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The functor of p-permutation modules for abelian groups

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ABSTRACT

Let k be a field of characteristic p, where p is a prime number, let $\mathrm{pp}_k(G)$ be the Grothendieck group of p-permutation kG-modules, where G is a finite group, and let $\mathbb{C}\,\mathrm{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{pp}_k(G)$. In this article, we find all the composition factors of the biset functor $\mathbb{C}\,\mathrm{pp}_k$ restricted to the category of abelian groups.

1. Introduction

Let k be an algebraically closed field of characteristic p, where p is a prime number.

Let G be a finite group and let $\mathrm{pp}_k(G)$ be the Grothendieck group of p-permutation kG-modules. If we tensor everything with $\mathbb{C}$ and if G varies, $\mathbb{C}\,\mathrm{pp}_k$ becomes a $\mathbb{C}$-linear biset functor (Definition 31).

Recall that the simple biset functors $S_{H,V}$ are parametrized by pairs (H, V), where H is a finite group and V a simple $\mathbb{C}\,\mathrm{Out}(H)$-module.

We want to describe the composition factors of the biset functor $\mathbb{C}\,\mathrm{pp}_k$ restricted to the category of abelian groups. This is similar in spirit to the work *The composition factors of the functor of permutation modules* [1], dealing with the subfunctor of permutation modules.

In order to work with abelian groups, we will define the tensor product of biset functors on groups of coprime order. Let $\mathcal{C}_{p\times p'}$ be the category whose objects are finite groups of the form $P \times Q$ where P is a p-group and Q is a p'-group (the morphisms are defined using bisets).

In this article, we will prove the following theorem:

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Theorem 1. *The composition factors of $\mathbb{C}\,pp_k$ on $\mathcal{C}_{p\times p'}$ are the simple functors $S_{C_p\times C_p\times C_m,\mathbb{C}_\xi}$ and $S_{C_m,\mathbb{C}_\xi}$, where (m,ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi:(\mathbb{Z}/m\mathbb{Z})^\star\to\mathbb{C}^\star$. Their multiplicity as composition factor is* 1.

Restricting to abelian groups, we obtain the following corollary:

Corollary 2. *If G is a finite abelian group and V is a simple $\mathbb{C}\,\mathrm{Out}(G)$-module then $S_{G,V}$ is a composition factor of $\mathbb{C}\,pp_k$ if and only if there exists a positive integer m prime to p and a primitive character $\xi:(\mathbb{Z}/m\mathbb{Z})^\star\to\mathbb{C}^\star$ such that $G\cong C_m$ or $G\cong C_p\times C_p\times C_m$, where V is the* 1*-dimensional module $\mathbb{C}_\xi$. Moreover, the multiplicity of $S_{G,V}$ as composition factor is* 1.

This article begins with some background results on biset functors and composition factors. Then we define the tensor product of biset functors on groups of coprime order and show that the tensor product of simple functors is also a simple functor. Using this and the composition factors of the Burnside functor $\mathbb{C}B$ and of the functor $\mathbb{C}R_k$ of ordinary representation obtained by Serge Bouc [5, Chapters 5 and 7], we will find the composition factors of the functor of p-permutation modules $\mathbb{C}\,pp_k$. So we will have the proof of Theorem 1. Then we will restrict the category $\mathcal{C}_{p\times p'}$ to abelian groups to obtain Corollary 2.

All groups are supposed finite, all vector spaces are finite dimensional and all modules are finitely generated left modules. Let G and H be finite groups. All G-sets and all (H,G)-bisets are finite. We denote by $[U]$ the isomorphism class of U (where U can be a group, a vector space, a module, a G-set, an (H,G)-biset, ...). In the sequel, we assume that k is an algebraically closed field of characteristic p, where for convenience, we allow p to be either a prime number or 0.

2. Background on biset functors

2.1. The category of biset functors

We begin with some facts on biset functors. See [5] for more details.

Definition 3. Let G and H be finite groups. Then $B(H,G)$ is the Grothendieck group of the isomorphism classes of finite (H,G)-bisets (for the disjoint union).

Notation 4. (See [5, Definition 2.4.9 and Notation 2.4.10].) Let G be a group. We denote by Id_G the (G,G)-biset G where the two actions are defined by left and right multiplication in G. We also denote by Id_G the image of Id_G in $B(G,G)$.

The product $\times_H$ can be extended to a map

$$\times_H : B(G,H)\times B(H,K)\to B(G,K)$$

such that $[V]\times_H[U]=[V\times_H U]$ for every (G,H)-biset V and every (H,K)-biset U.

Definition 5. (See [5, Definition 3.1.1].) We define the biset category $\underline{\mathrm{GrB}}$ of finite groups as follows:

- The objects of $\underline{\mathrm{GrB}}$ are all finite groups;
- If G and H are finite groups, then

$$\mathrm{Hom}_{\underline{\mathrm{GrB}}}(G,H)=B(H,G);$$

- If G, H and K are finite groups, then the composition $v\circ u$ is equal to $v\times_H u$, for all morphisms $u\in B(H,G)$ and for all morphisms $v\in B(K,H)$;
- For any finite group G, the identity morphism of G in $\underline{\mathrm{GrB}}$ is equal to Id_G.

Let R be a commutative noetherian ring with identity element.

Definition 6. (See [5, Definition 3.1.6].) We define the category $R\underline{\mathrm{GrB}}$ as follows:

- The objects of $R\underline{\mathrm{GrB}}$ are all finite groups;
- If G and H are finite groups, then

$$\mathrm{Hom}_{R\underline{\mathrm{GrB}}}(G,H) = R \otimes_{\mathbb{Z}} B(H,G);$$

- The composition of morphisms in $R\underline{\mathrm{GrB}}$ is the R-linear extension of the composition in $\underline{\mathrm{GrB}}$;
- For any finite group G, the identity morphism of G in $R\underline{\mathrm{GrB}}$ is equal to $\mathrm{Id}_R \otimes_{\mathbb{Z}} \mathrm{Id}_G$.

Definition 7. (See [5, Definition 3.2.2].) Let $\mathcal{D}$ be a preadditive subcategory of $\underline{\mathrm{GrB}}$. A *biset functor defined on* $\mathcal{D}$ with values in R-**mod** is an R-linear functor from $R\mathcal{D}$ to the category R-**mod** of all finitely generated R-modules.

Biset functors over $R\mathcal{D}$, with values in R-**mod**, are the objects of a category, denoted by $\mathcal{F}_{\mathcal{D},R}$, where morphisms are natural transformations of functors, and composition of morphisms is composition of natural transformations.

Proposition 8. *(See [5, Proposition 3.2.8].) Let $\mathcal{D}$ be a preadditive subcategory of $\underline{\mathrm{GrB}}$.*

1. *The category $\mathcal{F}_{\mathcal{D},R}$ is an R-linear abelian category: If $f: F \to F'$ is a morphism of biset functors, then for every object G of $\mathcal{D}$*

$$\big(\mathrm{Ker}(f)\big)(G) = \mathrm{Ker}\big(f(G)\big), \qquad \big(\mathrm{Coker}(f)\big)(G) = \mathrm{Coker}\big(f(G)\big).$$

2. *A sequence $0 \to F \xrightarrow{f} F' \xrightarrow{f'} F'' \to 0$ is an exact sequence of $\mathcal{F}_{\mathcal{D},R}$ if and only if for any object G of $\mathcal{D}$, the sequence*

$$0 \to F(G) \xrightarrow{f(G)} F'(G) \xrightarrow{f'(G)} F''(G) \to 0$$

is an exact sequence of R-modules.

Remark 9. For this proposition, we need the fact that R is noetherian.

