k_2 \triangleright m_0)_{-1} \otimes k_3) \\ \omega^{-1}((l_3 \triangleright (k_2 \triangleright m_0)_0)_{-1} \otimes l_4 \otimes k_4)(l_3 \triangleright (k_2 \triangleright m_0)_0)_0 \otimes l_5 k_5 \end{array}\right].$$

By applying $M \otimes \varepsilon_H$ on both sides, we arrive at (20). Moreover, by (9), we have $1_H \triangleright m = \tau(m) = m$ and

$$(h_1 \triangleright m)_{-1} h_2 \otimes (h_1 \triangleright m)_0 = \tau(h_1 m)_{-1} h_2 \otimes \tau(h_1 m)_0 = \tau\big((hm)_0\big)_{-1}(hm)_1 \otimes \tau\big((hm)_0\big)_0$$

$$\stackrel{(7)}{=} (hm)_{-1} \otimes \tau\big((hm)_0\big) = h_1 m_{-1} \otimes \tau(h_2 m_0) = h_1 m_{-1} \otimes (h_2 \triangleright m_0).$$

We have so proved that $G(M) \in {}^H_H \mathcal{YD}$. Now it is easy to verify that for every $g : M \to N \in {}^H_H \mathfrak{M}^H_H$, we have that $G(g) : M^{coH} \to N^{coH} \in {}^H_H \mathcal{YD}$. $\quad\square$

Proposition 3.8. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. (F, G) is an equivalence between ${}^H_H \mathfrak{M}^H_H$ and ${}^H_H \mathcal{YD}$, i.e. the morphisms ε_M and η_N of* Remark 2.7 *are in ${}^H_H \mathfrak{M}^H_H$ and in ${}^H_H \mathcal{YD}$ respectively, for each $M \in {}^H_H \mathfrak{M}^H_H$, $N \in {}^H_H \mathcal{YD}$.*

Proof. We already know that $\varepsilon_M \in {}^H \mathfrak{M}^H_H$. Let us check that ε_M is left H-linear

$$\varepsilon_M \mu_{M^{coH} \otimes H}(h \otimes m \otimes k)$$

$$= \varepsilon_M\big(h \cdot (m \otimes k)\big) \stackrel{(25)}{=} \varepsilon_M\big[\omega(h_1 \otimes m_{-1} \otimes k_1)(h_2 \triangleright m_0 \otimes h_3)k_2\big]$$

$$\stackrel{\varepsilon_M \text{ right lin}}{=} \omega(h_1 \otimes m_{-1} \otimes k_1)\varepsilon_M\big[(h_2 \triangleright m_0 \otimes h_3)\big]k_2 = \omega(h_1 \otimes m_{-1} \otimes k_1)\big[(h_2 \triangleright m_0)h_3\big]k_2$$

$$= \omega(h_1 \otimes m_{-1} \otimes k_1)\big[\tau(h_2 m_0)h_3\big]k_2 = \omega(h_1 \otimes m_{-1} \otimes k_1)\big[\tau(h_2 m_0)(h_3 m_1)\big]k_2$$

$$\stackrel{(8)}{=} \omega(h_1 \otimes m_{-1} \otimes k_1)(h_2 m_0)k_2 \stackrel{(3)}{=} h_1(m_0 k_1)\omega(h_2 \otimes m_1 \otimes k_2) = h(mk)$$

$$= \mu_M(H \otimes \varepsilon_M)(h \otimes m \otimes k).$$

Now let us check the compatibility of η with $\triangleright$. For $N \in {}^H_H \mathcal{YD}$ and $n \in N$,

$$\big[\mu^l_{(N \otimes H)^{coH}} \circ {}^H a_{H,N,H} \circ (H \otimes \eta_N)\big](h \otimes n)$$

$$= \big[\mu^l_{(N \otimes H)^{coH}} \circ {}^H a_{H,N,H}\big]\big(h \otimes (n \otimes 1_H)\big) = \omega^{-1}(h_1 \otimes n_{-1} \otimes 1_H)\mu^l_{(N \otimes H)^{coH}}\big(h_2 \otimes (n_0 \otimes 1_H)\big)$$

$$= \mu^l_{(N \otimes H)^{coH}}\big(h \otimes (n \otimes 1_H)\big) = \tau\big(h(n \otimes 1_H)\big) \overset{(25)}{=} \tau(h_1 \triangleright n \otimes h_2) \overset{(26)}{=} \tau\big((h_1 \triangleright n \otimes 1_H)h_2\big)$$

$$= \tau\big(\eta_N(h_1 \triangleright n)h_2\big) \overset{(9)}{=} \eta_N(h_1 \triangleright n)\varepsilon_H(h_2) = \eta_N(h \triangleright n).$$

So $\eta_N \in {}^H_H\mathcal{YD}$, for each $N \in {}^H_H\mathcal{YD}$. $\square$

4. Monoidal equivalences

In this section we prove that the equivalence between the categories ${}^H_H\mathfrak{M}^H_H$ and ${}^H_H\mathcal{YD}$ becomes monoidal if we equip ${}^H_H\mathfrak{M}^H_H$ with the tensor product $\otimes_H$ (or $\square_H$) and unit H. As a by-product we produce a monoidal equivalence between $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$ and $({}^H_H\mathfrak{M}^H_H, \square_H, H)$.

Lemma 4.1. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. The category* $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$ *is monoidal with respect to the following constraints*:

$$a_{U,V,W}\big((u \otimes_H v) \otimes_H w\big) = \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes_H (v_0 \otimes_H w_0)\omega(u_1 \otimes v_1 \otimes w_1),$$

$$l_U(h \otimes_H u) = hu,$$

$$r_U(u \otimes_H h) = uh.$$

Proof. See e.g. [2, Theorem 1.12]. $\square$

Lemma 4.2. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. Let* $U \in {}^H\mathfrak{M}^H$, $V \in {}^H\mathfrak{M}^H_H$. *Then* $(U \otimes V, \rho^l, \rho^r, \mu) \in {}^H\mathfrak{M}^H_H$ *with the following structures*:

$$\rho^l_{U \otimes V}(u \otimes v) = u_{-1}v_{-1} \otimes (u_0 \otimes v_0),$$

$$\rho^r_{U \otimes V}(u \otimes v) = (u_0 \otimes v_0) \otimes u_1 v_1,$$

$$\mu^r_{U \otimes V}\big((u \otimes v) \otimes h\big) = \omega^{-1}(u_{-1} \otimes v_{-1} \otimes h_1)u_0 \otimes v_0 h_2 \omega(u_1 \otimes v_1 \otimes h_3).$$

Proof. It is left to the reader. $\square$

Definition 4.3. We recall that a *lax monoidal functor*

$$(F, \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to \big(\mathcal{M}', \otimes', \mathbf{1}', a', l', r'\big)$$

between two monoidal categories consists of

- a functor $F : \mathcal{M} \to \mathcal{M}'$,
- a natural transformation $\phi_2(U, V) : F(U) \otimes' F(V) \to F(U \otimes V)$, with $U, V \in \mathcal{M}$, and
- a natural transformation $\phi_0 : \mathbf{1}' \to F(\mathbf{1})$ such that the diagram

$$
\begin{array}{ccccc}
(F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{\phi_2(U,V) \otimes' F(W)} & F(U \otimes V) \otimes' F(W) & \xrightarrow{\phi_2(U \otimes V, W)} & F((U \otimes V) \otimes W) \\
\Big\downarrow{\scriptstyle a'_{F(U),F(V),F(W)}} & & & & \Big\downarrow{\scriptstyle F(a_{U,V,W})} \\
F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{F(U) \otimes' \phi_2(V,W)} & F(U) \otimes' F(V \otimes W) & \xrightarrow{\phi_2(U, V \otimes W)} & F(U \otimes (V \otimes W))
\end{array}
$$

(28)

commutes and the following conditions are satisfied

$$F(l_U) \circ \phi_2(\mathbf{1}, U) \circ \big(\phi_0 \otimes F(U)\big) = l'_{F(U)}, \tag{29}$$

$$F(r_U) \circ \phi_2(U, \mathbf{1}) \circ \big(F(U) \otimes \phi_0\big) = r'_{F(U)}. \tag{30}$$

The morphisms $\phi_2(U, V)$ and ϕ_0 are called *structure morphisms*.

Colax monoidal functors are defined similarly but with the directions of the structure morphisms reversed. A strong monoidal functor or simply a *monoidal functor* is a lax monoidal functor with invertible structure morphisms.

Lemma 4.4. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. The functor* $F : {}^H_H\mathcal{YD} \to {}^H_H\mathfrak{M}^H_H$ *defines a monoidal functor* $F : ({}^H_H\mathcal{YD}, \otimes, \Bbbk) \to ({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$. *For* $U, V \in {}^H_H\mathcal{YD}$, *the structure morphisms are*

$$\varphi_2(U, V) : F(U) \otimes_H F(V) \to F(U \otimes V) \quad \text{and} \quad \varphi_0 : H \to F(\Bbbk)$$

which are defined, for every $u \in U$, $v \in V$, $h, k \in H$, *by*

$$\varphi_2(U, V)\big[(u \otimes h) \otimes_H (v \otimes k)\big] := \begin{bmatrix} \omega^{-1}(u_{-2} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \rhd v_0)_{-2} \otimes h_4 \otimes k_3)\omega(u_{-1} \otimes (h_3 \rhd v_0)_{-1} \otimes h_5k_4) \\ (u_0 \otimes (h_3 \rhd v_0)_0) \otimes h_6k_5 \end{bmatrix}$$

and

$$\varphi_0(h) := 1_\Bbbk \otimes h.$$

Moreover

$$\varphi_2(U, V)^{-1}\big((u \otimes v) \otimes k\big) = \omega^{-1}(u_{-1} \otimes v_{-1} \otimes k_1)(u_0 \otimes 1_H) \otimes_H (v_0 \otimes k_2). \tag{31}$$

Proof. Let us check that φ_0 is a morphism in ${}^H_H\mathfrak{M}^H_H$. Since $\varphi_0 = l_H^{-1} : H \to \Bbbk \otimes H$, i.e. the inverse of the left unit constraint in ${}^H\mathfrak{M}^H$, then φ_0 is in ${}^H\mathfrak{M}^H$ and it is invertible. It is easy to check that it is H-bilinear in ${}^H\mathfrak{M}^H$.

Let us consider now $\varphi_2(U, V)$.

By Lemma 3.4, for all $U, V \in {}^H_H\mathcal{YD}$, the map $\xi_{U, F(V)} : F(U) \otimes_H F(V) \to U \otimes F(V)$, is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$. By Lemma 3.5, $\alpha_{U, V} : U \otimes F(V) \to F(U \otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$, where $U \otimes F(V)$ has the structure described in Lemma 3.4 for $M = F(V)$.

