Journal of Algebra 437 (2015) 177–201

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Generic modules for gentle algebras

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A R T I C L E   I N F O

Article history:
Received 18 September 2013
Available online 25 May 2015
Communicated by Volodymyr Mazorchuk

Keywords:
Gentle algebras
Generic modules
Canonical decomposition

A B S T R A C T

In this article, we construct the generic modules in each irreducible component of variety of β-dimensional modules of a triangular gentle algebra. The construction is completely combinatorial and allows for determination of canonical decomposition of irreducible components as well as calculation of the dimension of the higher self-extension spaces for generic modules.

Introduction

Consider $\text{mod}(kQ, \beta)$, the variety of β-dimensional modules over a path algebra kQ. A decomposition $\beta = \beta(1) + \ldots + \beta(s)$ of β into smaller dimension vectors is called the canonical decomposition of β if there exists an open subset $U \subset \text{mod}(kQ, \beta)$ for which every module $V \in U$ has direct sum decomposition $V = V(1) \oplus \ldots \oplus V(s)$ where $V(i)$ is indecomposable of dimension $\beta(i)$ for each i. It was originally Kac who introduced this concept and showed that such a canonical decomposition (necessarily unique) exists for any dimension vector β. Later, Schofield [11] and Derksen and Weyman [5] gave independent algorithms for determining the canonical decomposition of a given

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[1] The author was partially supported by NSF grant number DMS-0801220.

http://dx.doi.org/10.1016/j.jalgebra.2015.03.035

dimension vector for any quiver Q, although a description of the corresponding open set U remains out of reach. The modules in this set will be called *generic* in this article.

For quivers with relations, the situation is more intricate. Module varieties need not be irreducible, so one needs to consider the canonical decomposition of a dimension vector and describe the generic modules with respect to a given irreducible component. While it is generally difficult to describe the irreducible components of module varieties, this problem can be solved for a certain class of zero-relation algebras, namely triangular gentle algebras.

In this article, we give an algorithm to describe the set U in irreducible components of module varieties for triangular gentle algebras. In particular, we construct, for each irreducible component of a module variety for a gentle algebra, a graph referred to as an *up-and-down graph*. From each such graph, a family of representations is constructed, and it is shown that the union of the orbits of these special representations is dense within the irreducible component. In particular, the up-and-down graph allows one to read the generic decomposition of, and describe the generic modules within any irreducible component. Its connected components count the indecomposable direct summands of the generic modules, and the type of each connected component (either a chain or cycle) describes the type of the corresponding indecomposable direct summand (as a string or band). This work is a generalization of Kraśkiewicz and Weyman (see [10]), in which the generic modules for the algebras $A(n)$ are constructed.

In [8], it is shown that for a quiver Q and dimension vector β, a decomposition $\beta = \beta(1) + \ldots + \beta(s)$ is the canonical decomposition of β if and only if $\beta(i)$ are Schur roots (i.e., the generic module in $\mathrm{mod}(kQ, \beta(i))$ is indecomposable) and there are no extensions between the generic modules of dimensions $\beta(i)$ and $\beta(j)$ for $i \neq j$. This result (with some modifications, recalled in Section 4) was extended to module varieties of finite dimensional associative algebras by Crawley-Boevey and Schröer in [3]. Furthermore (see [6,12]), if a β-dimensional module admits no self extensions, then its $\mathrm{GL}(\beta)$ orbit is open in its irreducible component. Thus the criterion used to determine the generic modules has interesting connections with tilting theory.

In the author's joint work with Calin Chindris [1], the generic modules (specifically, those which are band modules) herein described are put to use in showing that fields of rational invariants for irreducible components of modules varieties for gentle algebras are rational fields. In particular, the combinatorics of these specific modules, as well as their explicit projective presentations provide for simple construction of semi-invariant functions on the irreducible components. Ratios of semi-invariant functions of the same weight are rational invariants, and by evaluating on the set of generic modules it is possible to show that the rational invariants so-constructed are algebraically independent.

The paper is laid out as follows. In Section 1, the pertinent definitions are recalled, including the notion of a gentle algebra, quiver morphisms, and module varieties, concluding with the parametrization of irreducible components of module varieties for gentle algebras. In Section 2, we give the combinatorial construction of the generic modules, followed by a collection of some of their important properties in Section 3. Section 4 contains a statement of the two main theorems, the first asserting genericity of the modules constructed, and the second presenting the dimensions of various extension spaces. In Section 5, a minimal projective resolution of the modules is constructed and utilized to calculate the dimensions of the extension spaces referred to in the theorem from the previous section, thereby proving said theorem. Finally, in Section 6, the projective resolution is used to calculate the dimensions of spaces of higher self-extensions for the generic modules.

1. Preliminary definitions

Fix an algebraically closed field $k = \overline{k}$. A quiver $Q = (Q_0, Q_1)$ is a pair consisting of a set of vertices Q_0 and arrows Q_1. We denote by ta (resp. ha) the tail (resp. head) of the arrow a. A path p in Q is a sequence of arrows $a_s a_{s-1} \dots a_1$ such that $ha_i = ta_{i+1}$ for $i = 1, \dots, s-1$. A (finite-dimensional) representation V of Q is a pair $(\{V_x\}_{x \in Q_0}, \{V_a : V_{ta} \to V_{ha}\}_{a \in Q_1})$ where V_x are (finite-dimensional) vector spaces, and V_a are linear maps. A morphism of representations $\varphi : V \to W$ is a linear map $\varphi_x : V_x \to W_x$ for each vertex $x \in Q_0$ such that for each arrow $a \in Q_1$ the following square commutes:

$$\begin{tikzcd} V_{ta} \arrow[r, "\varphi_{ta}"] \arrow[d, "V_a"'] & W_{ta} \arrow[d, "W_a"] \\ V_{ha} \arrow[r, "\varphi_{ha}"'] & W_{ha} \end{tikzcd}$$

The vector $\underline{\dim} V$ with $(\underline{\dim} V)_x = \dim V_x$ is referred to as the dimension vector of V. The category of finite-dimensional representations of Q with morphisms so-defined is denoted by rep_Q.

The path algebra kQ of Q is the algebra whose basis consists of the set of paths p in Q (including the length-zero paths concentrated at each vertex) with multiplication defined by concatenation of paths. We denote by $\mathrm{mod}(kQ)$ the category of finitely generated left kQ-modules, and remark that the categories $\mathrm{mod}(kQ)$ and rep_Q are equivalent.

Given a two-sided ideal I of kQ, we call the pair (Q, I) a bound quiver. A representation V of a bound quiver (Q, I) is a representation of Q such that for $\rho \in I$, $V_\rho = 0$ (here V_ρ is the composition of the linear maps on the arrows of ρ, in the order prescribed by the path). The category $\mathrm{rep}_{(Q,I)}$ of (finite-dimensional) representations of the bound quiver is equivalent to $\mathrm{mod}(kQ/I)$, so we will make no distinction speaking of modules or representations.

Definition 1. A finite-dimensional k-algebra A is called *gentle* if it admits a presentation kQ/I satisfying the following properties:

- each vertex in Q is the source of at most two arrows, and the target of at most two arrows;

- for any arrow $b \in Q_1$, there is at most one arrow $a \in Q_1$ with $hb = ta$ and $ab \notin I$, and at most one arrow $c \in Q_1$ such that $hc = tb$ and $bc \notin I$;
- for any arrow $b \in Q_1$, there is at most one arrow $a \in Q_1$ with $hb = ta$ such that $ab \in I$, and at most one arrow $c \in Q_1$ with $hc = tb$ such that $ab \in I$;
- I is generated by paths of length 2.

The geometry of module varieties for gentle algebras can be best understood by recognizing them as products of varieties of complexes. This is achieved by "coloring" the quiver. A coloring c of Q is a surjective set map $c : Q_1 \to S$, with S some finite set of elements (whose elements are called colors) satisfying the property that the arrows in $c^{-1}(s)$ form a directed path in Q. Given a coloring c of Q, we denote by I_c the ideal generated by products of consecutive monochromatic arrows: $I_c = \langle ba \mid ha = tb \text{ and } c(a) = c(b)\rangle$. The bound quiver (Q, I_c) will often be denoted (Q, c). It can be easily shown that if kQ/I is a triangular gentle algebra (i.e., Q contains neither loops nor oriented cycles), then there is a coloring c on Q such that $I = I_c$ (see [2]).

1.1. Quiver morphisms

Let Γ and Q be quivers. A morphism of quivers $\pi : \Gamma \to Q$ is a pair of maps $\pi_0 : \Gamma_0 \to Q_0$ and $\pi_1 : \Gamma_1 \to Q_1$ such that π commutes with taking heads and tails of arrows. That is, $\pi_0(ha) = h(\pi_1(a))$ and $\pi_0(ta) = t(\pi_1(a))$. A morphism of quivers π gives rise to a pushforward map $\pi_* : \mathrm{rep}_\Gamma \to \mathrm{rep}_Q$ defined by

$$(\pi_*(V))_x = \bigoplus_{y \in \pi^{-1}(x)} V_y$$

$$(\pi_*(V))_a = \sum_{b \in \pi^{-1}(a)} V_b$$

For a quiver Γ, denote by $\mathbb{1}_\Gamma$ the representation of Γ with $(\mathbb{1}_\Gamma)_x = k$ for each $x \in Q_0$ and for each $a \in Q_1$, the map $(\mathbb{1}_\Gamma)_a : 1 \mapsto 1$ is simply multiplication by the unit. When Γ is an orientation of the diagram A_n, the module $\pi_*(\mathbb{1}_\Gamma)$ is called a *string module*. Alternatively, suppose that the connected components of Γ are orientations of the diagrams A_n and $\tilde{A}_n$. Let B be the set of connected components of type $\tilde{A}_n$, and for each $b \in B$ pick an arrow $\Theta'(b)$ in said component. For any $\underline{\lambda} = (\lambda_b) \in (k^*)^B$, denote by $\underline{\lambda}_\Gamma$ the representation of Γ with $(\underline{\lambda}_\Gamma)_x = k$ and

$$(\underline{\lambda}_\Gamma)_a : 1 \mapsto \begin{cases} \lambda_b & \text{if } a = \Theta'(b) \text{ for some } b \in B \\ 1 & \text{otherwise.} \end{cases}$$

If Γ consists of a single connected component of type $\tilde{A}_n$, then the modules $\pi_*(\underline{\lambda}_\Gamma)$ are called *band modules*.

1.2. Module varieties

Let $A = kQ/I$ be a bound quiver algebra, and $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector for Q. We denote by $\mathrm{mod}(A, \beta)$ the variety of β-dimensional A-modules, which we view as a closed subvariety of $\prod_{a \in Q_1} \mathrm{Hom}_k(k^{\beta_{ta}}, k^{\beta_{ha}})$. Notice that in general (if $I \neq 0$) these varieties need not be irreducible. While it can be quite difficult to determine these irreducible components, when $I = I_c$ for some coloring, this can be accomplished.

