Journal of Algebra 406 (2014) 321–336

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Invertible linear maps on Borel subalgebras preserving zero Lie products

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ARTICLE INFO

Article history:
Received 9 July 2009
Available online 25 March 2014
Communicated by Efim Zelmanov

MSC:
17B20
17B30
17B40

Keywords:
Simple Lie algebras
Borel subalgebras
Automorphisms of Lie algebras

ABSTRACT

Let $\mathfrak{g}$ be a simple Lie algebra of rank l over an algebraically closed field of characteristic zero, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. An invertible linear map φ on $\mathfrak{b}$ is said preserving zero Lie products in both directions if for $x, y \in \mathfrak{b}$, $[x, y] = 0$ if and only if $[\varphi(x), \varphi(y)] = 0$. In this paper, it is shown that an invertible linear map φ on $\mathfrak{b}$ preserving zero Lie products in both directions if and only if it is a composition of an inner automorphism, a graph automorphism, a scalar multiplication map and a diagonal automorphism, which extends the main result in [8] from a linear solvable Lie algebra to far more general cases.

1. Introduction

In this paper, the notations concerning Lie algebras mainly follow those of [1,2]. Let F be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a simple Lie algebra over F of rank l, $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\Phi \subseteq \mathfrak{h}^*$ the corresponding root system of $\mathfrak{g}$, Δ a fixed base of Φ, Φ^+ (resp., Φ^-) the set of positive (resp., negative) roots relative to Δ. The roots in Δ are called *simple*. Actually, Δ defines a partial order on Φ in such a way that $\beta \prec \alpha$ iff $\alpha - \beta$ is a sum of simple roots or $\beta = \alpha$. For $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \in \Phi$,

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[1] Supported by the National Natural Science Foundation of China (No. 11171343).

http://dx.doi.org/10.1016/j.jalgebra.2014.02.023
0021-8693/

the integer $ht\,\beta = \sum_{\alpha\in\Delta} k_\alpha$ is called the *height* of β. Φ has a unique maximal root and a unique maximal short root (if two root lengths occur), which we denote by θ and θ_s respectively. If Φ has two root lengths, we denote by Φ_l (resp., Φ_s) the subset of Φ consisting of long (resp., short) roots, and let $\Phi_l^+ = \Phi^+ \cap \Phi_l$, $\Phi_s^+ = \Phi^+ \cap \Phi_s$, $\Delta_l = \Delta \cap \Phi_l$, $\Delta_s = \Delta \cap \Phi_s$. For $\alpha \in \Phi^+$, set

$$X_\alpha = \{\beta \in \Phi^+ \mid \beta + \alpha \in \Phi^+\}; \qquad Y_\alpha = \{\beta \in \Phi^+ \mid \beta + \alpha \notin \Phi^+\}.$$

Then $X_\alpha \cup Y_\alpha = \Phi^+$ and $X_\alpha \cap Y_\alpha = \emptyset$. Let M_α be the number of elements in X_α. For a positive integer i, we set $\Phi^i = \{\alpha \in \Phi \mid ht\,\alpha = i\}$. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α, $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{b}$ is called the standard Borel subalgebra of $\mathfrak{g}$. For a positive integer k, let $\mathfrak{n}_k = \sum_{ht\,\beta\geqslant k} \mathfrak{g}_\beta$. For a subalgebra $\mathfrak{s}$ of $\mathfrak{b}$, a subset A of $\mathfrak{b}$, we denote by $C_\mathfrak{s}(A)$ the centralizer of A in $\mathfrak{s}$. $C_\mathfrak{b}(A)$ and $C_\mathfrak{b}(C_\mathfrak{b}(A))$ are abbreviated to $\{A\}'$ and $\{A\}''$, respectively. We denote by $ker\,\alpha$, for $\alpha \in \Phi$, the kernel of α in $\mathfrak{h}$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of $\mathfrak{g}_\alpha$, then there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,F)$ via

$$e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ forms a basis of $\mathfrak{g}$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a scalar multiple of $e_{\alpha+\beta}$ since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We define $N_{\alpha,\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, which is called the *structure constants* of $\mathfrak{g}$. A basis $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ of $\mathfrak{g}$ can be chosen such that all structure constants of $\mathfrak{g}$ are integers. In this case the basis is called a *Chevalley basis* of $\mathfrak{g}$. In the following of this paper, the set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ will always denote a Chevalley basis of $\mathfrak{g}$. For the base Δ of Φ, let $\mathfrak{d}_\Delta = \{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. Namely, $\beta(d_\alpha)$ takes the value 0 when $\beta \neq \alpha \in \Delta$ and takes the value 1 when $\beta = \alpha \in \Delta$. A symmetric bilinear form (,) is defined on the l-dimensional real vector space spanned by Φ, which is dual to the Killing form on $\mathfrak{g}$. For $\alpha, \beta \in \Phi$, let $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. If $\alpha \neq \pm\beta$, let p, q be the greatest non-negative integers for which $\beta - p\alpha, \beta + q\alpha \in \Phi$, then $\langle \beta, \alpha \rangle = p - q$, and $N_{\alpha,\beta} = \pm(p+1)$.

An invertible linear map φ on $\mathfrak{b}$ is called *preserving zero Lie products* in both directions if for any $x, y \in \mathfrak{b}$, $[x, y] = 0$ if and only if $[\varphi(x), \varphi(y)] = 0$. Note that the product of such two maps and the inverse of such a map are also such maps.

A lot of attention has been paid to linear preserver problem, which focuses on characterizing linear maps on matrix spaces or algebras that leave certain functions, subsets, relations, etc., invariant. The earliest paper on such problem dates back to 1897 (see [3]), and a great deal of effort has been devoted to the study of this type of question since then. One may consult the survey papers [4–7] for details. Here, let us particularly mention one type of classical example: linear maps preserving commutativity. The importance of this type example lies in the fact that the assumption of preserving commutativity of matrices can be considered as the assumption of preserving zero Lie products of the relative linear Lie algebra. Such type of linear preserver problem has been extensively studied on matrix algebras as well as on more general rings and operator algebras. For example, the commutativity preserving linear maps on triangular matrices was studied in [8]; the commutativity preserving linear maps on strictly triangular matrices was investigated in [9]; the nonlinear commutativity preserving maps on the algebra of full matrices was described by P. Šemrl in [10]. For more references about commutativity preserving maps on associated algebras one may consult the survey paper [11].

Recently, linear preserver problem concerning Lie algebras attracted more attention. Wong in [12] characterized the invertible linear maps on simple Lie algebras of linear types preserving zero Lie products. Padjavi and Šemrl in [13] characterized the non-linear maps on the general linear Lie algebra and the special linear Lie algebra which preserve solvability in both directions. The non-linear bijective maps on triangular matrices preserving Lie products were determined in [14]. The invertible linear maps on a finite dimensional complex simple Lie algebras preserving commutativity were characterized in [15]. A finite dimensional complex simple Lie algebra $\mathfrak{g}$ and its parabolic subalgebras were proved to be zero product determined in [16]. All quasi-automorphisms of $\mathfrak{g}$ are determined in [17]. Non-linear maps on a finite dimensional complex simple Lie algebra $\mathfrak{g}$ preserving Lie products and non-linear maps on the standard Borel subalgebra of $\mathfrak{g}$ that preserves ad-nilpotent ideals were respectively described in [18] and [19]. Invertible linear maps on a finite dimensional complex simple Lie algebra preserving solvability were studied in [20].

Preserver problems for algebraic groups and for simple Lie algebras usually goes in parallel (consult [21] and [22]). Bijective maps preserving commutators on certain classical groups were respectively studied in [23] and [24], except for the mapping induced by the field automorphism, the results in [23,24] are parallel to those on Lie algebras obtained in [18].

The Levi theorem tells us that every finite dimensional Lie algebra L is a semi-direct sum of a semisimple Lie algebra S and the maximal solvable ideal R of L, i.e., the radical of L. Semisimple Lie algebras over the field of complex numbers have been classified by Cartan in [25]. So the problem of classifying finite dimensional Lie algebras is reduced to the problem of classifying finite dimensional solvable Lie algebras. However, the classification of solvable Lie algebras is only complete for the case when the dimension is not bigger than six (see [26]). It seems to be impossible to classify solvable Lie algebras in an arbitrary large finite dimension. In view of this point, the structure of solvable Lie algebras seems much complicated than those of simple Lie algebras. In the present paper, we consider the invertible linear maps on a solvable Lie algebra, i.e., the standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The result of this paper can be viewed as a generalization of the main theorem in [8].

Before announcing the main result of this paper, we introduce several types of standard maps on $\mathfrak{b}$ preserving zero Lie products in both directions.

(i) For $\alpha \in \Phi^+$ and $t \in F$, te_α is nilpotent in $\mathfrak{b}$, so the map $\sigma_\alpha(t) = \exp(t\, ad\, e_\alpha)$ is an automorphism of $\mathfrak{b}$. We denote by U the group generated by the elements $\sigma_\alpha(t)$ for all $\alpha \in \Phi^+$, $t \in F$. Each element $\sigma \in U$ is called an *inner automorphism* of $\mathfrak{b}$.

