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The category of F-modules has finite global dimension

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ABSTRACT

Let R be a regular ring of characteristic $p > 0$. In [4], Hochster showed that the category of Lyubeznik's F_R-modules has enough injectives, so that every F_R-module has an injective resolution in this category. We show in this paper that under mild conditions on R, for example when R is essentially of finite type over an F-finite regular local ring, the category of F-modules has finite global dimension $d+1$ where $d = \dim R$. In [4], Hochster also showed that when M and N are F_R-finite F_R-modules, $\mathrm{Hom}_{F_R}(M, N)$ is finite. We show that in general $\mathrm{Ext}^1_{F_R}(M, N)$ is not necessarily finite.

1. Introduction

In [4], Hochster showed some properties of Lyubeznik's F-modules:

Theorem 1.1. *(See Theorem 3.1 in [4].) The category of F_R-modules over a Noetherian regular ring R of prime characteristic $p > 0$ has enough injectives, i.e., every F_R-module can be embedded in an injective F_R-module.*

Theorem 1.2. *(See Theorem 5.1 and Corollary 5.2(b) in [4].) Let R be a Noetherian regular ring of prime characteristic $p > 0$. Let M and N be F_R-finite F_R-modules. Then $\mathrm{Hom}_{F_R}(M, N)$ is a finite-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ and, hence, is a finite set. Moreover, when R is local, every F_R-finite F_R-module has only finitely many F_R-submodules.*

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The main purpose of this paper is to get some further results based on Hochster's results. In connection with Theorem 1.1, we prove the following (this can be viewed as an analogue of the corresponding statement for $\mathscr{D}$-modules in characteristic 0):

Theorem 1.3. *Let R be an F-finite regular ring of characteristic $p > 0$ such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$ (this holds if R is essentially of finite type over an F-finite regular local ring). Then the category of F_R-modules has finite global dimension $d+1$ where $d = \dim R$.*

Theorem 1.2 makes it quite natural to ask whether the higher Ext groups are also finite in this category (when M and N are F_R-finite F_R-modules). We show that in general this fails even for Ext^1:

Example 1.4. Let $(R, \mathfrak{m}, K)$ be a regular local ring of characteristic $p > 0$ and dimension $d \geqslant 1$, and let $E = E(R/\mathfrak{m})$ be the injective hull of the residue field. Then $\mathrm{Ext}^1_{F_R}(R, E) \neq 0$. Moreover, when K is infinite, $\mathrm{Ext}^1_{F_R}(R, E)$ is also infinite. In particular, E is not injective in the category of F_R-modules.

This paper is organized as follows. In Section 2 we review the definitions and basic properties of right $R\{F\}$-modules (i.e., Cartier modules) and Lyubeznik's F_R-modules, and we introduce the notion of *unit right $R\{F\}$-modules* which is motivated by the ideas in [2] and [1]. In Section 3 we prove Theorem 1.3, and we also obtain some results of independent interest on right $R\{F\}$-modules and unit right $R\{F\}$-modules. In Section 4 we show some (non)finiteness results on $\mathrm{Ext}^1_{F_R}(M, N)$ when M and N are F_R-finite F_R-modules. Examples will be given throughout.

2. Preliminaries

Throughout this paper, R will always denote a Noetherian regular ring of characteristic $p > 0$ and dimension d. We use $R^{(e)}$ to denote the target ring of the e-th Frobenius map $F^e : R \to R$. When M is an R-module and $x \in M$ is an element, we use $M^{(e)}$ to denote the corresponding module over $R^{(e)}$ and $x^{(e)}$ to denote the corresponding element in $M^{(e)}$. We shall let $F^e(-)$ denote the Peskine–Szpiro's Frobenius functor from R-modules to R-modules. In detail, $F^e(M)$ is given by base change to $R^{(e)}$ and then identifying $R^{(e)}$ with R. Note that by Kunz's result [5], we know that $R^{(e)}$ is faithfully flat as an R-module. We say R is *F-finite* if $R^{(1)}$ is finitely generated as an R-module. So for an F-finite regular ring, $R^{(1)}$ (and hence $R^{(e)}$ for every e) is finite and projective as an R-module.

We use $R\{F\}$ to denote the Frobenius skew polynomial ring, which is the noncommutative ring generated over R by the symbols $1, F, F^2, \ldots$ by requiring that $Fr = r^pF$ for $r \in R$. Note that $R\{F\}$ is always free as a left R-module and flat as a right R-module. When R is F-finite, $R\{F\}$ is projective as a right R-module (because $R^{(1)}$ is projective in this case). We say an R-module M is a *right* $R\{F\}$*-module* if it is a right module over the ring $R\{F\}$, or equivalently, there exists a morphism $\phi : M \to M$ such that for all $r \in R$ and $x \in M$, $\phi(r^p x) = r\phi(x)$ (the right action of F can be identified with ϕ). This morphism can be also viewed as an R-linear map $\phi : M^{(1)} \to M$. We note that a right $R\{F\}$-module is the same as a *Cartier module* defined in [1] (where it is defined for general Noetherian rings and schemes of characteristic $p > 0$).

We collect some definitions from [6]. These are the main objects that we shall study in this paper.

Definition 2.1. (See Definition 1.1 in [6].) An F_R*-module* is an R-module M equipped with an R-linear isomorphism $\theta : M \to F(M)$ which we call the structure morphism of M. A homomorphism of F_R-modules is an R-module homomorphism $f : M \to M'$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\theta"] & M' \arrow[d, "\theta'"] \\ F(M) \arrow[r, "F(f)"] & F(M') \end{tikzcd}$$

Definition 2.2. (See Definition 1.9 and Definition 2.1 in [6].) A generating morphism of an F_R-module M is an R-module homomorphism $\beta : M_0 \to F(M_0)$, where M_0 is some R-module, such that M is the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd} M_0 \arrow[r, "\beta"] \arrow[d, "\beta"] & F(M_0) \arrow[r, "F(\beta)"] \arrow[d, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] \arrow[d, "F^2(\beta)"] & \cdots \\ F(M_0) \arrow[r, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] & F^3(M_0) \arrow[r, "F^3(\beta)"] & \cdots \end{tikzcd}$$

and $\theta : M \to F(M)$, the structure isomorphism of M, is induced by the vertical arrows in this diagram. An F_R-module M is called F_R*-finite* if M has a generating morphism $\beta : M_0 \to F(M_0)$ with M_0 a finitely generated R-module.

Now we introduce the notion of *unit right* $R\{F\}$*-modules* which is an analogue of *unit left* $R\{F\}$*-modules* in [2]. This is a key concept in relating Lyubeznik's F_R-modules with right $R\{F\}$-modules. The ideas can be also found in Section 5.2 in [1]. We first recall the functor $F^!(-)$ in the case that R is regular and F-finite: for any R-module M, $F^!(M)$ is the R-module obtained by first considering $\mathrm{Hom}_R(R^{(1)}, M)$ as an $R^{(1)}$-module and then identifying $R^{(1)}$ with R. Remember that giving an R-module M a right $R\{F\}$-module structure is equivalent to giving an R-linear map $M^{(1)} \to M$. But this is the same as giving an $R^{(1)}$-linear map $M^{(1)} \to \mathrm{Hom}_R(R^{(1)}, M)$. Hence after identifying $R^{(1)}$ with R, we find that giving M a right $R\{F\}$-module structure is equivalent to giving a map $\tau : M \to F^! M$. Moreover, it is straightforward to check that a homomorphism of right $R\{F\}$-modules is an R-module homomorphism $g : M \to M'$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\tau"] & M' \arrow[d, "\tau'"] \\ F^! M \arrow[r, "F^!(g)"] & F^! M' \end{tikzcd}$$

Definition 2.3. A unit right $R\{F\}$-module is a right $R\{F\}$-module M such that the structure map $\tau : M \to F^! M$ is an isomorphism.