Thus $\alpha_{U, V}\xi_{U, F(V)} : F(U) \otimes_H F(V) \to F(U \otimes V)$ is a natural isomorphism in ${}^H_H\mathfrak{M}^H_H$. A direct computation shows that $\varphi_2(U, V) = \alpha_{U, V}\xi_{U, F(V)}$ and hence $\varphi_2(U, V)$ is a well-defined isomorphism in ${}^H_H\mathfrak{M}^H_H$. Moreover $\varphi_2(U, V)^{-1} = \xi_{U, F(V)}^{-1}\alpha_{U, V}^{-1}$ fulfills (31).

In order to check the commutativity of the diagram (28) it suffices to prove the following equality:

$$\big[\varphi_2^{-1}(U, V) \otimes_H F(W)\big]\varphi_2^{-1}(U \otimes V, W)F\big(a_{U, V, W}^{-1}\big)$$
$$= a_{F(U), F(V), F(W)}^{-1}\big[F(U) \otimes_H \varphi_2^{-1}(V, W)\big]\varphi_2^{-1}(U, V \otimes W).$$

Since these maps are right H-linear, it suffices to check this equality on elements of the form $(u \otimes (v \otimes w)) \otimes 1_H$, where $u \in U$, $v \in V$, $w \in W$. This computation and the ones of (29) and (30) are straightforward. $\square$

We now compute explicitly the braiding induced on ${}^H_H\mathfrak{M}^H_H$ through the functor F in Lemma 4.4, in case F comes out to be an equivalence i.e. when H has a preantipode.

Lemma 4.5. *Let* $(H, m, u, \Delta, \varepsilon, \omega, S)$ *be a dual quasi-bialgebra with a preantipode. Through the monoidal equivalence* (F, G) *we have that* $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$ *becomes a pre-braided monoidal category, with braiding defined as follows*:

$$c_{M,N}(m \otimes_H n) = \omega(m_{-2} \otimes \tau(n_0)_{-1} \otimes n_1)(m_{-1} \triangleright \tau(n_0)_0 \otimes_H m_0) \cdot n_2,$$

where $M, N \in {}_H^H\mathfrak{M}_H^H$ *and* $m \in M, n \in N$.

Proof. First of all, for any $U, V \in {}_H^H\mathcal{YD}$, let us consider the following composition:

$$\lambda_{U,V} := \left(F(U) \otimes_H F(V) \xrightarrow{\varphi_2(U,V)} F(U \otimes V) \xrightarrow{F(c_{U,V})} F(V \otimes U) \xrightarrow{\varphi_2^{-1}(V,U)} F(V) \otimes_H F(U)\right).$$

This map is right H-linear, so, if we compute

$$\lambda_{U,V}\left[(u \otimes h) \otimes_H (v \otimes 1_H)\right]$$

$$= \left[\begin{array}{c} \omega^{-1}(u_{-4} \otimes h_1 \otimes v_{-1})\omega(u_{-3} \otimes (h_2 \triangleright v_0)_{-1} \otimes h_3) \\ \omega^{-1}((u_{-2} \triangleright (h_2 \triangleright v_0)_0)_{-1} \otimes u_{-1} \otimes h_4)((u_{-2} \triangleright (h_2 \triangleright v_0)_0)_0 \otimes 1_H) \otimes_H u_0 \otimes h_5 \end{array}\right]$$

$$\overset{(20)}{=} \left((u_{-1}h_1) \triangleright v \otimes 1_H\right) \otimes_H (u_0 \otimes h_2),$$

we obtain

$$\lambda_{U,V}\left[(u \otimes h) \otimes_H (v \otimes k)\right]$$

$$= \lambda_{U,V}\left[(u \otimes h) \otimes_H (v \otimes 1_H) \cdot k\right]$$

$$= \omega(u_{-1}h_1 \otimes v_{-1} \otimes k_1)\lambda_{U,V}\left[\left[(u_0 \otimes h_2) \otimes_H (v_0 \otimes 1_H)\right] \cdot k_2\right]\omega^{-1}(h_3 \otimes 1_H \otimes k_3)$$

$$= \omega(u_{-1}h_1 \otimes v_{-1} \otimes k_1)\lambda_{U,V}\left[\left[(u_0 \otimes h_2) \otimes_H (v_0 \otimes 1_H)\right] \cdot k_2\right]$$

$$= \omega(u_{-1}h_1 \otimes v_{-1} \otimes k_1)\lambda_{U,V}\left[(u_0 \otimes h_2) \otimes_H (v_0 \otimes 1_H)\right] \cdot k_2$$

$$= \omega(u_{-2}h_1 \otimes v_{-1} \otimes k_1)\left[\left((u_{-1}h_2) \triangleright v_0 \otimes 1_H\right) \otimes_H (u_0 \otimes h_3)\right] \cdot k_2.$$

Now, using the map $\lambda_{U,V}$, we construct the braiding of ${}_H^H\mathfrak{M}_H^H$ in this way:

$$M \otimes_H N \xrightarrow{\epsilon_M^{-1} \otimes_H \epsilon_N^{-1}} FG(M) \otimes_H FG(N) \xrightarrow{\lambda_{G(M),G(N)}} FG(N) \otimes_H FG(M) \xrightarrow{\epsilon_N \otimes_H \epsilon_M} N \otimes_H M.$$

Therefore

$$(\epsilon_N \otimes_H \epsilon_M)\lambda_{G(M),G(N)}\left(\epsilon_M^{-1} \otimes_H \epsilon_N^{-1}\right)(m \otimes_H n)$$

$$= (\epsilon_N \otimes_H \epsilon_M)\lambda_{G(M),G(N)}\left\{\left[\tau(m_0) \otimes m_1\right] \otimes_H \left[\tau(n_0) \otimes n_1\right]\right\}$$

$$= \left[\begin{array}{c} \omega(\tau(m_0)_{-2}m_1 \otimes \tau(n_0)_{-1} \otimes n_1) \\ (\epsilon_N \otimes_H \epsilon_M)\{[((\tau(m_0)_{-1}m_2) \triangleright \tau(n_0)_0 \otimes 1_H) \otimes_H (\tau(m_0)_0 \otimes m_3)] \cdot n_2\} \end{array}\right]$$

$$\overset{(7)}{=} \left[\begin{array}{c} \omega(m_{-2} \otimes \tau(n_0)_{-1} \otimes n_1) \\ (\epsilon_N \otimes_H \epsilon_M)[(m_{-1} \triangleright \tau(n_0)_0 \otimes 1_H) \otimes_H (\tau(m_0) \otimes m_1)] \cdot n_2 \end{array}\right]$$

$$= \omega(m_{-2} \otimes \tau(n_0)_{-1} \otimes n_1)\left[m_{-1} \triangleright \tau(n_0)_0 \otimes_H \tau(m_0)m_1\right] \cdot n_2$$

$$\overset{(8)}{=} \omega(m_{-2} \otimes \tau(n_0)_{-1} \otimes n_1)\left[m_{-1} \triangleright \tau(n_0)_0 \otimes_H m_0\right] \cdot n_2. \quad \square$$

Next aim is to prove that the equivalence between the categories $_H^H\mathfrak{M}_H^H$ and $_H^H\mathcal{YD}$ becomes monoidal if we equip $_H^H\mathfrak{M}_H^H$ with the tensor product $\square_H$ and unit H.

4.6. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Note that, since H is an ordinary coalgebra, we have that $({}^H\mathfrak{M}^H, \square_H, H, b, r, l)$ is a monoidal category with constraints defined, for all $L, M, N \in {}^H\mathfrak{M}^H$, by

$$b_{L,M,N} : (L \square_H M) \square_H N \to L \square_H (M \square_H N) : (l \square_H m) \square_H n \mapsto l \square_H (m \square_H n),$$

$$r_M : M \square_H H \to M : m \square_H h \mapsto m\varepsilon_H(h),$$

$$l_M : H \square_H M \to M : h \square_H m \mapsto \varepsilon_H(h)m,$$

where, for sake of brevity we just wrote $m \square_H n$ in place of the more precise $\sum_i m^i \square_H n^i$.

We want to endow $_H^H\mathfrak{M}_H^H$ with a monoidal structure, following the dual version of [11] (see also [23, Definition 3.2]). The definition of the claimed structure is given in such a way that the forgetful functor $_H^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$ is a strict monoidal functor. Hence the constraints are induced by the ones of $^H\mathfrak{M}^H$ (i.e. $b_{L,M,N}$, l_M and r_M), and the tensor product is given by $M \square_H N$ with structures

$$\rho_{M\square_H N}^l(m \square_H n) = m_{-1} \otimes (m_0 \square_H n),$$

$$\rho_{M\square_H N}^r(m \square_H n) = (m \square_H n_0) \otimes n_1,$$

$$\mu_{M\square_H N}^l\big[h \otimes (m \square_H n)\big] = h \cdot (m \square_H n) = h_1 m \square_H h_2 n,$$

$$\mu_{M\square_H N}^r\big[(m \square_H n) \otimes h\big] = (m \square_H n) \cdot h = mh_1 \square_H nh_2.$$

The unit of the category is H endowed with the following structures:

$$\rho_H^l(h) = h_1 \otimes h_2, \qquad \rho_H^r(h) = h_1 \otimes h_2,$$

$$h \cdot l = hl, \qquad l \cdot h = lh.$$

The following result is similar to (2) in Lemma 3.4.

Lemma 4.7. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. For all $V \in {}_H^H\mathcal{YD}$ and $M \in {}_H^H\mathfrak{M}_H^H$, the map*

$$\beta_{V,M} : F(V) \square_H M \to V \otimes M : (v \otimes h) \square_H m \mapsto v\varepsilon(h) \otimes m$$

is a natural isomorphism in $_H^H\mathfrak{M}_H^H$ where $V \otimes M$ has the structures as in Lemma 3.4. The inverse of $\beta_{V,M}$ is given by

$$\beta_{V,M}^{-1} : V \otimes M \to (V \otimes H) \square_H M : v \otimes m \mapsto (v \otimes m_{-1}) \square_H m_0.$$

Proof. The proof is straightforward and is based on the fact that $(v \otimes h) \square_H m \in (V \otimes H) \square_H M$ implies

$$(v \otimes h) \otimes m = \big(v\varepsilon(h) \otimes m_{-1}\big) \otimes m_0. \qquad \square \tag{32}$$

Lemma 4.8. *(Cf. [23, Proposition 3.6].) Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The functor $F : {}_H^H\mathcal{YD} \to {}_H^H\mathfrak{M}_H^H$ defines a monoidal functor $F : ({}_H^H\mathcal{YD}, \otimes, \Bbbk) \to ({}_H^H\mathfrak{M}_H^H, \square_H, H)$. For $U, V \in {}_H^H\mathcal{YD}$, the structure morphisms are*

$$\psi_2(U, V) : F(U) \,\Box_H\, F(V) \to F(U \otimes V) \quad and \quad \psi_0 : H \to F(\Bbbk)$$

which are defined, for every $u \in U$, $v \in V$, $h, k \in H$, by

$$\psi_2(U, V)\big[(u \otimes h) \otimes (v \otimes k)\big] := \omega(u_{-1} \otimes v_{-1} \otimes k_1)u_0\varepsilon(h) \otimes v_0 \otimes k_2 \tag{33}$$

and

$$\psi_0(h) := 1_{\Bbbk} \otimes h.$$

Moreover

$$\psi_2(U, V)^{-1}\big((u \otimes v) \otimes h\big) = \omega^{-1}(u_{-1} \otimes v_{-2} \otimes h_1)(u_0 \otimes v_{-1}h_2) \otimes (v_0 \otimes h_3). \tag{34}$$