(ii) If ρ is a symmetry (nontrivial or trivial) of the Dynkin diagram of Φ, or equivalently, $\langle \rho(\alpha), \rho(\beta) \rangle = \langle \alpha, \beta \rangle$ for any $\alpha, \beta \in \Delta$, then ρ can be extended to an automorphism $\bar{\rho}$ of Φ in the way:

$$\sum_{\alpha \in \Delta} k_\alpha \alpha \in \Phi \mapsto \sum_{\alpha \in \Delta} k_\alpha \rho(\alpha).$$

Using $\bar{\rho}$ we define an automorphism φ_ρ of $\mathfrak{b}$ in the way:

$$\sum_{\alpha \in \Delta} a_\alpha h_\alpha + \sum_{\alpha \in \Phi^+} b_\alpha e_\alpha \mapsto \sum_{\alpha \in \Delta} a_\alpha h_{\rho(\alpha)} + \sum_{\alpha \in \Phi^+} b_\alpha r_\alpha e_{\bar{\rho}(\alpha)}, \quad a_\alpha, b_\alpha \in F,$$

where $r_\alpha = 1$ or -1 depending on α. φ_ρ is called a *graph automorphism* of $\mathfrak{b}$.

(iii) Let $P = \mathbb{Z}\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the elements of Φ. It is a free abelian group of rank l and has Δ as its basis. A homomorphism χ from the additive group P into the multiplicative group F^* of non-zero elements of F is called a *character* of P. Each character χ of P gives rise to an automorphism φ_χ of $\mathfrak{b}$, by

$$h + \sum_{\alpha \in \Phi^+} x_\alpha e_\alpha \mapsto h + \sum_{\alpha \in \Phi^+} x_\alpha \chi(\alpha) e_\alpha, \quad h \in \mathfrak{h},\ x_\alpha \in F.$$

φ_χ is called a *diagonal automorphism* of $\mathfrak{b}$.

(iv) For $c \in F^*$, define

$$\varphi_c : \mathfrak{b} \to \mathfrak{b}, \qquad x \mapsto cx, \quad \forall x \in \mathfrak{b}.$$

We call φ_c a *scalar multiplication map* on $\mathfrak{b}$.

It is clear that the inner automorphisms, the graph automorphisms, the diagonal automorphisms and the scalar multiplication maps on $\mathfrak{b}$ are respectively invertible linear maps on $\mathfrak{b}$ preserving zero Lie products in both directions.

Theorem 1.1. *Suppose rank $\mathfrak{g} = l \geqslant 2$. φ is an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions if and only if it is a composition of an inner automorphism, a graph automorphism, a scalar multiplication map and a diagonal automorphism.*

Remark 1.2.

(i) In view of the main theorem we see that an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions differs only slightly from an automorphism of $\mathfrak{b}$. Indeed, the group of all invertible linear maps on $\mathfrak{b}$ preserving zero Lie products in both directions is isomorphic to the direct product of $Aut\,\mathfrak{b}$ and F^*, where $Aut\,\mathfrak{b}$ denotes the automorphism group of $\mathfrak{b}$.

(ii) When $l = 1$, the corresponding result can be easily concluded from the main result of [8], so we leave this case.

2. Some elementary results

Let us start with some preliminary results on simple Lie algebras and irreducible root systems. In the following, Φ always denotes irreducible roots of rank l. If only one root length occurs, we view all roots in Φ to be long.

Lemma 2.1. *Let $h \in \mathfrak{h}$, $\alpha \in \Phi^+$. If $\beta(h) = 0$ for all $\beta \in Y_\alpha$, then $h = 0$.*

Proof. Let $\Delta_1 = \Delta \cap Y_\alpha$ and $\Delta_2 = \Delta \cap X_\alpha$. For any $\beta \in \Delta_2$, let k be the maximal positive integer such that $\beta + k\alpha \in \Phi^+$, i.e., $\beta + k\alpha \in \Phi^+$ and $\beta + (k+1)\alpha \notin \Phi^+$. Thus $(\beta + k\alpha)(h) = 0$. This follows that $\beta(h) = 0$ (note that $\alpha(h) = 0$). Therefore, $\beta(h) = 0$ for all $\beta \in \Delta$, forcing $h = 0$. $\square$

Lemma 2.2. *$\{e_\alpha\}'' = \mathfrak{g}_\alpha$ for any $\alpha \in \Phi^+$.*

Proof. Obviously, $\mathfrak{g}_\alpha \subseteq \{e_\alpha\}''$. Conversely, we first consider $\{e_\alpha\}'$. It's easy to see that $ker\,\alpha \oplus (\bigoplus_{\beta \in Y_\alpha} \mathfrak{g}_\beta) \subseteq \{e_\alpha\}'$. Suppose $h + n \in \{e_\alpha\}'$, where $h \in \mathfrak{h}$, $n \in \mathfrak{n}$. By $[h+n, e_\alpha] = \alpha(h)e_\alpha + [n, e_\alpha] = 0$, we have that $\alpha(h) = 0$ and $[n, e_\alpha] = 0$. Thus $h + n \in ker\,\alpha \oplus (\bigoplus_{\beta \in Y_\alpha} \mathfrak{g}_\beta)$. So $\{e_\alpha\}' = ker\,\alpha \oplus (\bigoplus_{\beta \in Y_\alpha} \mathfrak{g}_\beta)$. Suppose $h_0 + n_0 \in \{e_\alpha\}''$, where $h_0 \in \mathfrak{h}$, $n_0 \in \mathfrak{n}$. Then $[h_0 + n_0, e_\beta] = 0$ for all $\beta \in Y_\alpha$. This implies that $\beta(h_0) = 0$ for all $\beta \in Y_\alpha$, forcing $h_0 = 0$ (thanks to Lemma 2.1). Write n_0 as $n_0 = \sum_{\beta \in \Phi^+} a_\beta e_\beta$. For any $h \in ker\,\alpha$, by $[h, n_0] = 0$, we have that $\beta(h)a_\beta = 0$ for $\beta \in \Phi^+$. This implies that $ker\,\alpha \subseteq ker\,\beta$, when $a_\beta \neq 0$. So $\beta = \alpha$ when $a_\beta \neq 0$. Therefore, $n_0 = a_\alpha e_\alpha \in \mathfrak{g}_\alpha$. Hence $\{e_\alpha\}'' \subseteq \mathfrak{g}_\alpha$. Finally, $\{e_\alpha\}'' = \mathfrak{g}_\alpha$. $\square$

Lemma 2.3. *$\{d_\alpha\}'' = F\,d_\alpha$ for any $\alpha \in \Delta$.*

Proof. Obviously, $\{d_\alpha\}' = \mathfrak{h} \oplus (\bigoplus_{\beta(d_\alpha)=0} \mathfrak{g}_\beta)$. If $x = h_0 + n_0 \in \{d_\alpha\}''$, where $h_0 \in \mathfrak{h}$, $n_0 \in \mathfrak{n}$. Then $n_0 = 0$, since $[\mathfrak{h}, n_0] = 0$. Thus $x = h_0 \in \mathfrak{h}$. By $[h_0, \mathfrak{g}_\beta] = 0$ for all $\alpha \neq \beta \in \Delta$, we have that $h_0 \in F\,d_\alpha$. So $x \in F\,d_\alpha$ and $\{d_\alpha\}'' \subseteq F\,d_\alpha$. Another direction is clear. Therefore, $\{d_\alpha\}'' = F\,d_\alpha$. $\square$

Lemma 2.4. *Let $x = h_0 + n_0 \in \mathfrak{b}$, where $h_0 \in \mathfrak{h}$, $n_0 \in \mathfrak{n}$. If h_0 and n_0 are both nonzero, and $[h_0, n_0] = 0$, then the dimension of $\{x\}''$ is at least two.*

Proof. It suffices to prove that, for $y \in \mathfrak{b}$, if $[x, y] = 0$, then $[h_0, y] = [n_0, y] = 0$. Write n_0 as $n_0 = \sum_{\alpha \in \Phi^+} x_\alpha e_\alpha$, and suppose $y = h + n$, where $h \in \mathfrak{h}$, $n = \sum_{\alpha \in \Phi^+} y_\alpha e_\alpha \in \mathfrak{n}$. If we can show that $[h_0, n] = 0$ then the aim is achieved. Otherwise, if $[h_0, n] =$

$\sum_{\alpha\in\Phi^+} y_\alpha\alpha(h_0)e_\alpha \neq 0$, assume that α_0 is a positive root with the lowest height for which $y_{\alpha_0}\alpha_0(h_0) \neq 0$. It's clear that in the expression of $[h, n_0] = \sum_{\alpha\in\Phi^+} \alpha(h)x_\alpha e_\alpha$, the term $\alpha_0(h)x_{\alpha_0}e_{\alpha_0}$ is zero (note that $x_{\alpha_0} = 0$). On the other hand, in the expression of $[n_0, n] = \sum_{\alpha,\beta\in\Phi^+} x_\alpha y_\beta N_{\alpha,\beta}e_{\alpha+\beta}$, the coefficient of e_{α_0} is $\sum_{\alpha+\beta=\alpha_0} x_\alpha y_\beta N_{\alpha,\beta}$. We now intend to show that it takes the value 0. For $\alpha, \beta \in \Phi^+$ satisfying $\alpha + \beta = \alpha_0$, if $\alpha(h_0) \neq 0$, then $x_\alpha = 0$, thus $x_\alpha y_\beta = 0$. Suppose $\alpha(h_0) = 0$, then $\beta(h_0) \neq 0$ (since $\alpha_0(h_0) \neq 0$). If $x_\alpha = 0$ then $x_\alpha y_\beta = 0$. If $x_\alpha \neq 0$, for the aim to show $x_\alpha y_\beta = 0$, we need to show that $y_\beta = 0$. Otherwise, β is a positive root with the lower height than that of α_0 for which $\beta(h_0)y_\beta \neq 0$. This contradicts to the way of chosen of α_0. So we have that $\sum_{\alpha+\beta=\alpha_0} x_\alpha y_\beta N_{\alpha,\beta} = 0$. So $[x, y] = [h_0, n] - [h, n_0] + [n_0, n] \neq 0$, the absurd. □