Definition 10. (See [5, Definition 4.1.7].) A class $\mathcal{D}$ of finite groups is said to be *closed under taking subquotients* if any group isomorphic to a subquotient of an element of $\mathcal{D}$ is in $\mathcal{D}$.

A subcategory $\mathcal{D}$ of $\underline{\mathrm{GrB}}$ is called *replete* if it is a full subcategory whose class of objects is closed under taking subquotients.

A subcategory $\mathcal{C}$ of $R\underline{\mathrm{GrB}}$ is called *replete* if there exists a replete subcategory $\mathcal{D}$ of $\underline{\mathrm{GrB}}$ such that $\mathcal{C} = R\mathcal{D}$.

2.2. Simple functors and composition factors

Let R be a commutative ring with identity element and $\mathcal{D}$ be a replete subcategory of $\underline{\mathrm{GrB}}$ containing group isomorphisms. We set $\mathcal{C} = R\mathcal{D}$.

Now we recall some results on simple functors of the category $\mathcal{F}_{\mathcal{D},R}$. For more details, see [3,5,2].

Definition 11. A functor is said *simple* if it is non-zero and its only subfunctors are itself and the zero functor.

Remark 12. Proposition 4.2.2 of [5] implies that the restriction to a full subcategory of a simple functor is either zero or a simple functor.

Definition 13. (See [5, Example 3.3.5].) Let G be an object of $\mathcal{C}$ and V be a simple $R\,\mathrm{Out}(G)$-module. Then V is also an $\mathrm{End}_{\mathcal{C}}(G)$-module and we define the biset functor $L_{G,V}$ by:

- For all object H of $\mathcal{C}$, we set

$$L_{G,V}(H) = \mathrm{Hom}_{\mathcal{C}}(G,H) \otimes_{\mathrm{End}_{\mathcal{C}}(G)} V = RB(H,G) \otimes_{RB(G,G)} V.$$

- For all morphism $\varphi : H \to H'$ in $\mathcal{C}$, $L_{G,V}(\varphi) : L_{G,V}(H) \to L_{G,V}(H')$ is defined by

$$\begin{aligned} L_{G,V}(\varphi) : L_{G,V}(H) &\to L_{G,V}\bigl(H'\bigr), \\ f \otimes v &\mapsto (\varphi \circ f) \otimes v. \end{aligned}$$

We can notice that $L_{G,V}(G) \cong V$.

Proposition 14. *(See [5, Corollary 4.2.4].) Let G be an object of $\mathcal{C}$ and V be a simple $\mathrm{End}_{\mathcal{C}}(G)$-module. Then the functor $L_{G,V}$ has a unique proper maximal subfunctor $J_{G,V}$, and the quotient $S_{G,V} = L_{G,V}/J_{G,V}$ is a simple object of $\mathcal{F}_{\mathcal{D},R}$, such that $S_{G,V}(G) \cong V$.*

Remark 15. (See [5, Remark 4.2.6].) Let H be an object of $\mathcal{C}$. Then $J_{G,V}(H)$ is equal to the set of finite sums $\sum_{i=1}^{n} \varphi_i \otimes v_i$ in $L_{G,V}(H)$, where $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ and $v_i \in V$, such that $\sum_{i=1}^{n} (\psi \circ \varphi_i) \cdot v_i = 0$ for any $\psi \in \mathrm{Hom}_{\mathcal{C}}(H,G)$, where $(\psi \circ \varphi_i) \cdot v_i$ denotes the image of the element v_i of V under the action of the endomorphism $\psi \circ \varphi_i$ of G.

The simple objects of the category $\mathcal{F}_{\mathcal{D},R}$ are labeled by pairs (G,V), where G is a finite group and V a simple $R\,\mathrm{Out}(G)$-module. We denote by $S_{G,V}$ the simple functor associated to (G,V). If $F \in \mathcal{F}_{\mathcal{D},R}$ is a simple functor, then $F \cong S_{G,V}$ where G is the smallest group (unique up to isomorphism) such that $F(G) \neq \{0\}$ and $V = F(G)$. We can define a notion of isomorphism on those pairs such that two simple functors are isomorphic if and only if the corresponding pairs are isomorphic [5, Theorem 4.3.10].

Definition 16. Let F be a biset functor on $\mathcal{C}$. A simple functor S is a *composition factor* of F if there exist subfunctors $F' \subseteq F'' \subseteq F$ such that $F''/F' \cong S$.

Definition 17. Let G be an object of $\mathcal{C}$. We define the category $\mathcal{C}{\downarrow_G}$ as the full subcategory of $\mathcal{C}$ whose objects are subquotients of G.

Definition 18. (See [10, p. 20].) Let G be a fixed object of $\mathcal{C}$ and F be a biset functor on $\mathcal{C}$. The functor F has a *composition series over G* if there is a series of subfunctors

$$0 = T_0 \subseteq B_1 \subset T_1 \subseteq \cdots \subseteq B_m \subset T_m \subseteq B_{m+1} = F$$

such that:

- T_i/B_i is a simple functor, whose restriction to $\mathcal{C}{\downarrow_G}$ is non-zero for all $i = 1,\ldots,m$.
- $\mathrm{Res}^{\mathcal{C}}_{\mathcal{C}\downarrow_G}(B_{i+1}/T_i) = 0$ for all $i = 0,\ldots,m$.

If such a composition series exists, we will call the set of simple functors T_i/B_i together with their multiplicities the *composition factors of F over G*.

Remark 19. The composition factors of F over G correspond exactly to the composition factors of F which are non-zero on G. In particular, this allows us to have a notion of multiplicity for composition factors (thanks to the following proposition and theorem).

Proposition 20. *(See [10, Proposition 3.1].) Let G be a fixed object of $\mathcal{C}$ and F a biset functor on $\mathcal{C}$. If F has a composition series over G, then any other composition series over G of F has the same length and the composition factors on G (taken with multiplicities) are the same.*

Theorem 21. *We suppose that R is a field. Then every biset functor on $\mathcal{C}$ has a composition series over G, for every group G in $\mathcal{C}$.*

Proof. This is a consequence of Theorem 3.3, p. 22 of [10]. □

3. Tensor product of biset functors on groups of coprime order

Let k be an algebraically closed field. Let π be a set of prime numbers. Let $\mathcal{P}$ be the full subcategory of $k\underline{\mathrm{GrB}}$ whose objects are all finite π-groups and let $\mathcal{Q}$ be the full subcategory of $k\underline{\mathrm{GrB}}$ whose objects are all finite π'-groups. Moreover, let $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ be the full subcategory of $k\underline{\mathrm{GrB}}$ whose objects are all the groups of the form $P \times Q$ where P is an object of $\mathcal{P}$ and Q is an object of $\mathcal{Q}$. Those three categories are replete.

Let $F_{\mathcal{P}}$ be a biset functor on the category $\mathcal{P}$ and $F_{\mathcal{Q}}$ be a biset functor on the category $\mathcal{Q}$. The aim is to construct a biset functor $F = F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}}$ on the category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, which depends functorially on both $F_{\mathcal{P}}$ and $F_{\mathcal{Q}}$.

We start by defining F on the objects of $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$: Let $G = P \times Q$, where P is an object of $\mathcal{P}$ and Q is an object of $\mathcal{Q}$. We set

$$F(G) = F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q).$$

As $F_{\mathcal{P}}(P)$ and $F_{\mathcal{Q}}(Q)$ are finite dimensional k-vector spaces, $F(G)$ is also a finite dimensional k-vector space.