Remark 2.4. Similarly to Definition 2.2, we introduce the notion of generating morphism of unit right $R\{F\}$-modules. Let M_0 be a right $R\{F\}$-module with structure morphism $\tau_0 : M_0 \to F^!(M_0)$. Let M be the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd} M_0 \arrow[r, "\tau_0"] \arrow[d, "\tau_0"] & F^!(M_0) \arrow[r, "F^!(\tau_0)"] \arrow[d, "F^!(\tau_0)"] & (F^!)^2(M_0) \arrow[r, "(F^!)^2(\tau_0)"] \arrow[d, "(F^!)^2(\tau_0)"] & \cdots \\ F^!(M_0) \arrow[r, "F^!(\tau_0)"] & (F^!)^2(M_0) \arrow[r, "(F^!)^2(\tau_0)"] & (F^!)^3(M_0) \arrow[r, "(F^!)^3(\tau_0)"] & \cdots \end{tikzcd}$$

Since R is F-finite, it is easy to see that $F^!(-)$ commutes with direct limit. Hence $\tau : M \to F^! M$ induced by the vertical arrows in the above diagram is an isomorphism. M is a unit right $R\{F\}$-module.

For an F-finite regular ring R, any rank 1 projective module is a canonical module ω_R of R (we refer to [3] for a detailed definition of canonical module and dualizing complex). When R is local, $\omega_R = R$ is unique. It is easy to see that $F^! \omega_R$ is always a canonical module of R (see [3] for more general results). However, to the best of our knowledge, it is still unknown whether there always exists ω_R such that $F^! \omega_R \cong \omega_R$ for F-finite regular ring R. Nonetheless, it is true if either R is essentially of finite type over an F-finite regular local ring or R is sufficiently affine. We refer to Proposition 2.20 and Proposition 2.21 in [1] as well as [3] for more details on this question.

The next theorem is well known. It follows from duality theory in [3]. In the context of the Frobenius morphism it is explained in [1]. Since we need to use this repeatedly throughout the article, we give a short proof for completeness.

Theorem 2.5. *Let R be an F-finite regular ring such that there exists a canonical module ω_R with $F^! \omega_R \cong \omega_R$. Then the category of unit right $R\{F\}$-modules is equivalent to* *the category of F_R-modules. Moreover, the equivalence is given by tensoring with ω_R^{-1}, its inverse by tensoring with ω_R.*

Proof. We first note that, for any R-module M,

$$\big(\omega_R^{-1}\big)^{(1)} \otimes_{R^{(1)}} \operatorname{Hom}_R\big(R^{(1)}, M\big) \cong \big(\omega_R^{-1}\big)^{(1)} \otimes_{R^{(1)}} \operatorname{Hom}_R\big(R^{(1)}, \omega_R\big) \otimes_R \big(\omega_R^{-1} \otimes_R M\big).$$

Hence after identifying $R^{(1)}$ with R, the above equality becomes

$$\omega_R^{-1} \otimes_R F^! M \cong \omega_R^{-1} \otimes_R F^! \omega_R \otimes_R F\big(\omega_R^{-1} \otimes_R M\big) \cong F\big(\omega_R^{-1} \otimes_R M\big)$$

where the last equality is by our assumption $F^!\omega_R \cong \omega_R$. Now for any unit right $R\{F\}$-module M, we have an isomorphism $M \xrightarrow{\tau} F^! M$. Hence after tensoring with ω_R^{-1}, we get $\omega^{-1} \otimes_R M \xrightarrow{id \otimes_R \tau} \omega^{-1} \otimes_R F^! M \cong F(\omega_R^{-1} \otimes_R M)$. This shows that $\omega_R^{-1} \otimes_R M$ is an F_R-module with structure morphism θ given by $id \otimes_R \tau$. The converse can be proved similarly. $\square$

Throughout the rest of the paper, we will use Ext^i_R, $\operatorname{Ext}^i_{R\{F\}}$, $\operatorname{Ext}^i_{uR\{F\}}$, and $\operatorname{Ext}^i_{F_R}$ (respectively, id_R, $\mathrm{id}_{R\{F\}}$, $\mathrm{id}_{uR\{F\}}$, id_{F_R}) to denote the i-th Ext group (respectively, the injective dimension) computed in the category of R-modules, right $R\{F\}$-modules, unit right $R\{F\}$-modules, and F_R-modules.

We end this section by studying some examples of F_R-modules. The simplest example of an F_R-module is R equipped with structure isomorphism the identity map, that is, sending 1 in R to 1 in $F(R) \cong R$. Note that this corresponds to the unit right $R\{F\}$-module $\omega_R \cong F^!\omega_R$ under Theorem 2.5. Another important example is $E = E(R/\mathfrak{m})$, the injective hull of $R/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of R. We can give it a generating morphism $\beta : R/\mathfrak{m} \to F(R/\mathfrak{m})$ by sending $\bar{1}$ to $\overline{x_1^{p-1} \cdots x_d^{p-1}}$ (where $x_1, \ldots, x_d$ represents minimal generators of $\mathfrak{m}R_\mathfrak{m}$). We will call these structure isomorphisms of R and E the *standard* F_R-module structures on R and E. Note that in particular R and E with the standard F_R-module structures are F_R-finite F_R-modules. Now we provide a nontrivial example of an F_R-module:

Example 2.6. Let $R^\infty := \bigoplus_{i \in \mathbb{Z}} Rz_i$ denote the infinite direct sum of copies of R equipped with the F_R-module structure by setting

$$\theta : z_i \to z_{i+1}.$$

Then R^∞ is *not* F_R-finite. It is easy to see that we have a short exact sequence of F_R-modules:

$$0 \to R^\infty \xrightarrow{z_i \mapsto z_i - z_{i+1}} R^\infty \xrightarrow{z_i \mapsto 1} R \to 0$$

where the last R is equipped with the standard F_R-module structure.

We want to point out that the above sequence does not split in the category of F_R-modules. Suppose $g : R \to R^\infty$ is a splitting, say $g(1) = \{y_j\}_{j\in\mathbb{Z}} \neq 0$. Then a direct computation shows that $\theta(\{y_j\}) = \{y_j^p\}$, which is impossible by the definition of θ. Hence, by Yoneda's characterization of Ext groups, we know that $\mathrm{Ext}^1_{F_R}(R, R^\infty) \neq 0$.

3. The global dimension of Lyubeznik's F-modules

Our goal in this section is to prove Theorem 1.3. First we want to show that, when R is F-finite, the category of right $R\{F\}$-modules has finite global dimension $d+1$. We start with a lemma which is an analogue of Lemma 1.8.1 in [2].