Suppose now that $I = I_c$ for some coloring c of Q. A *rank function* r for β is a map $r : Q_1 \to \mathbb{Z}_{\geq 0}$ such that $r(a) \leq \min\{\beta_{ta}, \beta_{ha}\}$ and $r(a) + r(b) \leq \beta_x$ whenever a and b are arrows with $ha = tb = x$ and $c(a) = c(b)$. A rank function is called *maximal* if it is so with respect to the coordinate-wise partial order, namely $r \leq r'$ if and only if $r(a) \leq r'(a)$ for all $a \in Q_1$. Denote by $\mathrm{mod}(A, \beta, r)$ the closed subvariety of $\mathrm{mod}(A, \beta)$ consisting of modules V for which $\mathrm{rank}\, V_a \leq r(a)$.

Proposition 1. *(See [2].) With all assumptions as above, the collection of subvarieties* $\mathrm{mod}(A, \beta, r)$ *for* r *maximal constitutes the complete list of irreducible components of* $\mathrm{mod}(A, \beta)$.

The proof is essentially a corollary of the work of De Concini and Strickland [4] in which they show that irreducible components of varieties of complexes are parameterized by maximal ranks. We then notice that the module varieties $\mathrm{mod}(A, \beta)$ are products of varieties of complexes taken along each individual color. Details can be found in the article [2].

2. Up-and-down modules

For the remainder of the article, fix a triangular gentle algebra $A = kQ/I$ and a coloring $c : Q_1 \to S$ for which $I = I_c$. For each dimension vector β and each rank sequence r, we construct a module (or family of modules) in $\mathrm{mod}(A, \beta, r)$. This construction was inspired by the combinatorics arising from the calculation of the rings of semi-invariant functions in $k[\mathrm{mod}(A, \beta, r)]$.

Let $\mathfrak{X}$ denote the set of pairs $(x, s) \in Q_0 \times S$ such that there is an arrow of color s incident to x. Notice that since A is a gentle algebra, there are at most two elements in $\mathfrak{X}$ with first coordinate x for any $x \in Q_0$. A function $\epsilon : \mathfrak{X} \to \{\pm 1\}$ is called a *sign function* for A if $\epsilon(x, s) = -\epsilon(x, s')$ when $s \neq s'$. We now introduce the main combinatorial object.

Definition 2. Let $\Gamma(Q, c, \beta, r, \epsilon)$ be the quiver with vertices $\{v_i^x \mid x \in Q_0,\ i = 1, \ldots, \beta_x\}$ and arrows $\{f_j^a \mid a \in Q_1,\ j = 1, \ldots, r(a)\}$ where

$$tf_j^a = \begin{cases} v_j^{ta} & \text{if } \epsilon(ta, c(a)) = 1 \\ v_{\beta_{ta}-j+1}^{ta} & \text{if } \epsilon(ta, c(a)) = -1 \end{cases}$$

$$hf_j^a = \begin{cases} v^{ha}_{\beta_{ha}-j+1} & \text{if } \epsilon(ha, c(a)) = 1 \\ v^{ha}_j & \text{if } \epsilon(ha, c(a)) = -1 \end{cases}$$

The quiver $\Gamma(Q, c, \beta, r, \epsilon)$ is referred to as the *up-and-down graph*.

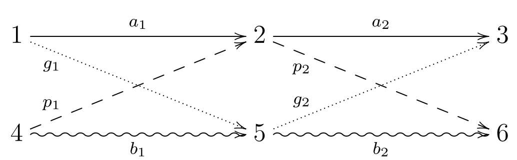
When the quiver and coloring are understood, they will be suppressed from the notation. There is an obvious morphism of quivers $\pi : \Gamma(\beta, r, \epsilon) \to Q$ with $\pi(v_i^x) = x$ and $\pi(f_i^a) = a$. By abuse of notation, we will write $c(f_i^a) = c(a)$ but note that it does not induce a coloring on $\Gamma(\beta, r, \epsilon)$ since the arrows of a single color may comprise more than a single path.

Remark 1. Notice that each vertex in $\Gamma(\beta, r, \epsilon)$ is incident to at most two arrows, which must be of different colors, so the connected components of $\Gamma(\beta, r, \epsilon)$ are orientations of A_n and $\tilde{A}_n$. We call these components *strings* and *bands*, respectively.

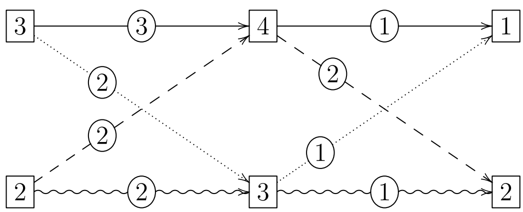
Recall that a sink (resp. source) in Q is a vertex which is not a tail (resp. not a head) of any arrow in Q. As before, denote by $B = B(\Gamma)$ the set of band components of $\Gamma(\beta, r, \epsilon)$. For each $b \in B$, chose a vertex $\Theta(b) \in \Gamma(\beta, r, \epsilon)$ which is a sink contained in B. Denote by $\Theta'(b)$ the arrow in $\Gamma(\beta, r, \epsilon)$ with $h(\Theta'(b)) = \Theta(b)$ and $\epsilon(\Theta(b), c(\Theta'(b))) = -1$, which is guaranteed to exist since a sink in a band component is necessarily the head of two arrows.

Definition 3. Fix the notation above. For any $\underline{\lambda} \in (k^*)^B$, the up-and-down module $M(Q, c, \beta, r, \epsilon, \underline{\lambda}, \Theta) \in \mathrm{mod}(kQ, \beta)$ is defined to be the module $\pi_*(\underline{\lambda}_{\Gamma(\beta,r,\epsilon)})$. As usual, we write $M(\beta, r, \epsilon, \underline{\lambda}, \Theta)$ when the quiver and coloring are understood.

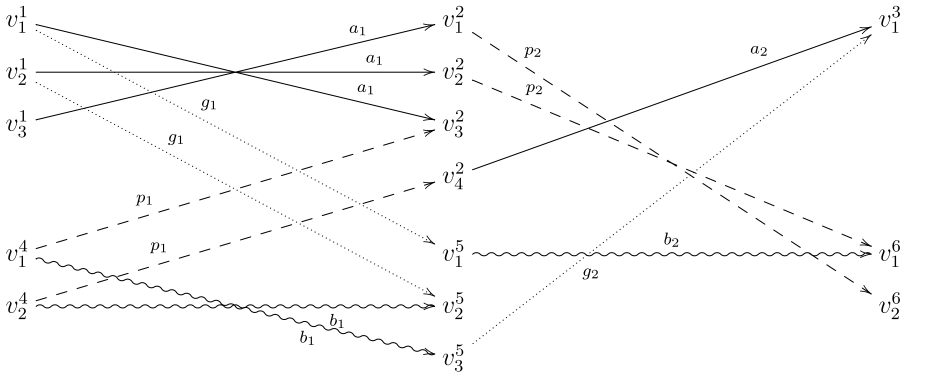
Example 1. Consider the quiver below with coloring indicated by type of arrow:



Let us say that the color of the arrow $\star_i$ is $\star$ in the above picture (so, for example, $c(a_1) = c(a_2) = a$). Let β, r be the pair depicted in the following diagram:



and $\epsilon^{-1}(1) = \{(1,g),(2,p),(3,g),(4,b),(5,b),(6,p)\}$ (so $\epsilon^{-1}(-1)$ is the complement in $\mathfrak{X}$). Then $\Gamma(Q,c,\beta,r,\epsilon)$ takes the following form:



For example, we know there are three arrows labeled $f_1^{a_1}, f_2^{a_1}, f_3^{a_1}$, with $t(f_1^{a_1}) = v_3^1$ and $h(f_1^{a_1}) = v_1^2$ since $\epsilon(1,r) = -1 = \epsilon(2,r)$.

Proposition 2. *The representation $M(Q,c,\beta,r,\epsilon,\underline{\lambda},\Theta)$ is a representation of (Q,I_c), so can be identified with a module (which we denote in the same way) over A. In particular, it is a direct sum of string and band modules.*

Proof. Abbreviate the module by M. Let e_i^x denote the basis element of M_x corresponding to the vertex v_i^x. We need only demonstrate that if $a, b \in Q_1$ are arrows with $ha = tb = x$ and $c(a) = c(b)$, $M_b \circ M_a = 0$. Suppose $\epsilon(x, c(a)) = 1$. Then the image of M_a is precisely the span of the basis elements e_i^x for which $\beta_x - r(a) + 1 \le i \le \beta_x$, while the kernel of M_b is precisely the span of the elements e_i^x for which $r(b) \le i \le \beta_x$. Since r is a rank sequence, $r(b) \le \beta_x - r(a) + 1$, so $\ker M_b \supset \operatorname{image} M_a$. When $\epsilon(x, c(a)) = -1$, the proof is essentially the same. That M is a direct sum of string and band modules is directly seen from Remark 1. □

In the subsequent sections, we will fix once-and-for-all a gentle algebra kQ/I_c (with associated coloring c), as well as a sign function ϵ, and distinguished function Θ for each up-and-down graph. We may therefore write $\Gamma(\beta, r)$ and $M(\beta, r, \underline{\lambda})$ without reference to these fixed elements.

3. Properties of up-and-down graphs

The goal will be to show that when r is a maximal rank function for β, the module $M(\beta, r, \epsilon, \underline{\lambda})$ is generic. The proof of this assertion employs homological considerations, most importantly calculation of the dimensions of $\mathrm{Ext}_A^1(M, M)$. This requires a projective resolution and various combinatorial data about these modules. We first collect some technical definitions and notation.

A vertex v_j^x in $\Gamma(\beta, r)$ is said to be *above* (resp. *below*) $v_{j'}^x$ if $j > j'$ (resp. $j < j'$). As in Example 1, we will depict the up-and-down graphs in the plane in such a way that above and below are literal. We denote by $S(\Gamma)$ and $T(\Gamma)$ the set of sources and sinks in $\Gamma(\beta, r)$, respectively. A source (resp. sink) incident to exactly i arrows will be called an *i-source* (resp. *i-target*), and the sets of these vertices will be denoted by $S^i(\Gamma)$ and $T^i(\Gamma)$ respectively. (This only defines four potentially non-empty sets, S^1, S^2, T^1, T^2 since $\Gamma(\beta, r)$ contains only strings and bands.)