Lemma 2.5. *Let $l \geqslant 2$ and $0 \neq h_0 \in \mathfrak{h}$. Then $dim\{h_0\}' \leqslant dim\, \mathfrak{b} - 2$.*

Proof. Firstly, $\{h_0\}' = \mathfrak{h} \oplus (\sum_{\alpha(h_0)=0} \mathfrak{g}_\alpha)$. Let $\Phi_0^+(h_0)$ be the subset of Φ^+ consisting of positive roots which take value 0 at h_0. Then $dim\{h_0\}' = l + |\Phi_0^+(h_0)|$ (where $|\Phi_0^+(h_0)|$ means the number of elements of $\Phi_0^+(h_0)$). $h_0 \neq 0$ implies that $\alpha(h_0) \neq 0$ for certain $\alpha \in \Delta$. If $\theta(h_0) \neq 0$, then the result is proved. If $\theta(h_0) = 0$, then there exists some $\alpha \neq \beta \in \Delta$ such that $\beta(h_0) \neq 0$. In this case, the assertion also holds. □

Lemma 2.6. *Let β be a long positive root, then β can be written in the form $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $\alpha_i \in \Delta$, $k = ht\,\beta$, such that α_1 is long and all partial sums $\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 1, 2, \ldots, k$, are positive roots.*

Proof. If Φ is of type G_2, the result can be checked directly to be true. For the left eight cases, we will give the proof by induction on $ht\,\beta$. If $ht\,\beta = 1$, the assertion naturally holds. Assume that the assertion holds for $ht\,\beta \leqslant m - 1$ and consider the case $ht\,\beta = m$. Write β in the form $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, $\alpha_i \in \Delta$, such that all partial sums $\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 1, 2, \ldots, m$, are roots. If α_1 is long, this expression has already satisfied what required. If α_1 is short, since β_1 is short and β_m is long, we assume that t ($1 \leqslant t \leqslant m - 1$) is the minimal positive integer such that β_t is short and β_{t+1} is long. Then $\beta_t - \alpha_{t+1}$ is also a long positive root. Now the height of $\beta_t - \alpha_{t+1}$ is smaller than that of β. By induction assumption, we know that $\beta_t - \alpha_{t+1}$ can be written as $\beta_t - \alpha_{t+1} = \alpha'_1 + \alpha'_2 + \cdots + \alpha'_{t-1}$ with $\alpha'_i \in \Delta$, such that α'_1 is long and all partial sums $\alpha'_1 + \alpha'_2 + \cdots + \alpha'_i$, $i = 1, 2, \ldots, t - 1$, are positive roots. Now β has a new expression $\beta = \alpha'_1 + \alpha'_2 + \cdots + \alpha'_{t-1} + \alpha_{t+1} + \alpha_{t+1} + \alpha_{t+2} + \cdots + \alpha_m$, which satisfies what required. □

Lemma 2.7. *Let α, β, η be roots, k a positive integer. If $\beta + k\alpha$ and $(\beta + k\alpha) + \eta$ are both roots, and $\beta + \eta \notin \Phi \cup \{0\}$, then $\eta + \alpha \in \Phi \cup \{0\}$.*

Proof. By assumption, we have $[e_\eta, [e_\alpha, [e_\alpha, [\ldots [e_\alpha, e_\beta] \ldots]]]] \neq 0$ (where the inner derivation $ad\, e_\alpha$ acts on e_β k times), and $[e_\eta, e_\beta] = 0$. Using Jacobi identity repeatedly, we have that $[e_\eta, e_\alpha] \neq 0$. Thus $\eta + \alpha \in \Phi \cup \{0\}$. □

Lemma 2.8. *Let Φ be an irreducible root system of type not G_2, $\beta \in \Phi^+$ with $ht\,\beta \geqslant 2$. Then there exists $\alpha \in \Delta$ such that $\beta - \alpha \in \Phi^+$ but $\beta + \alpha \notin \Phi^+$.*

Proof. We can find certain $\alpha_0 \in \Delta$ such that $(\beta, \alpha_0) > 0$. Otherwise, if $(\beta, \alpha) \leqslant 0$ for all $\alpha \in \Delta$, then $\Delta \cup \{\beta\}$ is linear independent, the absurd. By $(\beta, \alpha_0) > 0$ we have $\langle \beta, \alpha_0 \rangle \geqslant 1$. This implies that $\beta - \alpha_0 \in \Phi^+$ and $\beta + \alpha_0 \notin \Phi^+$ (note that Φ is not of type G_2). □

Lemma 2.9. *Let Φ be an irreducible root system of type not G_2, $\beta \in \Phi^+$, $\alpha \in \Delta$, $\gamma = \beta + \alpha \in \Phi^+$.*

(i) If β is long and γ is short, then $M_\beta \leqslant M_\gamma$.
(ii) If β is the unique maximal short root θ_s, then $M_\beta = M_\gamma + 1$.
(iii) If β, γ have the same root length, then $M_\beta = M_\gamma + 1$.
(iv) If $\gamma' = \gamma + \alpha$ is also a root, then $M_\beta = M_{\gamma'} + 1$.
(v) If β is short and γ is long, then $M_\beta \geqslant M_\gamma + 1$.

Proof. Let $\alpha \neq \eta \in X_\beta \setminus X_\gamma$, i.e., $\eta + \beta \in \Phi^+$ but $\eta + \gamma \notin \Phi^+$. We have, by Lemma 2.7, that $\eta - \alpha \in \Phi^+$. It's clear that $\eta - \alpha \in X_\gamma$. Similarly, if $\varepsilon \in X_\gamma \setminus X_\beta$, then we have that $\varepsilon + \alpha \in X_\beta$.

For (i), the fact that $\beta \in \Phi_l^+$, $\gamma \in \Phi_s^+$ implies that $\alpha \in \Delta_s$, $\langle \beta, \alpha \rangle = -2$ and $\alpha \in X_\beta \cap X_\gamma$. If $\eta \in X_\beta \setminus X_\gamma$, then we have, by above discussions, that $\eta - \alpha \in X_\gamma$. We now show that $\eta - \alpha \notin X_\beta$. Otherwise, $\beta + (\eta - \alpha) \in \Phi^+$. Thus $\langle \beta + \eta, \alpha \rangle = 1$ or 2 (recall that $\beta + \eta + \alpha \notin \Phi^+$). Since $\langle \beta, \alpha \rangle = -2$, we obtain $\langle \eta, \alpha \rangle \geqslant 3$, the absurd. Now we can set up an injective map from X_β to X_γ, which fixes each element in $X_\beta \cap X_\gamma$, and maps any $\eta \in X_\beta \setminus X_\gamma$ to $\eta - \alpha \in X_\gamma \setminus X_\beta$. Therefore, $M_\beta \leqslant M_\gamma$.

In case (ii), we see that $\alpha \in \Delta_s$, $\langle \beta, \alpha \rangle = 0$ and $\alpha \in X_\beta \setminus X_\gamma$. Let $\alpha \neq \eta \in X_\beta \setminus X_\gamma$, then we also have that $\eta - \alpha \in X_\gamma$. Actually, η is a short root, since $\eta + \beta$ is a long root. If $\eta - \alpha \in X_\beta$, then by $\beta + \eta - \alpha \in \Phi^+$ and $\beta + \eta + \alpha \notin \Phi^+$, we have $1 \leqslant \langle \beta + \eta, \alpha \rangle \leqslant 2$. Since $\beta + \eta$ is long and α is short, we obtain $\langle \beta + \eta, \alpha \rangle = 2$. This follows that $\langle \eta, \alpha \rangle = 2$, contradiction with the fact that η is short. So $\eta - \alpha \in X_\gamma \setminus X_\beta$. Conversely, let $\varepsilon \in X_\gamma \setminus X_\beta$, then we have that $\varepsilon + \alpha \in X_\beta$. If it also lies in X_γ, then we have $-2 \leqslant \langle \gamma + \varepsilon, \alpha \rangle \leqslant -1$, since $\gamma + \varepsilon + \alpha \in \Phi^+$ and $\gamma + \varepsilon - \alpha \notin \Phi^+$. Because $\gamma + \varepsilon$ is long and α is short, we see that $\langle \gamma + \varepsilon, \alpha \rangle = -2$. Thus $\langle \varepsilon, \alpha \rangle = -4$ (recall $\langle \gamma, \alpha \rangle = 2$), the absurd. Hence $\varepsilon + \alpha \in X_\beta \setminus X_\gamma$. Now we can set up a bijective map from $X_\beta \setminus \{\alpha\}$ to X_γ, which fixes each element in $X_\beta \cap X_\gamma$, and sends any $\alpha \neq \eta \in X_\beta \setminus X_\gamma$ to $\eta - \alpha \in X_\gamma \setminus X_\beta$. Therefore, $M_\beta = M_\gamma + 1$.