Proof. Since $\psi_0 = \varphi_0$ as in Lemma 4.4, we already know that ψ_0 is an isomorphism in $_H^H\mathfrak{M}_H^H$. Let us deal with $\psi_2(U, V)$. By Lemma 3.5, the map $\alpha_{U,V} : U \otimes F(V) \to F(U \otimes V)$ is a natural isomorphism in $_H^H\mathfrak{M}_H^H$, where $U \otimes F(V)$ has the structure described in Lemma 3.4 for $M = F(V)$. By Lemma 4.7, $\beta_{U,F(V)} = \beta : F(U) \,\Box_H\, F(V) \to U \otimes F(V)$ is a natural isomorphism in $_H^H\mathfrak{M}_H^H$, where $U \otimes F(V)$ has the structure described in Lemma 3.4 for $M = F(V)$. Hence it makes sense to consider the composition $\psi_2(U, V) := \alpha_{U,V} \circ \beta_{U,V\otimes H}$. Then $\psi_2(U, V)$ fulfills (33). It is clear that $\psi_2(U, V) : F(U) \,\Box_H\, F(V) \to F(U \otimes V)$ is a natural isomorphism in $_H^H\mathfrak{M}_H^H$ with inverse given by $\psi_2(U, V)^{-1} := \beta_{U,V\otimes H}^{-1} \circ \alpha_{U,V}^{-1}$. Moreover $\psi_2(U, V)^{-1}$ satisfies (34).

In order to check the commutativity of the diagram (28) it suffices to prove the following equality:

$$\big(\psi_2(U, V)^{-1} \otimes F(W)\big)\psi_2(U \otimes V, W)^{-1} F\big(a_{U,V,W}^{-1}\big)\big[(u \otimes (v \otimes w)) \otimes h\big]$$
$$= b_{F(U),F(V),F(W)}^{-1}\big[F(U) \otimes \psi_2(V, W)^{-1}\big]\psi_2(U, V \otimes W)^{-1}\big[(u \otimes (v \otimes w)) \otimes h\big].$$

By right H-linearity, it suffices to check the displayed equality for $h = 1_H$. The proof of this fact and of (29) and (30) is straightforward. □

If H has a preantipode, the functor F of Lemma 4.8 is an equivalence. As a consequence, its adjoint G is monoidal too. For future reference we include here its explicit monoidal structure.

Lemma 4.9. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. The right adjoint $G : {}_H^H\mathfrak{M}_H^H \to {}_H^H\mathcal{YD}$ of the functor F, defines a monoidal functor $G : ({}_H^H\mathfrak{M}_H^H, \Box_H, H) \to ({}_H^H\mathcal{YD}, \otimes, \Bbbk)$. For $M, N \in {}_H^H\mathfrak{M}_H^H$, the structure morphisms are*

$$\psi_2^G(M, N) : G(M) \otimes G(N) \to G(M \,\Box_H\, N) \quad and \quad \psi_0^G : \Bbbk \to G(H)$$

which are defined, for every $m \in M$, $n \in N$, $k \in H$, by

$$\psi_2^G(M, N)(m \otimes n) = mn_{-1} \,\Box_H\, n_0 \quad and \quad \psi_0^G(k) := k1_H.$$

Moreover, for all $m \in M$, $n \in N$,

$$\psi_2^G(M, N)^{-1}(m \,\Box_H\, n) = \tau(m) \otimes \tau(n).$$

Proof. Apply [3, Proposition 1.4] to the functor F. Then G is monoidal with structure morphisms

$$\psi_2^G(M, N) := G(\epsilon_M \,\square_H\, \epsilon_M) \circ G\big(\psi_2(GM, GN)^{-1}\big) \circ \eta_{GM \otimes GN},$$

$$\psi_0^G := G\big(\psi_0^{-1}\big) \circ \eta_{\Bbbk}.$$

A direct computation shows that they are the desired maps.

The inverse of $\psi_2^G(M, N)$ can be computed by

$$\psi_2^G(M, N)^{-1} := \eta_{GM \otimes GN}^{-1} \circ G\big(\psi_2(GM, GN)\big) \circ G\big(\epsilon_M^{-1} \,\square_H\, \epsilon_M^{-1}\big). \qquad \square$$

Remark 4.10. Consider the composition

$$\kappa = \kappa(U, V) := \psi_2(U, V)^{-1} \circ \varphi_2(U, V) : (U \otimes H) \otimes_H (V \otimes H) \to (U \otimes H) \,\square_H\, (V \otimes H).$$

We have

$$\kappa(U, V)\big[(u \otimes h) \otimes_H (v \otimes k)\big]$$

$$= \psi_2(U, V)^{-1} \varphi_2(U, V)\big[(u \otimes h) \otimes_H (v \otimes k)\big]$$

$$= \begin{bmatrix} \omega^{-1}(u_{-2} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright v_0)_{-2} \otimes h_4 \otimes k_3)\omega(u_{-1} \otimes (h_3 \triangleright v_0)_{-1} \otimes h_5k_4) \\ \psi_2(U, V)^{-1}[(u_0 \otimes (h_3 \triangleright v_0)_0) \otimes (h_6k_5)] \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{-1}(u_{-2} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright v_0)_{-2} \otimes h_4 \otimes k_3)\omega(u_{-1} \otimes (h_3 \triangleright v_0)_{-1} \otimes h_5k_4) \\ \omega^{-1}(u_{0-1} \otimes (h_3 \triangleright v_0)_{0-2} \otimes (h_6k_5)_1) \\ (u_{00} \otimes (h_3 \triangleright v_0)_{0-1}(h_6k_5)_2) \,\square_H\, ((h_3 \triangleright v_0)_{00} \otimes (h_6k_5)_3) \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{-1}(u_{-3} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright v_0)_{-4} \otimes h_4 \otimes k_3)\omega(u_{-2} \otimes (h_3 \triangleright v_0)_{-3} \otimes h_5k_4) \\ \omega^{-1}(u_{-1} \otimes (h_3 \triangleright v_0)_{-2} \otimes h_6k_5)(u_0 \otimes (h_3 \triangleright v_0)_{-1}(h_7k_6)) \,\square_H\, ((h_3 \triangleright v_0)_0 \otimes h_8k_7) \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright v_0)_{-2} \otimes h_4 \otimes k_3) \\ (u_0 \otimes (h_3 \triangleright v_0)_{-1}(h_5k_4)) \,\square_H\, ((h_3 \triangleright v_0)_0 \otimes h_6k_5) \end{bmatrix}$$

$$\overset{(3)}{=} \begin{bmatrix} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ (u_0 \otimes (((h_3 \triangleright v_0)_{-2}h_4)k_3)) \\ \omega^{-1}((h_3 \triangleright v_0)_{-1} \otimes h_5 \otimes k_4) \,\square_H\, ((h_3 \triangleright v_0)_0 \otimes h_6k_5) \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-2}k_1)\omega(h_2 \otimes v_{-1} \otimes k_2) \\ (u_0 \otimes (((h_3 \triangleright v_0)_{-1}h_4)k_3)) \,\square_H\, (((h_3 \triangleright v_0)_0 \otimes h_5) \cdot k_4) \end{bmatrix}$$

$$\overset{(22)}{=} \begin{bmatrix} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-3}k_1)\omega(h_2 \otimes v_{-2} \otimes k_2) \\ (u_0 \otimes ((h_3v_{-1})k_3)) \,\square_H\, (((h_4 \triangleright v_0) \otimes h_5) \cdot k_4) \end{bmatrix}$$

$$\overset{(3)}{=} \begin{bmatrix} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-3}k_1) \\ (u_0 \otimes (h_2(v_{-2}k_2))) \\ \omega(h_3 \otimes v_{-1} \otimes k_3) \,\square_H\, (((h_4 \triangleright v_0) \otimes h_5) \cdot k_4) \end{bmatrix}$$

$$\overset{(25)}{=} \omega^{-1}(u_{-1} \otimes h_1 \otimes v_{-2}k_1)\big(u_0 \otimes (h_2(v_{-1}k_2))\big) \,\square_H\, \big(h_3 \cdot (v_0 \otimes k_3)\big)$$

$$= (u_0 \otimes h_1) \cdot (v_{-1}k_1) \,\square_H\, h_3 \cdot (v_0 \otimes k_3)$$

$$= (u \otimes h)_0 \cdot (v \otimes k)_{-1} \,\square_H\, (u \otimes h)_1 \cdot (v \otimes k)_0$$

so that

$$\kappa(U, V)\big[(u \otimes h) \otimes_H (v \otimes k)\big] = (u \otimes h)_0 \cdot (v \otimes k)_{-1} \,\square_H\, (u \otimes h)_1 \cdot (v \otimes k)_0.$$

Thus, for $M, N \in {}_H^H\mathfrak{M}_H^H$, using that the counit ϵ is in ${}_H^H\mathfrak{M}_H^H$, one gets

$$\big[(\epsilon_M \,\square_H\, \epsilon_N) \circ \kappa\big(M^{coH}, N^{coH}\big) \circ \big(\epsilon_M^{-1} \otimes_H \epsilon_N^{-1}\big)\big](m \otimes_H n) = m_0 n_{-1} \,\square_H\, m_1 n_0.$$

We can also compute $\kappa(U, V)^{-1} := \varphi_2(U, V)^{-1} \circ \psi_2(U, V)$. We have

$$\kappa(U, V)^{-1}\big((u \otimes h) \,\square_H\, (v \otimes k)\big) = \big(u\varepsilon(h) \otimes 1_H\big) \otimes_H (v \otimes k).$$

We are now able to provide a monoidal equivalence between $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$ and $({}_H^H\mathfrak{M}_H^H, \square_H, H)$. This result is similar to [22, Corollary 6.1].