Let us call two paths γ, γ' in $\Gamma(\beta, r)$ parallel if $\pi(\gamma) = \pi(\gamma')$ (here we define the path $\pi(\gamma)$ to be the one obtained by applying the quiver morphism π to the sequence of arrows in γ). A path $\gamma = a_m a_{m-1} \dots a_1$ in $\Gamma(\beta, r)$ is called *target positive* (resp. *target negative*) if $\epsilon(ha_m, c(a_m)) = 1$ (resp. -1), and *source positive* (resp. *source negative*) if $\epsilon(ta_1, c(a_1)) = 1$ (resp. -1). The following lemma is utilized to determine the relative ordering of vertices on either end of parallel paths.

Lemma 1. *Suppose that γ is a path in $\Gamma(\beta, r)$, $h(\gamma) = v_j^x$ and $t(\gamma) = v_i^y$. Then the following hold.*

a. *If γ is target negative, and $v_{j'}^x$ is above v_j^x, then there is a path γ' parallel to γ with $h(\gamma') = v_{j'}^x$. Furthermore, $t(\gamma')$ is above (resp. below) $t(\gamma)$ if and only if γ is source positive (resp. negative).*
b. *Dually, if γ is target positive, and $v_{j'}^x$ is below v_j^x, then there is a path γ' parallel to γ with $h(\gamma') = v_{j'}^x$. In this case, $t(\gamma')$ is above (resp. below) $t(\gamma)$ if and only if γ is source positive (resp. negative).*
c. *If γ is source positive (resp. negative) and $v_{i'}^y$ is above (resp. below) v_i^y, then there is a path γ' parallel to γ.*

Proof. To any path $\gamma = a_m a_{m-1} \dots a_1$ in Γ, we can associate a sequence of pairs

$$((j_0, x_0), \dots, (j_m, x_m))$$

such that $h(a_i) = v_{j_i}^{x_i}$ and $t(a_1) = v_{j_0}^{x_0}$. Notice that if γ, γ' are two parallel paths, then for each i, the coordinates x_i coincide (since π commutes with taking heads and tails of arrows).

Suppose now that γ is target negative, and $v_{j'_m}^{x_m}$ is above $v_{j_m}^{x_m}$. In particular, $j'_m < j_m \le r(\pi(a_m))$, so by definition of Γ, there is an arrow $a'_m \in \Gamma_1$ with $h(a'_m) = v_{j'_m}^{x_m}$ that is parallel to a_m. Denote by $v_{j'_{m-1}}^{x_{m-1}}$ the tail of this arrow. If a_m is source positive (resp. negative) then so is a'_m (since they are parallel). So if a_m is source positive, $j'_{m-1} < j_{m-1} \le r(\pi(a_m))$, and if it is source negative, $j_{m-1} = \beta_{x_{m-1}} - j_m + 1$, and $j'_{m-1} = \beta_{x_{m-1}} - j'_m + 1$, i.e., $j'_{m-1} > j_{m-1}$. The argument of the analogous statement when γ is target positive and above is replaced with below is similar.

The statement is thus proven for paths of length one. Suppose now the lemma holds for the path $\tilde{\gamma} = a_m a_{m-1} \ldots a_2$, and consider the path $\gamma = (a_m \ldots a_2)a_1$ (which we assume again to be target negative, and $j'_m < j_m$). By inductive assumption, there is a path $\tilde{\gamma}'$ parallel to $\tilde{\gamma}$ with $t(\tilde{\gamma}') = v^{x_1}_{j'_1}$ and $h(\tilde{\gamma}') = v^{x_m}_{j'_m}$. If $\tilde{\gamma}$ is source positive (resp. negative), then $v^{x_1}_{j'_1}$ is above (resp. below) $v^{x_1}_{j_1}$. In either case, a_1 is of opposite target sign (since a_1, a_2 must be of different colors given they both contain $v^{x_1}_{j_1}$, and the sign function of arrows of different color incident to the same vertex must differ). Supposing $\tilde{\gamma}$ is source positive, then, a_1 is target negative. Therefore, by the base case, there is an arrow a'_1 parallel to a_1 with $h(a'_1) = v^{x_1}_{j'_1}$, which is above or below $v^{x_1}_{j_1}$ according to whether γ was source positive or negative (which is equivalent to a_1 being source positive or negative). The arguments for the analogous cases are similar. □

Lemma 1 can be iterated in an important way. Namely, suppose that we have paths in Γ

$$\begin{tikzcd} v^y_i \arrow[r, squiggly, "p"] \arrow[dr, squiggly, "q"] & v^{x_0}_{j_1} \\ & v^{x_1}_{j_1} \end{tikzcd} \qquad \begin{tikzcd} v^y_{i'} \arrow[r, squiggly, "p'"] \arrow[dr, squiggly, "q'"] & v^{x_0}_{j'_0} \\ & v^{x_1}_{j'_1} \end{tikzcd}$$

such that p, p' are parallel, and q, q' are parallel. Suppose that we would like to know whether the order of j_0 relative to j'_0 is the same as that of j_1 relative to j'_1. The order is reversed if and only if $\epsilon(p, x_0) = \epsilon(q, x_1)$. Indeed, we need only count the number of times that order is reversed, which is an occurrence of $\epsilon(\star, h\star) = \epsilon(\star, t\star)$ where $\star$ is either p or q, so that even parity overall implies preservation of order, and odd implies reversal. But $\epsilon(p, y_1) = -\epsilon(q, y_1)$, so modulo two, this count is the same as the number of sign changes in $\{\epsilon(p, x_0), \epsilon(q, x_1)\}$. This can be iterated again for longer sequences of paths.

We will also need some facts relating the rank functions to various configurations appearing in Γ.

Lemma 2. *Let $\Gamma(\beta, r)$ be an up-and-down graph, and consider the following configuration in Q:*

$$\begin{tikzcd} {} \arrow[dr, "a_1"] & & {} \\ & x \arrow[ur, "a_2"] \arrow[dr, dashed, "b_2"] & \\ {} \arrow[ur, dashed, "b_1"] & & {} \end{tikzcd}$$

Let $c(a_i) = a$ and $c(b_i) = b$.

a. *Suppose* $v_j^x \in S^2(\Gamma)$ *(resp.* $T^2(\Gamma)$*). Then* $r(a_2)+r(b_2) > \beta_y$ *(resp.* $r(a_1)+r(b_1) > \beta_y$*).*
b. *If* v_j^x *is incident to only one arrow of color a (say), then* $r(b_1) + r(b_2) < \beta_x$*.*
c. *There is an isolated vertex* v_j^y *in* Γ *if and only if* $\min\{r(a_1) + r(b_2)\} + \min\{r(a_2) + r(b_1)\} < \beta_y$ *(in particular, there are neither 2-sources nor 2-targets at y).*

Proof. We will illustrate (a). If v_j^y is a 2-source, then it is contained in an arrow f^{a_2} and an arrow f^{b_2}. Let us take $\epsilon(y, c(a_2)) = 1 = -\epsilon(y, c(b_2))$. Then by definition of Γ, $j \leq r(a_2)$ and $j \geq \beta_y - r(b_2) + 1$, so $r(a_2) + r(b_2) \geq \beta_y + 1$. □

Finally, we state a consequence for monochromatic paths of length three and the rank function.

Lemma 3. *Suppose that* r *is a maximal rank function for* β*, and* $a_3a_2a_1$ *is a monochromatic path in* Q*, with* $h(a_i) = x_i$*. Then at least one of the two equalities* $r(a_1) + r(a_2) = \beta_{x_1}$ *and* $r(a_2) + r(a_3) = \beta_{x_2}$ *must hold.*

Proof. Suppose neither equality holds. Since r is a rank function, $r(a_1) + r(a_2) < \beta_{x_1}$ and $r(a_2) + r(a_3) < \beta_{x_2}$. Notice that the rank function r' defined by $r'(a_2) = r(a_2) + 1$ and $r'(b) = r(b)$ otherwise is a rank function, and clearly $r' > r$, contradicting maximality. □

4. Main theorem

Theorem 1. *Fix* $A = kQ/I_c$ *a triangular gentle algebra with coloring c, a dimension vector* β*, and a maximal rank function* r*. Let* $B(\Gamma(\beta, r))$ *be the set of band components in* Γ*, and pick* Θ*,* ϵ *as before. Then we have the following:*

$$\operatorname{mod}(A, \beta, r) = \overline{\bigcup_{\underline{\lambda} \in (k^*)^B} \mathrm{GL}(\beta) M(\beta, r, \underline{\lambda})}$$

In particular $M(\beta, r, \underline{\lambda})$ *is the generic module in* $\operatorname{mod}(A, \beta, r)$*.*

Notice in particular that if there are no bands, the irreducible component is an orbit closure. The proof of the theorem relies heavily on the following theorem, whose proof will constitute the remainder of this article.

Theorem 2. *Suppose that* $\underline{\lambda} = (\lambda_b)$*,* $\underline{\nu} = (\nu_b) \in (k^*)^B$ *are two vectors with no common entries (i.e.,* $\lambda_b \neq \nu_{b'}$ *for all* $b \neq b' \in B$*), and* r *is a maximal rank function. Choose* $B(\Gamma(\beta, r))$*,* Θ*,* ϵ *as before. Then*

$$\dim \operatorname{Ext}^1_{kQ/I_c}(M(\beta, r, \underline{\lambda}), M(\beta, r, \underline{\nu})) = 0. \tag{1}$$

Furthermore, if $\Gamma(\beta, r)$ *consists of a single band component, for any* $\underline{\lambda} \in (k^*)^1$

$$\dim \mathrm{Ext}^i_{kQ/I_c}(M(\beta, r, \underline{\lambda}), M(\beta, r, \underline{\lambda})) = \begin{cases} 1 & \textit{if } i = 1 \\ 0 & \textit{if } i > 1 \end{cases} \tag{2}$$

Let us prove Theorem 1 assuming Theorem 2.

Proof of Theorem 1. Suppose that $M(\beta, r, \underline{\lambda}) = \bigoplus_{i=1}^t M(\beta(i), r(i), \lambda(i))$ is the indecomposable decomposition of $M(\beta, r, \underline{\lambda})$ arising from the decomposition of $\Gamma(\beta, r)$ into connected components (λ is no longer underlined in the decomposition since each of the components is at worst a band, and so has only one parameter). Let us recall a result of Crawley-Boevey and Schröer. Suppose that $C(i) \subset \mathrm{mod}(kQ/I, \beta(i))$ are irreducible components for some collection of dimension vectors $\beta(i)$, $i = 1, \ldots, t$, and denote by β the sum $\sum_i \beta(i)$. Define by $C(1) \oplus \ldots \oplus C(t)$ the subset of $\mathrm{mod}(A, \beta)$ given by the set of all $\mathrm{GL}(\beta)$ orbits of direct sums $M(1) \oplus \ldots \oplus M(t)$ with $M(i) \in C(i)$. Then $\overline{C(1) \oplus \ldots \oplus C(t)}$ is an irreducible component of $\mathrm{mod}(A, \beta)$ if and only if $\mathrm{ext}^1_A(C(i), C(j)) = 0$ for all $i \neq j$ (see [3, Theorem 1.2]). Here $\mathrm{ext}^1_A(C(i), C(j))$ denotes $\min\{\dim \mathrm{Ext}^1_A(M, M') \mid M \in C(i), M' \in C(j)\}$.

