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Group rings that are exact

Yaroslav Shitov

National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

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ABSTRACT

A ring R is left exact if, for every finitely generated left submodule $S \subset R^n$, every left R-linear function from S to R extends to a left R-linear function from R^n to R. The class of exact rings generalizes that of self-injective rings and has been introduced in a recent paper by Wilding, Johnson, and Kambites. In our paper we show that the group ring of a group G over a ring R is left exact if and only if R is left exact and G is locally finite.

1. Introduction

The word *ring* will stand throughout our paper for an associative ring with a unity (which is different from zero), and R will denote such a ring. Recall that R is called *left self-injective* if it is injective as a left module over itself, that is, any left R-linear function from a submodule X of a left R-module Y to R can be extended to a left R-linear function from Y to R.

In a recent paper [3] Wilding, Johnson, and Kambites introduced a more general class of rings which they called *exact* rings. For any m-by-n matrix A over R, they define the left R-module $\mathrm{Row}(A)$ as the set of all vectors $x \in R^{1\times n}$ that can be written as

E-mail address: yaroslav-shitov@yandex.ru.

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$x = uA$ for some vector $u \in R^{1\times m}$. The right R-module $\mathrm{Col}(A)$ is defined as the set of all vectors $y \in R^{m\times 1}$ that have the form $y = Av$ for some $v \in R^{n\times 1}$. As Theorem 3.4 of [3] shows, the ring R is *exact* if and only if, for every matrix $A \in R^{m\times n}$, every left R-linear function from $\mathrm{Row}(A)$ to R extends to a left R-linear function from $R^{1\times n}$ to R, and every right R-linear function from $\mathrm{Col}(A)$ to R extends to a right R-linear function from $R^{m\times 1}$ to R.

In our paper, we will separate between the concepts of left and right exactness. Noting that a left (right, respectively) submodule $S \subset R^n$ is finitely generated if and only if $S = \mathrm{Row}(A)$ ($S = \mathrm{Col}(A)$, respectively) for some matrix A over R, we can rewrite the definition of exactness as follows.

Definition 1.1. A ring R is *left exact* if, for every finitely generated left submodule $S \subset R^n$ and every left R-linear function $\varphi : S \to R$, there is a left R-linear function $\psi : R^n \to R$ satisfying $\varphi(x) = \psi(x)$ for any $x \in S$.

Replacing the word 'left' by 'right' everywhere in Definition 1.1, we obtain the definition of right exactness. As noted above, the authors of [3] call a ring exact if and only if is both left and right exact in our notation. Similarly, we will call a ring *self-injective* if it is both left and right self-injective.

As Definition 1.1 shows, we can get the definition of exactness from that of self-injectivity by requiring X to be a finitely generated submodule of a free module $Y = R^n$. So any left self-injective ring must also be left exact, and in [3] the question has been asked whether the converse is true or not.

Question 1.2. (See [3, Section 8].) Does there exist a ring which is exact but not self-injective?

In our paper we examine the concept of exactness by applying it to studying the group rings. Recall that the group ring of a group G over R is the set $R[G]$ of all formal sums $\rho = \sum_{h\in G} \rho_h\, h$ whose *coefficients* ρ_h belong to R and whose *support* (the set of all $h \in G$ satisfying $\rho_h \neq 0$) is finite. The sum $\rho' + \rho''$ is defined by $[\rho' + \rho'']_h = \rho'_h + \rho''_h$, and the product $\rho'\rho''$ by $[\rho'\rho'']_h = \sum_{g\in G} \rho'_{g^{-1}}\rho''_{gh}$. An element $\rho_e \in R$ is called the *constant term* of ρ if e is the unity of G, and we identify the subring of $R[G]$ formed by the constant terms with the ring R.

The goal of our paper is to obtain a characterization of exact group rings similar to the following which is known for self-injective rings.

Theorem 1.3. *(See [1].) Let $\mathbb{F}$ be a field and G a group. The ring $\mathbb{F}[G]$ is left self-injective if and only if G is finite.*

Namely, the main result of our paper will state that $R[G]$ is left exact if and only if R is left exact and G is locally finite. This result will imply a positive answer for

Question 1.2, and we will also provide an explicit example of a ring which is exact but neither left nor right self-injective.

2. Group rings of locally finite groups

Recall that a group G is called *locally finite* if any finite subset of G generates a finite subgroup in G. We will show that the group ring of a locally finite group over a left exact ring is left exact itself. The following lemma shows how exactness can be characterized in terms of finitely generated subrings.