Lemma 4.11. *Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. The identity functor on ${}_H^H\mathfrak{M}_H^H$ defines a monoidal functor $E : ({}_H^H\mathfrak{M}_H^H, \otimes_H, H) \to ({}_H^H\mathfrak{M}_H^H, \square_H, H)$. For $M, N \in {}_H^H\mathfrak{M}_H^H$, the structure morphisms are*

$$\vartheta_2(M, N) : E(M) \,\square_H\, E(V) \to E(M \otimes_H N) \quad and \quad \vartheta_0 : H \to E(H) = H$$

which are defined, for every $m \in M, n \in N, h \in H$, by

$$\vartheta_2(M, N)(m \,\square_H\, n) := \tau(m) \otimes_H n \quad and \quad \vartheta_0(h) := h.$$

Moreover

$$\vartheta_2(M, N)^{-1}(m \otimes_H n) = m_0 n_{-1} \,\square_H\, m_1 n_0, \tag{35}$$

$$\vartheta_2(FU, FV) = \varphi_2(U, V)^{-1} \circ \psi_2(U, V). \tag{36}$$

Proof. Using the map κ of Remark 4.10, for each $M, N \in {}_H^H\mathfrak{M}_H^H$, we set

$$\vartheta_2(M, N) := (\epsilon_M \otimes_H \epsilon_N) \circ \kappa\big(M^{coH}, N^{coH}\big)^{-1} \circ \big(\epsilon_M^{-1} \,\square_H\, \epsilon_N^{-1}\big).$$

Clearly, by Remark 4.10, $\vartheta_2(M, N)^{-1}$ fulfills (35). Moreover, using (8), one gets

$$\vartheta_2(M, N)(m \,\square_H\, n) = \tau(m) \otimes_H n.$$

It is straightforward to check that ϑ_2^{-1} makes commutative the diagram (28) and that (29) and (30) hold. Let us check that (36) holds

$$\vartheta_2(FU, FV) = (\epsilon_{FU} \otimes_H \epsilon_{FV}) \circ \kappa(GFU, GFV)^{-1} \circ \big(\epsilon_{FU}^{-1} \,\square_H\, \epsilon_{FV}^{-1}\big)$$

$$= (\epsilon_{FU} \otimes_H \epsilon_{FV}) \circ \varphi_2(GFU, GFV)^{-1} \circ \psi_2(GFU, GFV) \circ \big(\epsilon_{FU}^{-1} \,\square_H\, \epsilon_{FV}^{-1}\big)$$

$$= \begin{bmatrix} (\epsilon_{FU} \otimes_H \epsilon_{FV}) \circ \varphi_2(GFU, GFV)^{-1} \circ F(\eta_U \otimes \eta_V) \\ F(\eta_U^{-1} \otimes \eta_V^{-1}) \circ \psi_2(GFU, GFV) \circ (\epsilon_{FU}^{-1} \,\square_H\, \epsilon_{FV}^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} (\epsilon_{FU} \otimes_H \epsilon_{FV}) \circ (F\eta_U \otimes F\eta_V) \circ \varphi_2(U, V)^{-1} \\ \psi_2(U, V) \circ (F\eta_U^{-1} \otimes F\eta_V^{-1}) \circ (\epsilon_{FU}^{-1} \,\square_H\, \epsilon_{FV}^{-1}) \end{bmatrix} = \varphi_2(U, V)^{-1} \circ \psi_2(U, V). \quad \square$$

The following result is similar to [23, Proposition 3.11].

Corollary 4.12. *Let* $(H, m, u, \Delta, \varepsilon, \omega)$ *be a dual quasi-bialgebra. The identity functor on* ${}_H^H\mathfrak{M}_H^H$ *defines a monoidal functor* $\Xi : ({}_H^H\mathfrak{M}_H^H, \square_H, H) \to ({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$. *For* $M, N \in {}_H^H\mathfrak{M}_H^H$, *the structure morphisms are*

$$\gamma_2(M, N) : \Xi(M) \otimes_H \Xi(V) \to \Xi(M \square_H N) \quad and \quad \gamma_0 : H \to \Xi(H)$$

which are defined by $\gamma_2(M, N) := \vartheta_2^{-1}(M, N)$ *and* $\gamma_0 := \vartheta_0^{-1}$ *using Lemma* 4.11.

Proof. It follows by [3, Proposition 1.4]. □

Next, we include a technical result that will be used in Section 5.

Lemma 4.13. *Let* $(\mathcal{M}, \otimes, \mathbf{1})$ *be a monoidal category which is abelian.*

(1) *Let* A *be an algebra in* $\mathcal{M}$. *Assume that the tensor functors are additive and right exact* (*see* [2, Theorem 1.12]). *Then the forgetful functor*

$$D : ({}_A\mathcal{M}_A, \otimes_A, A) \to (\mathcal{M}, \otimes, \mathbf{1})$$

is a lax monoidal functor with structure morphisms

$$\zeta_2(M, N) : D(M) \otimes D(N) \to D(M \otimes_A N) \quad and \quad \zeta_0 : \mathbf{1} \to D(A),$$

where ζ_2 *is the canonical epimorphism and* ζ_0 *is the unity of* A.

(2) *Let* C *be a coalgebra in* $\mathcal{M}$. *Assume that the tensor functors are additive and left exact. Then the forgetful functor*

$$D : ({}^C\mathcal{M}^C, \square_C, C) \to (\mathcal{M}, \otimes, \mathbf{1})$$

is a colax monoidal functor with structure morphisms

$$\zeta_2(M, N) : D(M \square_C N) \to D(M) \otimes D(N) \quad and \quad \zeta_0 : D(C) \to \mathbf{1},$$

where ζ_2 *is the canonical monomorphism and* ζ_0 *is the counit of* C.

Proof. (1) From [2, 1.11], for all $M, N, S \in {}_A\mathcal{M}_A$, we deduce

$$D\left({}^A a_{M,N,S}^A\right) \circ \zeta_2(M \otimes_A N, S) \circ \left[\zeta_2(M, N) \otimes D(S)\right]$$
$$= \zeta_2(M, N \otimes_A S) \circ \left[D(M) \otimes \zeta_2(N, S)\right] \circ a_{M,N,S}.$$

Moreover, for all $M \in {}_A\mathcal{M}_A$, we have

$$D\left({}^A l_M^A\right) \circ \zeta_2(A, M) \circ \left[\zeta_0 \otimes D(M)\right] = {}^A l_M^A \circ \zeta_2(A, M) \circ (\zeta_0 \otimes M)$$
$$= \mu_M^l \circ (u_A \otimes M) = l_M.$$

Similarly $D\left({}^A r_M^A\right) \circ \zeta_2(M, A) \circ [D(M) \otimes \zeta_0] = r_M$. We have so proved that D is a lax monoidal functor.
  (2) It follows by dual arguments. □

5. The main results: bosonization

5.1. Let H be a Hopf algebra, let A be a bialgebra and let $\sigma : H \to A$ and $\pi : A \to H$ be morphisms of bialgebras such that $\pi \sigma = \mathrm{Id}$. In this case A is called a bialgebra with projection onto H and $A \in {}^H_H \mathfrak{M}^H_H$ through

$$\rho^r(a) = a_1 \otimes \pi(a_2), \qquad \rho^l(a) = \pi(a_1) \otimes a_2,$$

$$\mu^r(a \otimes h) = a\sigma(h), \qquad \mu^l(h \otimes a) = \sigma(h)a.$$

Define now the map $\tau : A \to A : a \mapsto a_1 \sigma S(a_2)$. It can be proved that $\mathrm{Im}\,\tau = A^{coH} =: R$ and, when H is the coradical of A, that R is connected. Indeed it is well-known that R becomes a connected bialgebra in the pre-braided monoidal category ${}^H_H \mathcal{YD}$ of Yetter–Drinfeld modules over H (cf. [19]).

Now, from the fact that (F, G) is an equivalence we know that $\epsilon_A : R \otimes H \to A$ is an isomorphism. Conversely, it can be proved that, given a Hopf algebra H and a braided bialgebra R in ${}^H_H \mathcal{YD}$, we can endow $R \otimes H$ with a bialgebra structure and define two bialgebras morphisms σ and π such that $\pi \sigma = \mathrm{Id}_H$, see [19]. This bialgebra is called *Radford–Majid Bosonization* (or Radford biproduct) and permits to classify different kinds of bialgebras as "compositions" (crossed product) of different objects in the same category.

The main aim of this section is to extend the results above to the setting of dual quasi-bialgebras.

Theorem 5.2. *Let* $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ *be a dual quasi-bialgebra.*
Let $(R, \mu_R, \rho_R, \Delta_R, \varepsilon_R, m_R, u_R)$ *be a bialgebra in* ${}^H_H \mathcal{YD}$ *and use the following notations*

$$h \triangleright r := \mu_R(h \otimes r), \qquad r_{-1} \otimes r_0 := \rho_R(r),$$

$$r \cdot_R s := m_R(r \otimes s), \qquad 1_R := u_R(1_{\Bbbk}),$$

$$r^1 \otimes r^2 := \Delta_R(r).$$

Let us consider on $B := F(R) = R \otimes H$ *the following structures:*

$$m_B\big[(r \otimes h) \otimes (s \otimes k)\big] = \left[\begin{array}{c} \omega_H^{-1}(r_{-2} \otimes h_1 \otimes s_{-2}k_1)\omega_H(h_2 \otimes s_{-1} \otimes k_2), \\ \omega_H^{-1}[(h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3]\omega_H(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6 k_5 \end{array} \right],$$

$$u_B(k) = k 1_R \otimes 1_H,$$

$$\Delta_B(r \otimes h) = \omega_H^{-1}\big(r^1_{-1} \otimes r^2_{-2} \otimes h_1\big) r^1_0 \otimes r^2_{-1} h_2 \otimes r^2_0 \otimes h_3,$$

$$\varepsilon_B(r \otimes h) = \varepsilon_R(r)\varepsilon_H(h),$$

$$\omega_B\big((r \otimes h) \otimes (s \otimes k) \otimes (t \otimes l)\big) = \varepsilon_R(r)\varepsilon_R(s)\varepsilon_R(t)\omega_H(h \otimes k \otimes l).$$

Then $(B, \Delta_B, \varepsilon_B, m_B, u_B, \omega_B)$ *is a dual quasi-bialgebra.*

Proof. It is analogue to [9, Lemma 3.1]. □

Remark 5.3. The Referee pointed out to our attention that the coalgebra structure of $F(R)$ in the previous result is a smash coproduct one, see [8, Definition 3.4].

Definition 5.4. With hypotheses and notations as in Theorem 5.2, the bialgebra B will be called the *bosonization of R by H* and denoted by $R \# H$.

Definition 5.5. Let $(H, m, u, \Delta, \varepsilon, \omega)$ and $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ be dual quasi-bialgebras, and suppose there exist morphisms of dual quasi-bialgebras

$$\sigma : H \to A \quad \text{and} \quad \pi : A \to H$$

such that $\pi\sigma = \mathrm{Id}_H$. Then (A, H, σ, π) is called a *dual quasi-bialgebra with a projection onto H*.

Proposition 5.6. *Keep the hypotheses and notations of* Theorem 5.2. *Then* $(R \# H, H, \sigma, \pi)$ *is a dual quasi-bialgebra with projection onto H where*

$$\sigma : H \to R \# H, \quad \sigma(h) := 1_R \# h, \qquad \pi : R \# H \to H, \quad \pi(r \# h) := \varepsilon_R(r)h.$$

Proof. It is straightforward. $\quad\square$

Next aim is to characterize dual quasi-bialgebras with a projection onto a dual quasi-bialgebra with a preantipode as bosonizations.