To apply this theorem, then, we need only show that $\bigcup_{\lambda(i) \in k^*} \mathrm{GL}(\beta(i))M(\beta(i), r(i), \lambda(i))$ is dense within $\mathrm{mod}(A, \beta(i), r(i))$. When $M(\beta(i), r(i), \lambda(i))$ is an indecomposable string (that is, there is no parameter $\lambda(i)$), Eq. (1) of Theorem 2 shows that

$$\mathrm{Ext}^1_A(M(\beta(i), r(i)), M(\beta(i), r(i))) = 0.$$

Therefore by Gabriel [7], $M(\beta(i), r(i))$ has an open orbit. In particular $\mathrm{mod}(A, \beta(i), r(i)) = \overline{\mathrm{GL}(\beta(i))M(\beta(i), r(i))}$. Now assume that $M(\beta(i), r(i), \lambda(i))$ is an indecomposable band module, and let us abbreviate this module by M_λ. Following Kraft [9, 2.7], there is an embedding

$$T_{M_\lambda}(\mathrm{mod}(A, \beta(i), r(i)))/T_{M_\lambda}(\mathrm{GL}(\beta(i))M_\lambda) \hookrightarrow \mathrm{Ext}^1_A(M_\lambda, M_\lambda)$$

where $T_M(X)$ denotes the tangent space to M inside of X. By Eq. (2), the dimension of the target space is 1, so

$$\dim T_{M_\lambda}(\mathrm{mod}(A, \beta(i), r(i))) - \dim T_{M_\lambda}(\mathrm{GL}(\beta(i))M_\lambda) \leq 1.$$

The crucial fact is that $\mathrm{mod}(A, \beta(i), r(i))$ is smooth at M_λ. This is so because $\mathrm{mod}(A, \beta(i), r(i))$ is isomorphic to a product of varieties of complexes taken along each full colored path. Under a particular choice of this isomorphism, M_λ is the image of the product of complexes which have dense orbits in their varieties, which are therefore smooth. As a result we have that

$$\dim(\mathrm{mod}(A, \beta(i), r(i))) - \dim \mathrm{GL}(\beta(i))M_\lambda \leq 1.$$

If the difference is 0, we are done. Otherwise, the difference is one, so $\overline{\mathrm{GL}(\beta(i))M_\lambda}$ is a codimension one subvariety. Now it is clear that $\varphi : k^* \times \mathrm{GL}(\beta(i)) \to \mathrm{mod}(A, \beta(i), r(i))$ given by $(\lambda, g) \mapsto gM_\lambda$ is an injective regular morphism, so $\overline{\bigcup_{\lambda(i)\in k^*} \mathrm{GL}(\beta(i))M_\lambda} = \mathrm{mod}(A, \beta(i), r(i))$. This concludes the proof. □

In order to prove Theorem 2, we construct an explicit projective resolution of the representations $M(\beta, r, \underline{\lambda})$.

5. Minimal projective resolution

To construct the projective resolution in the most general way possible, we will need a fair bit of notation. In order to balance this need, let us fix the quiver Q, the coloring c, the sign function ϵ, the function Θ, and the parameters r and β. We write $M_{\underline{\lambda}}$ for the representation $M(Q, c, \beta, r, \epsilon, \underline{\lambda}, \Theta)$. Denote by $S^0(\Gamma)$ the set of isolated vertices in $\Gamma(\beta, r)$. For each vertex $v_j^x \in T^2(\Gamma)$, we have two distinguished paths $l^+(v_j^x)$ and $l^-(v_j^x)$ defined by the following two conditions:

i. $tl^\delta(v_j^x)$ is a source in Γ;
ii. $\epsilon(x, c(a)) = \delta$ where f_j^a is the arrow in $l^\delta(v_j^x)$ terminating at v_j^x.

Consider a vertex $v_j^x \in S^1$, and let $f_{j'}^a$ be the arrow incident to v_j^x. Denote by $[v_j^x]_0$ the arrow in $Q_1 \setminus a$ with $t[v_j^x]_0 = x$, if it exists. Similarly, consider $v_j^x \in T^1$, and $f_{j'}^a$ the arrow incident to v_j^x. Denote by $[v_j^x]_0$ the arrow in Q_1 with $t[v_j^x]_0 = x$ and $c([v_j^x]_0) \neq c(a)$. In either case, we define $[v_j^x]_i$ inductively by the conditions that $h[v_j^x]_{i-1} = t[v_j^x]_i$ and $c([v_j^x]_{i-1}) = c([v_j^x]_i)$.

Finally, if $v_j^x \in S^0$ is an isolated vertex, for $\delta \in \{-1, 1\}$ we denote by $[v_j^x]_0^\delta$ the arrow (if it exists) defined by the conditions $t[v_j^x]_0^\delta = x$ and $\epsilon(x, c([v_j^x]_0^\delta)) = \delta$. Recursively define $[v_j^x]_i^\delta$ to be the arrow with $t[v_j^x]_i^\delta = h[v_j^x]_{i-1}^\delta$ and $c([v_j^x]_i^\delta) = c([v_j^x]_{i-1}^\delta)$. (For ease of notation, we will often write $\delta = +$ or $-$ in lieu of $1, -1$, respectively.)

For any vertex v_j^x in Γ, we will write $P(v_j^x)$ for the indecomposable projective left A-module $P(x)$, which is the projective cover of the simple module with support at the vertex x. If any of the arrows $[v_j^x]_i$ or $[v_j^x]_i^\delta$ fails to exist, we simply write $\emptyset$ in lieu of an arrow. In this case, if $P(h[v_j^x]_i)$ or $P(h[v_j^x]_i^\delta)$ appears in the projective resolution it is taken to be the zero object in $\mathrm{mod}(A)$.

We are now ready to build the terms of the projective resolution of the representation $M_{\underline{\lambda}}$. For $i \geq 0$, define the modules $\mathbf{F}_i(M_{\underline{\lambda}})$ as follows:

$$\mathbf{F}_0(M_{\underline{\lambda}}) = \bigoplus_{v_j^x \text{ a source in } \Gamma} P(v_j^x)$$

$$\mathbf{F}_1(M_{\underline{\lambda}}) = \bigoplus_{v_j^x \in T^2} P(v_j^x) \oplus \bigoplus_{v_j^x \in S^1 \cup T^1} P(h[v_j^x]_0) \oplus \bigoplus_{v_j^x \in S^0} \left(P(h[v_j^x]_0^+) \oplus P(h[v_j^x]_0^-)\right)$$

$$\mathbf{F}_i(M_{\underline{\lambda}}) = \bigoplus_{v_j^x \in S^1 \cup T^1} P(h[v_j^x]_{i-1}) \oplus \bigoplus_{v_j^x \in S^0} \left(P(h[v_j^x]_{i-1}^+) \oplus P(h[v_j^x]_{i-1}^-)\right) \qquad i > 1$$

We will define the maps $\partial(M_{\underline{\lambda}})_i : \mathbf{F}_{i+1}(M_{\underline{\lambda}}) \to \mathbf{F}_i(M_{\underline{\lambda}})$ component-wise in the following way. The symbol δ can stand either for $+1$ or -1.

$\boxed{\partial(M_{\underline{\lambda}})_0}$ Write $\Lambda(v_j^x) = \lambda_b$ if $v_j^x = \Theta(b)$ for some $b \in B$ and $\Lambda(v_j^x) = 1$ otherwise. When $v_j^x \in T^2(\Gamma)$, the component of $\partial(M_{\underline{\lambda}})_0$ from $P(v_j^x)$ to $P(v_i^y)$ is defined by

$$\begin{cases} \Lambda(v_j^x)\pi(l^+(v_j^x)) & \text{if } v_i^y = t(l^+(v_j^x)) \\ -\pi(l^-(v_j^x)) & \text{if } v_i^y = t(l^-(v_j^x)) \\ 0 & \text{otherwise} \end{cases}$$

If $v_j^x \in S^1$ (in particular, v_j^x is a source) the component of $\partial(M_{\underline{\lambda}})_0$ from $P(h[v_j^x]_0)$ to $P(v_j^x)$ is simply multiplication by the arrow $[v_j^x]_0$. When $v_j^x \in T^1$, let p be the longest path in Γ terminating at v_j^x, whose tail $t(p) := v_i^y$ is by necessity a source. Then the component of $\partial(M_{\underline{\lambda}})_0$ from $P(h[v_j^x]_0)$ to $P(v_i^y)$ is defined by multiplication by the path $p \cdot [v_j^x]_0$. Finally, when $v_j^x \in S^0$, the component of $\partial(M_{\underline{\lambda}})_0$ from $P(h[v_j^x]_0^\delta)$ to $P(v_j^x)$ is multiplication by the arrow $[v_j^x]_0^\delta$. All other components not defined in this paragraph are then taken to be zero.

$\boxed{\partial(M_{\underline{\lambda}})_i}$ For $i > 0$, we define $\partial(M_{\underline{\lambda}})_i : \mathbf{F}_{i+1}(M_{\underline{\lambda}})_{i+1} \to \mathbf{F}_i(M_{\underline{\lambda}})_i$ component-wise as well. For $v_j^x \in S^1 \cup T^1$, we take the component of $\partial(M_{\underline{\lambda}})_0$ from $P(h[v_j^x]_i)$ to $P(h[v_j^x]_{i-1})$ to be multiplication by $[v_j^x]_i$. When $v_j^x \in S^0$, take the component of $\partial(M_{\underline{\lambda}})_0$ from $P(h[v_j^x]_i^\delta)$ to $P(h[v_j^x]_{i-1}^\delta)$ to be multiplication by $[v_j^x]_i^\delta$. As before, all other components not defined herein are taken to be zero.