Lemma 3.1. *Let R be a regular ring and let M be a right $R\{F\}$-module, so that there is an R-linear map $\phi : M^{(1)} \to M$ (so for every i, we get an R-linear map $\phi^i : M^{(i)} \to M$ by composing ϕ i times). Then we have an exact sequence of right $R\{F\}$-modules*

$$0 \to M^{(1)} \otimes_R R\{F\} \xrightarrow{\alpha} M \otimes_R R\{F\} \xrightarrow{\beta} M \to 0$$

where for every $x^{(1)} \in M^{(1)}$,

$$\alpha\big(x^{(1)} \otimes F^i\big) = \phi\big(x^{(1)}\big) \otimes F^i - x \otimes F^{i+1}$$

and for every $y \in M$,

$$\beta\big(y \otimes F^i\big) = \phi^i\big(y^{(i)}\big).$$

Proof. It is clear that every element in $M^{(1)} \otimes_R R\{F\}$ (resp. $M \otimes_R R\{F\}$) can be written uniquely as a finite sum $\sum x_i^{(1)} \otimes F^i$ where $x_i^{(1)} \in M^{(1)}$ (resp. $x_i \in M$) because $R\{F\}$ is free as a left R-module (this verifies that our maps α and β are well-defined). It is straightforward to check that α, β are morphisms of right $R\{F\}$-modules and that $\beta \circ \alpha = 0$ and β is surjective (because $\beta(y \otimes 1) = \phi^0(y) = y$). So it suffices to show α is injective and $\ker(\beta) \subseteq \mathrm{im}(\alpha)$.

Suppose $\alpha(\sum x_i^{(1)} \otimes F^i) = 0$. By definition of α we get $\sum(\phi(x_i^{(1)}) - x_{i-1}) \otimes F^i = 0$. Hence by uniqueness we get $\phi(x_i^{(1)}) = x_{i-1}$ for all i. Hence $x_i = 0$ for all i (because it is a finite sum). This proves α is injective.

Now suppose $\beta(\sum_{i=0}^n y_i \otimes F^i) = 0$. We want to find x_i such that

$$\alpha\left(\sum_{i=0}^n x_i^{(1)} \otimes F^i\right) = \sum_{i=0}^n y_i \otimes F^i. \tag{3.1.1}$$

By definition of β we know that $\sum_{i=0}^n \phi^i(y_i^{(i)}) = 0$. Now one can check that

$$\begin{aligned}
x_0 &= -\big(y_1 + \phi\big(y_2^{(1)}\big) + \cdots + \phi^{n-1}\big(y_n^{(n-1)}\big)\big), \\
x_2 &= -\big(y_2 + \phi\big(y_3^{(1)}\big) + \cdots + \phi^{n-2}\big(y_n^{(n-2)}\big)\big), \\
&\ \vdots \\
x_{n-1} &= -y_n, \\
x_n &= 0,
\end{aligned}$$

is a solution of (3.1.1). This proves $\ker(\beta) \subseteq \operatorname{im}(\alpha)$. □

In [2], a similar two-step resolution is proved for left $R\{F\}$-modules (see Lemma 1.8.1 in [2]). And using the two-step resolution it is proved in [2] that the category of left $R\{F\}$-modules has Tor-dimension at most $d+1$ (see Corollary 1.8.4 in [2]). We want to mimic the strategy and prove the corresponding results for right $R\{F\}$-modules. And we can actually improve the result: we show that when R is F-finite, the category of right $R\{F\}$-modules has finite *global* dimension *exactly* $d+1$.

Theorem 3.2. *Let R be an F-finite regular ring of dimension d. Then the category of right $R\{F\}$-modules has finite global dimension $d+1$.*

Proof. We first note that for every right $R\{F\}$-module M with structure map $\tau : M \to F^!M$, a projective resolution of M *in the category of R-modules* can be given a structure of right $R\{F\}$-modules such that it becomes an exact sequence of right $R\{F\}$-modules. This is because we can lift the natural map $\tau : M \to F^!M$ to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_k \arrow[r] \arrow[d, dotted] & P_{k-1} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dotted] & P_0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & F^!(P_k) \arrow[r] & F^!(P_{k-1}) \arrow[r] & \cdots \arrow[r] & F^!(P_1) \arrow[r] & F^!(P_0) \arrow[r] & F^!(M) \arrow[r] & 0
\end{tikzcd}$$

because we can always lift a map from a complex of projective modules to an acyclic complex ($F^!(-)$ is an exact functor when R is F-finite).

By Lemma 3.1, we have an exact sequence of right $R\{F\}$-modules

$$0 \to M^{(1)} \otimes_R R\{F\} \xrightarrow{\alpha} M \otimes_R R\{F\} \xrightarrow{\beta} M \to 0. \tag{3.2.1}$$

Now we tensor the above (3.2.1) with the projective resolution of M over R, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_k \arrow[r] & P_{k-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0 \\
0 \arrow[r] & P_k\otimes_R R\{F\} \arrow[r] \arrow[u] & P_{k-1}\otimes_R R\{F\} \arrow[r] \arrow[u] & \cdots \arrow[r] & P_1\otimes_R R\{F\} \arrow[r] \arrow[u] & P_0\otimes_R R\{F\} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_k^{(1)}\otimes_R R\{F\} \arrow[r] \arrow[u, "\alpha_k"] & P_{k-1}^{(1)}\otimes_R R\{F\} \arrow[r] \arrow[u, "\alpha_{k-1}"] & \cdots \arrow[r] & P_1^{(1)}\otimes_R R\{F\} \arrow[r] \arrow[u, "\alpha_1"] & P_0^{(1)}\otimes_R R\{F\} \arrow[r] \arrow[u, "\alpha_0"] & 0
\end{tikzcd} \tag{3.2.2}$$

The first line is a projective resolution of M over R. And by the above discussion we can give each P_i a right $R\{F\}$-module structure such that it is an exact sequence of right $R\{F\}$-modules. The second line (resp. the third line) is obtained from the first line by tensoring with $R\{F\}$ (resp. applying $^{(1)}$ and then tensoring with $R\{F\}$). Each column is the map described in Lemma 3.1. In particular, all columns are exact sequences of right $R\{F\}$-modules.

Let $C_\bullet$ be the complex of the third line and $D_\bullet$ be the complex of the second line of (3.2.2). The homology of the mapping cone of $C_\bullet \to D_\bullet$ is the same as the homology of the quotient complex $D_\bullet/C_\bullet$, which is the first line in (3.2.2). Hence the mapping cone is acyclic. Since each P_i is projective as an R-module, we know that P_i is a direct summand of a free R-module G. So $P_i^{(1)}$ is a direct summand of $G^{(1)}$. Since R is F-finite, $G^{(1)}$ is projective as an R-module, so $P_i^{(1)}$ is also projective as an R-module. Hence $P_i^{(1)}\otimes_R R\{F\}$ and $P_i \otimes_R R\{F\}$ are projective as right $R\{F\}$-modules for every i. Thus the mapping cone of $C_\bullet \to D_\bullet$ gives a right $R\{F\}$-projective resolution of M. We note that this resolution has length $k+1$. Since we can always take a projective resolution of M of length $k \leqslant d$, the right $R\{F\}$-projective resolution we obtained has length $\leqslant d+1$.