Now we have to define $F(\mu)$ when μ is an element of $kB(H,G)$ (where P and P' are objects of $\mathcal{P}$ and Q and Q' are objects of $\mathcal{Q}$ such that $G = P \times Q$ and $H = P' \times Q'$). In order to do this, we need the following proposition:

Proposition 22. *Let G, G', H and H' be finite groups. If U is a (G,G')-biset and V is an (H,H')-biset, then $U \times V$ is a $(G \times H, G' \times H')$-biset for the structure given by*

$$(g,h)\cdot(u,v)\cdot\big(g',h'\big) = \big(gxg',hyh'\big),$$

for all $g \in G$, for all $g' \in G'$, for all $h \in H$, for all $h' \in H'$, for all $u \in U$ and for all $v \in V$. The correspondence $(U,V) \mapsto U \times V$ induces a bilinear map from $B(G,G') \times B(H,H')$ to $B(G \times H, G' \times H')$, hence a linear map

$$\varepsilon : B\big(G,G'\big) \otimes_{\mathbb{Z}} B\big(H,H'\big) \to B\big(G \times H, G' \times H'\big),$$

which is an injective $\mathbb{Z}$-module homomorphism preserving identity elements. If $G \times G'$ and $H \times H'$ have coprime order, this map is an isomorphism.

Proof. It is a generalization of Proposition 2.5.14 b), pp. 38–39 of [5]. The proof is analogous. The fact that the correspondence induces a bilinear map also comes from Lemma 8.1.2, p. 135 of [5]. □

Remark 23. The previous proposition remains true if we replace $\mathbb{Z}$ by k. We obtain a homomorphism of k-vector spaces

$$\varepsilon : kB\big(G, G'\big) \otimes_k kB\big(H, H'\big) \to kB\big(G \times H, G' \times H'\big)$$

which becomes an isomorphism if $G \times G'$ and $H \times H'$ have coprime order. If moreover $G = G'$ and $H = H'$, it becomes an isomorphism of k-algebras.

Let $G = P \times Q$ and $H = P' \times Q'$, where P and P' are objects of $\mathcal{P}$ and Q and Q' are objects of $\mathcal{Q}$. Let u be an element of $\mathbb{C}B(P \times Q, P' \times Q')$. Then we define $F(u) : F(G) \to F(H)$ as the map

$$\sum_{i=1}^{n} \lambda_i F_{\mathcal{P}}(u_{\mathcal{P},i}) \otimes_k F_{\mathcal{Q}}(u_{\mathcal{Q},i}) : F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \to F_{\mathcal{P}}\big(P'\big) \otimes_k F_{\mathcal{Q}}\big(Q'\big),$$

where $\varepsilon^{-1}(u) = \sum_{i=1}^{n} \lambda_i u_{\mathcal{P},i} \otimes_k u_{\mathcal{Q},i}$, with $u_{\mathcal{P},i} \in kB(P, P')$ and $u_{\mathcal{Q},i} \in kB(Q, Q')$, for all $i = 1, \ldots, n$.

Proposition 24.

1. *The above constructions define a biset functor $F = F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}}$ on $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$.*
2. *The assignment $(F_{\mathcal{P}}, F_{\mathcal{Q}}) \mapsto F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}}$ is a k-bilinear functor from $\mathcal{F}_{\mathcal{P},k} \times \mathcal{F}_{\mathcal{Q},k}$ to $\mathcal{F}_{\mathcal{C}_{\mathcal{P},\mathcal{Q}},k}$.*

Proof. 1. Since ε preserves identity elements, it is easy to check that $F(\mathrm{Id}_G) = \mathrm{Id}_{F(G)}$ for all objects G of $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$.

It remains to check that $F(u_1 \circ u_2) = F(u_1) \circ F(u_2)$ for all $u_1 \in kB(P'' \times Q'', P' \times Q')$ and for all $u_2 \in kB(P' \times Q', P \times Q)$. By k-linearity, it is enough to verify this in the case of transitive bisets. Let P, P' and P'' be objects of $\mathcal{P}$, Q, Q' and Q'' be objects of $\mathcal{Q}$, U be a transitive $(P'' \times Q'', P' \times Q')$-biset and V be a transitive $(P' \times Q', P \times Q)$-biset. Then, by Proposition 22, there exists a (P'', P')-biset $U_{\mathcal{P}}$, a (P', P)-biset $V_{\mathcal{P}}$, a (Q'', Q')-biset $U_{\mathcal{Q}}$ and a (Q', Q)-biset $V_{\mathcal{Q}}$ such that

$$U = U_{\mathcal{P}} \times U_{\mathcal{Q}} \quad \text{and} \quad V = V_{\mathcal{P}} \times V_{\mathcal{Q}}.$$

Then we have

$$\begin{aligned}
F\big([U]\big) \circ F\big([V]\big) &= \big(F_{\mathcal{P}}\big([U_{\mathcal{P}}]\big) \otimes_k F_{\mathcal{Q}}\big([U_{\mathcal{Q}}]\big)\big) \circ \big(F_{\mathcal{P}}\big([V_{\mathcal{P}}]\big) \otimes_k F_{\mathcal{Q}}\big([V_{\mathcal{Q}}]\big)\big) \\
&= \big(F_{\mathcal{P}}\big([U_{\mathcal{P}}]\big) \circ F_{\mathcal{P}}\big([V_{\mathcal{P}}]\big)\big) \otimes_k \big(F_{\mathcal{Q}}\big([U_{\mathcal{Q}}]\big) \circ F_{\mathcal{Q}}\big([V_{\mathcal{Q}}]\big)\big) \\
&= F_{\mathcal{P}}\big([U_{\mathcal{P}} \times_{P'} V_{\mathcal{P}}]\big) \otimes_k F_{\mathcal{Q}}\big([U_{\mathcal{Q}} \times_{Q'} V_{\mathcal{Q}}]\big) \\
&= F\big([U \times_{P' \times Q'} V]\big)
\end{aligned}$$

because there exists an isomorphism of $(P'' \times Q'', P \times Q)$-bisets

$$(U_{\mathcal{P}} \times_{P'} V_{\mathcal{P}}) \times (U_{\mathcal{Q}} \times_{Q'} V_{\mathcal{Q}}) \cong (U_{\mathcal{P}} \times U_{\mathcal{Q}}) \times_{P' \times Q'} (V_{\mathcal{P}} \times V_{\mathcal{Q}}).$$

2. We define a map $\nu : \mathcal{F}_{\mathcal{P},k} \times \mathcal{F}_{\mathcal{Q},k} \to \mathcal{F}_{\mathcal{C}_{\mathcal{P},\mathcal{Q}},k}$. On the objects, ν is defined by $\nu(F_{\mathcal{P}}, F_{\mathcal{Q}}) = F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}}$. Let $\varphi : F_{\mathcal{P}} \to F'_{\mathcal{P}}$ and $\psi : F_{\mathcal{Q}} \to F'_{\mathcal{Q}}$ be two morphisms of $\mathcal{F}_{\mathcal{P},k}$ and $\mathcal{F}_{\mathcal{Q},k}$ respectively. Then $\nu(\varphi, \psi)$ is defined by

$$\nu(\varphi, \psi)(P, Q) = \varphi(P) \otimes \psi(Q),$$

for any object P of $\mathcal{P}$ and any object Q of $\mathcal{Q}$. This defines a k-bilinear functor. We leave the details to the read. $\square$

Proposition 25. *Let $F_{\mathcal{Q}}$ be a biset functor on the category $\mathcal{Q}$. Then the functor $F_{\mathcal{P}} \mapsto F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}}$ is an exact functor from $\mathcal{F}_{\mathcal{P},k}$ to $\mathcal{F}_{\mathcal{C}_{\mathcal{P}\times\mathcal{Q}},k}$.*