Finally, it is quite natural to define a morphism $\varepsilon : \mathbf{F}_0(M_{\underline{\lambda}}) \to M_{\underline{\lambda}}$ in the following way. Fix a basis for each of the one-dimensional vector spaces $(\underline{\lambda}_\Gamma)_{v_j^x}$, and denote by e_j^x the basis element in $(M_{\underline{\lambda}})_x$ corresponding to 1 in this space. If v_j^x is a source, then we define $\varepsilon(e)$ of the generator e of $P(v_j^x)$ to be e_j^x and extend A-linearly. In particular, given a path p in Q starting at x considered as an element of $P(v_j^x)$, $\varepsilon(p) = (M_{\underline{\lambda}})_p e_j^x$. So if there is a path p' in Γ with $\pi(p') = p$ and $tp' = v_j^x$, then $\varepsilon(p) = \Lambda(p) e_i^y \in (M_{\underline{\lambda}})_y$ (where e_i^y is the basis element of M_λ corresponding to the vertex $v_i^y := hp'$ and $\Lambda(p)$ is defined by λ_b if p passes through $\Theta'(b)$ for some band b) and 1 otherwise.

Proposition 3. *The complex*

$$\ldots \to \mathbf{F}_2(M_{\underline{\lambda}}) \xrightarrow{\partial(M_{\underline{\lambda}})_1} \mathbf{F}_1(M_{\underline{\lambda}}) \xrightarrow{\partial(M_{\underline{\lambda}})_0} \mathbf{F}_0(M_{\underline{\lambda}}) \xrightarrow{\varepsilon} M_{\underline{\lambda}} \to 0$$

is a minimal projective resolution of $M_{\underline{\lambda}}$ in mod(A).

Proof. First, the surjectivity of ε is clear. M is a finitely generated module generated by the elements e_j^x which are sources, and $\varepsilon^{-1}(e_j^x)$ contains a generator of $P(v_j^x)$.

To show that $\mathbf{F}_\bullet$ is a complex, note that there are two types of elements in the image of ∂_0. Suppose $v_j^x \in T^2(\Gamma)$, let $p_+ = l^+(v_j^x)$ and $p_- = l^-(v_j^x)$, and $v_{i_+}^{y+} = tp_+$, $v_{i_-}^{y-} = tp_-$. If $v_j^x = \Theta(b)$, then $(\varepsilon \circ \partial_0)(v_j^x) = \varepsilon(\lambda_b \pi(l^+(v_j^x)) - \pi(l^-(v_j^x))) = \lambda_b (M_{\underline{\lambda}})_{\pi(p_+)}(e_{i_+}^{y+}) - (M_{\underline{\lambda}})_{\pi(p_-)}(e_{i_-}^{y-}) = \lambda_b e_j^x - \lambda_b e_j^x = 0$. If v_j^x is not $\Theta(b)$ for any band, then the calculation is the same with λ_b replaced by 1.

For the other types of vertices, we have specifically chosen arrows $[v_j^x]_0^\delta$ which act trivially on the basis element in $M_{\underline{\lambda}}$ corresponding to v_j^x, so the composition with ε will indeed yield zero. Furthermore, $[v_j^x]_i^\delta \circ [v_j^x]_{i-1}^\delta = 0$ since both have the same color. Thus, the sequence is indeed a complex.

Now we consider exactness at $\mathbf{F}_0(M_{\underline{\lambda}})$. As above, if v_j^x is a vertex in Γ, let e_j^x denote the basis element of $M_{\underline{\lambda}}$ corresponding to 1 in the one-dimensional vector space at the vertex v_j^x in $(\underline{\lambda}_\Gamma)$. Thus, the collection $\{e_j^x : v_j^x \in \Gamma_0\}$ forms a basis for $M_{\underline{\lambda}}$. Recall that $P(v_j^x)$ has a basis consisting of paths p in Q, not containing subpaths that are in the ideal I, with $tp = x$. We will write these elements as (p, v_j^x) to emphasize the projective from which they came. Call such a pair (p, v_j^x) *compatible* if in $\pi^{-1}(p)$, there is a path q such that $tq = v_j^x$ (and incompatible otherwise). Notice that $\varepsilon(p, v_j^x) \neq 0$ if and only if (p, v_j^x) is compatible. This is clear, since

$$\varepsilon(p, v_j^x) = (M_{\underline{\lambda}})_p e_j^x = \begin{cases} \Lambda(p) e_i^y & \text{if } hq = v_i^y, tq = v_j^x \text{ for some path } q \text{ in } \Gamma \\ 0 & \text{otherwise} \end{cases}$$

and $\Lambda(p) \neq 0$. Let (p, v_j^x) be incompatible. Let p' be the initial subpath of maximal length of p such that (p', v_j^x) is compatible, which is guaranteed to exist since v_j^x itself is a (lazy) path in Γ. Denote by v_i^y the head of this subpath. We distinguish two cases. If $v_i^y \in S^0 \cup S^1 \cup T^1$, then $[v_i^y]_0^\delta p'$ is still a subpath of p for one choice of δ, where we also include the potential that δ does not appear, as in the case of $v_i^y \in T^1$. This holds since the only other concatenable arrow would extend p' to an element of the ideal I by definition of a gentle algebra. Thus, $p = p''[v_i^y]_0^\delta p'$, which is in the image $P(h[v_i^y]_0^\delta)$ under $\partial(M_{\underline{\lambda}})_0$. On the other hand, suppose that $v_i^y \in T^2$, and let a be the next arrow in p after p' (i.e., ap' is still a subpath of p). By abuse of notation, we will write v_i^y for the generator of $P(v_i^y)$. Recall that $\partial(M_{\underline{\lambda}})_0 : v_i^y \mapsto \Lambda(p)\pi(l^+(v_i^y)) - \pi(l^-(v_i^y))$. Assume without loss of generality that $p' = \pi(l^+(v_i^y))$ (it must be one of the two paths). Then $\partial(M_{\underline{\lambda}})_0 : a \mapsto \Lambda(p)ap' - 0$ since $a\pi(l^-(v_i^y))$ must be in the ideal I, again by definition of the gentle algebra. Since $\Lambda(p) \neq 0$, we have that all incompatible pairs (p, v_j^x) are in the image of $\partial(M_{\underline{\lambda}})_0$.

Let τ be a linear combination of compatible paths $\sum \gamma_{(p,v_j^x)}(p, v_j^x)$.

$$\begin{aligned} \varepsilon(\tau) &= \sum_{\substack{(p,v_j^x) \text{ compatible} \\ v_j^x \in S(\Gamma)}} \gamma_p \varepsilon(p, v_j^x) \\ &= \sum \gamma_{(p,v_j^x)} (M_{\underline{\lambda}})_p e_j^x \end{aligned}$$

For each path p in Q in the above sum, there is a *unique* path $q \in \pi^{-1}(p)$ defined by the property that $tq = v_j^x$. Take $v_i^y = hq$, so that $(M_{\underline{\lambda}})_p e_j^x = \Lambda(p) e_i^y$ where $\Lambda(p) = \lambda_b$ if p contains the arrow $\pi(\Theta(b'))$ for some band b, and 1 otherwise. When v_j^x and v_i^y are as above, we will write $v_j^x \leq_p v_i^y$ as a relation. Therefore, we can rewrite the sum above by collecting coefficients of the e_i^y as follows:

$$\varepsilon(\tau) = \sum_{v_i^y} \left(\sum_{(p,v_j^x): v_j^x \leq_p v_i^y} \gamma_{(p,v_j^x)} \Lambda(p) \right) e_i^y$$

Since the e_i^y form a basis of $M_{\underline{\lambda}}$, the coefficients in the above expansion must equal zero. Again we distinguish two cases: either one or two arrows terminate at v_i^y. If there is only one arrow, then there is exactly one compatible pair (p, v_j^x) with $v_j^x \leq_p v_i^y$, so $\gamma_{(p,v_j^x)} = 0$. If there are two such arrows, then the paths $l^-(v_i^y) := p_-$ and $l^+(v_i^y) := p_+$ are the unique paths in Γ terminating at v_i^y and starting at a source, v_- and v_+, respectively. Thus, we have $\Lambda(p_-)\gamma_{(p_-,v_-)} = -\Lambda(p_+)\gamma_{(p_+,v_+)} = 0$. In particular, τ has the summand

$$\gamma_{(p_+,v_+)}(p_+, v_+) + \gamma_{(p_-,v_-)}(p_-, v_-) = \frac{\gamma_{(p_+,v_+)}}{\Lambda(p_-)} \left(\Lambda(p_-) \cdot (p_+, v_+) - \Lambda(p_+) \cdot (p_-, v_-)\right).$$

Now if v_i^x is not of the form $\Theta(b)$, then $\Lambda(p_-) = \Lambda(p_+) = 1$, while if v_i^x is of the form $\Theta(b)$, then $\Lambda(p_+) = \lambda_b$ and $\Lambda(p_-) = 1$, so in either case the above element is in the image of $\partial(M_{\underline{\lambda}})_0$, namely it is the image of $\frac{\gamma_{(p_+,v_+)}}{\Lambda(p_-)} v_i^y$.

Finally, we note that each component of $\partial(M_{\underline{\lambda}})_i$ is in the Jacobson radical of A, so $\mathbf{F}_\bullet$ is indeed minimal. $\square$

Now that we have a projective resolution of $M_{\underline{\lambda}}$, we will apply the functor $\mathrm{Hom}_A(-, M_{\underline{\nu}})$ to it and calculate the dimensions of the extension groups. Recall that the module $M_{\underline{\nu}}$ is the pushdown of a module $\underline{\nu}_\Gamma$, which assigns to each vertex v_i^x the one dimensional vector space k, and $\pi_*(\underline{\lambda}_\Gamma)_x = \bigoplus_{j=1,\ldots,\beta_x} k$. In this way, we can take $\{v_j^x : j = 1, \ldots, \beta_x\}$ to be the corresponding basis of $(M_{\underline{\nu}})_x$. Furthermore, $\mathrm{Hom}(P_x, M_{\underline{\nu}}) \cong (M_{\underline{\nu}})_x$, so for each v_j^x let us denote the basis of $\mathrm{Hom}_A(P(v_j^x), M_{\underline{\nu}})$ by $\{v_j^x \boxtimes v_{j'}^x : j' = 1, \ldots, \beta_x\}$, and write $M_{\underline{\nu}}(v_j^x)$ as shorthand for this space to indicate the projective from which it arose. The space $\mathrm{Hom}_A(P(h[v_j^x]_i^\delta), M_{\underline{\nu}})$ will be denoted by $M_{\underline{\nu}}([v_j^x]_i^\delta)$, and its basis by $\{v_j^x \boxtimes v_{j'}^{h[v_j^x]_i^\delta} : j' = 1, \ldots, \beta_{h[v_j^x]_i^\delta}\}$. The benefit of this cumbersome notation is that one can keep track of the projective in the first step of the resolution that is responsible for the appearance of these particular basis elements.