Lemma 5.7. *Let* $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ *and* $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ *be dual quasi-bialgebras such that* (A, H, σ, π) *is a dual quasi-bialgebra with a projection onto H. Then A is an object in* $_H^H\mathfrak{M}_H^H$ *through*

$$\rho_A^r(a) = a_1 \otimes \pi(a_2), \qquad \rho_A^l(a) = \pi(a_1) \otimes a_2,$$
$$\mu_A^r(a \otimes h) = a\sigma(h), \qquad \mu_A^l(h \otimes a) = \sigma(h)a.$$

Proof. It is straightforward. $\quad\square$

Theorem 5.8. *Let* $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ *and* $(H, m_H, u_H, \Delta_H, \varepsilon_H, \omega_H)$ *be dual quasi-bialgebras such that* (A, H, σ, π) *is a dual quasi-bialgebra with a projection onto H. Assume that H has a preantipode S. For all* $a, b \in A$, *we set* $a_1 \otimes a_2 := \Delta_A(a)$ *and* $ab = m_A(a \otimes b)$. *Then, for all* $a \in A$ *we have*

$$\tau(a) := \omega_A\big[a_1 \otimes \sigma S\pi(a_3)_1 \otimes a_4\big]a_2\sigma S\pi(a_3)_2$$

and $R := G(A)$ *is a bialgebra* $((R, \mu_R, \rho_R), m_R, u_R, \Delta_R, \varepsilon_R, \omega_R)$ *in* $_H^H\mathcal{YD}$ *where, for all* $r, s \in R, h \in H, k \in \Bbbk$, *we have*

$$h \triangleright r := \mu_R(h \otimes r) := \tau\big[\sigma(h)r\big], \qquad r_{-1} \otimes r_0 := \rho_R(r) := \pi(r_1) \otimes r_2,$$
$$m_R(r \otimes s) := rs, \qquad u_R(k) := k1_A,$$
$$r^1 \otimes r^2 := \Delta_R(r) := \tau(r_1) \otimes \tau(r_2), \qquad \varepsilon_R(r) := \varepsilon_A(r).$$

Moreover there is a dual quasi-bialgebra isomorphism $\epsilon_A : R \# H \to A$ *given by*

$$\epsilon_A(r \otimes h) = r\sigma(h), \qquad \epsilon_A^{-1}(a) = \tau(a_1) \otimes \pi(a_2).$$

Proof. We have

$$\rho_A^r(a_1) \otimes a_2 = a_1 \otimes \pi(a_2) \otimes a_3 = a_1 \otimes \rho_A^l(a_2)$$

so that $\Delta_A(a) \in A \,\square_H\, A$ for all $a \in A$. Let $\overline{\Delta}_A : A \to A \,\square_H\, A$ be the corestriction of Δ_A to $A \,\square_H\, A$. Using that $\omega_H = \omega_A(\pi \otimes \pi \otimes \pi)$, we obtain

$$m_A \circ \left(A \otimes \mu_A^l\right) \circ {}^H a_{A,H,A}^H = m_A \circ \left(\mu_A^r \otimes A\right).$$

Denote by $\chi_{X,Y} : X \otimes Y \to X \otimes_H Y$ the canonical projection, for all X, Y objects in ${}_H^H \mathfrak{M}_H^H$.

Since $(A \otimes_H A, \chi_{A,A})$ is the coequalizer of $((A \otimes \mu_A^l)^H a_{A,H,A}^H, (\mu_A^r \otimes A))$, we get that m_A factors through to a map $m_A' : A \otimes_H A \to A$ such that $m_A' \circ \chi_{A,A} = m_A$. Consider the canonical map $\vartheta_2(M, N) : M \,\square_H\, N \to M \otimes_H N$ of Lemma 4.11 defined by $\vartheta_2(M, N)(m \,\square_H\, n) := \tau(m) \otimes_H n$ and let $\overline{m}_A := m_A' \circ \vartheta_2(A, A)$. Then

$$\overline{m}_A(a \,\square_H\, b) = m_A'\left(\tau(a) \otimes_H b\right) = \tau(a)b.$$

Note that, by Lemma 2.12, the map $\tau : A \to A^{coH}$ is defined, for all $a \in A$, by

$$\begin{aligned}
\tau(a) &= \omega_H\left[a_{-1} \otimes S(a_1)_1 \otimes a_2\right] a_0 S(a_1)_2 \\
&= \omega_H\left[\pi(a_1) \otimes S\pi(a_3)_1 \otimes \pi(a_4)\right] a_2 \sigma\left[S\pi(a_3)_2\right] \\
&= \omega_H\left[\pi(a_1) \otimes \pi\sigma\left[S\pi(a_3)_1\right] \otimes \pi(a_4)\right] a_2 \sigma\left[S\pi(a_3)_2\right] \\
&= \omega_A\left[a_1 \otimes \sigma\left[S\pi(a_3)_1\right] \otimes a_4\right] a_2 \sigma\left[S\pi(a_3)_2\right] \\
&= \omega_A\left[a_1 \otimes \sigma S\pi(a_3)_1 \otimes a_4\right] a_2 \sigma S\pi(a_3)_2.
\end{aligned}$$

It is straightforward to prove that $(A, \overline{\Delta}_A, \overline{\varepsilon}_A := \pi)$ is a coalgebra in $({}_H^H \mathfrak{M}_H^H, \square_H, H)$.

One checks that (A, m_A', σ) is an algebra in $({}_H^H \mathfrak{M}_H^H, \otimes_H, H)$.

Now, by applying [3, Proposition 1.5] to the monoidal functor $E : ({}_H^H \mathfrak{M}_H^H, \otimes_H, H) \to ({}_H^H \mathfrak{M}_H^H, \square_H, H)$ of Lemma 4.11 we have that $(E(A), m_{E(A)}, u_{E(A)})$ is an algebra in $({}_H^H \mathfrak{M}_H^H, \square_H, H)$ where

$$m_{E(A)} = E\left(m_A'\right) \circ \vartheta_2(A, A) \quad \text{and} \quad u_{E(A)} = E(\sigma) \circ \vartheta_0.$$

It is clear that $(E(A), m_{E(A)}, u_{E(A)}) = (A, \overline{m}_A, \overline{u}_A = \sigma)$. Thus $(A, \overline{m}_A, \overline{u}_A)$ is an algebra in $({}_H^H \mathfrak{M}_H^H, \square_H, H)$.

Now, we apply [3, Proposition 1.5] to the functor $G : {}_H^H \mathfrak{M}_H^H \to {}_H^H \mathcal{YD}$ of Lemma 4.9. Set $R := G(A) = A^{coH}$. Then R is both an algebra and a coalgebra in ${}_H^H \mathcal{YD}$ through

$$m_R := G(\overline{m}_A) \circ \psi_2^G(A, A), \qquad u_R := G(\overline{u}_A) \circ \psi_0^G,$$

$$\Delta_R := \psi_2^G(A, A)^{-1} \circ G(\overline{\Delta}_A), \qquad \varepsilon_R := \left(\psi_0^G\right)^{-1} \circ G(\overline{\varepsilon}_A).$$

Explicitly, for all $r, s \in R$, $k \in \Bbbk$

$$m_R(r \otimes s) = \tau(rs_{-1})s_0 \overset{(9)}{=} r\varepsilon_H(s_{-1})s_0 = rs,$$

$$u_R(k) = G(\overline{u}_A)\psi_0^G(k) = \overline{u}_A(k1_H) = k\sigma(1_H) = k1_A,$$

$$\Delta_R(r) = \tau(r_1) \otimes \tau(r_2),$$

$$\varepsilon_R(r) = \left(\psi_0^G\right)^{-1} G(\overline{\varepsilon}_A)(r) = \left(\psi_0^G\right)^{-1}\pi(r) = \pi(r) = \varepsilon_A(r_1)\pi(r_2) = \varepsilon_A(r_0)r_1 = \varepsilon_A(r)1_H.$$

We will use the following notations for all $r, s \in R$,

$$r \cdot_R s := m_R(r \otimes s), \qquad 1_R := u_R(1_\Bbbk).$$

Now, by [3, Corollary 1.7], we have that $\epsilon_A : FG(A) \to A$ is an algebra and a coalgebra isomorphism in $({}^H_H \mathfrak{M}^H_H, \square_H, H)$. Let us write the algebra and coalgebra structure of $FG(A) = R \otimes H$. By construction [3, Proposition 1.5], we have

$$\overline{m}_{F(R)} := F(m_R) \circ \psi_2(R, R) : F(R) \square_H F(R) \to F(R),$$

$$\overline{u}_{F(R)} := F(u_R) \circ \psi_0 : H \to F(R),$$

$$\overline{\Delta}_{F(R)} := \psi_2(R, R)^{-1} \circ F(\Delta_R) : F(R) \to F(R) \square_H F(R),$$

$$\overline{\varepsilon}_{F(R)} := \psi_0^{-1} \circ F(\varepsilon_R) : F(R) \to H.$$

Explicitly we have

$$\overline{m}_{F(R)}\big((r \otimes h) \square_H (s \otimes k)\big) = \omega(r_{-1} \otimes s_{-1} \otimes k_1) r_0 \varepsilon(h) \cdot_R s_0 \otimes k_2,$$

$$\overline{u}_{F(R)}(h) = F(u_R)\psi_0(h) = 1_R \otimes h,$$

$$\overline{\Delta}_{F(R)}(r \otimes h) = \omega^{-1}\big(r^1_{-1} \otimes r^2_{-2} \otimes h_1\big)\big(r^1_0 \otimes r^2_{-1} h_2\big) \square_H \big(r^2_0 \otimes h_3\big),$$

$$\overline{\varepsilon}_{F(R)}(r \otimes h) = \psi_0^{-1} F(\varepsilon_R)(r \otimes h) = \psi_0^{-1}\big(\varepsilon_R(r) \otimes h\big) = \varepsilon_R(r)h.$$

In view of 4.6, the forgetful functor $({}^H_H \mathfrak{M}^H_H, \square_H, H) \to ({}^H \mathfrak{M}^H, \square_H, H)$ is a strict monoidal functor. Being $\epsilon_A : (F(R), \overline{\Delta}_{F(R)}, \overline{\varepsilon}_{F(R)}) \to (A, \overline{\Delta}_A, \overline{\varepsilon}_A = \pi)$ a coalgebra morphism in $({}^H_H \mathfrak{M}^H_H, \square_H, H)$, we have that $\epsilon_A : (F(R), \overline{\Delta}_{F(R)}, \overline{\varepsilon}_{F(R)}) \to (A, \overline{\Delta}_A, \overline{\varepsilon}_A = \pi)$ is a coalgebra morphism in $({}^H \mathfrak{M}^H, \square_H, H)$. Apply Lemma 4.13 to the case $(\mathcal{M}, \otimes, \mathbf{1}) = (\mathfrak{M}, \otimes, \Bbbk)$ and $C = H$. Let $j_{X,Y} : X \square_H Y \to X \otimes Y$ be the canonical map. Then $\epsilon_A : (F(R), j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)}, \varepsilon_H \circ \overline{\varepsilon}_{F(R)}) \to (A, j_{A,A} \circ \overline{\Delta}_A, \varepsilon_H \circ \overline{\varepsilon}_A)$ is a coalgebra morphism in $(\mathfrak{M}, \otimes, \Bbbk)$. In other words it is an ordinary coalgebra morphism. Note that $(A, j_{A,A} \circ \overline{\Delta}_A, \varepsilon_H \circ \overline{\varepsilon}_A) = (A, \Delta_A, \varepsilon_A)$. Set $(\Delta_{F(R)}, \varepsilon_{F(R)}) := (j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)}, \varepsilon_H \circ \overline{\varepsilon}_{F(R)})$. Let us compute explicitly these maps. We have

$$\Delta_{F(R)}(r \otimes h) = (j_{F(R),F(R)} \circ \overline{\Delta}_{F(R)})(r \otimes h) = \omega^{-1}\big(r^1_{-1} \otimes r^2_{-2} \otimes h_1\big)\big(r^1_0 \otimes r^2_{-1} h_2\big) \otimes \big(r^2_0 \otimes h_3\big),$$

$$\varepsilon_{F(R)}(r \otimes h) = (\varepsilon_H \circ \overline{\varepsilon}_{F(R)})(r \otimes h) = \varepsilon_R(r)\varepsilon_H(h).$$

Thus $\epsilon_A : (F(R), \Delta_{F(R)}, \varepsilon_{F(R)}) \to (A, \Delta_A, \varepsilon_A)$ is an ordinary coalgebra morphism.