Proof. Let $0 \to F_{\mathcal{P}} \xrightarrow{\mu} F'_{\mathcal{P}} \xrightarrow{\eta} F''_{\mathcal{P}} \to 0$ be an exact sequence of biset functors on the category $\mathcal{P}$ and let $F_{\mathcal{Q}}$ be a biset functor on the category $\mathcal{Q}$. We can define an exact sequence of biset functors on $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$:

$$0 \to F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \xrightarrow{\mu\otimes_k \mathrm{Id}_{F_{\mathcal{Q}}}} F'_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \xrightarrow{\eta\otimes_k \mathrm{Id}_{F_{\mathcal{Q}}}} F''_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \to 0.$$

We begin by defining $\mu \otimes_k \mathrm{Id}_{F_{\mathcal{Q}}}$ and verifying that this is a natural transformation. Let $G = P \times Q$, where P is an object of $\mathcal{P}$ and Q is an object of $\mathcal{Q}$. We define $(\mu \otimes_k \mathrm{Id}_{F_{\mathcal{Q}}})(G) : (F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}})(G) \to (F'_{\mathcal{P}} \otimes_k F_{\mathcal{Q}})(G)$ by

$$(\mu \otimes_k \mathrm{Id}_{F_{\mathcal{Q}}})(P \times Q) = \mu(P) \otimes_k \mathrm{Id}_{F_{\mathcal{Q}}(Q)} : F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \to F'_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q).$$

It is easy to check that it is a natural transformation, by verifying that the diagrams involved in this property are commutative for transitive bisets, using the fact that we have k-linear maps. The map $\eta \otimes_k \mathrm{Id}_{F_{\mathcal{Q}}}$ is defined in the same way.

Now we have to verify that we really obtain an exact sequence. By Proposition 8, it is enough to verify that it is an exact sequence when evaluating in a group G of $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$. Let $G = P \times Q$, where P is an object of $\mathcal{P}$ and Q is an object of $\mathcal{Q}$. We have to check that the following sequence of $\mathbb{C}$-vector spaces is exact:

$$0 \to F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \to F'_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \to F''_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \to 0.$$

But, by Proposition 8, the following sequence is exact:

$$0 \to F_{\mathcal{P}}(P) \xrightarrow{\mu(P)} F'_{\mathcal{P}}(P) \xrightarrow{\eta(P)} F''_{\mathcal{P}}(P) \to 0,$$

which implies that the previous sequence is also exact. $\square$

3.1. The product of simple functors

Let P be an object of $\mathcal{P}$, Q be an object of $\mathcal{Q}$, V be a simple $k\,\mathrm{Out}(P)$-module and W be a simple $k\,\mathrm{Out}(Q)$-module.

Remark 26.

1. When P and Q are finite groups of coprime orders, there is a natural isomorphism $\mathrm{Out}(P \times Q) \cong \mathrm{Out}(P) \times \mathrm{Out}(Q)$.
2. Since k is an algebraically closed field, the simple modules for the algebra $k\,\mathrm{Out}(P) \otimes_k k\,\mathrm{Out}(Q) \cong k\,\mathrm{Out}(P \times Q)$ are exactly the modules $V \otimes_k W$, where V is a simple $k\,\mathrm{Out}(P)$-module, and W is a simple $k\,\mathrm{Out}(Q)$-module.

We want to show the following theorem:

Theorem 27. *We consider $S_{P,V}$ as a simple functor on $\mathcal{P}$ and $S_{Q,W}$ as a simple functor on $\mathcal{Q}$. The functor $S_{P,V} \otimes_k S_{Q,W}$ is, on the category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, isomorphic to the simple functor $S_{P\times Q,V\otimes_k W}$.*

In order to prove this theorem, we first need the following proposition:

Proposition 28. *We consider $L_{P,V}$ as a functor on $\mathcal{P}$ and $L_{Q,W}$ as a functor on $\mathcal{Q}$. The functor $L_{P,V} \otimes_k L_{Q,W}$ is, on the category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, isomorphic to the functor $L_{P\times Q,V\otimes_k W}$.*

Proof. Let G be an object of $\mathcal{P}$ and H be an object of $\mathcal{Q}$. Then

$$(L_{P,V} \otimes_k L_{Q,W})(G \times H) = \bigl(kB(G,P) \otimes_{kB(P,P)} V\bigr) \otimes_k \bigl(kB(H,Q) \otimes_{kB(Q,Q)} W\bigr)$$

and

$$\begin{aligned}(L_{P\times Q,V\otimes_k W})(G \times H) &= kB(G \times H, P \times Q) \otimes_{kB(P\times Q,P\times Q)} (V \otimes_k W)\\ &\cong \bigl(kB(G,P) \otimes_k kB(H,Q)\bigr) \otimes_{kB(P,P)\otimes_k kB(Q,Q)} (V \otimes_k W).\end{aligned}$$

Those are isomorphic k-vector spaces, by the map $\varphi(G \times H)$ defined by

$$\begin{aligned}&\varphi(G \times H) : (L_{P,V} \otimes_k L_{Q,W})(G \times H) \to L_{P\times Q,V\otimes_k W}(G \times H),\\ &(u_1 \otimes_{kB(P,P)} v) \otimes_k (u_2 \otimes_{kB(Q,Q)} w) \mapsto \varepsilon(u_1 \otimes_k u_2) \otimes_{kB(P\times Q,P\times Q)} (v \otimes_k w).\end{aligned}$$

This induces a natural transformation $\varphi : L_{P,V} \otimes_k L_{Q,W} \to L_{P\times Q,V\otimes_k W}$ and completes the proof of the proposition. $\square$

Now we have the following commutative diagram:

$$\begin{tikzcd} L_{P,V} \otimes_k L_{Q,W} \arrow[r, two heads, "proj\otimes_k proj"] \arrow[d, "\varphi"] \arrow[dr, two heads, "\psi"] & S_{P,V} \otimes_k S_{Q,W} \\ L_{P\times Q,V\otimes_k W} \arrow[r, two heads, "proj"] & S_{P\times Q,V\otimes_k W}. \end{tikzcd}$$

We set $\psi = proj \circ \varphi : L_{P,V} \otimes_k L_{Q,W} \to S_{P\times Q,V\otimes_k W}$.

Lemma 29. *The kernel of $proj \otimes_k proj$ is contained in the kernel of ψ.*

Proof. By Proposition 8, it is enough to prove it for a group $G \times H$, where G is an object of $\mathcal{P}$ and H is an object of $\mathcal{Q}$. Moreover,

$$\mathrm{Ker}(proj \otimes_k proj) = J_{P,V} \otimes_k L_{Q,W} + L_{P,V} \otimes_k J_{Q,W}$$

and Remark 15 gives a description of $J_{P,V}(G)$ and $J_{Q,W}(H)$.