Applying $\mathrm{Hom}_A(-, M_{\underline{\nu}})$ to the resolution then yields the complex

$$C_0 \xrightarrow{\mathrm{Hom}(\partial_0, M_{\underline{\nu}})} C_1 \xrightarrow{\mathrm{Hom}(\partial_1, M_{\underline{\nu}})} C_2 \xrightarrow{\mathrm{Hom}(\partial_2, M_{\underline{\nu}})} \ldots \tag{3}$$

where

$$C_0 = \bigoplus_{v_j^x \text{ a source in } \Gamma} M_{\underline{\nu}}(v_j^x)$$

$$C_1 = \bigoplus_{v_j^x \in T^2} M_{\underline{\nu}}(v_j^x) \oplus \bigoplus_{v_j^x \in S^1 \cup T^1} M_{\underline{\nu}}([v_j^x]_i) \oplus \bigoplus_{v_j^x \in S^0} \big(M_{\underline{\nu}}([v_j^x]_0^+) \oplus M_{\underline{\nu}}([v_j^x]_0^-)\big)$$

$$C_i = \bigoplus_{v_j^x \in S^1 \cup T^1} M_{\underline{\nu}}([v_j^x]_{i-1}) \oplus \bigoplus_{v_j^x \in S^0} \big(M_{\underline{\nu}}([v_j^x]_{i-1}^+) \oplus M_{\underline{\nu}}([v_j^x]_{i-1}^-)\big) \qquad i > 1$$

In order to calculate the homology of this complex, we construct a graph $\mathbb{EXT}$ whose vertices correspond to the fixed basis described above. We partition the vertices into subsets $\mathbb{EXT}(i)$ for $i \geq 0$ which we call *levels*, whose vertices correspond to the basis of C_i.

Definition 4. Let $\mathbb{EXT}(l)$ be the sets defined as:

$$\begin{aligned}
\mathbb{EXT}(0) &:= \{v_j^x \boxtimes v_{j'}^x : \ v_j^x \text{ a source in } \Gamma, \ j' = 1, \dots, \beta_x\} \\
\mathbb{EXT}(1) &:= \{v_j^x \boxtimes v_{j'}^x : \ v_j^x \in T^2, \ j' = 1, \dots, \beta_x\} \\
&\qquad \cup \{v_j^x \boxtimes v_{j'}^{h[v_j^x]_0} : \ v_j^x \in S^1 \cup T^1, \ j' = 1, \dots, \beta_{h[v_j^x]_0}\} \\
&\qquad \cup \{v_j^x \boxtimes v_{j'}^{h[v_j^x]_0^t} : \ v_j^x \in S^0, \ j' = 1, \dots, \beta_{h[v_j^x]_0^t}, \ t = -, +\} \\
\mathbb{EXT}(l) &:= \{v_j^x \boxtimes v_{j'}^{h[v_j^x]_{l-1}} : \ v_j^x \in T^1 \cup S^1, \ j' = 1, \dots, h[v_j^x]_{l-1}\} \\
&\qquad \cup \{v_j^x \boxtimes v_{j'}^{h[v_j^x]_{l-1}^t} : \ v_j^x \in S^0, \ j' = 1, \dots, \beta_{h[v_j^x]_{l-1}^t}, \ t = +, -\} \quad l > 1
\end{aligned}$$

The graph $\mathbb{EXT}$ is, by definition, the digraph with vertices $\bigcup_{l \geq 0} \mathbb{EXT}(l)$ and arrows $v_i^x \boxtimes v_j^y \to v_{i'}^{x'} \boxtimes v_{j'}^{y'}$ if the coefficient of $v_{i'}^{x'} \boxtimes v_{i'}^{y'}$ in

$$\mathrm{Hom}(\partial(M_{\underline{\lambda}})_l, M_{\underline{\nu}})(v_i^x \boxtimes v_j^y)$$

is non-zero. In particular arrows that start in level $\mathbb{EXT}(l)$ end in level $\mathbb{EXT}(l+1)$. It will be convenient to label such an arrow by said coefficient.

The digraph $\mathbb{EXT}$ can be thought of as the graph of each map $\mathrm{Hom}(\partial(M_{\underline{\lambda}})_l, M_{\underline{\nu}})$. In order to show that there is no homology in the first degree, we need to collect some properties of this graph.

5.1. Properties of the $\mathbb{EXT}$ graph

The absence of homology in the first degree will follow from a number of considerations. First we show that there are no isolated vertices at level $\mathbb{EXT}(1)$, each vertex in this level is then shown to be incident to at most two arrows. This breaks the kernel of $\mathrm{Hom}(\partial(M_{\underline{\lambda}})_l, M_{\underline{\nu}})$ into pieces which lie in strings or bands, and then these components are shown to be surjective.

Proposition 4. *Let $\mathbb{EXT}$ be the graph given above. Then we have the following:*

E1: *There is an arrow*

$$\mathbb{EXT}(0) \ni v_j^x \boxtimes v_{j'}^x \to v_i^y \boxtimes v_{i'}^y \in \mathbb{EXT}(1)$$

if $v_j^x \in S^2$, $v_i^y \in T^2$ such that $v_j^x \xrightarrow{p} v_i^y$ and $v_{j'}^x \xrightarrow{p'} v_{i'}^y$ are parallel paths in Γ.

E2: *Suppose $v_i^y \in T^1$, $v_j^x = l^{\pm}(v_i^y)$, and p is the path from v_j^x to v_i^y. Then there is an edge*

$$\mathbb{EXT}(0) \ni v_j^x \boxtimes v_{j'}^x \to v_i^y \boxtimes v_{i'}^{y'} \in \mathbb{EXT}(1)$$

if there is a path $v_{j'}^x \xrightarrow{p'} v_{i'}^{y'}$ in Γ with $\pi(p') = [v_i^y]_0 \pi(p)$. Furthermore, there is an arrow

$$\mathbb{EXT}(l) \ni v_j^x \boxtimes v_{j'}^{h[v_i^y]_{l-1}} \to v_i^y \boxtimes v_{i'}^{h[v_i^y]_l} \in \mathbb{EXT}(l+1)$$

if in Γ there is an edge $v_{i'}^{h[v_i^y]_{l-1}} \xrightarrow{a} v_{j'}^{h[v_i^y]_l}$ such that $\pi(a) = [v_i^y]_l$.

E3: *Similarly, if $v_i^y \in S^1$, then there is an edge*

$$\mathbb{EXT}(0) \ni v_j^y \boxtimes v_{j'}^y \to v_i^y \boxtimes v_{i'}^{h[v_i^y]_0} \in \mathbb{EXT}(1)$$

if there is an edge $v_{j'}^y \xrightarrow{a} v_{i'}^{h[v_i^y]_0}$ with $\pi(a) = [v_j^y]_0$. Furthermore, an arrow

$$\mathbb{EXT}(l) \ni v_j^x \boxtimes v_{j'}^{h[v_j^x]_{l-1}} \to v_j^x \boxtimes v_{i'}^{h[v_j^x]_l} \in \mathbb{EXT}(l+1)$$

if there is an edge $v_{j'}^{h[v_j^x]_{l-1}} \xrightarrow{a} v_{i'}^{h[v_j^x]_l}$ in Γ with $\pi(a) = [v_j^x]_l$.

E4: *Finally, if v_j^x is an isolated vertex in Γ, then there are arrows*

$$\mathbb{EXT}(0) \ni v_j^x \boxtimes v_{j'}^x \to v_j^x \boxtimes v_{i'}^{h[v_j^x]_0^{\pm}} \in \mathbb{EXT}(1)$$

$$\mathbb{EXT}(l) \ni v_j^x \boxtimes v_{j'}^{h[v_j^x]_{l-1}^{\pm}} \to v_j^x \boxtimes v_{i'}^{h[v_j^x]_l^{\pm}} \in \mathbb{EXT}(l+1)$$

if there is an arrow $v_{j'}^{h[v_j^x]_{l-1}^{\pm}} \xrightarrow{a} v_{i'}^{h[v_j^x]_l^{\pm}}$ in Γ with $\pi(a) = [v_j^x]_l^{\pm}$ (here we take $h[v_j^x]_{-1}^{\pm} = x$).

Lemma 4. *There are no isolated vertices in level $\mathbb{EXT}(1)$.*

Proof. Suppose $v_j^x \boxtimes v_{j'}^x \in \mathbb{EXT}(1)$ (that is, $v_j^x \in T^2$). If $j' < j$ (resp. $j' > j$), take p to be the path from $v_i^y := lp^-(v_j^x)$ (resp. $v_i^y := lp^+(v_j^x)$) to v_j^x. Then by Lemma 1 there is a path p' terminating at $v_{j'}^x$ parallel to p. Denote by $v_{i'}^y = tp'$. Then there is an arrow $v_i^y \boxtimes v_{i'}^y \to v_j^x \boxtimes v_{j'}^x$ by property E1 of Proposition 4.

Suppose now that $v_j^x \in T^1$ and $[v_j^x]_0$ exists, so that $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0} \in \mathbb{EXT}(1)$. Assume without loss of generality that $\epsilon(x, c[v_j^x]_0) = 1$ and $\epsilon(h[v_j^x]_0, c([v_j^x]_0)) = -1$. Let p be the path of maximum length terminating at v_j^x, and $tp = v_i^y$ (which is a source in Γ). Let $[v_j^x]_{-1}$ be the arrow (if it exists) with $c([v_j^x]_{-1}) = c([v_j^x]_0)$ and $h[v_j^x]_{-1} = x$. By Lemma 2(b), $r([v_j^x]_{-1}) + r([v_j^x]_0) < \beta_x$, so by Lemma 3, $r([v_j^x]_0) + r([v_j^x]_1) = \beta_{h[v_j^x]_0}$. In particular, $v_{j'}^{h[v_j^x]_0}$ is the head of an arrow $f_{j'}^{[v_j^x]_0}$ (if $j' \leq r([v_j^x]_0) < j$) or the tail of an arrow $f^{[v_j^x]_1}$ otherwise. In the former case, there is a path p' terminating at $v_{j'}^{h[v_j^x]_0}$ with $\pi(p') = [v_j^x]_0\pi(p)$, so that there is an arrow $v_i^y \boxtimes tp' \to v_j^x \boxtimes v_{j'}^{h[v_j^x]_0}$. In the latter case, there is an arrow $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0} \to v_j^x \boxtimes hf^{[v_j^x]_1}$.

An analogous proof shows that if $v_j^x \in S^1$, then $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0}$ is not isolated.