In case (iii), $\langle \beta, \alpha \rangle = -1$ and $\alpha \in X_\beta \setminus X_\gamma$. Let $\alpha \neq \eta \in X_\beta \setminus X_\gamma$, then $\eta - \alpha \in X_\gamma$. If $\eta - \alpha \in X_\beta$, we also have that $1 \leqslant \langle \beta + \eta, \alpha \rangle \leqslant 2$. Then by $\langle \beta, \alpha \rangle = -1$, we have that $\langle \eta, \alpha \rangle = 2$, and $\langle \beta + \eta, \alpha \rangle = 1$. So $\eta - 2\alpha \in \Phi^+$, $\gamma + (\eta - 2\alpha) = \beta + \eta - \alpha \in \Phi^+$ and $\beta + (\eta - 2\alpha) \notin \Phi^+$, i.e., $\eta - 2\alpha \in X_\gamma \setminus X_\beta$. Conversely, let $\varepsilon \in X_\gamma \setminus X_\beta$, then we have that $\varepsilon + \alpha \in X_\beta$. If it also lies in X_γ, then we also have that $-2 \leqslant \langle \gamma + \varepsilon, \alpha \rangle \leqslant -1$. By $\langle \gamma, \alpha \rangle = \langle \beta + \alpha, \alpha \rangle = 1$, we have $\langle \varepsilon, \alpha \rangle = -2$ and $\langle \gamma + \varepsilon, \alpha \rangle = -1$. So $\varepsilon + 2\alpha \in X_\beta \setminus X_\gamma$. Now we can set up a bijective map τ from $X_\beta \setminus \{\alpha\}$ to X_γ, in such a way that $\tau(\eta) = \eta$

for $\eta \in X_\beta \cap X_\gamma$; $\tau(\eta) = \eta - \alpha$ for η (distinct with α) in $X_\beta \setminus X_\gamma$ satisfying $\eta - \alpha \notin X_\beta$; and $\tau(\eta) = \eta - 2\alpha$ for η (distinct with α) in $X_\beta \setminus X_\gamma$ satisfying $\eta - \alpha \in X_\beta$. Therefore, $M_\beta = M_\gamma + 1$.

In case (iv), β is long and α is short, $\langle \beta, \alpha \rangle = -2$, and $\beta + \alpha$ is a short root, $\beta + 2\alpha$ is a long root. It's easy to see that $\alpha \in X_\beta \setminus X_{\gamma'}$. Let $\alpha \neq \eta \in X_\beta \setminus X_{\gamma'}$, then $\eta - \alpha \in \Phi^+$. If η is long, then $\langle \eta, \alpha \rangle = 2, \langle \beta + \eta, \alpha \rangle = 0$. This says that $\eta - 2\alpha \in \Phi^+$. It's not difficult to verify that $\eta - 2\alpha \in X_{\gamma'} \setminus X_\beta$. If η is short, then $\langle \eta, \alpha \rangle = 1$, $\langle \beta + \eta, \alpha \rangle = -1$. This says that $\eta - \alpha \in \Phi^+$. Then we have $\eta - \alpha \in X_{\gamma'} \setminus X_\beta$. Conversely, let $\varepsilon \in X_{\gamma'} \setminus X_\beta$, then we see $\varepsilon + \alpha \in X_\beta$. If ε is long then $\langle \varepsilon, \alpha \rangle = -2$ and $\langle \gamma' + \varepsilon, \alpha \rangle = 0$. Thus $\varepsilon + 2\alpha \in \Phi^+$. Furthermore, we have that $\varepsilon + 2\alpha \in X_\beta \setminus X_{\gamma'}$. If ε is short then $\langle \varepsilon, \alpha \rangle = -1$ and $\langle \gamma' + \varepsilon, \alpha \rangle = 1$. Thus $\varepsilon + \alpha \in \Phi^+$. Furthermore, we have that $\varepsilon + \alpha \in X_\beta \setminus X_{\gamma'}$. Now we can set up a bijective map τ from $X_\beta \setminus \{\alpha\}$ to $X_{\gamma'}$ in such a way that, $\tau(\eta) = \eta$ for $\eta \in X_\beta \cap X_{\gamma'}$; $\tau(\eta) = \eta - 2\alpha$ for $\alpha \neq \eta \in X_\beta \setminus X_{\gamma'}$ being a long root; $\tau(\eta) = \eta - \alpha$ for $\alpha \neq \eta \in X_\beta \setminus X_{\gamma'}$ being a short root. Finally, we also have $M_\beta = M_{\gamma'} + 1$.

In case (v), $\langle \beta, \alpha \rangle = 0$ and $\beta - \alpha$ is a long root. Denote $\beta - \alpha$ by β_1. Then we have, by (i), that $M_{\beta_1} \leqslant M_\beta$, and by (iv), that $M_{\beta_1} = M_\gamma + 1$. Hence $M_\beta \geqslant M_\gamma + 1$. $\square$

Lemma 2.10. *Let Φ be an irreducible root system with only one root length. Then for any $\beta \in \Phi^+$, $M_\beta = ht\,\theta - ht\,\beta$.*

Proof. We give the proof by induction on $ht\,\beta$. If $ht\,\beta = ht\,\theta$, the result obviously holds. Assume that the assertion holds for $ht\,\beta = k$ $(2 \leqslant k \leqslant ht\,\theta)$, and consider the case that $ht\,\beta = k - 1$. Find $\alpha \in \Delta$ such that $\gamma = \beta + \alpha \in \Phi^+$. It follows from (iii) of Lemma 2.9 that $M_\beta = M_\gamma + 1$. By induction assumption, $M_\gamma = ht\,\theta - ht\,\gamma$. Therefore, $M_\beta = ht\,\theta - ht\,\beta$. $\square$

Lemma 2.11. *Let Φ be an irreducible root system with two root lengths and suppose it is not of type G_2.*

(i) If $\beta \in \Phi^+$ is short, then $M_\beta = ht\,\theta - ht\,\beta$. In particular, $M_\beta = ht\,\theta - 1$ for any $\beta \in \Delta_s$.
(ii) M_α are equal for all $\alpha \in \Delta_l$; and $M_\alpha < ht\,\theta - 1$ for any $\alpha \in \Delta_l$.
(iii) If $\beta \in \Phi^+$ is a long root with $ht\,\beta \geqslant 2$, then $M_\beta < M_\alpha$ for any $\alpha \in \Delta_l$.
(iv) If $\beta \in \Phi^+$ is long, then $M_\beta \leqslant ht\,\theta - ht\,\beta$.

Proof. We give the proof of (i) by steps.

Step 1. If $\theta_s \prec \beta \in \Phi^+$ and $\theta_s \neq \beta$, then $M_\beta = ht\,\theta - ht\,\beta$.

We use induction on $ht\,\beta$ to complete the proof of this step. If $ht\,\beta = ht\,\theta$, i.e., $\beta = \theta$, the result naturally holds. Assume that the assertion holds when $ht\,\beta = k$. For the case that $ht\,\beta = k - 1$. Find $\alpha \in \Delta$ such that $\gamma = \beta + \alpha$ is a root. Note that β and γ are both long roots. So by (iii) of Lemma 2.9, we have that $M_\beta = M_\gamma + 1$. By induction assumption, $M_\gamma = ht\,\theta - ht\,\gamma$. Therefore, $M_\beta = ht\,\theta - ht\,\beta$.

Step 2. $M_{\theta_s} = ht\,\theta - ht\,\theta_s$.

For θ_s, we can find some $\alpha \in \Delta$ such that $\gamma = \beta + \alpha$ is a root. Then we have, by (ii) of Lemma 2.9, that $M_{\theta_s} = M_\gamma + 1 = ht\,\theta - ht\,\gamma + 1 = ht\,\theta - ht\,\theta_s$.

Step 3. For any short positive root β, $M_\beta = ht\,\theta - ht\,\beta$.

For θ_s, one will see (by directly check for Φ of different types) that there exist simple roots $\alpha_1, \alpha_2, \ldots, \alpha_t$, $t = ht\,\theta_s - ht\,\beta$ such that $\theta_s = \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_t$, and all partial sums $\beta_i = \beta + \alpha_1 + \cdots + \alpha_i$, $i = 0, 1, \ldots, t$, are short roots (where β_0 and β_t refer to β and θ_s, respectively). Thus by (iii) of Lemma 2.9, we have that

$$\begin{aligned} M_\beta &= M_{\beta_1} + 1 = M_{\beta_2} + 2 = \cdots = M_{\beta_t} + t \\ &= ht\,\theta - ht\,\theta_s + t = ht\,\theta - ht\,\beta. \end{aligned}$$

For (ii), set $\lambda = \sum_{\alpha \in \Delta_l} \alpha$, then λ is a long root. Thus we have, by (iii) of Lemma 2.9, that $M_\alpha = M_\lambda + ht\,\lambda - 1$ for any $\alpha \in \Delta_l$. This says that all M_α are equal for $\alpha \in \Delta_l$. We can write θ in the form $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, $m = ht\,\theta$, $\alpha_i \in \Delta$, such that α_1 is long and all partial sums $\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 1, 2, \ldots, m$, are roots (thanks to Lemma 2.6). Note that there exists certain α_i (in the expression of θ as the linear combination of simple roots) whose length is short. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ are all long and α_k is short. Then β_{k-1} is long and β_k is short. Then by Lemma 2.9, we know that $M_{\beta_{k-1}} \leqslant M_{\beta_k}$. Furthermore,

$$M_{\alpha_1} = M_{\beta_{k-1}} + k - 2 \leqslant M_{\beta_k} + k - 2 = ht\,\theta - ht\,\beta_k + k - 2 = ht\,\theta - 2.$$

Thus $M_\alpha < ht\,\theta - 1$ for all $\alpha \in \Delta_l$.