Let $n \in \mathbb{N}$, $u_i \in \mathrm{Hom}_{\mathcal{P}}(G,P)$ and $v_i \in V$ such that $\sum_{i=1}^n (\alpha \times_G u_i) \cdot v_i = 0$ for all $\alpha \in \mathrm{Hom}_{\mathcal{P}}(P,G)$ (that is $\sum_{i=1}^n u_i \otimes v_i \in J_{P,V}(G)$). Let $u \otimes w \in L_{Q,W}$. We have to verify that $\psi(G \times H)((\sum_{i=1}^n u_i \otimes v_i) \otimes_k (u \otimes w)) = 0$, which is equivalent to checking that $\varphi(G \times H)((\sum_{i=1}^n u_i \otimes v_i) \otimes_k (u \otimes w)) \in J_{P\times Q,V\otimes_k W}(G \times H)$. But

$$\varphi(G \times H)\left(\left(\sum_{i=1}^n u_i \otimes v_i\right) \otimes_k (u \otimes w)\right) = \sum_{i=1}^n \varepsilon(u_i \otimes_k u) \otimes (v_i \otimes_k w).$$

Let U be a transitive $(P \times Q, G \times H)$-biset. Then there exists a (P,G)-biset $U_{\mathcal{P}}$ and a (Q,H)-biset $U_{\mathcal{Q}}$ such that $U = U_{\mathcal{P}} \times U_{\mathcal{Q}}$. Then

$$\begin{aligned}\sum_{i=1}^{n}\big([U]\times_{G\times H}\varepsilon(u_i\otimes_k u)\big)(v_i\otimes_k w) &= \sum_{i=1}^{n}\big(\varepsilon\big(([U_{\mathcal{P}}]\times_G u_i)\otimes_k([U_{\mathcal{Q}}]\times_H u)\big)\big)(v_i\otimes_k w)\\ &= \underbrace{\sum_{i=1}^{n}\big([U_{\mathcal{P}}]\times_G u_i\big)(v_i)}_{=0}\otimes_k\big([U_{\mathcal{Q}}]\times_H u\big)(w)\\ &= 0.\end{aligned}$$

By k-linearity, this implies that

$$\varphi(G\times H)\left(\left(\sum_{i=1}^{n}u_i\otimes v_i\right)\otimes_k(u\times w)\right)\in J_{P\times Q,V\otimes_k W}(G\times H).$$

Similarly, we can show that

$$\varphi(G\times H)\big(L_{P,V}(G)\otimes_k J_{Q,W}(H)\big)\subseteq J_{P\times Q,V\otimes_k W}(G\times H).$$

This completes the proof of the lemma. □

As $\mathrm{Ker}(\mathit{proj}\otimes_k\mathit{proj})\subseteq\mathrm{Ker}\,\psi$, there exists a natural transformation

$$\mu: S_{P,V}\otimes_k S_{Q,W}\to S_{P\times Q,V\otimes_k W}.$$

We will show that it is an isomorphism. It is enough to show it for a group $G\times H$ of $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$.

Lemma 30. *The k-vector space $S_{P\times Q,V\otimes_k W}(G\times H)$ is zero if and only if $S_{P,V}(G)$ or $S_{Q,W}(H)$ is zero.*

Proof. If $S_{P,V}(G)$ or $S_{Q,W}(H)$ is zero, then $S_{P,V}(G)\otimes_k S_{Q,W}(H)=0$. As $\mu(G\times H)$ is surjective (because φ and *proj* are surjective and so is ψ), we have that $S_{P\times Q,V\otimes_k W}(G\times H)=0$.

Conversely, suppose that $S_{P\times Q,V\otimes_k W}(G\times H)=0$. If $S_{P,V}(G)=0$, we have finished, so we can suppose that $S_{P,V}(G)\neq 0$, that is $J_{P,V}(G)\subsetneq L_{P,V}(G)$. Therefore, there exists $n\in\mathbb{N}$, $\varphi_i\in kB(G,P)$, $v_i\in V$ for all $1\leqslant i\leqslant n$ and $\rho\in kB(P,G)$ such that $\sum_{i=1}^{n}(\rho\times_G\varphi_i)(v_i)\neq 0$. We will show that $S_{Q,W}(H)=0$. It is enough to show that the elements $u\otimes w$, where $u\in kB(H,Q)$ and $w\in W$, are in $J_{Q,W}(H)$, because, this implies that $J_{Q,W}(H)=L_{Q,W}(H)$ and so $S_{Q,W}(H)=0$. Let $u\in kB(H,Q)$ and $w\in W$. For all $\widetilde{u}\in kB(Q,H)$, as $\sum_{i=1}^{n}\varepsilon(\varphi_i\otimes_k u)\otimes(v_i\otimes_k w)$ is in $L_{P\times Q,V\otimes_k W}(G\times H)=J_{P\times Q,V\otimes_k W}(G\times H)$:

$$\begin{aligned}0 &= \sum_{i=1}^{n}\big(\varepsilon(\rho\otimes_k\widetilde{u})\varepsilon(\varphi_i\otimes_k u)\big)(v_i\otimes_k w)\\ &= \sum_{i=1}^{n}\varepsilon\big((\rho\times_G\varphi_i)\otimes_k(\widetilde{u}\times_H u)\big)(v_i\otimes_k w)\\ &= \sum_{i=1}^{n}\big((\rho\times_G\varphi_i)(v_i)\big)\otimes_k\big((\widetilde{u}\times_H u)(w)\big)\\ &= \underbrace{\left(\sum_{i=1}^{n}(\rho\times_G\varphi_i)(v_i)\right)}_{\neq 0}\otimes_k\big((\widetilde{u}\times_H u)(w)\big).\end{aligned}$$

Consequently, we must have that $(\widetilde{u} \times_H u)(w) = 0$. As $\widetilde{u}$ is arbitrary, this implies that $u \otimes_k w \in J_{Q,W}(H)$. This completes the proof of the lemma. □

Now we can suppose that $S_{P\times Q, V\otimes_k W}(G \times H)$ and $S_{P,V}(G) \otimes_k S_{Q,W}(H)$ are non-zero. But then, by Corollary 4.2.4, p. 58 of [5], we know that $S_{P,V}(G)$ is a simple $kB(G,G)$-module, $S_{Q,W}(H)$ is a simple $kB(H,H)$-module and $S_{P\times Q, V\otimes_k W}(G \times H)$ is a simple $kB(G \times H, G \times H)$-module. As k is algebraically closed, the tensor product of two simple modules on finite dimensional k-algebras is a simple module on the tensor product of the two algebras (Proposition 3.56, p. 65 of [7] and Schur's lemma [8, Lemma 27.3, p. 181]), so $S_{P,V}(G) \otimes_k S_{Q,W}(H)$ is a simple $kB(G,G) \otimes_k kB(H,H)$-module. But $kB(G,G) \otimes_k kB(H,H) \cong kB(G \times H, G \times H)$ as k-algebras (Remark 23), so we can consider $S_{P,V}(G) \otimes_k S_{Q,W}(H)$ as a simple $kB(G \times H, G \times H)$-module. Now we have a non-zero homomorphism of modules between two simple $kB(G \times H, G \times H)$-modules, which implies that it is an isomorphism by Schur's lemma [8, Lemma 27.3, p. 181]. Now we have proved Theorem 27.

4. The biset functor of p-permutation modules

Now, suppose that k is an algebraically closed field of characteristic p, where p is a prime number. First, we need to define $\mathbb{C}\,\mathrm{pp}_k$ and two other biset functors:

Definition 31. Let G be a finite group. We define $\mathrm{pp}_k(G)$ as the Grothendieck group of the set of isomorphism classes of p-permutation kG-modules (i.e. direct sums of indecomposable trivial source kG-modules) with respect to direct sums. For every (H,G)-biset U we define

$$\mathrm{pp}_k\big([U]\big) : \mathrm{pp}_k(G) \to \mathrm{pp}_k(H),$$
$$[M] \mapsto [kU \otimes_{kG} M]$$

for every trivial source kG-module M. This extends by $\mathbb{C}$-linearity to a map $\mathbb{C}\,\mathrm{pp}_k([U]) : \mathbb{C}\,\mathrm{pp}_k(G) \to \mathbb{C}\,\mathrm{pp}_k(H)$, where $\mathbb{C}\,\mathrm{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{pp}_k(G)$.