Finally, suppose that v_j^x is an isolated vertex in Γ such that at least one of $[v_j^x]_0^\pm$ exists (otherwise there is no vertex with first component v_j^x in $\mathbb{EXT}(1)$). Consider $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0^\pm} \in \mathbb{EXT}(1)$, and let $[v_j^x]_{-1}^\pm$ be the arrow in Q_1 such that $c([v_j^x]_{-1}^\pm) = c([v_j^x]_0)$ and $h[v_j^x]_{-1}^\pm = x$. From Lemma 2(c), $r([v_j^x]_{-1}^\pm) + r([v_j^x]_0^\pm) < \beta_x$, so according to Lemma 3, $r([v_j^x]_0^\pm) + r([v_j^x]_1^\pm) = \beta_{h[v_j^x]_0^\pm}$. Thus, $v_{j'}^{h[v_j^x]_0^\pm}$ is either contained in an arrow $f^{[v_j^x]_0^\pm}$ or $f^{[v_j^x]_1^\pm}$. In the former case, $v_j^x \boxtimes tf^{[v_j^x]_0^\pm} \to v_j^x \boxtimes v_{j'}^{h[v_j^x]_0^\pm}$ is an arrow in $\mathbb{EXT}$, and in the latter case, $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0^\pm} \to v_j^x \boxtimes hf^{[v_j^x]_1^\pm}$ is in $\mathbb{EXT}$. $\square$

The following lemma shows that $\mathbb{EXT}$ splits into string and band components, and that the band components occur between levels $\mathbb{EXT}(0)$ and $\mathbb{EXT}(1)$.

Lemma 5. *All vertices in $\mathbb{EXT}(1)$ and $\mathbb{EXT}(2)$ are contained in at most two arrows. All vertices labeled $v_j^x \boxtimes v_{j'}^{h[v_j^x]_l}$ for $l \geq 0$ (and therefore all vertices in level $\mathbb{EXT}(l)$ for $l > 1$) are contained in at most one arrow.*

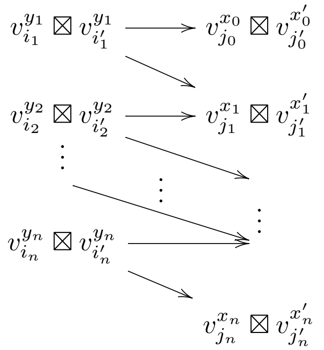
Proof. Recall from property E2 of Proposition 4 that $v_j^x \boxtimes v_{j'}^{h[v_j^x]_0}$ shares an arrow either with $v_j^x \boxtimes v_{j''}^x$ or $v_j^x \boxtimes v_{j''}^{h[v_j^x]_1}$ (exclusively) whenever v_j^x is either isolated or a one-source/target. Otherwise, suppose that $v_j^x \in T^2$, and consider the vertex $v_j^x \boxtimes v_{j'}^x$. If $v_i^y \boxtimes v_{i'}^y \to v_j^x \boxtimes v_{j'}^x$, then $v_i^x = tl^\delta(v_j^x)$ (for some $\delta \in \{+, -\}$) and there is a path p in Γ parallel to $tl^\delta(v_j^x)$ terminating at $v_{j'}^x$ in Γ. For each δ there is only one such path, so indeed the bound holds. $\square$

A particular consequence of the preceding lemmas is that the kernel of the map $\mathrm{Hom}(\partial(M_{\underline{\lambda}})_1, M_{\underline{\nu}})$ is spanned by the basis elements corresponding to the vertices in

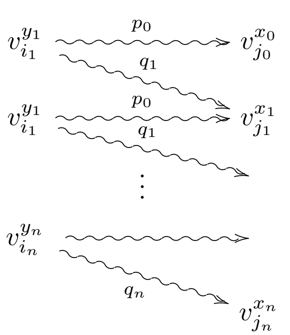
$\mathbb{EXT}(1)$ which are not contained in an arrow to $\mathbb{EXT}(2)$. As a result of Lemma 4, none of these are isolated, so we need only show that the restriction of the map to the subspace spanned by those elements who are contained in an arrow from $\mathbb{EXT}(1)$ is surjective. Via this restriction, we can consider the string and band components of this graph separately, first showing that the string components correspond to surjections, and then that the band components do.

Lemma 6. *No string component in* $\mathbb{EXT}$ *has both endpoints in* $\mathbb{EXT}(1)$.

Proof. Suppose that there is a string in $\mathbb{EXT}$ with one endpoint $v_{j_0}^{x_0'} \boxtimes v_{j_0'}^{x_0'} \in \mathbb{EXT}(1)$ and containing the substring

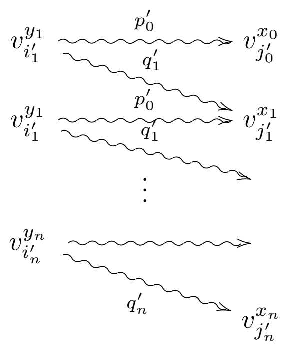


where $v_{i_t}^{y_t} \boxtimes v_{i_t'}^{y_t} \in \mathbb{EXT}(0)$ and $v_{j_t}^{x_t} \boxtimes v_{j_s'}^{x_t} \in \mathbb{EXT}(1)$. We need only show that the string does not end at $v_{j_n}^{x_n} \boxtimes v_{j_n'}^{x_n}$. By definition of the digraph $\mathbb{EXT}$, we must have paths



in Γ. We break the consideration into four cases, depending on the type of the vertices $v_{j_0}^{x_0}, v_{j_n}^{x_n'}$. If a is the color of the arrow in a path p starting or ending at the vertex $z \in \Gamma$, we will write $\epsilon(z, p)$ for the sign $\epsilon(z, a)$.

Case 1: Suppose $v_{j_0}^{x_0}, v_{j_n}^{x_n} \in T^2$. In particular, there are paths in Γ



for which p_i' and q_i' are parallel to p_i and q_i, respectively. Since $v_{j_n}^{x_n}$ is a 2-target, one of $l^+(v_{j_n}^{x_n})$ or $l^-(v_{j_n}^{x_n})$ exists and is not p_n, let us call this q_0; similarly there is a path $q_0 \neq p_0$ which is one of $l^+(v_{j_0}^{x_0})$ or $l^-(v_{j_0}^{x_0})$.

Suppose first that $j_0' < j_0$. If $v_{j_0'}^{x_0}$ is above $v_{j_0}^{x_0}$, then $\epsilon(x_0, p_0) = -1$. Otherwise, by Lemma 1, there would exist a path q_0' terminating at $v_{j_0'}^{x_0}$ parallel to q_0, so by definition $tq_0 \boxtimes tq_0' \to v_{j_0}^{x_0} \boxtimes v_{j_0'}^{x_0}$ would be an arrow in $\mathbb{EXT}$, contradicting the assumption that this vertex is an endpoint of the string.

Now suppose that $v_{j_n'}^{x_n}$ is below $v_{j_n}^{x_n}$, so $\epsilon(x_n, q_n) = \epsilon(x_n, q_n') = -1$ (by applying induction and Lemma 1), and so $\epsilon(x_n, p_n) = 1$. Again, applying the lemma, there is a path p_n' parallel to p_n terminating at $v_{j_n'}^{x_n}$, thus an arrow $tp_n \boxtimes tp_n' \to v_{j_n}^{x_n} \boxtimes v_{j_n'}^{x_n}$, the latter vertex is not the other endpoint of string.

The argument follows similarly for different possibilities for i_0', i_n' relative to i_0, i_n.

Case 2: Suppose now that $v_{j_0}^{x_0} \in T^2$ and $v_{j_n}^{x_n} \in T^1$. Given that $v_{j_n}^{x_n} \boxtimes v_{j_n'}^{x_n'}$, Proposition 4 E2 implies the existence of an arrow $a \in \Gamma_1$ with $h(a) = v_{j_n'}^{x_n'}$ and $\pi(a) = [v_{j_n}^{x_n}]_1$. Denote by $v_{(j_n')^-}^{x_n} = t(a)$. In particular, $\pi(q_n') = \pi(aq_n)$.

Furthermore, since $v_{j_0}^{x_0} \in T^2$, there exists a longest non-trivial path q_0 distinct from p_0 with head $v_{j_0}^{x_0}$ (in particular, with $\epsilon(q_0, x_0) = -\epsilon(p_0, x_0)$).

Suppose that $j_0' < j_0$. In this case, $\epsilon(q_0, x_0) = 1$ (otherwise there would be a path q_0' with $\pi(q_0) = \pi(q_0')$ and $hq_0' = v_{j_0'}^{x_0'}$ giving rise to a second arrow $\to v_{j_0}^{x_0} \boxtimes v_{j_0'}^{x_0'}$ in $\mathbb{EXT}$, a contradiction). Therefore, by definition, $\epsilon(p_0, x_0) = -1$. Now we iterate Lemma 1: suppose, for example, that $(j_n')^- < j_n$, i.e., the orientation has been preserved, so $\epsilon(q_n, x_n) = 1$. But since aq_n is a path, we must have $\epsilon(a, x_n) = -1$, implying the existence of a path a' parallel to a with $ta' = v_{j_n}^{x_n}$, contradicting the assumption that $v_{j_n}^{x_n} \in T^1$. If $(j_n')^- > j_n$, the argument is replicated changing signs, again yielding a contradiction.

If $j_0' > j_0$, then $\epsilon(q_0, x_0) = -1$. Again, by definition, $\epsilon(p_0, x_0) = 1$ and the same argument is replicated changing the signs to yield a contradiction.

Case 3: Finally, suppose that both $v_{j_0}^{x_0}, v_{j_n}^{x_n} \in T^1$. In this case, we have arrows $a_0 := [v_{j_0}^{x_0}]_1$ and $a_n := [v_{j_n}^{x_n}]_1$ in Γ with heads $v_{j_0'}^{x_0'}$ and $v_{j_n'}^{x_n'}$, respectively. Denote by $(j_0')^-$ and $(j_n')^-$, respectively, the lower indices of the tails of these arrows, $v_{(j_0')^-}^{x_0}$, $v_{(j_n')^-}^{x_n}$.

Suppose that $(j_0')^- < j_0$. In this case, $\epsilon(a_0, x_0) = 1$ (or else $v_{j_0}^{x_0}$ would be the tail of an arrow parallel to a_0, contradicting its membership in T^1). Since $a_0 p_0$ is a path, $\epsilon(p_0, x_0) = -1$.

If $(j_n')^- < j_n$, that is, the order has been preserved, then $\epsilon(q_n, x_n) = 1$, and since $a_n q_n$ is a path, $\epsilon(a_n, x_n) = -1$. But this would imply the existence of an arrow a_n' parallel to a_n with $ta_n' = v_{j_n}^{x_n}$, contradicting its membership in T^1. Similarly, if $(j_n')^- > j_n$, the order has been reversed, so $\epsilon(q_n, x_n) = -1$ and $\epsilon(a_n, x_n) = 1$, implying existence of the troubling arrow a_n' with tail $v_{j_n}^{x_n}$, again a contradiction.