We have already seen that the global dimension of right $R\{F\}$-modules is $\leqslant d+1$. Now we let M and N be two right $R\{F\}$-modules with trivial right F-action (i.e., the structure maps of M and N are the zero maps). I claim that in this case, we have

$$\operatorname{Ext}^j_{R\{F\}}(M,N) = \operatorname{Ext}^j_R(M,N) \oplus \operatorname{Ext}^{j-1}_R\bigl(M^{(1)},N\bigr). \tag{3.2.3}$$

To see this, we look at (3.2.2) applied to M with trivial right F-action. It is clear that in this case each P_i in the first line of (3.2.2) also has trivial right $R\{F\}$-module structure. So as described in Lemma 3.1, we have

$$\alpha_j\bigl(x^{(1)} \otimes F^i\bigr) = -x \otimes F^{i+1} \tag{3.2.4}$$

for every $x^{(1)} \otimes F^i \in P_j^{(1)} \otimes_R R\{F\}$. The key observation is that, since N has trivial right F-action, when we apply $\operatorname{Hom}_{R\{F\}}(-,N)$ to $\alpha_j : P_j^{(1)} \otimes_R R\{F\} \to P_j \otimes_R R\{F\}$, the dual map $\alpha_j^\vee$ is the zero map (one can check this by a direct computation using (3.2.4)). Hence when we apply $\operatorname{Hom}_{R\{F\}}(-,N)$ to the mapping cone of $C_\bullet \to D_\bullet$, the j-th cohomology is the same as the direct sum of the j-th cohomology of $\mathrm{Hom}_{R\{F\}}(C_\bullet[-1], N)$ and the j-th cohomology of $\mathrm{Hom}_{R\{F\}}(D_\bullet, N)$. That is,

$$\mathrm{Ext}^j_{R\{F\}}(M, N) = H^j\big(\mathrm{Hom}_{R\{F\}}(D_\bullet, N)\big) \oplus H^j\big(\mathrm{Hom}_{R\{F\}}\big(C_\bullet[-1], N\big)\big). \tag{3.2.5}$$

But for every right $R\{F\}$-module N, $\mathrm{Hom}_{R\{F\}}(- \otimes_R R\{F\}, N) \cong \mathrm{Hom}_R(-, N)$. So applying $\mathrm{Hom}_{R\{F\}}(-, N)$ to $D_\bullet$ and $C_\bullet$ are the same as applying $\mathrm{Hom}_R(-, N)$ to

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to 0$$

and

$$0 \to P_k^{(1)} \to P_{k-1}^{(1)} \to \cdots \to P_0^{(1)} \to 0,$$

which are R-projective resolutions of M and $M^{(1)}$ respectively. Hence we know that

$$\begin{aligned} &H^j\big(\mathrm{Hom}_{R\{F\}}(D_\bullet, N)\big) \oplus H^j\big(\mathrm{Hom}_{R\{F\}}\big(C_\bullet[-1], N\big)\big) \\ &\quad = \mathrm{Ext}^j_R(M, N) \oplus \mathrm{Ext}^{j-1}_R\big(M^{(1)}, N\big). \end{aligned} \tag{3.2.6}$$

Now (3.2.3) follows from (3.2.5) and (3.2.6).

In particular, we can take two R-modules M and N such that $\mathrm{Ext}^d_R(M^{(1)}, N) \neq 0$ (for example, take $N = R$ and $M = R/(x_1, \dots, x_d)$ where $x_1, \dots, x_d$ is a regular sequence in R). Applying (3.2.3) to $j = d + 1$ gives

$$\mathrm{Ext}^{d+1}_{R\{F\}}(M, N) = \mathrm{Ext}^d_R\big(M^{(1)}, N\big) \neq 0.$$

Hence the global dimension of right $R\{F\}$-modules is at least $d+1$. Since we have already shown it is bounded by $d+1$, this completes the proof that the global dimension of right $R\{F\}$-modules is exactly $d + 1$. □

We can use the method in the proof of Theorem 3.2 to compute some Ext groups in the category of right $R\{F\}$-modules. Below we give an example which is a key ingredient when we show that the global dimension of F_R-module is $d + 1$.

Example 3.3. Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Let $\omega_R^\infty := \bigoplus_{j\in\mathbb{Z}} \omega_R z_j$ be an infinite direct sum of ω_R. We give ω_R^∞ the right $R\{F\}$-module structure by setting $\tau : \omega_R^\infty \to F^!(\omega_R^\infty) \cong \omega_R^\infty$ such that

$$\tau(yz_j) = yz_{j+1}$$

for every $y \in \omega_R$. It is clear that ω_R^∞ is in fact a unit right $R\{F\}$-module, and it corresponds to the F_R-module R^∞ described in Example 2.6 under Theorem 2.5.

Lemma 3.4. *With the same notations as in Example 3.3, we have*

$$\operatorname{id}_{R\{F\}} \omega_R^\infty = d+1. \tag{3.4.1}$$

Proof. We first notice that the right $R\{F\}$-module structure on ω_R defined by $\omega_R \cong F^!\omega_R$ induces a canonical map $\phi : \omega_R^{(1)} \to \omega_R$, which is a generator of the free $R^{(1)}$-module $\operatorname{Hom}_R(\omega_R^{(1)}, \omega_R) \cong R^{(1)}$. That is, any map in $\operatorname{Hom}_R(\omega_R^{(1)}, \omega_R)$ can be expressed as $\phi(r^{(1)} \cdot -)$ for some $r^{(1)} \in R^{(1)}$ (we refer to [1] for more details on this).

Next we fix $x_1, \ldots, x_d$ a regular sequence in R. We note that

$$\tilde{\phi} := \phi\big(\big(x_1^{(1)} \cdots x_d^{(1)}\big)^{p-1} \cdot -\big) \in \operatorname{Hom}_R\big(\omega_R^{(1)}, \omega_R\big)$$

satisfies $\tilde{\phi}((x_1^{(1)}, \ldots, x_d^{(1)})\omega_R^{(1)}) \subseteq (x_1, \ldots, x_d)\omega_R$, so it induces a map

$$\big(\omega_R/(x_1, \ldots, x_d)\omega_R\big)^{(1)} \to \omega_R/(x_1, \ldots, x_d)\omega_R.$$

That is, $\tilde{\phi}$ gives $\omega_R/(x_1, \ldots, x_d)\omega_R$ a right $R\{F\}$-module structure. It is clear that we can lift this map $\tilde{\phi}$ to the Koszul complex $K_\bullet(x_1, \ldots, x_d; \omega_R)$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & \omega_R^{(1)} \arrow[r] \arrow[d, dotted, "\phi"] & \cdots \arrow[r] & (\omega_R^d)^{(1)} \arrow[r] \arrow[d, dotted] & \omega_R^{(1)} \arrow[r] \arrow[d, "\tilde{\phi}"] & (\omega_R/(x_1, \ldots, x_d)\omega_R)^{(1)} \arrow[r] \arrow[d, "\tilde{\phi}"] & 0 \\
0 \arrow[r] & \omega_R \arrow[r] & \cdots \arrow[r] & \omega_R^d \arrow[r] & \omega_R \arrow[r] & \omega_R/(x_1, \ldots, x_d)\omega_R \arrow[r] & 0
\end{tikzcd}$$

Chasing through the diagram, one can check that the induced map on the last spot of the above commutative diagram is exactly the map ϕ (the generator of $\operatorname{Hom}_R(\omega_R^{(1)}, \omega_R)$).