Now we can define $\mathbb{C}\,\mathrm{pp}_k(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H,G)$. Let $u = \sum_{i=1}^n \lambda_i [U_i]$ where $\lambda_i \in \mathbb{C}$ and U_i is an (H,G)-biset, for every $i = 1, \dots, n$. Then $\mathbb{C}\,\mathrm{pp}_k(u) = \sum_{i=1}^n \lambda_i \mathbb{C}\,\mathrm{pp}_k([U_i])$. This defines a structure of biset functor $\mathbb{C}\,\mathrm{pp}_k$.

Definition 32. Let G be a finite group and $B(G)$ be the Grothendieck group of the set of isomorphism classes of finite G-sets (for disjoint union). Then $B(G)$ is a ring (called the Burnside ring of G), where the multiplication is defined by

$$[U] \cdot [V] = [U \times V]$$

for all G-sets U and V (extended to $B(G)$ by bilinearity).

Let G and H be two finite groups. For every (finite) (H,G)-biset U, we can define the following map:

$$B\big([U]\big) : B(G) \to B(H),$$
$$[V] \mapsto [U \times_G V]$$

for every (finite) G-set V.

This map can be extended by $\mathbb{C}$-linearity to a map $\mathbb{C}B([U]) : \mathbb{C}B(G) \to \mathbb{C}B(H)$, where $\mathbb{C}B(G) = \mathbb{C} \otimes_{\mathbb{Z}} B(G)$. As for $\mathbb{C}\,\mathrm{pp}_k$, we can now define $\mathbb{C}B(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H,G)$. This defines a structure of biset functor $\mathbb{C}B$.

Remark 33. Let G be a finite group. The group $B(G)$ is isomorphic to the group $B(G, \mathbf{1})$.

Definition 34. (See [5, Notation 5.2.2 and Definition 5.4.6].) If G is a finite group, and N is a normal subgroup of G, denote by $m_{G,N}$ the rational number defined by

$$m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X|\mu(X,G),$$

where μ is the Möbius function of the poset of subgroups of G.

A finite group G is called a B-*group* if for any non-trivial normal subgroup N of G, the constant $m_{G,N}$ is equal to zero.

Theorem 35. *(See [5, Remark 5.5.2].) Let $\mathcal{D}$ be a replete subcategory of $\underline{\mathrm{GrB}}$. The composition factors of the Burnside functor $\mathbb{C}B$ on $\mathbb{C}\mathcal{D}$ are exactly the functors $S_{G,\mathbb{C}}$, where G is an object of $\mathcal{D}$ which is a B-group.*

Definition 36. (See [5, Notation 7.1.1].) Let G be a finite p'-group. Denote by $R_k(G)$ the Grothendieck group of the category of finite dimensional kG-modules (for exact sequences). If H is another finite p'-group and U is an (H,G)-biset, denote by $R_k([U]) : R_k(G) \to R_k(H)$ the group homomorphism defined by

$$R_k\big([U]\big)\big([E]\big) = [kU \otimes_{kG} E],$$

for all kG-modules E. By $\mathbb{Z}$-linearity, we can extend this definition to $R_k(G)$.

As for $\mathbb{C}\,\mathrm{pp}_k$ and $\mathbb{C}B$, this construction can be extend by $\mathbb{C}$-linearity to a biset functor $\mathbb{C}R_k$ defined on $\mathbb{C}\mathcal{D}$ with values in $\mathbb{C}$-**mod**, where $\mathcal{D}$ is the full subcategory of $\underline{\mathrm{GrB}}$ of all finite p'-groups.

Definition 37. (See [5, Definition 7.3.1].) Let $m \in \mathbb{N}^\star$. A character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$ is called *primitive* if it cannot be factored through any quotient $(\mathbb{Z}/n\mathbb{Z})^\star$ of $(\mathbb{Z}/m\mathbb{Z})^\star$, where n is a proper divisor of m.

Theorem 38. *The functor $\mathbb{C}R_k$ is a semisimple object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ (where $\mathcal{D}$ is the full subcategory of $\underline{\mathrm{GrB}}$ of all finite p'-group). More precisely*

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S_{C_m,\mathbb{C}_\xi},$$

where (m,ξ) runs through the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$ and $\mathbb{C}_\xi$ is the vector space $\mathbb{C}$ on which the group $\mathrm{Out}(C_m) \cong (\mathbb{Z}/m\mathbb{Z})^\star$ acts via ξ.

Proof. This is a consequence of [5, Corollary 7.3.5] and the fact that when k is an algebraically closed field of positive characteristic p, the restrictions of the functors $\mathbb{C}R_k$ and $\mathbb{C}R_\mathbb{C}$ to p'-groups are isomorphic biset functors. $\square$

Let $\mathcal{C}_p$ be the full subcategory of $\mathbb{C}\underline{\mathrm{GrB}}$ whose objects are all the finite p-groups and let $\mathcal{C}_{p'}$ be the full subcategory of $\mathbb{C}\underline{\mathrm{GrB}}$ whose objects are all the finite p'-groups. Let moreover $\mathcal{C}_{p\times p'}$ be the full subcategory of $\mathbb{C}\underline{\mathrm{GrB}}$ whose objects are all the groups of the form $P \times Q$ where P is a finite p-group and Q is a finite p'-group. We will now prove to the following theorem:

Theorem 39. *The biset functor $\mathbb{C}\,\mathrm{pp}_k$ on the category $\mathcal{C}_{p\times p'}$ is isomorphic to the functor $\mathbb{C}B \otimes_\mathbb{C} \mathbb{C}R_k$, where $\mathbb{C}B$ is considered as a functor on $\mathcal{C}_p$ and $\mathbb{C}R_k$ as a functor on $\mathcal{C}_{p'}$.*

Let P be a finite p-group and Q be a finite p'-group. We define $\mu(P\times Q)$ by

$$\begin{aligned}\mu(P\times Q):\mathbb{C}B(P)\otimes\mathbb{C}R_k(Q)&\to\mathbb{C}\,\mathrm{pp}_k(P\times Q),\\ [X]\otimes_{\mathbb{C}}[V]&\mapsto[kX\otimes_k V]\end{aligned}$$

for all P-set X and for all kQ-module V. The action of $P\times Q$ on $kX\otimes_k V$ is such that P acts on kX and Q acts on V. The $k(P\times Q)$-module $kX\otimes_k V$ is a p-permutation module because V is a p-permutation kQ-module. We extend the definition of $\mu(P\times Q)$ to $\mathbb{C}B(P)\otimes\mathbb{C}R_k(Q)$ by $\mathbb{C}$-linearity. It is a bijective $\mathbb{C}$-linear map: we are going to prove that its inverse is the following map:

$$\begin{aligned}\mu^{-1}(P\times Q):\mathbb{C}\,\mathrm{pp}_k(P\times Q)&\to\mathbb{C}B(P)\otimes\mathbb{C}R_k(Q),\\ [M]&\mapsto[P/D]\otimes[V],\end{aligned}$$

for all indecomposable p-permutation $k(P\times Q)$-module, where D is a vertex of M and V is a simple kQ-module such that $M=\mathrm{Ind}_{D\times Q}^{P\times Q}\mathrm{Inf}_{Q}^{D\times Q}V$.