For (iii), we write β in the form $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $k = ht\,\beta$, $\alpha_i \in \Delta$, such that α_1 is long and all partial sums $\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 1, 2, \ldots, k$, are roots (thanks to Lemma 2.6). If all β_i, $i = 1, 2, \ldots, k$, are long roots, then we have, by (iii) of Lemma 2.9, that $M_\beta = M_{\alpha_1} - (k-1) < M_{\alpha_1}$. If certain β_i is short, we use induction on $ht\,\beta$ to show that $M_\beta < M_{\alpha_1}$. In this case, we may assume that β_{i-1} is short but $\beta_i, \ldots, \beta_k$ are all long. Then $\beta_i - 2\alpha_i$ is a long root. Denote it by γ. Then we have, by Lemma 2.9, that $M_{\beta_i} = M_\gamma - 1$. Note that γ is a long root. So we have, by induction assumption, that $M_\gamma \leqslant M_{\alpha_1}$. Therefore $M_\beta = M_{\beta_i} - (k-i) \leqslant M_{\alpha_1} - 1 - (k-i) < M_{\alpha_1}$.

For (iv), we write θ in the form $\theta = \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $k = ht\,\theta - ht\,\beta$, $\alpha_i \in \Delta$, such that all partial sums $\beta_i = \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 0, 1, 2, \ldots, k$, are roots (β_0 and β_k refer to β and θ respectively). If all β_i, $i = 0, 1, \ldots, k$, are long roots, then we have, by (iii) of Lemma 2.9, that $M_\beta = ht\,\theta - ht\,\beta$. If certain β_i is short, we use decreasing induction on $ht\,\beta$ to complete the proof. In this case, we assume that $\beta_0, \beta_1, \ldots, \beta_{i-1}$ are long but β_i is short. Then $\gamma = \beta_{i-1} + 2\alpha_i$ is also a long root. By (iv) of Lemma 2.9, we have that $M_{\beta_{i-1}} = M_\gamma + 1$. By induction assumption, $M_\gamma \leqslant ht\,\theta - ht\,\gamma$. So $M_{\beta_{i-1}} \leqslant ht\,\theta - ht\,\beta_{i-1} - 1$. Therefore, $M_\beta = M_{\beta_{i-1}} + i - 1 \leqslant ht\,\theta - ht\,\beta - 1 < ht\,\theta - ht\,\beta$. $\square$

Let $\mathfrak{d}$ be a subspace of $\mathfrak{h}$, and define $\Phi_0(\mathfrak{d})$ to be the subset of Φ consisting of $\alpha \in \Phi$ satisfying $\alpha(d) = 0$ for all $d \in \mathfrak{d}$, $\Phi_0^+(\mathfrak{d}) = \Phi_0(\mathfrak{d}) \cap \Phi^+$. It's easy to see that, if $\Phi_0(\mathfrak{d})$ is not the empty set, then it forms a new root system in the subspace spanned by $\Phi_0(\mathfrak{d})$. Now put $U(\mathfrak{d})$ to be the subgroup of U generated by all $\sigma_\alpha(t)$ for $\alpha \in \Phi_0^+(\mathfrak{d})$, $t \in F$. Obviously, $\sigma(d) = d$ for any $\sigma \in U(\mathfrak{d})$ and $d \in \mathfrak{d}$.

Lemma 2.12. *For any fixed $x \in \{\mathfrak{d}\}'$, we can find certain $\sigma \in U(\mathfrak{d})$ such that $\sigma(x) = h+n$, where $h \in \mathfrak{h}$, $n \in C_{\mathfrak{n}}(\mathfrak{d})$ and $[h,n] = 0$.*

Proof. It's clear that $\{\mathfrak{d}\}' = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Phi_0^+(\mathfrak{d})} \mathfrak{g}_\alpha)$. Write x in the form $x = h + \sum_{\alpha\in\Phi_0^+(\mathfrak{d})} a_\alpha e_\alpha$, $a_\alpha \in F$, $h \in \mathfrak{h}$. For a positive integer i, set $\Phi_*^i = \{\alpha \in \Phi^i \mid \alpha(h) \neq 0\}$; $\Phi_0^i = \{\alpha \in \Phi^i \mid \alpha(h) = 0\}$. Take $\sigma_1 = \prod_{\alpha\in\Phi_*^1\cap\Phi_0^+(\mathfrak{d})} \sigma_\alpha(\alpha(h)^{-1}a_\alpha)$, where the product is taken according to any fixed order. Then

$$\sigma_1(x) \equiv h + \sum_{\alpha\in\Phi_0^1\cap\Phi_0^+(\mathfrak{d})} a_\alpha e_\alpha \pmod{\mathfrak{n}_2}.$$

Now suppose

$$\sigma_1(x) \equiv h + \sum_{\alpha\in\Phi_0^1\cap\Phi_0^+(\mathfrak{d})} a_\alpha e_\alpha + \sum_{\alpha\in\Phi^2\cap\Phi_0^+(\mathfrak{d})} b_\alpha e_\alpha \pmod{\mathfrak{n}_3},$$

and take $\sigma_2 = \prod_{\alpha\in\Phi_*^2\cap\Phi_0^+(\mathfrak{d})} \sigma_\alpha(\alpha(h)^{-1}b_\alpha)$. Then

$$(\sigma_2\sigma_1)(x) \equiv h + \sum_{\alpha\in\Phi_0^1\cap\Phi_0^+(\mathfrak{d})} a_\alpha e_\alpha + \sum_{\alpha\in\Phi_0^2\cap\Phi_0^+(\mathfrak{d})} b_\alpha e_\alpha \pmod{\mathfrak{n}_3}.$$

Continuing such process, after $ht\,\theta$ steps, we can choose $\sigma_1, \sigma_2, \dots, \sigma_m$ ($m = ht\,\theta$) such that $(\sigma_m \cdots \sigma_2\sigma_1)(x) = h + n$, where $n \in C_{\mathfrak{n}}(\mathfrak{d})$ and $[h,n] = 0$, as desired. □

Let φ be an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions. Then for an arbitrary subset A of $\mathfrak{b}$, it is clear that $\varphi(\{A\}') = \{\varphi(A)\}'$ and $\varphi(\{A\}'') = \{\varphi(A)\}''$.

Lemma 2.13. *Suppose rank $\mathfrak{g} \geqslant 2$, then $\varphi(\mathfrak{g}_\theta) = \mathfrak{g}_\theta$.*

Proof. Firstly, we can choose $\sigma \in U$, by Lemma 2.12, such that $(\sigma \circ \varphi)(e_\theta) = h + n$, where $h \in \mathfrak{h}$, $n \in \mathfrak{n}$ and $[h,n] = 0$. It follows from $\{e_\theta\}'' = \mathfrak{g}_\theta$ that $\{h+n\}''$, as the image of $\{e_\theta\}''$ under $\sigma \circ \varphi$, is one dimensional. Thus $h = 0$ or $n = 0$ (recall Lemma 2.4). If $n = 0$, then $\{h+n\}' = \{h\}'$ has the dimension $\leqslant dim\,\mathfrak{b} - 2$ (recall Lemma 2.5). However the dimension of $\{e_\theta\}'$ is $dim\,\mathfrak{b} - 1$, the absurd. So $h = 0$ and $n \neq 0$. Suppose $n = \sum_{\alpha\in\Phi^+} a_\alpha e_\alpha$. If there exist distinct positive roots α, β such that a_α and a_β both are nonzero. Then we have, by $C_{\mathfrak{h}}(n) \subseteq ker\,\alpha \cap ker\,\beta$, that $dim\,C_{\mathfrak{h}}(n) \leqslant l - 2$. Since

$$dim\,C_{\mathfrak{h}}(n) = dim\big(\{n\}' \cap \mathfrak{h}\big) = dim\{n\}' + dim\,\mathfrak{h} - dim\big(\{n\}' + \mathfrak{h}\big),$$

we have that $dim\{n\}' \leqslant dim\, C_{\mathfrak{h}}(n) + dim\, \mathfrak{b} - l \leqslant dim\, \mathfrak{b} - 2$, which contradicts with the fact that $dim\{e_\theta\}' = dim\, \mathfrak{b} - 1$. Now we see that there exists certain $\beta \in \Phi^+$ such that $n \in \mathfrak{g}_\beta$. Since $dim\{n\}' = dim\{e_\theta\}' = dim\, \mathfrak{b} - 1$ and $\{n\}' = ker\, \beta \oplus C_{\mathfrak{n}}(e_\beta)$, we have that $C_{\mathfrak{n}}(e_\beta) = \mathfrak{n}$, which shows that $\beta = \theta$. Now we see that $(\sigma \circ \varphi)(\mathfrak{g}_\theta) = \mathfrak{g}_\theta$. It follows that $\varphi(\mathfrak{g}_\theta) = \mathfrak{g}_\theta$, since $\sigma^{-1}(\mathfrak{g}_\theta) = \mathfrak{g}_\theta$. $\square$

Lemma 2.14. *Let $rank\, \mathfrak{g} \geqslant 2$, φ be an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions. There exists $\sigma \in U$ such that $(\sigma \circ \varphi)(\mathfrak{h}) = \mathfrak{h}$.*