Now we apply (3.2.2) to $M = \omega_R/(x_1, \ldots, x_d)\omega_R$ with structure map $\tilde{\phi}$ and let the first line in (3.2.2) be the Koszul complex $K_\bullet(x_1, \ldots, x_d; \omega_R)$. The above argument shows that the induced right $R\{F\}$-module structure on $P_d = \omega_R$ is given by the canonical map $\phi : \omega_R^{(1)} \to \omega_R$ (i.e., it corresponds to $\omega_R \cong F^!\omega_R$). As in Theorem 3.2, the mapping cone of $C_\bullet \to D_\bullet$ is a right $R\{F\}$-projective resolution of $\omega_R/(x_1, \ldots, x_d)\omega_R$ of length $d+1$, and the tail of this resolution is

$$0 \to \omega_R^{(1)} \otimes_R R\{F\} \xrightarrow{h} \omega_R \otimes_R R\{F\} \oplus \big(\omega_R^d\big)^{(1)} \otimes_R R\{F\} \to \cdots \tag{3.4.2}$$

where we have

$$h\big(y^{(1)} \otimes F^i\big) = (-1)^d\big(y \otimes F^{i+1} - \phi\big(y^{(1)}\big) \otimes F^i\big) \oplus \big(x_1^{(1)}y^{(1)}, \ldots, x_d^{(1)}y^{(1)}\big) \otimes F^i \tag{3.4.3}$$

for every $y \in \omega_R$. Now we apply $\operatorname{Hom}_{R\{F\}}(-, \omega_R^\infty)$ to (3.4.2) and identify $\operatorname{Hom}_R(-, \omega_R^\infty) = \operatorname{Hom}_{R\{F\}}(- \otimes_R R\{F\}, \omega_R^\infty)$, we get

$$0 \leftarrow \operatorname{Hom}_R\big(\omega_R^{(1)}, \omega_R^\infty\big) \xleftarrow{h^\vee} \operatorname{Hom}_R\big(\omega_R, \omega_R^\infty\big) \oplus \operatorname{Hom}_R\big(\omega_R^{(1)}, \omega_R^\infty\big)^d \leftarrow \cdots. \tag{3.4.4}$$

Since $\operatorname{Hom}_R(\omega_R^{(1)}, \omega_R) \cong R^{(1)}$ and $\operatorname{Hom}_R(\omega_R, \omega_R) = R$, we can rewrite (3.4.4) as

$$0 \leftarrow \bigoplus_{j\in\mathbb{Z}} R^{(1)} \xleftarrow{h^\vee} \left(\bigoplus_{j\in\mathbb{Z}} R\right) \oplus \left(\bigoplus_{j\in\mathbb{Z}} R^{(1)}\right)^d \leftarrow \cdots.$$

And after a careful computation using (3.4.3) and the right $R\{F\}$-module structure of ω_R^∞, we have

$$h^\vee\bigl(\{s_j\} \oplus \bigl(\{t_{1j}^{(1)}\}, \ldots, \{t_{dj}^{(1)}\}\bigr)\bigr) = \left\{(-1)^d\bigl(\bigl(s_j^{(1)}\bigr)^p - s_{j-1}^{(1)}\bigr) + \sum_{i=1}^d x_i^{(1)} t_{ij}^{(1)}\right\}_{j\in\mathbb{Z}} \tag{3.4.5}$$

where $\{s_j\}$ denotes an element in $\bigoplus_{j\in\mathbb{Z}} R$ and $(\{t_{1j}^{(1)}\}, \ldots, \{t_{dj}^{(1)}\})$ denotes an element in $(\bigoplus_{j\in\mathbb{Z}} R^{(1)})^d$. The key point here is that $h^\vee$ is *not* surjective. To be more precise, I claim $(-1)^{d-1}z_0 = (\ldots, 0, (-1)^{d-1}, 0, 0, \ldots)$ (i.e., the element in $\bigoplus_{j\in\mathbb{Z}} R^{(1)}$ with 0-th entry $(-1)^{d-1}$ and other entries 0) is not in the image of $h^\vee$. This is because $\sum_{i=1}^d x_i^{(1)} t_{ij}^{(1)}$ can only take values in $\bigoplus_{j\in\mathbb{Z}}(x_1^{(1)}, \ldots, x_d^{(1)})$, so if $z_0 \in \operatorname{im} h^\vee$, then mod $\bigoplus_{j\in\mathbb{Z}}(x_1^{(1)}, \ldots, x_d^{(1)})$, we know by (3.4.5) that $(\overline{s_j}^{(1)})^p - \overline{s_{j-1}}^{(1)} = 0$ for $j \neq 0$ and $(\overline{s_0}^{(1)})^p - \overline{s_{-1}}^{(1)} = -1$. And it is straightforward to see that a solution $\{s_j\}_{j\in\mathbb{Z}}$ to this system must satisfy $\overline{s_j} = 0$ when $j \geqslant 0$ and $\overline{s_j} = \bar{1}$ when $j < 0$ where $\bar{s}$ denotes the image of $s \in R$ mod $(x_1, \ldots, x_d)$. So there is no solution in $\bigoplus_{j\in\mathbb{Z}} R$, since $\overline{s_j} = \bar{1}$ for every $j < 0$ implies there has to be infinitely many nonzero s_j.

Hence we get

$$\operatorname{Ext}^{d+1}_{R\{F\}}\bigl(\omega_R/(x_1, \ldots, x_d)\omega_R, \omega_R^\infty\bigr) \cong \operatorname{coker} h^\vee \neq 0. \tag{3.4.6}$$

Combining (3.4.6) with Theorem 3.2 completes the proof of the lemma. □

Remark 3.5. One might hope that $\operatorname{id}_{R\{F\}} \omega_R = d + 1$ by the same type computation used in Lemma 3.4. But there is a small gap when doing this. The problem is, when we apply $\operatorname{Hom}_{R\{F\}}(-, \omega_R)$ to (3.4.2) and compute $\operatorname{coker} h^\vee$, we get

$$\operatorname{Ext}^{d+1}_{R\{F\}}\bigl(\omega_R/(x_1, \ldots, x_d)\omega_R, \omega_R\bigr) = \operatorname{coker} h^\vee \cong \frac{R}{(x_1, \ldots, x_d) + \{r^p - r\}_{r\in R}}. \tag{3.5.1}$$

So if the set $\{r^p - r\}_{r\in R}$ can take all values of R (this happens, for example when $(R, \mathfrak{m})$ is a complete regular local ring with algebraically closed residue field, see Remark 4.6), then $\operatorname{Ext}^{d+1}_{R\{F\}}(\omega_R/(x_1, \ldots, x_d)\omega_R, \omega_R) = 0$. So we cannot get the desired result in this way. However, we do get from (3.5.1) that if $(R, \mathfrak{m}, K)$ is an F-finite regular local ring with $K \cong R/\mathfrak{m}$ a *finite* field, then $\operatorname{id}_{R\{F\}} R = d + 1$.

Now we prove our main result. We start by proving that the Ext groups are the same no matter one computes in the category of unit right $R\{F\}$-modules or the category of right $R\{F\}$-modules. We give two proofs of this result, the second proof is suggested by the referee, which in fact shows a stronger result.