The same argument can be repeated under the assumption $(j_0')^- > j_0$ by changing signs. □

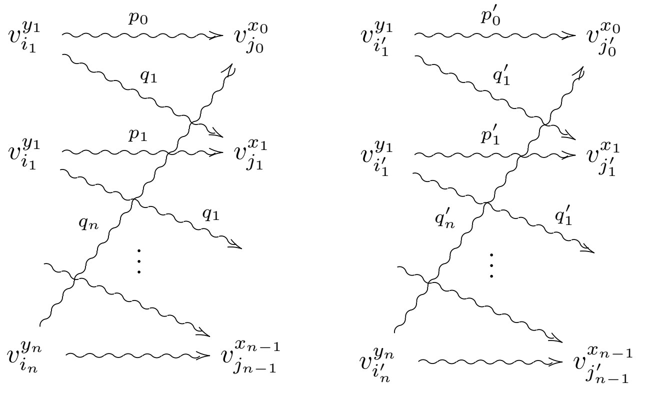
5.1.1. Homology and the $\mathbb{EXT}$ graph

The above results can be interpreted in the context of the maps $\mathrm{Hom}(\partial(M_{\underline{\lambda}})_1, M_{\underline{\nu}})$ and $\mathrm{Hom}(\partial(M_{\underline{\lambda}})_0, M_{\underline{\nu}})$, which we will denote h_1 and h_0, respectively. By Lemmas 4 and 5, the kernel of h_1 is spanned by the basis elements contained in an arrow whose tail is in $\mathbb{EXT}(0)$. Finally, Lemma 6 implies that there are only three types of connected components of the $\mathbb{EXT}$ graph between levels $\mathbb{EXT}(0)$ and $\mathbb{EXT}(1)$: strings with both endpoints in $\mathbb{EXT}(0)$, strings with one endpoint in $\mathbb{EXT}(1)$ and the other in $\mathbb{EXT}(0)$, and cycles. On the level of the map h_0, strings with both endpoints in $\mathbb{EXT}(0)$ show that the subspace spanned by those vertices in $\mathbb{EXT}(1)$ in such a connected component is in the image of h_0. Similarly, strings with one endpoint in $\mathbb{EXT}(0)$ and one in $\mathbb{EXT}(1)$ represent square blocks of h_0 which are either upper- or lower-triangular with non-zero diagonal entries (after permutation of the basis elements), so again the subspace spanned by the vertices in $\mathbb{EXT}(1)$ contained in such connected components is in the image of h_0.

Therefore, it suffices to show that the restriction of h_0 to the subspaces spanned by vertices contained in cyclic connected components in $\mathbb{EXT}(0)$ and $\mathbb{EXT}(1)$ is an isomorphism.

Lemma 7. *Suppose the connected component C_i of the EXT-graph is cyclic. Then the restriction of h_0 to C_i is an isomorphism onto its image.*

Proof. If C_i is a cyclic connected component, then there are paths p_i, q_i, p_i', q_i' in Γ such that p_i and p_i' are parallel, q_i and q_i' are parallel, and sit in the following configuration:



In particular, these are bands in Γ of the same shape (possibly the same band, in case $i'_k = i_k$ and $j'_k = j_k$). By definition of h_1, the matrix of h_0 on C_i, projected to its image, is of the form

$$\begin{bmatrix} 1 & -\lambda_b & 0 & \dots & 0 \\ 0 & \pm 1 & \pm 1 & \dots & 0 \\ 0 & 0 & \pm 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \pm 1 & 0 & \dots & \dots & \pm 1 \end{bmatrix}$$

where one of the diagonal entries is $\nu_{b'}$ (with b' corresponding to the second band), and in each row there is exactly one positive and one negative entry. Adding each column (or $1/\nu_{b'}$ of the column containing $\nu_{b'}$) to its successor, and then the last column to the first yields an upper triangular matrix with $1 - \frac{\lambda_b}{\nu_{b'}}$ as the first entry, ± 1 on all but one other diagonal entry, which is $\nu_{b'}$. So the determinant is $\pm(\lambda_b - \nu_{b'})$. By assumption, $\underline{\lambda}$ and $\underline{\nu}$ have no common components, so h_0 restricted to C_i is non-singular. $\square$

This concludes the proof of Eq. (1) of Theorem 2. The second part of the theorem is the calculation of $\operatorname{Ext}^1_A(M_{\underline{\lambda}}, M_{\underline{\lambda}})$ when $M_{\underline{\lambda}}$ is an indecomposable band. Hence, $\underline{\lambda} = (\lambda) \in k^1$. We have already seen that $M_{\underline{\lambda}}$ has projective dimension one, so $\operatorname{Hom}(\mathbf{F}_1(M_{\underline{\lambda}}), M_{\underline{\lambda}}) = \ker \operatorname{Hom}(\partial(M_{\underline{\lambda}})_2, M_{\underline{\lambda}})$. By assumption, there is only one cyclic component of Γ, and therefore only one cyclic component C of the $\mathbb{EXT}$ graph. The other components represent surjective maps when restricted to the span of the vertices in $\mathbb{EXT}(1)$ contained in these components. By the above discussion, h_0 restricted to C is not an isomorphism with the span of the vertices in $\mathbb{EXT}(1)$ contained in the band component C (since $\det h_1|_C = \pm(\lambda - \lambda) = 0$). However, in this case, the result of the column reduction yields an upper triangular matrix with 0 as the first entry, and ± 1, $\pm\lambda$ as the rest of the diagonal entries. This has rank $n-1$, and so indeed $\ker \operatorname{Hom}(\delta_2, M_{\underline{\lambda}}) / \operatorname{image} \operatorname{Hom}(\partial(M_{\underline{\lambda}})_1, M_{\underline{\lambda}})$ is one-dimensional.

6. Higher extension groups

The graphical representation given above can be used to calculate higher extension groups. For each vertex $v_j^x \in S^1 \cup T^1$, let $X_{j,x}$ be the complex

$$(M_{\underline{\lambda}})_x \xrightarrow{[v_j^x]_1} (M_{\underline{\lambda}})_{h[v_j^x]_1} \xrightarrow{[v_j^x]_2} (M_{\underline{\lambda}})_{h[v_j^x]_2} \xrightarrow{[v_j^x]_3} \cdots.$$

Furthermore, if $v_j^x \in S^0$, let $X_{j,x}^+$ be the complex

$$(M_{\underline{\lambda}})_x \xrightarrow{[v_j^x]_1^+} (M_{\underline{\lambda}})_{h[v_j^x]_1^+} \xrightarrow{[v_j^x]_2^+} (M_{\underline{\lambda}})_{h[v_j^x]_2^+} \xrightarrow{[v_j^x]_3^+} \cdots,$$

and analogously for $X_{j,x}^-$. Let $h^i(X)$ be the dimension of the i-th homology space of the complex X.

Corollary 1. *Let $\Gamma(Q, c, \beta, r, \epsilon)$ be an up-and-down graph for kQ/I_c a gentle string algebra. Then*

$$\dim_k \operatorname{Ext}_A^i(M(\beta, r, \underline{\lambda}), M(\beta, r, \underline{\nu})) = \sum_{v_j^x \in S^1 \cup T^1} h^i(X_{j,x}) + \sum_{v_j^x \in S^0} \left(h^i(X_{j,x}^+) + h^i(X_{j,x}^-)\right).$$

Thus, in order to sound the dimensions of the spaces of higher extensions between the generic modules, it suffices to measure the failure of the complexes along a given colored path to be exact at various locations.

6.1. Example

We finish by exhibiting the $\mathbb{EXT}$ graph for Example 1. Recall that we chose $\Theta(b) = v_1^6$ for the band component, taking $\underline{\lambda} = (\lambda) \in k^1$. By Proposition 3, the projective resolution of the representation in the example is given by

$$M_{\underline{\lambda}} \leftarrow P_1^3 \oplus P_4^2 \xleftarrow{\partial_0} P_2 \oplus P_3 \oplus P_5^2 \oplus P_6 \xleftarrow{\partial_1} P_3$$

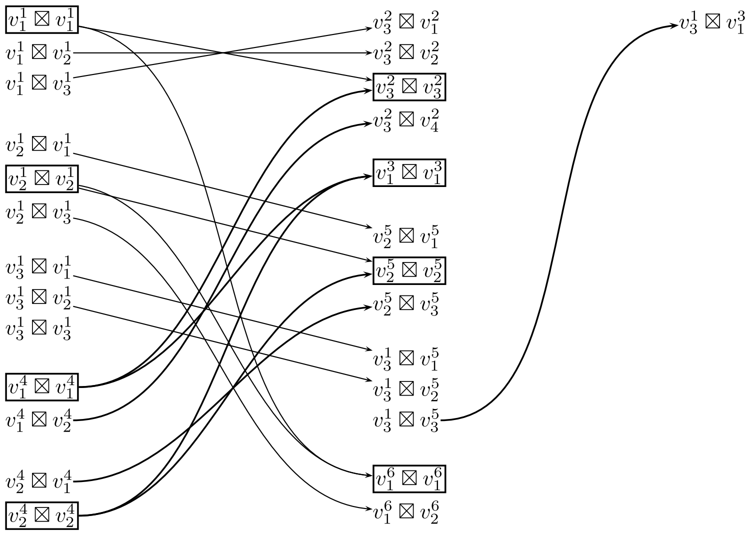
where

$$\partial_0 = \begin{bmatrix} -a_1 & 0 & 0 & 0 & -\lambda b_2 g_1 \\ 0 & 0 & -g_1 & 0 & p_2 a_1 \\ 0 & 0 & 0 & g_1 & 0 \\ p_1 & -g_2 b_1 & 0 & 0 & 0 \\ 0 & a_2 p_1 & b_1 & 0 & 0 \end{bmatrix} \qquad \partial_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_2 \\ 0 \end{bmatrix}$$

The associated $\mathbb{EXT}$ graph is obtained by applying $\operatorname{Hom}(-, M_\nu)$ to the resolution, so we have the complex:

$$(M_\nu)_1^3 \oplus (M_\nu)_4^2 \xrightarrow{\operatorname{Hom}(\partial_0, M_\nu)} (M_\nu)_2 \oplus (M_\nu)_3 \oplus (M_\nu)_5^2 \oplus (M_\nu)_6 \xrightarrow{\operatorname{Hom}(\partial_1, M_\nu)} (M_\nu)_3$$

The $\mathbb{EXT}$ graph is depicted below, with the vertices lying in a cyclic component of the graph boxed.



Acknowledgments

The author would like to thank Jerzy Weyman for many helpful insights in the preparation of this article. In addition, discussions with Ryan Kinser and Kavita Sutar proved very helpful. Finally, the referee's report was thoughtful and thorough, and the level of detail has greatly improved the presentation of the paper.

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