Being $\epsilon_A : (F(R), \overline{m}_{F(R)}, \overline{u}_{F(R)}) \to (A, \overline{m}_A, \overline{u}_A = \sigma)$ an algebra morphism in $({}^H_H \mathfrak{M}^H_H, \square_H, H)$, then, in view of Lemma 4.11,

$$\epsilon_A : \big(F(R), \varXi(\overline{m}_{F(R)}) \circ \gamma_2(F(R), F(R)), \varXi(\overline{u}_{F(R)}) \circ \gamma_0\big) \to \big(A, \varXi(\overline{m}_A) \circ \gamma_2(A, A), \varXi(\overline{u}_A) \circ \gamma_0\big)$$

is an algebra morphism in $({}^H_H \mathfrak{M}^H_H, \otimes_H, H)$. Note that

$$\varXi(\overline{m}_A) \circ \gamma_2(A, A) = \overline{m}_A \circ \vartheta_2^{-1}(A, A) = m'_A,$$

$$\varXi(\overline{u}_A) \circ \gamma_0 = \overline{u}_A = \sigma$$

so that

$$\left(A, \varXi(\overline{m}_A) \circ \gamma_2(A, A), \varXi(\overline{u}_A) \circ \gamma_0\right) = \left(A, m_A', \sigma\right).$$

Set $(m_{F(R)}', u_{F(R)}') := (\varXi(\overline{m}_{F(R)}) \circ \gamma_2(F(R), F(R)), \varXi(\overline{u}_{F(R)}) \circ \gamma_0)$. We have

$$
\begin{aligned}
&m_{F(R)}'\left((r \otimes h) \otimes_H (s \otimes k)\right) \\
&\quad = \left[\varXi(\overline{m}_{F(R)}) \circ \gamma_2\left(F(R), F(R)\right)\right]\left((r \otimes h) \otimes_H (s \otimes k)\right) \\
&\quad = \overline{m}_{F(R)}\left[(r \otimes h)_0 (s \otimes k)_{-1} \otimes_H (r \otimes h)_1 (s \otimes k)_0\right] \\
&\quad = \overline{m}_{F(R)}\left[(r \otimes h_1)(s_{-1}k_1) \otimes_H h_2(s_0 \otimes k_2)\right] \\
&\quad = \omega^{-1}[r_{-1} \otimes h_1 \otimes s_{-2}k_1]\overline{m}_{F(R)}\left[r_0 \otimes \left[h_2(s_{-1}k_2)\right] \otimes_H h_3(s_0 \otimes k_3)\right] \\
&\quad = \left[\begin{array}{c} \omega^{-1}[r_{-1} \otimes h_1 \otimes s_{-3}k_1]\omega(h_3 \otimes s_{-1} \otimes k_3)\omega^{-1}((h_4 \triangleright s_0)_{-1} \otimes h_5 \otimes k_4) \\ \overline{m}_{F(R)}[r_0 \otimes [h_2(s_{-2}k_2)] \otimes_H (h_4 \triangleright s_0)_0 \otimes h_6 k_5] \end{array}\right] \\
&\quad = \left[\begin{array}{c} \omega^{-1}[r_{-2} \otimes h_1 \otimes s_{-3}k_1]\omega(h_3 \otimes s_{-1} \otimes k_3)\omega^{-1}((h_4 \triangleright s_0)_{-2} \otimes h_5 \otimes k_4) \\ \omega(r_{-1} \otimes (h_4 \triangleright s_0)_{-1} \otimes h_6 k_5)r_0 \varepsilon_H[h_2(s_{-2}k_2)] \cdot_R (h_4 \triangleright s_0)_0 \otimes h_7 k_6 \end{array}\right] \\
&\quad = \left[\begin{array}{c} \omega^{-1}[r_{-2} \otimes h_1 \otimes s_{-2}k_1]\omega(h_2 \otimes s_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3)\omega(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6 k_5 \end{array}\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
&m_{F(R)}'\left((r \otimes h) \otimes_H (s \otimes k)\right) \\
&\quad = \left[\begin{array}{c} \omega^{-1}[r_{-2} \otimes h_1 \otimes s_{-2}k_1]\omega(h_2 \otimes s_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3)\omega(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6 k_5 \end{array}\right].
\end{aligned}
$$

Moreover

$$u_{F(R)}'(h) = \left[\varXi(\overline{u}_{F(R)}) \circ \gamma_0\right](h) = \overline{u}_{F(R)}(h) = 1_R \otimes h.$$

Apply Lemma 4.13 to the case $(\mathcal{M}, \otimes, \mathbf{1}) = ({}^H\mathfrak{M}^H, \otimes, \Bbbk)$ and $A = H$. Then

$$\epsilon_A : \left(F(R), m_{F(R)}' \circ \chi_{F(R),F(R)}, u_{F(R)}' \circ u_H\right) \to \left(A, m_A' \circ \chi_{A,A}, \sigma \circ u_H\right)$$

is an algebra homomorphism in $({}^H\mathfrak{M}^H, \otimes, \Bbbk)$. Note that $(A, m_A' \circ \chi_{A,A}, \sigma \circ u_H) = (A, m_A, u_A)$. Moreover, if we set $(m_{F(R)}, u_{F(R)}) := (m_{F(R)}' \circ \chi_{F(R),F(R)}, u_{F(R)}' \circ u_H)$, we get

$$
\begin{aligned}
&m_{F(R)}\left((r \otimes h) \otimes (s \otimes k)\right) \\
&\quad = \left[\begin{array}{c} \omega^{-1}[r_{-2} \otimes h_1 \otimes s_{-2}k_1]\omega(h_2 \otimes s_{-1} \otimes k_2) \\ \omega^{-1}((h_3 \triangleright s_0)_{-2} \otimes h_4 \otimes k_3)\omega(r_{-1} \otimes (h_3 \triangleright s_0)_{-1} \otimes h_5 k_4) \\ r_0 \cdot_R (h_3 \triangleright s_0)_0 \otimes h_6 k_5 \end{array}\right].
\end{aligned}
$$

Moreover

$$u_{F(R)}(k) = 1_R \otimes k.$$

Thus $\epsilon_A : (F(R), m_{F(R)}, u_{F(R)}) \to (A, m_A, u_A)$ is an algebra isomorphism in $({}^H\mathfrak{M}^H, \otimes, \Bbbk)$ and $\epsilon_A : (F(R), \Delta_{F(R)}, \varepsilon_{F(R)}) \to (A, \Delta_A, \varepsilon_A)$ is an ordinary coalgebra isomorphism. Thus

$$m_A \circ (\epsilon_A \otimes \epsilon_A) = \epsilon_A \circ m_{F(R)}, \qquad \epsilon_A \circ u_{F(R)} = u_A,$$

$$(\epsilon_A \otimes \epsilon_A) \circ \Delta_{F(R)} = \Delta_A \circ \epsilon_A, \qquad \varepsilon_A \circ \epsilon_A = \varepsilon_{F(R)},$$

so that $m_{F(R)}$, $u_{F(R)}$, $\Delta_{F(R)}$, $\varepsilon_{F(R)}$ are exactly the morphisms induced by m_A, u_A, Δ_A, ε_A via the vector space isomorphism $\epsilon_A : F(R) \to A$. Let $\omega_{F(R)}$ be the map induced by ω_A via the vector space isomorphism ϵ_A i.e.

$$\omega_{F(R)} := \omega_A \circ (\epsilon_A \otimes \epsilon_A \otimes \epsilon_A) : F(R) \otimes F(R) \otimes F(R) \to \Bbbk.$$

Then $\epsilon_A : (F(R), \Delta_{F(R)}, \varepsilon_{F(R)}, m_{F(R)}, u_{F(R)}, \omega_{F(R)}) \to (A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ is clearly an isomorphism of dual quasi-bialgebras. Since, for all $r \in R$, we have $\pi(r) = \varepsilon_A(r_1)\pi(r_2) = \varepsilon_A(r)1_H$, then, for $r, s, t \in R$, $h, k, l \in H$, we get

$$\begin{aligned}
&\omega_{F(R)}\big[(r \otimes h) \otimes (s \otimes k) \otimes (t \otimes l)\big] \\
&= \omega_A\big(r\sigma(h) \otimes s\sigma(k) \otimes t\sigma(l)\big) = \omega_H\big[\pi\big(r\sigma(h)\big) \otimes \pi\big(s\sigma(k)\big) \otimes \pi\big(t\sigma(l)\big)\big] \\
&= \omega_H\big[\pi(r)h \otimes \pi(s)k \otimes \pi(t)l\big] = \omega_H\big[\varepsilon_A(r)h \otimes \varepsilon_A(s)k \otimes \varepsilon_A(t)l\big] \\
&= \varepsilon_A(r)\varepsilon_A(t)\varepsilon_A(s)\omega_H(h \otimes k \otimes l)
\end{aligned}$$

so that

$$\omega_{F(R)}\big[(r \otimes h) \otimes (s \otimes k) \otimes (t \otimes l)\big] = \varepsilon_A(r)\varepsilon_A(t)\varepsilon_A(s)\omega_H(h \otimes k \otimes l).$$