Theorem 3.6. *Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Let M, N be two unit right $R\{F\}$-modules. Then we have $\operatorname{Ext}^i_{uR\{F\}}(M,N) \cong \operatorname{Ext}^i_{R\{F\}}(M,N)$ for every i. In particular, the category of unit right $R\{F\}$-modules and the category of F_R-modules has finite global dimension $\leqslant d+1$.*

Proof. First we note that by Theorem 3.2 and Theorem 2.5, it is clear that we only need to show $\operatorname{Ext}^i_{uR\{F\}}(M,N) \cong \operatorname{Ext}^i_{R\{F\}}(M,N)$ for M, N two unit right $R\{F\}$-modules. Below we give two proofs of this fact.

First proof: We use Yoneda's characterization of Ext^i (*cf.* Chapter 3.4 in [7]). Note that this is the same as the derived functor Ext^i whenever the abelian category has enough injectives or enough projectives, hence holds for both the category of unit right $R\{F\}$-modules and the category of right $R\{F\}$-modules (unit right $R\{F\}$-modules has enough injectives by Theorem 1.1 and Theorem 2.5). An element in $\operatorname{Ext}^i_{uR\{F\}}(M,N)$ (resp. $\operatorname{Ext}^i_{R\{F\}}(M,N)$) is an equivalence class of exact sequences of the form

$$\xi : 0 \to N \to X_1 \to \cdots \to X_i \to M \to 0$$

where each X_i is a unit right $R\{F\}$-module (resp. right $R\{F\}$-module) and the maps are maps of unit right $R\{F\}$-modules (resp. maps of right $R\{F\}$-modules). The equivalence relation is generated by the relation $\xi_X \sim \xi_Y$ if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_i \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From this characterization of Ext^i it is clear that we have a well-defined map

$$\iota : \operatorname{Ext}^i_{uR\{F\}}(M,N) \to \operatorname{Ext}^i_{R\{F\}}(M,N)$$

taking an equivalence class of an exact sequence of unit right $R\{F\}$-modules to the same exact sequence but viewed as an exact sequence in the category of right $R\{F\}$-modules.

Conversely, if we have an element in $\operatorname{Ext}^i_{R\{F\}}(M,N)$, say ξ, we have an exact sequence of right $R\{F\}$-modules, this induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F^!(N) \arrow[r] \arrow[d, "\cong"] & F^!(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & F^!(X_i) \arrow[r] \arrow[d] & F^!(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (F^!)^2(N) \arrow[r] \arrow[d, "\cong"] & (F^!)^2(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (F^!)^2(X_i) \arrow[r] \arrow[d] & (F^!)^2(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
& {} & {} & & {} & {} &
\end{tikzcd}$$

Taking direct limits for columns and noticing that M, N are unit right $R\{F\}$-modules, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & \varinjlim (F^!)^e(X_1) \arrow[r] & \cdots \arrow[r] & \varinjlim (F^!)^e(X_i) \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \tag{3.6.1}$$

Since the functor $F^!(-)$ and the direct limit functor are both exact, the bottom sequence is still exact and hence it represents an element in $\mathrm{Ext}^i_{uR\{F\}}(M,N)$ (note that each $\varinjlim(F^!)^e(X_j)$ is a unit right $R\{F\}$-module by Remark 2.4). We call this element ξ'. Then we have a map

$$\eta : \mathrm{Ext}^i_{R\{F\}}(M,N) \xrightarrow{\xi \mapsto \xi'} \mathrm{Ext}^i_{uR\{F\}}(M,N).$$

This map is well-defined because it is easy to check that if $\xi_1 \sim \xi_2$, then we also have $\xi'_1 \sim \xi'_2$. It is also straightforward to check that ι and η are inverses of each other. Obviously $\eta \circ \iota([\xi]) = [\xi]$ and $\iota \circ \eta([\xi]) = [\xi'] = [\xi]$, where the last equality is by (3.6.1) (which shows that $\xi \sim \xi'$, and hence they represent the same equivalence class in $\mathrm{Ext}^i_{R\{F\}}(M,N)$).

Second proof: By Theorem 1.1 and Theorem 2.5 we know that the category of unit right $R\{F\}$-module has enough injectives. Now we show that every injective object in the category of unit right $R\{F\}$ modules is in fact injective in the category of right $R\{F\}$-modules. To see this, let I be a unit right $R\{F\}$-injective module. It is enough to show that whenever we have $0 \to I \to W$ for some right $R\{F\}$-module W, the sequence splits. But $0 \to I \to W$ induces the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\cong"] & W \arrow[d] \\
0 \arrow[r] & F^!(I) \arrow[r] \arrow[d, "\cong"] & F^!(W) \arrow[d] \\
0 \arrow[r] & (F^!)^2(I) \arrow[r] \arrow[d, "\cong"] & (F^!)^2(W) \arrow[d] \\
 & {} & {}
\end{tikzcd}$$

Taking direct limit for the columns we get

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\cong"] & W \arrow[d] \\
0 \arrow[r] & I \arrow[r] & \varinjlim (F^!)^e(W)
\end{tikzcd} \tag{3.6.2}$$

We still have exactness because the functor $F^!(-)$ and the direct limit functor are both exact. We also note that $\varinjlim (F^!)^e(W)$ is a unit right $R\{F\}$-module by Remark 2.4. Now since I is injective in the category of unit right $R\{F\}$-modules, we know that the bottom map $0 \to I \to \varinjlim (F^!)^e(W)$ splits as unit right $R\{F\}$-modules, so it also splits as right $R\{F\}$-modules. But now composing with the commutative diagram (3.6.2) shows that the map $0 \to I \to W$ splits as right $R\{F\}$-modules.

Now it is clear that $\mathrm{Ext}^i_{uR\{F\}}(M,N) \cong \mathrm{Ext}^i_{R\{F\}}(M,N)$. Because one can take an injective resolution of N in the category of unit right $R\{F\}$-modules:

$$0 \to N \to I_0 \to I_1 \to \cdots. \tag{3.6.3}$$

By the above argument this can be also viewed as an injective resolution in the category of right $R\{F\}$-modules. Since applying $\mathrm{Hom}_{R\{F\}}(M,-)$ and $\mathrm{Hom}_{uR\{F\}}(M,-)$ to (3.6.3) are obviously the same, we know that $\mathrm{Ext}^i_{uR\{F\}}(M,N) \cong \mathrm{Ext}^i_{R\{F\}}(M,N)$. $\square$

Theorem 3.7. *Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Then the category of unit right $R\{F\}$-modules and the category of Lyubeznik's F_R-modules both have finite global dimension $d+1$.*

Proof. By Theorem 2.5, it suffices to show that the category of unit right $R\{F\}$-modules has finite global dimension $d+1$. By Theorem 3.6, we know that the global dimension is at most $d+1$.