Proposition 40. *Such a D and such a V always exist.*

Proof. By the classification of the p-permutation indecomposable modules (Theorem 3.2 of [6] or Theorem 27.10 of [9]), there exists a vertex D of M and a projective indecomposable $kN_P(D)/D\times Q$-module W such that M is a direct summand of

$$\mathrm{Ind}_{N_P(D)\times Q}^{P\times Q}\mathrm{Inf}_{N_P(D)/D\times Q}W.$$

But then, there exists a simple kQ-module V such that $W=\mathrm{Ind}_Q^{N_P(D)/D\times Q}V$ and so M is a direct summand of

$$\mathrm{Ind}_{N_P(D)\times Q}^{P\times Q}\mathrm{Inf}_{N_P(D)/D\times Q}\mathrm{Ind}_Q^{N_P(D)/D\times Q}V.$$

But $\mathrm{Ind}_{N_P(D)\times Q}^{P\times Q}\mathrm{Inf}_{N_P(D)/D\times Q}\mathrm{Ind}_Q^{N_P(D)/D\times Q}V$ is indecomposable (Corollary 6.11, [4]) and so equal to M. But then

$$\begin{aligned}M&\cong\mathrm{Ind}_{N_P(D)\times Q}^{P\times Q}\mathrm{Inf}_{N_P(D)/D\times Q}\mathrm{Ind}_Q^{N_P(D)/D\times Q}V\\ &\cong\mathrm{Ind}_{N_P(D)\times Q}^{P\times Q}\mathrm{Ind}_{D\times Q}^{N_P(D)/D\times Q}\mathrm{Inf}_Q^{D\times Q}V\\ &\cong\mathrm{Ind}_{D\times Q}^{P\times Q}\mathrm{Inf}_Q^{D\times Q}V\end{aligned}$$

which concludes the proof. □

Lemma 41. *If D is a subgroup of P and V is a kQ-module, then $\mathrm{Ind}_{D\times Q}^{P\times Q}\mathrm{Inf}_Q^{D\times Q}V$ is isomorphic to the $k(P\times Q)$-module $k(P/D)\otimes_k V$.*

Proof. We have

$$\mathrm{Ind}_{D\times Q}^{P\times Q}\mathrm{Inf}_Q^{D\times Q}V\cong k(P\times Q)\otimes_{k(D\times Q)}V.$$

Now we define the following homomorphism of $k(P\times Q)$-modules:

$$\varphi : k(P/D) \otimes_k V \to k(P \times Q) \otimes_{k(D \times Q)} V,$$
$$\sum_{i=1}^{n} \lambda_i(p_i D) \otimes_k v \mapsto \sum_{i=1}^{n} \lambda_i(p_i, 1) \otimes_{k(D \times Q)} v$$

for all $\sum_{i=1}^{n} \lambda_i(p_i D) \in k(P/D)$ and for all $v \in V$. It is easy to check that this map is well defined and a surjective homomorphism of $k(P \times Q)$-modules. It remains to check that it's bijective. But

$$\begin{aligned}\dim_k\bigl(k(P/D) \otimes_k V\bigr) &= [P : D] \cdot \dim_k V \\ &= [P \times Q : D \times Q] \cdot \dim_k V \\ &= \dim_k\bigl(k(P \times Q) \otimes_{k(D \times Q)} kQ \otimes_{kQ} V\bigr). \qquad \square\end{aligned}$$

Proposition 42. *The application $\mu^{-1}(P \times Q)$ is the inverse of the application $\mu(P \times Q)$.*

Proof. Let M be an indecomposable p-permutation $k(P \times Q)$-module. Let D be a vertex of M and V be a simple kQ-module such that $M = \mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Inf}_Q^{D \times Q} V$. Then

$$\begin{aligned}\mu(P \times Q)\mu^{-1}(P \times Q)\bigl([M]\bigr) &= \mu(P \times Q)\bigl([P/D] \otimes [V]\bigr) \\ &= \bigl[k(P/D) \otimes_k V\bigr] \\ &= \bigl[\mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Inf}_Q^{D \times Q} V\bigr] \\ &= [M].\end{aligned}$$

Conversely, let D be a subgroup of P and V be a simple kQ-module. Then

$$\mu(P \times Q)\bigl([P/D] \otimes_k [V]\bigr) = \bigl[k(P/D) \otimes_k V\bigr]$$

so if we prove that D is a vertex of $k(P/D) \otimes_k V$ then

$$\mu^{-1}(P \times Q)\mu(P \times Q)\bigl([P/D] \otimes_k [V]\bigr) = [P/D] \otimes_k [V].$$

As $M = k(P/D) \otimes_k V \cong \mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Inf}_Q^{D \times Q} V$ is a direct summand of

$$\begin{aligned}\mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Inf}_Q^{D \times Q} kQ &\cong \mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Inf}_Q^{D \times Q} \mathrm{Ind}_{\mathbf{1}}^{Q} k \\ &\cong \mathrm{Ind}_{D \times Q}^{P \times Q} \mathrm{Ind}_D^{D \times Q} \mathrm{Inf}_{\mathbf{1}}^{D} k \\ &\cong \mathrm{Ind}_D^{P \times Q} \mathrm{Inf}_{\mathbf{1}}^{D} k \\ &\cong \mathrm{Ind}_D^{P \times Q} k,\end{aligned}$$

M is relatively D-projective. So some vertex of M must be a subgroup of D, say $\widetilde{D}$. Then M is a direct summand of $\mathrm{Ind}_{\widetilde{D}}^{P \times Q} k$, so $\mathrm{Res}_D^{P \times Q} M$ is a direct summand of $\mathrm{Res}_D^{P \times Q} \mathrm{Ind}_{\widetilde{D}}^{P \times Q} k$. But

$$
\begin{aligned}
\operatorname{Res}_D^{P\times Q} M &= \operatorname{Res}_D^{P\times Q}\big(k(P/D)\otimes_k V\big)\\
&\cong \bigoplus_{i=1}^{\dim_k V} \operatorname{Res}_D^P k(P/D)\\
&\cong k \oplus (\text{other summands}),
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Res}_D^{P\times Q}\operatorname{Ind}_{\widetilde{D}}^{P\times Q} k &\cong \sum_{x\in[D\backslash P\times Q/\widetilde{D}]} \operatorname{Ind}_{D\cap {}^x\widetilde{D}}^D \operatorname{Conj}(x)\operatorname{Res}_{D^x\cap\widetilde{D}}^{\widetilde{D}} k\\
&\cong \sum_{x\in[D\backslash P\times Q/\widetilde{D}]} \operatorname{Ind}_{D\cap {}^x\widetilde{D}}^D k.
\end{aligned}
$$

So k is a direct summand of $\operatorname{Ind}_{D\cap {}^x\widetilde{D}}^D k$ for some $x\in P\times Q$. This is impossible except if $D=\widetilde{D}$ because the kD-module k has vertex D. $\square$

It remains to check that μ is a natural transformation between $\mathbb{C}B\otimes_{\mathbb{C}}\mathbb{C}R_k$ and $\mathbb{C}\,\mathrm{pp}_k$. Thanks to the $\mathbb{C}$-linearity of μ, it is enough to verify it for the element of a basis of $\mathbb{C}B(P\times Q, P'\times Q')$, where P and P' are finite p-groups and Q and Q' are finite p'-groups. Let U be a transitive $(P\times Q, P'\times Q')$-biset. We have to check that the following diagram commutes

$$
\begin{tikzcd}
\mathbb{C}B(P)\otimes_{\mathbb{C}}\mathbb{C}R_k(Q) \arrow[r, "\mu(P\times Q)"] \arrow[d, "(\mathbb{C}B\otimes\mathbb{C}R_k)([U])"'] & \mathbb{C}\,\mathrm{pp}_k(P\times Q) \arrow[d, "\mathbb{C}\,\mathrm{pp}_k([U])"] \\
\mathbb{C}B(P')\otimes_{\mathbb{C}}\mathbb{C}R_k(Q') \arrow[r, "\mu(P'\times Q')"'] & \mathbb{C}\,\mathrm{pp}_k(P'\times Q').
\end{tikzcd}
$$