Proof. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, and let $\mathfrak{d}_\Delta = \{d_{\alpha_1}, d_{\alpha_2}, \ldots, d_{\alpha_l}\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. Now consider the action of φ on $d_{\alpha_1}, d_{\alpha_2}, \ldots, d_{\alpha_l}$ in turn. Suppose that $\varphi(d_{\alpha_1}) = h_1 + v_1$, $h_1 \in \mathfrak{h}$, $v_1 \in \mathfrak{n}$. By Lemma 2.12, we can find $\sigma_1 \in U$ such that $(\sigma_1 \circ \varphi)(d_{\alpha_1}) = h_1 + n_1$, where $n_1 \in \mathfrak{n}$ and $[h_1, n_1] = 0$. Since $\{d_{\alpha_1}\}'' = F d_{\alpha_1}$ is one dimensional, we see that $\{h_1 + n_1\}''$ is also one dimensional. Thus either $h_1 = 0$ or $n_1 = 0$ (using Lemma 2.4). Since $\mathfrak{g}_\theta$ is not contained in $\{d_{\alpha_1}\}'$ and $\mathfrak{g}_\theta$ is stable under $\sigma_1 \circ \varphi$, we know that $\mathfrak{g}_\theta \nsubseteq \{h_1 + n_1\}'$. This shows that $n_1 = 0$ and $h_1 \neq 0$. Denote $\sigma_1 \circ \varphi$ by φ_1. Suppose that $\varphi_1(d_{\alpha_2}) = h_2 + v_2$, $h_2 \in \mathfrak{h}$, $v_2 \in \mathfrak{n}$. Then $h_2 + v_2 \in \{Fh_1\}'$. By Lemma 2.12, we can find $\sigma_2 \in U(Fh_1)$ such that $(\sigma_2 \circ \varphi_1)(d_{\alpha_2}) = h_2 + n_2$, where $n_2 \in C_{\mathfrak{n}}(Fh_1)$ and $[h_2, n_2] = 0$. Similar as above, we obtain that $n_2 = 0$ and $h_2 \neq 0$. Denote $\sigma_2 \circ \varphi_1$ by φ_2. Note that $\varphi_2(d_{\alpha_j}) = h_j$, $j = 1, 2$. Generally, suppose that we have found $\sigma_1 \in U$, $\sigma_2 \in U(Fh_1), \ldots, \sigma_{k-1} \in U(\sum_{i=1}^{k-2} Fh_i)$ such that

$$(\sigma_{k-1} \circ \sigma_{k-2} \circ \cdots \circ \sigma_1 \circ \varphi)(d_{\alpha_i}) = h_i \in \mathfrak{h}, \quad i = 1, 2, \ldots, k-1.$$

Denote $\sigma_{k-1} \circ \sigma_{k-2} \circ \cdots \circ \sigma_1 \circ \varphi$ by φ_{k-1}. We now intend to find $\sigma_k \in U(\sum_{i=1}^{k-1} Fh_i)$ such that $(\sigma_k \circ \varphi_{k-1})(d_{\alpha_j}) \in \mathfrak{h}$ for $j = 1, 2, \ldots, k$. By $[d_{\alpha_k}, d_i] = 0$, for $i = 1, 2, \ldots, k-1$, we know that $[\varphi_{k-1}(d_{\alpha_k}), \sum_{i=1}^{k-1} Fh_i] = 0$. Thus $\varphi_{k-1}(d_{\alpha_k}) \in \{\sum_{i=1}^{k-1} Fh_i\}'$. Suppose $\varphi_{k-1}(d_{\alpha_k}) = h_k + v_k$, where $h_k \in \mathfrak{h}$, $v_k \in C_{\mathfrak{n}}(\sum_{i=1}^{k-1} Fh_i)$. By Lemma 2.12, we can find $\sigma_k \in U(\sum_{i=1}^{k-1} Fh_i)$ such that $(\sigma_k \circ \varphi_{k-1})(d_{\alpha_k}) = h_k + n_k$, where $n_k \in C_{\mathfrak{n}}(\sum_{i=1}^{k-1} Fh_i)$ satisfying $[h_k, n_k] = 0$. Discussing as the first step, we also have $n_k = 0$. So $(\sigma_k \circ \varphi_{k-1})(d_{\alpha_k}) = h_k \in \mathfrak{h}$. Note that σ_k fixes each h_i for $i = 1, 2, \ldots, k-1$. So $(\sigma_k \circ \varphi_{k-1})(d_{\alpha_i}) = h_i$, $i = 1, 2, \ldots, k$. Finally, we have by induction that there exist $\sigma_1, \sigma_2, \ldots, \sigma_l \in U$, such that

$$(\sigma_l \circ \sigma_{l-1} \circ \cdots \circ \sigma_1 \circ \varphi)(d_{\alpha_i}) = h_i \in \mathfrak{h}, \quad i = 1, 2, \ldots, l.$$

Set $\sigma = \sigma_l \circ \sigma_{l-1} \circ \cdots \circ \sigma_1$ we have that $(\sigma \circ \varphi)(\mathfrak{h}) = \mathfrak{h}$. $\square$

Lemma 2.15. *Let φ be an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions. If $\varphi(\mathfrak{h}) = \mathfrak{h}$, then $\varphi(\mathfrak{n}) = \mathfrak{n}$.*

Proof. For any $\alpha \in \Phi^+$, we can find $\sigma \in U$ such that $(\sigma \circ \varphi)(e_\alpha) = h + n$, where $h \in \mathfrak{h}$, $n \in \mathfrak{n}$ satisfy $[h, n] = 0$. Since $\{e_\alpha\}''$ is one dimensional we know that $\{h + n\}''$ is also one dimensional. Thus we know, by Lemma 2.4, that either $h = 0$ or $n = 0$. If $n = 0$ then $(\varphi^{-1} \circ \sigma^{-1})(h) = e_\alpha$, which is impossible (recall $\varphi(\mathfrak{h}) = \mathfrak{h}$). So $h = 0$ and $n \neq 0$. Then $(\sigma \circ \varphi)(e_\alpha) \in \mathfrak{n}$. Obviously, $\sigma^{-1}(\mathfrak{n}) = \mathfrak{n}$, from which it follows that $\varphi(e_\alpha) \in \mathfrak{n}$. Therefore, $\varphi(\mathfrak{n}) = \mathfrak{n}$. □

Lemma 2.16. *Let φ be an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions. If $\varphi(\mathfrak{h}) = \mathfrak{h}$, then for any given $\alpha \in \Phi^+$, there exists $\beta \in \Phi^+$ such that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$.*

Proof. It's easy to see that

$$dim\, C_{\mathfrak{h}}\big(\varphi(e_\alpha)\big) = dim\, C_{\mathfrak{h}}(e_\alpha) = dim(ker\, \alpha) = l - 1.$$

Write $\varphi(e_\alpha)$ as $\varphi(e_\alpha) = \sum_{\beta \in \Phi^+} a_\beta e_\beta$ (recall that $\varphi(\mathfrak{n}) = \mathfrak{n}$). If there exist distinct positive roots β_1, β_2 such that a_{β_1}, a_{β_2} both are nonzero, then by $C_{\mathfrak{h}}(\varphi(e_\alpha)) \subseteq ker\, \beta_1 \cap ker\, \beta_2$, we have that $dim\, C_{\mathfrak{h}}(\varphi(e_\alpha)) \leqslant l - 2$, the absurd. So only one $\beta \in \Phi^+$ such that $a_\beta \neq 0$. Therefore, $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$. □

If φ stabilizes $\mathfrak{h}$, then by Lemma 2.16, φ induces a permutation ρ on Φ^+ in the way that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\rho(\alpha)}$. The following lemma about the properties of ρ is clear.

Lemma 2.17.

(i) $(\rho(\alpha), \rho(\beta)) = 0$ *if and only if* $(\alpha, \beta) = 0$ *for* $\alpha, \beta \in \Delta$.
(ii) $\rho(X_\alpha) = X_{\rho(\alpha)}$*;* $\rho(Y_\alpha) = Y_{\rho(\alpha)}$*, for any* $\alpha \in \Phi^+$.
(iii) $M_{\rho(\alpha)} = M_\alpha$ *for any* $\alpha \in \Phi^+$. □

Lemma 2.18. *ρ stabilizes Δ and $\langle \rho(\alpha), \rho(\beta) \rangle = \langle \alpha, \beta \rangle$ for any $\alpha, \beta \in \Delta$. In other words, ρ induces a symmetry of the Dynkin diagram of Φ.*

Proof. We give the proof for Φ of different types.

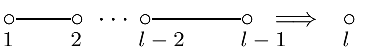
Case 1. Φ has only one root length.

In this case, $M_\beta = ht\,\theta - 1$ if and only if $\beta \in \Delta$. Thus we have, by Lemma 2.17, that $\rho(\Delta) = \Delta$. It's clear that $\langle \rho(\alpha), \rho(\beta) \rangle = \langle \alpha, \beta \rangle$ for any $\alpha, \beta \in \Delta$ (using (i) of Lemma 2.17).

In the following we consider the case that Φ has two different root lengths.

Case 2. Φ is of type B_l $(l \geqslant 3)$.

Let the Dynkin diagram of B_l type root system be

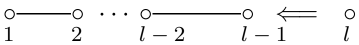


In this case, $\theta = \alpha_1 + \sum_{i=2}^{l} 2\alpha_i$ and $\theta_s = \sum_{i=1}^{l} \alpha_i$. We know that $M_\beta = ht\,\theta - 1$ if and only if $\beta = \alpha_l$. So ρ fixes α_l. Let $\Psi_1 = \{\beta \in \Phi^+ \mid M_\beta = ht\,\theta - 2\}$. It is stable under ρ.