Now let ω_R^∞ be the unit right $R\{F\}$-module described in Example 3.3. If the global dimension is $\leqslant d$, then we know that ω_R^∞ has a unit right $R\{F\}$-injective resolution of length $d' \leqslant d$:

$$0 \to \omega_R^\infty \to I_0 \to I_1 \to \cdots \to I_{d'} \to 0. \tag{3.7.1}$$

But by the argument in the second proof of Theorem 3.6, we know that each I_j is injective in the category of right $R\{F\}$-modules. So (3.7.1) can be viewed as an injective resolution of ω_R^∞ in the category of right $R\{F\}$-modules. And hence $\mathrm{id}_{R\{F\}}\,\omega_R^\infty \leqslant d$, which contradicts Lemma 3.4. □

Remark 3.8. It is clear from Theorem 2.5 and the above proof of Theorem 3.7 that

$$\mathrm{id}_{F_R} R^\infty = \mathrm{id}_{uR\{F\}}\,\omega_R^\infty = d+1.$$

4. Non-finiteness of $\mathrm{Ext}^1_{F_R}$

In this section we study the group $\mathrm{Ext}^1_{F_R}(M,N)$ when M, N are F_R-finite F_R-modules. We prove that when $(R,\mathfrak{m},K)$ is a regular local ring, $\mathrm{Ext}^1_{F_R}(M,N)$ is finite when $K = R/\mathfrak{m}$ is separably closed and M is supported only at $\mathfrak{m}$. However, we provide examples to show that in general $\mathrm{Ext}^1_{F_R}(M,N)$ is not necessarily a finite set. We begin with some lemmas.

Lemma 4.1. *(See Proposition 3.1 in [6].) Let S be a regular ring of characteristic $p>0$ and let $R \to S$ be a surjective homomorphism with kernel $I \subseteq R$. There exists an equivalence of categories between F_R-modules supported on $\operatorname{Spec} S = V(I) \subseteq \operatorname{Spec} R$ and F_S-modules. Under this equivalence the F_R-finite F_R-modules supported on $\operatorname{Spec} S = V(I) \subseteq \operatorname{Spec} R$ correspond to the F_S-finite F_S-modules.*

Lemma 4.2. *(See Theorem 4.2(c)(e) in [4].) Let K be a separably closed field. Then every F_K-finite F_K-module is isomorphic with a finite direct sum of copies of K with the standard F_K-module structure. Moreover, $\mathrm{Ext}^1_{F_K}(K,K)=0$.*

Lemma 4.3. *Let $(R,\mathfrak{m},K)$ be a regular local ring with K separably closed. Then every F_R-finite F_R-module supported only at $\mathfrak{m}$ is isomorphic (as an F_R-module) with a finite direct sum of copies of $E = E(R/\mathfrak{m})$ (where E is equipped with the standard F_R-module structure). Moreover, $\mathrm{Ext}^1_{F_R}(E,E)=0$.*

Proof. This is clear from Lemma 4.1 (applied to $S = K$ and $I = \mathfrak{m}$) and Lemma 4.2 because it is straightforward to check that the standard F_R-module structure on E corresponds to the standard F_K-module structure on K via Lemma 4.1. □

Theorem 4.4. *Let $(R, \mathfrak{m}, K)$ be a regular local ring such that K is separably closed and let M, N be F_R-finite F_R-modules. Then $\mathrm{Ext}^1_{F_R}(M, N)$ is finite if M is supported only at $\mathfrak{m}$.*

Proof. Since K is separably closed, by Lemma 4.3 we know that M is a finite direct sum of copies of E in the category of F_R-modules. So it suffices to show that $\mathrm{Ext}^1_{F_R}(E, N)$ is finite. For every exact sequence of F_R-finite F_R-modules

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

the long exact sequence of Ext gives

$$\mathrm{Ext}^1_{F_R}(E, N_1) \to \mathrm{Ext}^1_{F_R}(E, N_2) \to \mathrm{Ext}^1_{F_R}(E, N_3).$$

So we immediately reduce to the case that N is simple (since R is local, every F_R-finite F_R-module has finite length by Theorem 3.2 in [6]).

We want to show that $\mathrm{Ext}^1_{F_R}(E, N)$ is finite when N is simple. There are two cases: $\mathrm{Ass}_R(N) = \mathfrak{m}$ or $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$ (by Theorem 2.12(b) in [6]). If $\mathrm{Ass}_R(N) = \mathfrak{m}$, then $N \cong E$ as F_R-modules by Lemma 4.3. So $\mathrm{Ext}^1_{F_R}(E, N) = \mathrm{Ext}^1_{F_R}(E, E) = 0$ by Lemma 4.3.

If $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, by Yoneda's characterization of Ext groups, it suffices to show that we only have a finite number of isomorphism classes of short exact sequences

$$0 \to N \to L \to E \to 0$$

of F_R-modules. We first show the number of choices of isomorphism classes for L is finite. Say $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, we have $P \in \mathrm{Ass}_R(L) \subseteq \{P, \mathfrak{m}\}$. If $\mathrm{Ass}_R(L) = \{P, \mathfrak{m}\}$, then $H^0_{\mathfrak{m}}(L) \neq 0$ and it does not intersect N. So $H^0_{\mathfrak{m}}(L) \oplus N$ is an F_R-submodule of L. Hence we must have $L \cong H^0_{\mathfrak{m}}(L) \oplus N \cong E \oplus N$ since L has length 2 as an F_R-module. If $\mathrm{Ass}_R(L) = \{P\}$, we can pick $x \in \mathfrak{m} - P$. Localizing at x gives a short exact sequence

$$0 \to N_x \to L_x \to E_x \to 0.$$

But $E_x = 0$, so we get $N_x \cong L_x$ as F_R-module. Since x is not in P, we have $L \hookrightarrow L_x$ as F_R-module. That is, L is isomorphic to an F_R-submodule of L_x, hence is isomorphic to an F_R-submodule of N_x. But N_x is F_R-finite by Proposition 2.9(b) in [6], so it only has finitely many F_R-submodules by Theorem 1.2. This proves that the number of choices of isomorphism classes for L is finite.

Because the number of choices of isomorphism classes for L is finite, and for each F_R-finite F_R-module L, $\mathrm{Hom}_{F_R}(N, L)$ is always finite by Theorem 1.2. It follows that the number of isomorphism classes of short exact sequences $0 \to N \to L \to E \to 0$ is finite. $\square$

If M is an F_R-module with structure morphism θ_M, for every $x \in M$ we use x^p to denote $\theta_M^{-1}(1 \otimes x)$. Notice that when $M = R$ with the standard F_R-module structure, this is exactly the usual meaning of x^p. We let G_M denote the set $\{x^p - x \mid x \in M\}$. It is clear that G_M is an abelian subgroup of M.