Note that $(F(R), \Delta_{F(R)}, \varepsilon_{F(R)}, m_{F(R)}, u_{F(R)}, \omega_{F(R)}) = R \# H$ once proved that $(R, m_R, u_R, \Delta_R, \varepsilon_R)$ is a bialgebra in the monoidal category $({}^H_H\mathcal{YD}, \otimes, \Bbbk)$. It remains to prove that m_R and u_R are coalgebra maps. Since ${}^H_H\mathcal{YD}$ is a pre-braided monoidal category and $(R, \Delta_R, \varepsilon_R)$ is a coalgebra in this category, then we can define two morphisms $\Delta_{R \otimes R}$ and $\varepsilon_{R \otimes R}$ in ${}^H_H\mathcal{YD}$ such that $(R \otimes R, \Delta_{R \otimes R}, \varepsilon_{R \otimes R})$ is a coalgebra in ${}^H_H\mathcal{YD}$ too. We have

$$\Delta_{R \otimes R} := a_{R,R,R \otimes R}^{-1} \circ (R \otimes a_{R,R,R}) \circ (R \otimes c_{R,R} \otimes R) \circ \big(R \otimes a_{R,R,R}^{-1}\big) \circ a_{R,R,R \otimes R} \circ (\Delta_R \otimes \Delta_R),$$

$$\varepsilon_{R \otimes R} := \varepsilon_R \otimes \varepsilon_R.$$

Explicitly $\Delta_{R \otimes R}$ satisfies

$$\Delta_{R \otimes R}(r \otimes s) = \begin{bmatrix} \omega^{-1}(r_{-2}^1 \otimes r_{-5}^2 \otimes s_{-2}^1 s_{-4}^2)\omega(r_{-4}^2 \otimes s_{-1}^1 \otimes s_{-3}^2) \\ \omega^{-1}[(r_{-3}^2 \rhd s_0^1)_{-2} \otimes r_{-2}^2 \otimes s_{-2}^2]\omega(r_{-1}^1 \otimes (r_{-3}^2 \rhd s_0^1)_{-1} \otimes r_{-1}^2 s_{-1}^2) \\ [r_0^1 \otimes (r_{-3}^2 \rhd s_0^1)_0] \otimes (r_0^2 \otimes s_0^2) \end{bmatrix}. \quad (37)$$

In order to prove that m_R is a morphism of coalgebras in ${}^H_H\mathcal{YD}$, we have to check the following equality

$$(m_R \otimes m_R)\Delta_{R \otimes R} = \Delta_R m_R.$$

Since we already obtained that $B := F(R)$ is a dual quasi-bialgebra, we know that

$$\Delta_B\big[(r \otimes 1_H) \cdot_B (s \otimes 1_H)\big] = (r \otimes 1_H)_1 \cdot_B (s \otimes 1_H)_1 \otimes (r \otimes 1_H)_2 \cdot_B (s \otimes 1_H)_2.$$

By applying $R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H$ on both sides we get

$$(r \cdot_R s)^1 \otimes (r \cdot_R s)^2 = \begin{bmatrix} \omega^{-1}((r^1)_{-2} \otimes (r^2_{-1})_1 \otimes (s^1)_{-2}(s^2_{-1})_1) \\ \omega((r^2_{-1})_2 \otimes (s^1)_{-1} \otimes (s^2_{-1})_2) \\ \omega^{-1}[((r^2_{-1})_3 \rhd (s^1)_0)_{-2} \otimes (r^2_{-1})_4 \otimes (s^2_{-1})_3] \\ \omega((r^1)_{-1} \otimes ((r^2_{-1})_3 \rhd (s^1)_0)_{-1} \otimes (r^2_{-1})_5(s^2_{-1})_4) \\ (r^1)_0 \cdot_R ((r^2_{-1})_3 \rhd (s^1)_0)_0 \end{bmatrix} \otimes \big(r^2_0 \cdot_R s^2_0\big)$$

$$= \begin{bmatrix} \omega^{-1}(r^1_{-2} \otimes r^2_{-5} \otimes s^1_{-2}s^2_{-4})\omega(r^2_{-4} \otimes s^1_{-1} \otimes s^2_{-3}) \\ \omega^{-1}[(r^2_{-3} \rhd s^1_0)_{-2} \otimes r^2_{-2} \otimes s^2_{-2}] \\ \omega(r^1_{-1} \otimes (r^2_{-3} \rhd s^1_0)_{-1} \otimes r^2_{-1}s^2_{-1}) \\ r^1_0 \cdot_R (r^2_{-3} \rhd s^1_0)_0 \otimes (r^2_0 \cdot_R s^2_0) \end{bmatrix}$$

$$= (m_R \otimes m_R)\Delta_{R \otimes R}(r \otimes s).$$

The compatibility of m_R with ε_R and the fact that u_R is a coalgebra morphism can be easily proved.   □

6. Applications

Here we collect some applications of the results of the previous sections.

6.1. The associated graded coalgebra

Example 6.1. Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \omega_A)$ be a dual quasi-bialgebra with the dual Chevalley property i.e. such that the coradical H of A is a dual quasi-subbialgebra of A. Since A is an ordinary coalgebra, we can consider the associated graded coalgebra

$$\text{gr}\, A := \bigoplus_{n \in \mathbb{N}} \text{gr}^n A \quad \text{where } \text{gr}^n A := \frac{A_n}{A_{n-1}}.$$

Here $A_{-1} := \{0\}$ and, for all $n \geq 0$, A_n is the nth term of the coradical filtration of A. The coalgebra structure of $\text{gr}\, A$ is given as follows. The nth graded component of the counit is the map $\varepsilon^n_{\text{gr}\, A} : A_n/A_{n-1} \to \Bbbk$ defined by setting

$$\varepsilon^n_{\text{gr}\, A}(x + A_{n-1}) = \delta_{n,0}\varepsilon_A(x).$$

The nth graded component of comultiplication is the map

$$\Delta^n_{\text{gr}\, A} : \text{gr}^{a+b} A \to \bigoplus_{a+b=n,\, a,b \geq 0} \text{gr}^a A \otimes \text{gr}^b A$$

defined as the diagonal map of the family $(\Delta^{a,b}_{\text{gr}\, A})_{a+b=n,\, a,b \geq 0}$ where

$$\Delta^{a,b}_{\text{gr}\, A} : \text{gr}^{a+b} A \to \text{gr}^a A \otimes \text{gr}^b A, \qquad \Delta^{a,b}_{\text{gr}\, A}(x + A_{a+b-1}) = (x_1 + A_{a-1}) \otimes (x_2 + A_{b-1}).$$

Proposition 6.2. *Let A be a dual quasi-bialgebra with the dual Chevalley property. Then*

$$(\mathrm{gr}\, A, m_{\mathrm{gr}\, A}, u_{\mathrm{gr}\, A}, \Delta_{\mathrm{gr}\, A}, \varepsilon_{\mathrm{gr}\, A}, \omega_{\mathrm{gr}\, A})$$

is a dual quasi-bialgebra where the graded components of the structure maps are given by the maps

$$m_{\mathrm{gr}\, A}^{a,b} : \mathrm{gr}^a A \otimes \mathrm{gr}^b A \to \mathrm{gr}^{a+b} A, \qquad u_{\mathrm{gr}\, A}^n : \Bbbk \to \mathrm{gr}^n A,$$

$$\Delta_{\mathrm{gr}\, A}^{a,b} : \mathrm{gr}^{a+b} A \to \mathrm{gr}^a A \otimes \mathrm{gr}^b A, \qquad \varepsilon_{\mathrm{gr}\, A}^n : \mathrm{gr}^n A \to \Bbbk,$$

$$\omega_{\mathrm{gr}\, A}^{a,b,c} : \mathrm{gr}^a A \otimes \mathrm{gr}^b A \otimes \mathrm{gr}^c A \to \Bbbk,$$

defined by

$$m_{\mathrm{gr}\, A}^{a,b}\big[(x + A_{a-1}) \otimes (y + A_{b-1})\big] := xy + A_{a+b-1}, \qquad u_{\mathrm{gr}\, A}^n(k) := \delta_{n,0} 1_A + A_{-1} = \delta_{n,0} 1_A,$$

$$\Delta_{\mathrm{gr}\, A}^{a,b}(x + A_{a+b-1}) := (x_1 + A_{a-1}) \otimes (x_2 + A_{b-1}), \qquad \varepsilon_{\mathrm{gr}\, A}^n(x + A_{n-1}) := \delta_{n,0} \varepsilon_A(x),$$

$$\omega_{\mathrm{gr}\, A}^{a,b,c}\big[(x + A_{a-1}) \otimes (y + A_{b-1}) \otimes (z + A_{c-1})\big] := \delta_{a,0}\delta_{b,0}\delta_{c,0}\omega_A(x \otimes y \otimes z).$$

Here $\delta_{i,j}$ denotes the Kronecker delta.

Proof. The proof of the facts that $m_{\mathrm{gr}\, A}$ and $u_{\mathrm{gr}\, A}$ are well-defined, are coalgebra maps and that $m_{\mathrm{gr}\, A}$ is unitary is analogous to the classical case, and depend on the fact that the coradical filtration is an algebra filtration. This can be proved mimicking [18, Lemma 5.2.8]. The cocycle condition and the quasi-associativity of $m_{\mathrm{gr}\, A}$ are straightforward. □

Proposition 6.3. *Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Then $(\mathrm{gr}\, A, H, \sigma, \pi)$ is a dual quasi-bialgebra with projection onto H, where*

$$\sigma : H \to \mathrm{gr}\, A : h \mapsto h + A_{-1},$$

$$\pi : \mathrm{gr}\, A \to H : a + A_{n-1} \mapsto \delta_{n,0} a \quad \text{for all } a \in A_n.$$

Proof. It is straightforward. □

Corollary 6.4. *Let A be a dual quasi-bialgebra with the dual Chevalley property and coradical H. Assume that H has a preantipode. Then there is a bialgebra R in ${}^H_H\mathcal{YD}$ such that $\mathrm{gr}\, A$ is isomorphic to $R \# H$ a dual quasi-bialgebra.*

Proof. It follows by Proposition 6.3 and Theorem 5.8. □

Definition 6.5. Following [1, Definition, p. 659], the bialgebra R in ${}^H_H\mathcal{YD}$ of Corollary 6.4, is called the *diagram* of A.

6.2. On pointed dual quasi-bialgebras

We conclude this section considering the pointed case.

Lemma 6.6. *Let G be a monoid and consider the monoid algebra $H := \Bbbk G$. Suppose there is a map $\omega \in (H \otimes H \otimes H)^*$ such that (H, ω) is a dual quasi-bialgebra. Then (H, ω) has a preantipode S if and only if G is a group. In this case*

$$S(g) = \left[\omega \left(g \otimes g^{-1} \otimes g \right) \right]^{-1} g^{-1}.$$

Proof. Suppose that S is a preantipode for (H, ω). Since H is a cocommutative ordinary bialgebra, by Theorem 2.16, we have that $\Bbbk G$ is an ordinary Hopf algebra, where the antipode is defined, for all $g \in G$, by

$$s(g) := S(g)_1 \omega \left[g \otimes S(g)_2 \otimes g \right].$$

Moreover one has $S = \varepsilon S * s$. Now, since $\Bbbk G$ is a Hopf algebra, one has that the set of grouplike elements in $\Bbbk G$, namely G itself, form a group, where $g^{-1} := s(g)$, for all $g \in G$.