Then by Lemmas 2.9 and 2.11, we know $\Psi_1 = \{\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \alpha_{l-1} + \alpha_l\}$. It's easy to see that $\Psi_1 \cap Y_{\alpha_l} = \{\alpha_1, \alpha_2, \dots, \alpha_{l-2}\}$ and $\Psi_1 \cap X_{\alpha_l} = \{\alpha_{l-1}, \alpha_{l-1} + \alpha_l\}$. They are both stable under ρ. Now let $\Psi_2 = \{\beta \in \Phi^+ \mid M_\beta \leqslant l-2\}$. Actually, Ψ_2 consists of the roots of height $\geqslant l+1$. Since $\Psi_2 \cap X_{\alpha_{l-1}+\alpha_l} = \emptyset$, $\theta_s + \alpha_l \in \Psi_2 \cap X_{\alpha_{l-1}}$, it is concluded that ρ fixes α_{l-1} and $\alpha_{l-1} + \alpha_l$, respectively. By $\rho(\alpha_{l-1}) = \alpha_{l-1}$, we conclude that $\rho(\alpha_{l-2}) = \alpha_{l-2}$, since $\alpha \in \Delta_l$ satisfies $(\alpha, \alpha_{l-1}) \neq 0$ iff $\alpha = \alpha_{l-2}$ or $\alpha = \alpha_{l-1}$. Then, by this way, we will have that $\rho(\alpha_{l-3}) = \alpha_{l-3}, \dots, \rho(\alpha_2) = \alpha_2$, $\rho(\alpha_1) = \alpha_1$. Now we see that ρ fixes each element in Δ.

Case 3. Φ is of type C_l ($l \geqslant 2$).

Let the Dynkin diagram of C_l type root system be



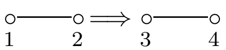
In this case, $\theta = \alpha_l + \sum_{i=1}^{l-1} 2\alpha_i$ and $\theta_s = \theta - \alpha_1$. By Lemmas 2.11 and 2.17, we see that ρ stabilizes Δ_s, which is $\{\alpha_1, \alpha_2, \dots, \alpha_{l-1}\}$. Let Ψ_1 be the subset of Φ^+ consisting of β satisfying $M_\beta = 1$. It is stable under ρ. Actually, Ψ_1 has only two roots, they are $\theta - \alpha_1$ and $\theta - 2\alpha_1$. Note that, for $\gamma \in \Phi^+$, $\Psi_1 \subseteq X_\gamma$ if and only if $\gamma = \alpha_1$. This forces that $\rho(\alpha_1) = \alpha_1$. Obviously, for $\alpha \in \Delta_s$ (distinct with α_1), $(\alpha_1, \alpha) \neq 0$ if and only if $\alpha = \alpha_2$. By this we conclude that $\rho(\alpha_2) = \alpha_2$. Similarly, we will further have that $\rho(\alpha_3) = \alpha_3, \dots, \rho(\alpha_{l-1}) = \alpha_{l-1}$. Let Ψ_2 be the subset of Φ^+ consisting of the roots β satisfying $M_\beta = l-1$. It is also stabilized by ρ. We know, by Lemma 2.9, that Ψ_2 consists of α_l together with all short roots of height l. It is not difficult to see that $\Psi_2 \cap X_{\alpha_{l-1}} = \{\alpha_l, \gamma\}$, where $\gamma = \sum_{i=1}^{l} \alpha_l$. Thus ρ stabilizes the subset $\{\alpha_l, \gamma\}$ of Ψ_2. Now we intend to show that ρ actually fixes α_l and γ, respectively. Otherwise, if ρ permutes α_l and γ nontrivially, namely, $\varphi(\mathfrak{g}_{\alpha_l}) = \mathfrak{g}_\gamma$ and $\varphi(\mathfrak{g}_\gamma) = \mathfrak{g}_{\alpha_l}$. By $\varphi(\mathfrak{g}_{\alpha_i}) = \mathfrak{g}_{\alpha_i}$ for $i = 1, 2, \dots, l-1$ and $\varphi(\mathfrak{h}) = \mathfrak{h}$, we see that $\varphi(F\, d_{\alpha_l}) = F\, d_{\alpha_l}$, since $F\, d_{\alpha_l} = \bigcap_{i=1}^{l-1} C_{\mathfrak{h}}(\mathfrak{g}_{\alpha_i})$. Suppose $\varphi(e_{\alpha_l}) = ae_\gamma$, $\varphi(e_\gamma) = be_{\alpha_l}$ and $\varphi(d_{\alpha_l}) = cd_{\alpha_l}$. Now it follows from $[e_{\alpha_l} + d_{\alpha_l}, e_{\gamma-\alpha_l} - N_{\alpha_l,\gamma-\alpha_l} e_\gamma] = 0$ that

$$\left[ae_\gamma + cd_{\alpha_l}, \varphi(e_{\gamma-\alpha_l}) - N_{\alpha_l,\gamma-\alpha_l} be_{\alpha_l}\right] = \left[ae_\gamma + cd_{\alpha_l}, \varphi(e_{\gamma-\alpha_l})\right] - bcN_{\alpha_l,\gamma-\alpha_l} e_{\alpha_l} = 0.$$

But it is impossible (note that $\varphi(e_{\gamma-\alpha_l}) \notin \mathfrak{g}_{\alpha_l}$). So $\rho(\alpha_l) = \alpha_l$. Finally, we see that ρ fixes each element in Δ.

Case 4. Φ is of type F_4.

Let the Dynkin diagram of F_4 type root system be



In this case, $\Delta_s = \{\alpha_3, \alpha_4\}$, which is stabilized by ρ (thanks to Lemmas 2.11 and 2.17). Let

$$\Psi_1 = \{\beta \in \Phi^+ \mid M_\beta = 1\}; \qquad \Psi_2 = \{\beta \in \Phi^+ \mid M_\beta = 2\}.$$

Thus Ψ_1 and Ψ_2 are both stable under ρ. Actually, Ψ_1 and Ψ_2 both consist of just one root, they respectively are $\theta - \alpha_1$ and $\theta - \alpha_1 - \alpha_2$. One easily sees that $X_{\theta-\alpha_1} = \{\alpha_1\}$

and $X_{\theta-\alpha_1-\alpha_2} = \{\alpha_2, \alpha_1+\alpha_2\}$. Thus we know that $\rho(\alpha_1) = \alpha_1$ and ρ stabilizes the subset $\{\alpha_2, \alpha_1+\alpha_2\}$. Since $M_{\alpha_2} = M_{\alpha_1+\alpha_2}+1$, we see that ρ also fixes α_2. Finally, it's easy to see that ρ fixes α_3 and α_4, respectively.

Case 5. Φ is of type G_2.

In this case, let $\alpha_1 \in \Delta$ be long and $\alpha_2 \in \Delta$ be short. Then $\theta = 2\alpha_1+3\alpha_2$ and $\theta_s = \alpha_1+2\alpha_2$. Denote $\theta-\alpha_1$ by θ_1. Let

$$\Psi_1 = \{\beta \in \Phi^+ \mid M_\beta = 1\}; \qquad \Psi_2 = \{\beta \in \Phi^+ \mid M_\beta = 3\}.$$

Then $\rho(\Psi_i) = \Psi_i$, $i = 1, 2$. Actually, Ψ_1 and Ψ_2 both consist of just one root, they respectively are θ_1 and α_2. This follows that $\rho(\alpha_2) = \alpha_2$ and $\rho(\theta_1) = \theta_1$. By $X_{\theta_1} = \{\alpha_1\}$ we have that $\rho(\alpha_1) = \alpha_1$. $\square$

3. The proof of the main theorem

Let φ be an invertible linear map on $\mathfrak{b}$ preserving zero Lie products in both directions. By Lemma 2.14, there exists $\sigma \in U$ such that $(\sigma\circ\varphi)(\mathfrak{h}) = \mathfrak{h}$. Denote $\sigma\circ\varphi$ by φ_1. Then, by Lemma 2.18, φ_1 induces a symmetry ρ (trivial or nontrivial) on the Dynkin diagram of Φ. Using ρ we construct the graph automorphism φ_ρ of $\mathfrak{b}$. Then $\varphi_\rho^{-1}\circ\varphi_1$ stabilizes each root space $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$. Denote $\varphi_\rho^{-1}\circ\varphi_1$ by φ_2.

For any given $\alpha \in \Delta$, since

$$F\,d_\alpha = \bigcap_{\beta\in\Delta\setminus\{\alpha\}} ker\,\beta = \bigcap_{\beta\in\Delta\setminus\{\alpha\}} C_{\mathfrak{h}}(\mathfrak{g}_\beta);$$
$$\varphi_2\big(C_{\mathfrak{h}}(\mathfrak{g}_\beta)\big) = C_{\mathfrak{h}}\big(\varphi_2(\mathfrak{g}_\beta)\big) = C_{\mathfrak{h}}(\mathfrak{g}_\beta), \quad \text{for } \beta \in \Delta,$$

we have that $\varphi_2(F\,d_\alpha) = F\,d_\alpha$ for any $\alpha \in \Delta$. Now suppose $\varphi_2(d_\alpha) = a_\alpha d_\alpha$ for $\alpha \in \Delta$. We wish to show that all a_α, $\alpha \in \Delta$, are equal. Write θ as the linear combination of the simple roots: $\theta = \sum_{\alpha\in\Delta} k_\alpha\alpha$, where all k_α are positive integers. We know that $C_{\mathfrak{h}}(\mathfrak{g}_\theta) = \{\sum_{\alpha\in\Delta} x_\alpha d_\alpha \in \mathfrak{h} \mid \sum_{\alpha\in\Delta} k_\alpha x_\alpha = 0\} = ker\,\theta$ is an $l-1$ dimensional subspace of $\mathfrak{h}$. If $\sum_{\alpha\in\Delta} k_\alpha x_\alpha = 0$, then $\sum_{\alpha\in\Delta} x_\alpha d_\alpha \in C_{\mathfrak{h}}(\mathfrak{g}_\theta)$. Thus