Lemma 2.1. *Assume that, for every finite subset F of R, there is a left exact ring S which includes F and is a subring of R. Then R is left exact.*

Proof. Consider a left R-module M generated by vectors $v_1, \dots, v_k$ from $R^{n\times 1}$ and a left R-linear function $\varphi : M \to R$. By the assumption of the lemma, R has a left exact subring S that contains, for any i, all the coordinates of v_i and the element $\varphi(v_i)$. Since S is left exact, the restriction of φ to the S-module $Sv_1 + \dots + Sv_k$ extends to a left S-linear function $\psi : S^{n\times 1} \to S$.

Denote the element $\psi(0, \dots, 1, \dots, 0)$ (where unity is in the jth position) by σ_j and set $\sigma = (\sigma_1, \dots, \sigma_n)$. Then we have $\psi(x) = \sigma \cdot x$ for every $x \in S^n$, and also $\psi(v_i) = \varphi(v_i)$ for every i. Now let us define $\Psi(y)$ as $\sigma \cdot y$ for every $y \in R^n$; then we have $\Psi(v_i) = \psi(v_i) = \varphi(v_i)$ for every i. Since Ψ is left R-linear by its definition, we see that also $\Psi(w) = \varphi(w)$ for every $w \in M$. $\square$

From Theorem 5.3 of [3] it follows that the group ring of a finite group over an exact (that is, both left and right exact) ring is exact itself. The following lemma generalizes this statement and demonstrates that it is actually possible to prove separate versions for left and right exactness.

Lemma 2.2. *If R is a left exact ring and G a finite group, then $R[G]$ is left exact.*

Proof. *Step 1*. Denote $\mathcal{R} = R[G]$. Consider a left $\mathcal{R}$-module S generated by a finite set Σ of vectors from $\mathcal{R}^n$ and a left $\mathcal{R}$-linear function $\varphi : S \to \mathcal{R}$.

Step 2. Denote by $\varphi_1 : S \to R$ the function sending a vector $x \in S$ to the constant term of $\varphi(x)$. Since φ is left R-linear, so is φ_1.

Step 3. Since φ is left $\mathcal{R}$-linear we have $[\varphi(x)]_h = \varphi_1(h^{-1}x)$ for each $h \in G$, and as such $\varphi(x) = \sum_{h\in G} \varphi_1(h^{-1}x)\, h$ for every $x \in S$.

Step 4. Note that the vectors $g\,v$ (where g runs over G and v over Σ) generate S as an R-module, and that $\mathcal{R}^n$ is isomorphic to $R^{|G|n}$ as an R-module. Since R is left exact, there is then a left R-linear function $\psi_1 : \mathcal{R}^n \to R$ satisfying $\varphi_1(x) = \psi_1(x)$ for every $x \in S$.

Step 5. Define the function $\Psi : \mathcal{R}^n \to \mathcal{R}$ by $\Psi(y) = \sum_{h\in G} \psi_1(h^{-1}y)\, h$.

Step 6. Let us note that, for a fixed element $g \in G$, it holds that

$$\Psi(gy) = \sum_{h\in G} \psi_1\bigl(h^{-1}gy\bigr)h = \sum_{\gamma\in G} \psi_1\bigl(\gamma^{-1}y\bigr)g\gamma = g\Psi(y).$$

Step 7. By Step 4, ψ_1 is left R-linear, and then so is Ψ by Step 5. From Step 6 it then follows that Ψ is left $\mathcal{R}$-linear. Steps 4 and 5 also show that $\Psi(x) = \sum_{h\in G} \varphi_1(h^{-1}x)\,h$ for $x \in S$, and so we have $\Psi(x) = \varphi(x)$ for every $x \in S$ by Step 3. □

Lemma 2.1 and Lemma 2.2 imply the main result of this section.

Theorem 2.3. *Let R be a left exact ring and G a locally finite group. Then $R[G]$ is left exact.*

Proof. For any finite subset $S \subset R[G]$, the union J of all the supports of elements from S is a finite set. By the assumption of the theorem, J generates a finite subgroup G_0 in G, so the subring $R[G_0]$ of $R[G]$ is left exact by Lemma 2.2. Since $R[G_0]$ includes S, the result follows from Lemma 2.1. □

3. The main results

Using the technique similar to that developed in [1] we will prove that a group G is necessarily locally finite if $R[G]$ is left exact. The following lemma is fairly easy and has been used in [1] without a proof; we provide its proof for completeness.