Now, since s is an anti-coalgebra map, we have

$$S(g)_1 \otimes S(g)_2 = \varepsilon S(g) s(g)_1 \otimes s(g)_2 = \varepsilon S(g) s(g) \otimes s(g) = S(g) \otimes g^{-1}$$

so that $s(g) = S(g)_1 \omega[g \otimes S(g)_2 \otimes g] = S(g) \omega(g \otimes g^{-1} \otimes g)$. Hence $S(g) = [\omega(g \otimes g^{-1} \otimes g)]^{-1} g^{-1}$.

The other implication is trivial (see [4, Example 3.14]). □

The motivation for the previous result is Corollary 6.9 below.

Proposition 6.7. *Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Then the set of grouplike elements $\mathbb{G}(A)$ of A is a monoid and the monoid algebra $\Bbbk \mathbb{G}(A)$ is a dual quasi-subbialgebra of A.*

Proof. It is straightforward. □

Corollary 6.8. *Let $(A, m, u, \Delta, \varepsilon, \omega)$ be a pointed dual quasi-bialgebra. Then $A_0 = \Bbbk \mathbb{G}(A)$ is a dual quasi-subbialgebra of A.*

Proof. By Remark 2.20, $A_0 = \Bbbk \mathbb{G}(A)$. In view of Proposition 6.7, we conclude. □

Corollary 6.9. *Let $(A, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a pointed dual quasi-Hopf algebra. Then $\mathbb{G}(A)$ is a group and $A_0 = \Bbbk \mathbb{G}(A)$ is a dual quasi-Hopf algebra with respect to the induced structures.*

Proof. Set $G := \mathbb{G}(A)$. By Corollary 6.8, $A_0 = \Bbbk G$ is a dual quasi-subbialgebra of A. It remains to check that the antipode on A induces an antipode on A_0. We have

$$\Delta s(g) = s(g_2) \otimes s(g_1) = s(g) \otimes s(g),$$

$$\varepsilon s(g) = \varepsilon(g) = 1,$$

i.e. $s(g) \in G$, for any $g \in G$. Let $s_0, \alpha_0, \beta_0, \omega_0, m_0, u_0, \Delta_0, \varepsilon_0$ be the induced maps from $s, \alpha, \beta, \omega, m, u, \Delta, \varepsilon$, respectively. It is then clear from the definition that A_0, with respect to these structures, is a dual quasi-Hopf algebra. Since any dual quasi-Hopf algebra has a preantipode, by Lemma 6.6, G is a group. □

Pointed dual quasi-Hopf algebras have been investigated also in [12, p. 2] under the name of pointed Majid algebras. In view of Corollary 6.9, which seems to be implicitly assumed in [12, p. 2], we can apply Corollary 6.4 to obtain the following result.

Theorem 6.10. *Let A be a pointed dual quasi-Hopf algebra. Then* $\mathrm{gr}\, A$ *is isomorphic to* $R \# \Bbbk \mathbb{G}(A)$ *as dual quasi-bialgebra where R is the diagram of A.*

Acknowledgment

Appendix A. The weak right center

Definition A.1. (See [7, Section 1.5].) Let $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. The weak right center $\mathcal{W}_r(\mathcal{M})$ of $\mathcal{M}$ is a category defined as follows. An object in $\mathcal{W}_r(\mathcal{M})$ is a pair $(V, c_{-,V})$, where V is an object of $\mathcal{M}$ and $c_{-,V}$ is a family of morphisms in $\mathcal{M}$, $c_{X,V} : X \otimes V \to V \otimes X$, defined for any object X in $\mathcal{M}$, which is natural in the first entry, such that, for all $X, Y \in \mathcal{M}$ we have

$$a_{V,X,Y}^{-1} \circ c_{X \otimes Y, V} \circ a_{X,Y,V}^{-1} = (c_{X,V} \otimes Y) \circ a_{X,V,Y}^{-1} \circ (X \otimes c_{Y,V}) \tag{38}$$

and such that $r_V \circ c_{\mathbf{1},V} = l_V$. A morphism $f : (V, c_{-,V}) \to (W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{M}$ such that, for each $X \in \mathcal{M}$ we have

$$(f \otimes X) \circ c_{X,V} = c_{X,W} \circ (X \otimes f).$$

$\mathcal{W}_r(\mathcal{M})$ becomes a monoidal category with unit $(\mathbf{1}, l^{-1} \circ r)$ and tensor product

$$(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V \otimes W})$$

where, for all $X \in \mathcal{M}$, the morphism $c_{X,V \otimes W} : X \otimes (V \otimes W) \to (V \otimes W) \otimes X$ is defined by

$$c_{X,V \otimes W} := a_{V,W,X}^{-1} \circ (V \otimes c_{X,W}) \circ a_{V,X,W} \circ (c_{X,V} \otimes W) \circ a_{X,V,W}^{-1}.$$

The constraints are the same of $\mathcal{M}$ viewed as morphisms in $\mathcal{W}_r(\mathcal{M})$. Moreover the monoidal category $\mathcal{W}_r(\mathcal{M})$ is pre-braided, with braiding

$$c_{(V,c_{-,V}),(W,c_{-,W})} : (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V})$$

given by $c_{V,W}$.

Theorem A.2. *Let H be a dual quasi-bialgebra. The categories $\mathcal{W}_r(^H \mathfrak{M})$ and $^H_H \mathcal{YD}$ are isomorphic, where $^H \mathfrak{M}$ is regarded as a monoidal category as in Section* 2.

Proof. The proof is analogue to [5, Theorem 3.5]. $\quad\square$

A.1. Example: the group algebra

We now investigate the category of Yetter–Drinfeld modules over a particular dual quasi-Hopf algebra.

Let G be a group. Let $\theta : G \times G \times G \to \Bbbk^* := \Bbbk \backslash \{0\}$ be a normalized 3-cocycle on the group G in the sense of [15, Example 2.3.2, p. 54] i.e. a map such that, for all $g, h, k, l \in G$

$$\theta(g, 1_G, h) = 1,$$

$$\theta(h, k, l)\theta(g, hk, l)\theta(g, h, k) = \theta(g, h, kl)\theta(gh, k, l).$$

Then θ can be extended by linearity to a reassociator $\omega : \Bbbk G \otimes \Bbbk G \otimes \Bbbk G \to \Bbbk$ making $\Bbbk G$ a dual quasi-bialgebra with usual underlying algebra and coalgebra structures. This dual quasi-bialgebra is denoted by $\Bbbk^\theta G$. Note that in particular $\Bbbk^\theta G$ is an ordinary bialgebra but with nontrivial reassociator. In particular it is associative as an algebra. Let us investigate the category $^{\Bbbk^\theta G}_{\Bbbk^\theta G}\mathcal{YD}$ of Yetter–Drinfeld module over $\Bbbk^\theta G$.

Definition A.3. Let $\theta : G \times G \times G \to \Bbbk^*$ be a normalized 3-cocycle on a group G. The category of cocycle crossed left G-modules (G, θ)-Mod is defined as follows. An object in (G, θ)-Mod is a pair $(V, \blacktriangleright)$, where $V = \bigoplus_{g \in G} V_g$ is a G-graded vector space endowed with a map $\blacktriangleright : G \times V \to V$ such that, for all $g, h, l \in H$ and $v \in V$, we have

$$h \blacktriangleright V_g \in V_{hgh^{-1}}, \tag{39}$$

$$h \blacktriangleright (l \blacktriangleright v) = \frac{\theta(hlgl^{-1}h^{-1}, h, l)\theta(h, l, g)}{\theta(h, lgl^{-1}, l)}(hl) \blacktriangleright v, \tag{40}$$

$$1_H \blacktriangleright v = v. \tag{41}$$

A morphism $f : (V, \blacktriangleright) \to (V', \blacktriangleright')$ in (G, θ)-Mod is a morphism $f : V \to V'$ of G-graded vector spaces such that, for all $h \in H$, $v \in V$, we have $f(h \blacktriangleright v) = h \blacktriangleright' f(v)$.

The following result is inspired by [16, Proposition 3.2].

Proposition A.4. *Let $\theta : G \times G \times G \to \Bbbk^*$ be a normalized 3-cocycle on a group G. Then the category $^{\Bbbk^\theta G}_{\Bbbk^\theta G}\mathcal{YD}$ is isomorphic to (G, θ)-Mod.*

Proof. Set $H := \Bbbk^\theta G$ and let $(V, \rho_V, \triangleright) \in {}^H_H\mathcal{YD}$. Then (V, ρ_V) is an object in $^{\Bbbk G}\mathfrak{M}$. Hence, see e.g. [18, Example 1.6.7], we have that $V = \bigoplus_{g \in G} V_g$ where $V_g = \{v \in V \mid \rho_V(v) = g \otimes v\}$. Define the map $\blacktriangleright : G \times V \to V$, by setting $g \blacktriangleright v := g \triangleright v$. It is easy to prove that the assignments

$$(V, \rho_V, \triangleright) \mapsto \left(V = \bigoplus_{g \in G} V_g, \blacktriangleright\right), \quad f \mapsto f$$

define a functor $L : {}^H_H\mathcal{YD} \to (G, \theta)$-Mod. Conversely, let $(V = \bigoplus_{g \in G} V_g, \blacktriangleright)$ be an object in (G, θ)-Mod. Then $\blacktriangleright$ can be extended by linearity to a map $\triangleright : \Bbbk G \otimes V \to V$. Define $\rho_V : V \to \Bbbk G \otimes V$, by setting $\rho_V(v) = g \otimes v$ for all $v \in V_g$. Therefore, the assignments

$$\left(V = \bigoplus_{g \in G} V_g, \blacktriangleright\right) \mapsto (V, \rho_V, \triangleright), \quad f \mapsto f$$

define a functor $R : (G, \theta)$-Mod $\to {}^H_H\mathcal{YD}$. It is clear that $LR = \mathrm{Id}$ and $RL = \mathrm{Id}$. $\quad\square$

A.5. As a consequence of the previous result, the pre-braided monoidal structure on $^{\Bbbk^\theta G}_{\Bbbk^\theta G}\mathcal{YD}$ induces a pre-braided monoidal structure on (G, θ)-Mod as follows. The unit is $\Bbbk$ regarded as a G-graded vector space whose homogeneous components are all zero excepted the one corresponding to 1_G. Moreover $h \blacktriangleright k = \varepsilon_H(h)k$ for all $h \in H$, $k \in \Bbbk$. The tensor product is defined by

$$(V, \blacktriangleright) \otimes (W, \blacktriangleright) = (V \otimes W, \blacktriangleright)$$

where

$$(V \otimes W)_g = \bigoplus_{h \in H} (V_h \otimes W_{h^{-1}g})$$

and, for all $v \in V_g$, $w \in W_l$, we have

$$h \blacktriangleright (v \otimes w) = \frac{\theta(hgh^{-1}, hlh^{-1}, h)\theta(h, g, l)}{\theta(hgh^{-1}, h, l)} (h \blacktriangleright v) \otimes (h \blacktriangleright w).$$

The constraints are the same of $^H\mathfrak{M}$ viewed as morphisms in $^H_H\mathcal{YD}$.

The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given, for all $v \in V_g$, $w \in W_l$, by

$$c_{V,W}(v \otimes w) = (g \blacktriangleright w) \otimes v.$$

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