There exists a (transitive) (P,P')-biset $U_{\mathcal{P}}$ and a (transitive) (Q,Q')-biset $U_{\mathcal{Q}}$ such that $U=U_{\mathcal{P}}\times U_{\mathcal{Q}}$ (Proposition 22). It is enough to verify that the diagram commutes for the elements of a basis of $\mathbb{C}B(P)\otimes_{\mathbb{C}}\mathbb{C}R_k(Q)$. So let X be a P-set and V a kQ-module. Then

$$
\begin{aligned}
\mathbb{C}\,\mathrm{pp}_k\big([U]\big)\circ\mu(P\times Q)\big([X]\otimes_{\mathbb{C}}[V]\big) &= \mathbb{C}\,\mathrm{pp}_k\big([U]\big)\big([kX\otimes_k V]\big)\\
&= \big[kU\otimes_{k(P\times Q)}(kX\otimes_k V)\big]\\
&= \big[(kU_{\mathcal{P}}\otimes_k kU_{\mathcal{Q}})\otimes_{k(P\times Q)}(kX\otimes_k V)\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu\big(P'\times Q'\big)\circ(\mathbb{C}B\otimes_{\mathbb{C}}\mathbb{C}R_k)\big([U]\big)\big([X]\otimes_{\mathbb{C}}[V]\big)\\
&\quad=\mu\big(P'\times Q'\big)\circ\big(\mathbb{C}B\big([U_{\mathcal{P}}]\big)\otimes\mathbb{C}R_k\big([U_{\mathcal{Q}}]\big)\big)\big([X]\otimes_{\mathbb{C}}[V]\big)\\
&\quad=\mu\big(P'\times Q'\big)\big([U_{\mathcal{P}}\times_P X]\otimes_{\mathbb{C}}[kU_{\mathcal{Q}}\otimes_{kQ} V]\big)\\
&\quad=\big[k(U_{\mathcal{P}}\times_P X)\otimes_k(kU_{\mathcal{Q}}\otimes_{kQ} V)\big]\\
&\quad=\big[(kU_{\mathcal{P}}\otimes_{kP} kX)\otimes_k(kU_{\mathcal{Q}}\otimes_{kQ} V)\big].
\end{aligned}
$$

Given the definition of the action of $P \times Q$ on $kU_{\mathcal{P}} \otimes_k kU_{\mathcal{Q}}$ and $kX \otimes_k V$, it is easy to check that there is an isomorphism of $k(P' \times Q')$-modules

$$(kU_{\mathcal{P}} \otimes_k kU_{\mathcal{Q}}) \otimes_{k(P\times Q)} (kX \otimes_k V) \cong (kU_{\mathcal{P}} \otimes_{kP} kX) \otimes_k (kU_{\mathcal{Q}} \otimes_{kQ} V).$$

This completes the proof of Theorem 39.

Now on $\mathcal{C}_{p'}$, $\mathbb{C}R_k$ decomposes as a direct sum of simple functors:

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi},$$

where (m,ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$. Consequently, on $\mathcal{C}_{p\times p'}$, we have the following decomposition:

$$\mathbb{C}\,\mathrm{pp}_k \cong \bigoplus_{(m,\xi)} \mathbb{C}B \otimes_{\mathbb{C}} S_{C_m,\mathbb{C}_\xi},$$

where (m,ξ) runs over the same set of pairs. So, to find the composition factors of $\mathbb{C}\,\mathrm{pp}_k$ on $\mathcal{C}_{p\times p'}$, it is enough to find the composition factors of $\mathbb{C}B \otimes_{\mathbb{C}} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ on $\mathcal{C}_{p\times p'}$, for every positive integer m prime to p and for every primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$. By 5.6.9 “The case of p-groups”, p. 94 of [5], we have the following exact sequence, on $\mathcal{C}_p$:

$$0 \to S_{C_p\times C_p,\mathbb{C}} \to \mathbb{C}B \to S_{\mathbf{1},\mathbb{C}} \to 0.$$

But now, applying Proposition 25, we have the following exact sequence (on $\mathcal{C}_{p\times p'}$):

$$0 \to S_{C_p\times C_p,\mathbb{C}} \otimes_{\mathbb{C}} S_{C_m,\mathbb{C}_\xi} \to \mathbb{C}B \otimes_{\mathbb{C}} S_{C_m,\mathbb{C}_\xi} \to S_{\mathbf{1},\mathbb{C}} \otimes_{\mathbb{C}} S_{C_m,\mathbb{C}_\xi} \to 0.$$

We can now apply Theorem 27 and we obtain the following exact sequence:

$$0 \to S_{C_p\times C_p\times C_m,\mathbb{C}_\xi} \to \mathbb{C}B \otimes_{\mathbb{C}} S_{C_m,\mathbb{C}_\xi} \to S_{C_m,\mathbb{C}_\xi} \to 0$$

on $\mathcal{C}_{p\times p'}$, where the $\mathbb{C}\,\mathrm{Out}(C_p \times C_p \times C_m)$-module $\mathbb{C}_\xi$ is defined as follows: the action of $\mathrm{Out}(C_p \times C_p)$ is trivial and $\mathrm{Out}(C_m)$ acts through ξ (by Remark 26 $\mathrm{Out}(C_p \times C_p \times C_m) \cong \mathrm{Out}(C_p \times C_p) \times \mathrm{Out}(C_m)$, so we can describe the two actions separately).

Now the composition factors of $\mathbb{C}B \otimes_{\mathbb{C}} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}$ on $\mathcal{C}_{p\times p'}$ are $S_{C_p\times C_p\times C_m,\mathbb{C}_\xi}$ and $S_{C_m,\mathbb{C}_\xi}$. Consequently, we have proved the following theorem:

Theorem 43. *The composition factors of $\mathbb{C}\,\mathrm{pp}_k$ on $\mathcal{C}_{p\times p'}$ are the simple functors $S_{C_p\times C_p\times C_m,\mathbb{C}_\xi}$ and $S_{C_m,\mathbb{C}_\xi}$, where (m,ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$. Their multiplicity as composition factor is* 1.

As corollary, we obtain:

Corollary 44. *If G is a finite abelian group and V is a simple $\mathbb{C}\,\mathrm{Out}(G)$-module then $S_{G,V}$ is a composition factor of $\mathbb{C}\,\mathrm{pp}_k$ on $\mathbb{C}\underline{\mathrm{GrB}}$ if and only if there exists a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \to \mathbb{C}^\star$ such that $G \cong C_m$ or $G \cong C_p \times C_p \times C_m$, where V is the module $\mathbb{C}_\xi$. Moreover, the multiplicity of $S_{G,V}$ as composition factor is* 1.

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References

[1] M. Baumann, The composition factors of the functor of permutation modules, J. Algebra 344 (2011) 284–295.
[2] M. Baumann, Le foncteur de bi-ensembles des modules de p-permutation, PhD thesis, Ecole Polytechnique Fédérale de Lausanne, 2012.
[3] S. Bouc, Foncteurs d'ensembles munis d'une double action, J. Algebra (1996) 664–736.
[4] S. Bouc, Résolutions de foncteurs de Mackey, Proc. Sympos. Pure Math. 63 (1998) 31–83.
[5] S. Bouc, Biset Functors for Finite Groups, Springer, Berlin, 2010.
[6] M. Broué, On Scott modules and p-permutation modules: An approach through the Brauer morphism, Proc. Amer. Math. Soc. 93 (1985) 401–408.
[7] C.W. Curtis, I. Reiner, Methods of Representation Theory: With Applications to Finite Groups and Orders, vol. 1, Wiley, New York, 1981.
[8] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley, New York, 1988.
[9] J. Thévenaz, G-Algebras and Modular Representation Theory, Clarendon, Oxford, 1995.
[10] P. Webb, Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces, J. Pure Appl. Algebra (1993) 265–304.