$$\sum_{\alpha\in\Delta} a_\alpha x_\alpha d_\alpha = \varphi_2\left(\sum_{\alpha\in\Delta} x_\alpha d_\alpha\right) \in \varphi_2\big(C_{\mathfrak{h}}(\mathfrak{g}_\theta)\big) = C_{\mathfrak{h}}\big(\varphi_2(\mathfrak{g}_\theta)\big) = C_{\mathfrak{h}}(\mathfrak{g}_\theta) = ker\,\theta,$$

from which it follows that $\sum_{\alpha\in\Delta} a_\alpha k_\alpha x_\alpha = 0$. So the equation (in variants x_α, $\alpha \in \Delta$)

$$\sum_{\alpha\in\Delta} k_\alpha x_\alpha = 0$$

and the equations

$$\begin{cases} \sum_{\alpha\in\Delta} k_\alpha x_\alpha = 0 \\ \sum_{\alpha\in\Delta} a_\alpha k_\alpha x_\alpha = 0 \end{cases}$$

have the same solutions. So all $\frac{a_\alpha k_\alpha}{k_\alpha}$ $(= a_\alpha)$ are equal for $\alpha \in \Delta$. Now we denote the same value a_α by c, and using it we construct the multiplication map φ_c. Then we see that $\varphi_c^{-1} \circ \varphi_2$ fixes each element in $\mathfrak{h}$. Denote $\varphi_c^{-1} \circ \varphi_2$ by φ_3.

For $\beta \in \Phi^+$, it's clear that $\mathfrak{g}_\beta = C_\mathfrak{n}(ker\,\beta)$. By applying φ_3, we have that $\varphi_3(\mathfrak{g}_\beta) = \mathfrak{g}_\beta$ for any $\beta \in \Phi^+$. Now suppose that $\varphi_3(e_\alpha) = b_\alpha e_\alpha$ for $\alpha \in \Delta$, and define $\chi : P = \mathbb{Z}\Phi \to F^*$, by $\sum_{\alpha \in \Delta} k_\alpha \alpha \mapsto \prod_{\alpha \in \Delta} b_\alpha^{k_\alpha}$. Then χ is an F-character of P. Using it we construct the diagonal automorphism φ_χ of $\mathfrak{b}$. Then $\varphi_\chi^{-1} \circ \varphi_3$ will further fix each e_α for $\alpha \in \Delta$. Denote $\varphi_\chi^{-1} \circ \varphi_3$ by φ_4.

If Φ is not of type G_2, we wish to show, by induction on $ht\,\beta$, that φ_4 fixes all e_β for $\beta \in \Phi^+$. If $ht\,\beta = 1$, the result has been shown to be true. Assume that $\varphi_4(e_\beta) = e_\beta$ for $\beta \in \Phi^+$ with $ht\,\beta = k-1$ (where $2 \leqslant k \leqslant ht\,\theta$). Let $ht\,\beta = k$. Since Φ is not of type G_2, we can find $\alpha \in \Delta$ such that $\gamma = \beta - \alpha \in \Phi^+$, but $\beta + \alpha \notin \Phi^+$ (recall Lemma 2.8). Choose $h \in \mathfrak{h}$ such that $\gamma(h) = 0$ and $\beta(h) = -N_{\alpha,\gamma}$. Then by $[e_\alpha + h, e_\gamma + e_\beta] = 0$ and $\varphi_4(e_\gamma) = e_\gamma$, we have that $[e_\alpha + h, e_\gamma + \varphi_4(e_\beta)] = 0$. This implies that $\varphi_4(e_\beta) = e_\beta$. We have by induction that $\varphi_4(e_\beta) = e_\beta$ for all $\beta \in \Phi^+$. If Φ is of type G_2, let $\theta_1 = \theta - \alpha_1$. We know that $N_{\alpha_2,\alpha_1} = \delta$, $N_{\alpha_2,\alpha_1+\alpha_2} = 2\delta$, $N_{\alpha_2,\alpha_1+2\alpha_2} = 3\delta$, where $\delta = 1$, or -1. Let $h_0 = -\delta d_{\alpha_2}$. Then $(\alpha_1 + k\alpha_2)(h_0) = -k\delta$. It follows from

$$[e_{\alpha_2} + h_0, e_{\alpha_1} + e_{\alpha_1+\alpha_2} + e_{\alpha_1+2\alpha_2} + e_{\alpha_1+3\alpha_2}] = 0$$

that

$$\big[e_{\alpha_2} + h_0, e_{\alpha_1} + \varphi_4(e_{\alpha_1+\alpha_2}) + \varphi_4(e_{\alpha_1+2\alpha_2}) + \varphi_4(e_{\alpha_1+3\alpha_2})\big] = 0.$$

This implies that $\varphi(e_{\alpha_1+k\alpha_2}) = e_{\alpha_1+k\alpha_2}$, for $k = 1, 2, 3$. Now consider the action of φ_4 on e_θ. Choose $h \in \mathfrak{h}$ such that $\theta_1(h) = 0$ and $\theta(h) = -N_{\alpha_1,\theta_1}$. Then by $[h + e_{\alpha_1}, e_{\theta_1} + e_\theta] = 0$, we obtain $[h + e_{\alpha_1}, e_{\theta_1} + \varphi_4(e_\theta)] = 0$. So $\varphi_4(e_\theta) = e_\theta$. Now we have that $\varphi_4(e_\beta) = e_\beta$ for all $\beta \in \Phi^+$.

Till now, we have shown that φ_4 acts as the identity on $\mathfrak{b}$. Finally, we know that the original invertible linear map φ on $\mathfrak{b}$ is a composition of an inner automorphism, a graph automorphism, a scalar multiplication map and a diagonal automorphism. □

Acknowledgment

The author is grateful to the referees for their valuable comments, corrections and suggestions.

References

[1] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, 1972.
[2] R.W. Carter, Simple Groups of Lie Type, Wiley Interscience, New York, 1972.
[3] G. Frobenius, Uber die Darstellung der endlichen Gruppen durch lineare Substitutioen, Sitzungsber. Deutsch. Akad. Wiss. Berlin (1897).
[4] C.K. Li, N.K. Tsing, Linear preserver problem: a brief introduction and some special techniques, Linear Algebra Appl. 162–164 (1992) 217–235.

[5] S. Pierce, et al., A survey of linear preserver problems, Linear Multilinear Algebra 33 (1992) 1–192.
[6] C.K. Li, S. Pierce, Linear preserver problem, Amer. Math. Monthly 108 (2001) 591–605.
[7] M. Marcus, Linear operations of matrices, Amer. Math. Monthly 69 (1962) 837–847.
[8] L.W. Marcoux, A.R. Sourour, Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras, Linear Algebra Appl. 288 (1999) 89–104.
[9] Y. Cao, Z. Chen, C. Huang, Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space, Linear Algebra Appl. 350 (2002) 41–66.
[10] P. Šemrl, Non-linear commutativity preserving maps, Acta Sci. Math. (Szeged) 71 (2005) 781–819.
[11] P. Šemrl, Commutativity preserving maps, Linear Algebra Appl. 429 (2008) 1051–1070.
[12] W.J. Wong, Maps on simple algebras preserving zero products, II: Lie algebras of linear type, Pacific J. Math. 92 (1981) 469–487.
[13] H. Radjavi, P. Šemrl, Non-linear maps preserving solvability, J. Algebra 280 (2004) 624–634.
[14] G. Dolinar, Maps on upper triangular matrices preserving Lie products, Linear Multilinear Algebra 55 (2007) 191–198.
[15] D. Wang, Z. Chen, Invertible linear maps on simple Lie algebras preserving commutativity, Proc. Amer. Math. Soc. 139 (2011) 3881–3893.
[16] D. Wang, X. Yu, Z. Chen, A class of zero product determined Lie algebras, J. Algebra 331 (2011) 145–151.
[17] D. Wang, Z. Chen, Quasi-automorphisms of Lie algebras, Comm. Algebra 39 (2011) 2388–2395.
[18] D. Wang, Y. Zhao, Z. Chen, Non-linear maps on simple Lie algebras preserving bracket product, Comm. Algebra 39 (2011) 424–434.
[19] D. Wang, Y. Zhao, Z. Chen, Non-linear maps on a Borel subalgebras preserving ad-nilpotent ideals, Comm. Algebra 39 (2011) 1519–1527.
[20] D. Wang, S. Ou, X. Yu, Invertible linear maps on simple Lie algebras preserving solvability, Comm. Algebra 42 (2014) 1368–1378.
[21] J.E. Humphreys, Linear Algebraic Groups, Springer-Verlag, Berlin, 1975.
[22] F. Grunewald, V. Platonov, Rigidity results for groups with radical, cohomology of finite groups and arithmeticity problems, Duke Math. J. 100 (1999) 321–358.
[23] D. Wang, S. Ou, W. Zhang, Bijective maps preserving commutators on a solvable classical group, Sci. China Ser. A 53 (2010) 1723–1730.
[24] S. Ou, Bijective maps on standard Borel subgroup of symplectic group preserving commutators, Front. Math. China 7 (2012) 497–512.
[25] E. Cartan, Sur la structure des groups de transformations finis et continus (Thesis, Nony, Paris), in: Oeuvres Completes, Partie I, Tome 1, 1894, pp. 137–287.
[26] W.A. de Graaf, Classification of solvable Lie algebras, Experiment. Math. 14 (2005) 15–25.