Lemma 3.1. *Let G' be a subset of a group G. If there is some nonzero $a \in R[G]$ satisfying $a(1 - g') = 0$ for all $g' \in G'$ then the subgroup Γ generated by G' is finite.*

Proof. Let $\Sigma \subset G$ be the support of a, then Σ has finite cardinality n. For any $g \in G$, consider a function $\varphi_g : \Sigma \to G$ sending an $h \in \Sigma$ to hg. If $a = ag$ then the supports of ag and ag^{-1} both equal Σ, and so in this case φ_g is a permutation on Σ. Therefore φ_γ is a permutation on Σ for every $\gamma \in \Gamma$. It is clear that $\varphi_{\gamma'} = \varphi_{\gamma''}$ implies $\gamma' = \gamma''$, so the cardinality of Γ does not exceed $n!$. □

To prove the main result, we will also need a description of the annihilators of finitely generated ideals of exact rings. The following lemma has been obtained as a generalization of Lemma 3 from [2], but I subsequently learned that it can be deduced from the considerations of Section 4 from [3].

Lemma 3.2. *If R is a left exact ring, then every proper finitely generated right ideal has a nonzero left annihilator.*

Proof. Consider a right ideal $I = s_1R + \cdots + s_nR \subset R$ and the left submodule $S \subset R^n$ consisting of the vectors $(rs_1, \ldots, rs_n)$ with $r \in R$. Define the function ϕ from R to S by $\phi(r) = (rs_1, \ldots, rs_n)$.

If the left annihilator of I is $\{0\}$, then ϕ is an isomorphism of left R-modules. In this case, the inverse function ϕ^{-1} extends to a left R-linear function $\psi : R^n \to R$ since R is left exact. In particular, we have $\psi(s_1, \ldots, s_n) = \psi(\phi(1)) = 1$. Denoting the element $\psi(0, \ldots, 1, \ldots, 0)$ (where unity is in the jth position) by σ_j, we get also $\psi(s_1, \ldots, s_n) = s_1\sigma_1 + \cdots + s_n\sigma_n \in I$, so that $1 \in I$ and $I = R$. □

Lemmas 3.1 and 3.2 imply the converse of the result of Theorem 2.3.

Theorem 3.3. *If $R[G]$ is left exact, then G is locally finite.*

Proof. For $G' = \{g_1, \ldots, g_n\}$ a finite subset of G, consider the right ideal $(1 - g_1)R[G] + \cdots + (1 - g_n)R[G]$ (which is proper being a subset of the augmentation ideal). By Lemma 3.2, there is a nonzero $a \in R[G]$ satisfying $a(1 - g') = 0$ for every $g \in G'$. The subgroup generated by G' is then finite by Lemma 3.1. □

It also turns out that if $R[G]$ is left exact, then so is R.

Lemma 3.4. *If $R[G]$ is left exact, then R is left exact.*

Proof. Consider a left R-module S generated by vectors $s_1, \ldots, s_k$ from R^n and a left R-linear function $\varphi : S \to R$. Denote $\mathcal{R} = R[G]$ and let $\mathcal{S} \subset \mathcal{R}^n$ be the left $\mathcal{R}$-module generated by $s_1, \ldots, s_k$ over $\mathcal{R}$. Then the function $\Phi : \mathcal{S} \to \mathcal{R}$ sending $\sum_{g\in G} g\sigma_g$ to $\sum_{g\in G} \varphi(\sigma_g)g$ is left $\mathcal{R}$-linear (here σ_g denote arbitrary elements from S). Since $\mathcal{R}$ is left exact, there is a left $\mathcal{R}$-linear function $\Psi : \mathcal{R}^n \to \mathcal{R}$ which coincides with Φ on $\mathcal{S}$. Let $\psi : R^n \to R$ be the function sending $r \in R^n$ to the sum of the coefficients of $\Psi(r)$. Since Ψ is left R-linear, so is ψ. For $s \in S$ we have $\Psi(s) = \Phi(s) = \varphi(s) \in R$, so that $\psi(s) = \varphi(s)$. □

Theorems 2.3 and 3.3 together with Lemma 3.4 imply the main result of our paper.

Theorem 3.5. *$R[G]$ is left exact if and only if R is left exact and G is locally finite.*

Proof. If $R[G]$ is left exact, then G is locally finite by Theorem 3.3 and R is left exact by Lemma 3.4. On the other hand, Theorem 2.3 shows that $R[G]$ is always left exact when R is left exact and G is locally finite. □

Let us also show how Theorems 1.3 and 3.5 allow us to obtain a positive answer for Question 1.2. Basic results of linear algebra imply that any field $\mathbb{F}$ is exact as a ring; consider also any infinite direct sum G of finite groups, which is locally finite. Then the group ring $\mathbb{F}[G]$ is both left and right exact by Theorem 3.5 and neither left nor right self-injective as Theorem 1.3 shows.

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