Theorem 4.5. *Let R be a regular ring. Giving R the standard F_R-module structure, we have $\operatorname{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group for every F_R-module M.*

Proof. By Yoneda's characterization of Ext groups, an element in $\operatorname{Ext}^1_{F_R}(R, M)$ can be represented by an exact sequence of F_R-modules

$$0 \to M \to L \to R \to 0.$$

It is clear that $L \cong M \oplus R$ as R-module. Moreover, one can check that the structure isomorphism θ_L composed with $\theta_M^{-1} \oplus \theta_R^{-1}$ defines an isomorphism

$$M \oplus R \xrightarrow{\theta_L} F(M) \oplus F(R) \xrightarrow{\theta_M^{-1} \oplus \theta_R^{-1}} M \oplus R$$

which sends (y, r) to $(y + rz, r)$ for every $(y, r) \in M \oplus R$ and for some $z \in M$. Hence, giving a structure isomorphism of L is equivalent to giving some $z \in M$. Therefore, θ_L is determined by an element $z \in M$. Two exact sequences with structure isomorphism θ_L, θ'_L are in the same isomorphism class if and only if there exists a map $g : L \to L$, sending (y, r) to $(y + rx, r)$ for some $x \in M$ such that

$$(1 \otimes g) \circ \theta_L = \theta'_L \circ g.$$

Now we apply $\theta_M^{-1} \oplus \theta_R^{-1}$ on both sides. If θ_L, θ'_L are determined by z_1 and z_2 respectively, a direct computation gives that

$$\left(\theta_M^{-1} \oplus \theta_R^{-1}\right) \circ (1 \otimes g) \circ \theta_L(y, r) = \left(y + rz_1 + rx^p, r\right)$$

while

$$\left(\theta_M^{-1} \oplus \theta_R^{-1}\right) \circ \theta'_L \circ g(y, r) = (y + rz_2 + rx, r).$$

So θ_L and θ'_L are in the same isomorphism class if and only if there exists $x \in M$ such that

$$z_2 - z_1 = x^p - x.$$

So $\operatorname{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group. $\square$

Before we use Theorem 4.5 to study examples, we make the following remark. I would like to thank the referee for his suggestions on this remark.

Remark 4.6.

(1) In the case that R is a regular ring which is F*-finite* and *local*. By Theorem 2.5, we can identify the category of F_R-modules with the category of unit right $R\{F\}$-modules ($\omega_R = R$ is unique). And by Theorem 3.6, we can compute $\operatorname{Ext}^1_{F_R}(R,M) \cong \operatorname{Ext}^1_{uR\{F\}}(R,M) \cong \operatorname{Ext}^1_{R\{F\}}(R,M)$ by taking the right $R\{F\}$-projective resolution of R and then applying $\operatorname{Hom}_{R\{F\}}(-,M)$. Note that one right $R\{F\}$-projective resolution of R is given by

$$0 \to R^{(1)} \otimes_R R\{F\} \to R\{F\} \to R \to 0$$

as in Lemma 3.1. Thus in this case one can give another proof of Theorem 4.5, we leave the details to the reader.

(2) When $(R,\mathfrak{m})$ is a strict Henselian local ring (e.g., $(R,\mathfrak{m})$ is a complete local ring with algebraically closed residue field), the Artin–Schreier sequence

$$0 \to \mathbb{F}_p \to R \xrightarrow{x^p - x} R \to 0$$

is exact in the Zariski topology, which shows that $G_R = R$, and hence $\operatorname{Ext}^1_{F_R}(R,R) = 0$ when R is a strict Henselian local ring. In particular, applying this to $R = K$ a separably closed field, we recover Lemma 4.2.

Now we give some examples to show that, in general, $\operatorname{Ext}^1_{F_R}(R,M) \cong M/G_M$ is not necessarily finite, even in simple cases.

Example 4.7. Let $R = k(t)$ or $k[t]_{(t)}$ with k an algebraically closed field. We will prove that $\operatorname{Ext}^1_{F_R}(R,R)$ is infinite in both cases. By Theorem 4.5, it suffices to show that for $a, b \in k$ ($a, b \neq 0$ in the second case), $\frac{1}{t-a}$ and $\frac{1}{t-b}$ are different in R/G_R whenever $a \neq b$. Otherwise there exists $\frac{h(t)}{g(t)} \in R$ with $h(t), g(t) \in k[t]$ ($g(t)$ is not divisible by t in the second case) and $\gcd(h(t), g(t)) = 1$ such that

$$\frac{1}{t-a} - \frac{1}{t-b} = \frac{h(t)^p}{g(t)^p} - \frac{h(t)}{g(t)}$$

which gives

$$\frac{a-b}{t^2 - (a+b)t + ab} = \frac{h(t)^p - h(t) \cdot g(t)^{p-1}}{g(t)^p}. \tag{4.7.1}$$

Since $\gcd(h(t), g(t)) = 1$, $\gcd(h(t)^p - h(t) \cdot g(t)^{p-1}, g(t)^p) = 1$. So from (4.7.1) we know that $g(t)^p | (t^2 - (a+b)t + ab)$. This is clearly impossible.

Example 4.8. Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension $d \geqslant 1$. Let $E = E(R/\mathfrak{m})$ be the injective hull of the residue field. We will show that $\operatorname{Ext}^1_{F_R}(R,E)$ is not zero and is in fact infinite when K is infinite. In particular, $E = E(R/\mathfrak{m})$, though injective as an R-module, is *not* injective as an F_R-module (with its standard F_R-structure) when $\dim R \geqslant 1$.

Recall that $E = \varinjlim_n \frac{R}{(x_1^n,\ldots,x_d^n)}$. So every element z in E can be expressed as $(r; x_1^n, \ldots, x_d^n)$ for some $n \geqslant 1$ (which means z is the image of r in the n-th piece in this direct limit system). By Theorem 4.5, $\mathrm{Ext}^1_{F_R}(R, E) \cong E/G_E$. I claim that any two different socle elements u_1, u_2 are different in E/G_E. Suppose this is not true, we have:

$$u_1 - u_2 = z^p - z \tag{4.8.1}$$

in E. Since $u_1 - u_2$ is a nonzero element in the socle of E, we may write $u_1 - u_2 = (\lambda; x_1, \ldots, x_d)$ for some $\lambda \neq 0$ in K. Say $z = (r; x_1^n, \ldots, x_d^n)$ with n minimum. Then (4.8.1) will give

$$\left(r; x_1^n, \ldots, x_d^n\right) = (\lambda; x_1, \ldots, x_d) + \left(r^p; x_1^{np}, \ldots, x_d^{np}\right).$$

This will give us

$$r^p + \lambda(x_1 \cdots x_d)^{np-1} - r(x_1 \cdots x_d)^{np-n} \in \left(x_1^{np}, \ldots, x_d^{np}\right). \tag{4.8.2}$$

If $n = 1$, then $0 \neq z \in \mathrm{Soc}(E)$, hence r is a nonzero unit in R. But (4.8.2) shows that $r^p \in (x_1, \ldots, x_d)$ which is a contradiction.

If $n \geqslant 2$, we have $np-1 \geqslant np-n \geqslant p$. We know from (4.8.2) that for every $1 \leqslant i \leqslant d$, we have $r^p \in (x_1^{np}, \ldots, x_{i-1}^{np}, x_i^p, x_{i+1}^{np}, \ldots, x_d^{np})$. Hence $r \in (x_1^n, \ldots, x_{i-1}^n, x_i, x_{i+1}^n, \ldots, x_d^n)$ for every $1 \leqslant i \leqslant d$. Taking their intersection, we get that $r \in (x_1 \cdots x_d, x_1^n, \ldots, x_d^n)$. That is, mod $(x_1^n, \ldots, x_d^n)$, we have $r = (x_1 \cdots x_d) r_0$. But then we have $z = (r_0; x_1^{n-1}, \ldots, x_d^{n-1})$ contradicting our choice of n.

Therefore we have proved that any two different socle elements u_1, u_2 are different in $\mathrm{Ext}^1_{F_R}(R, E) = E/G_E$. This shows that $\mathrm{Ext}^1_{F_R}(R, E) \neq 0$ and is infinite when K is infinite.

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