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The $A + XB[X]$ construction from Prüfer v-multiplication domains

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ABSTRACT

Let $A \subseteq B$ be an extension of integral domains, X be an indeterminate over B, and $R = A + XB[X]$. We prove that if B is t-flat over A, then R is a PvMD if and only if A is a PvMD and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a t-splitting set of ideals of A. We also prove that R is a GGCD domain if and only if A is a GGCD domain and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A. Finally, we use this result to recover that R is a GCD domain if and only if A is a GCD domain and $B = A_S$ for some splitting set S of A.

0. Introduction

Let D be an integral domain, $qf(D)$ be the quotient field of D, S be a (saturated) multiplicative set of D, X be an indeterminate over D, and $D^{(S)} = D + XD_S[X]$; so $D[X] \subseteq D^{(S)} \subseteq D + XK[X]$, where $K = qf(D)$. In particular, if S is the set of units

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of D (resp., $S = D \setminus \{0\}$), then $D^{(S)} = D[X]$ (resp., $D^{(S)} = D + XK[X]$). We plan to include, in Sections 0.1 and 0.2, a sufficient introduction to the terminology used in this paper and in this introduction. If needed the readers may read Sections 0.1 and 0.2 first, for a better understanding.

Let $T = \bigoplus_{n\in\mathbb{N}} R_n$ be a nontrivial graded integral domain graded by $\mathbb{N}$, the monoid of nonnegative integers. Then T is a Prüfer domain if and only if R_0 is a Prüfer domain and $T \cong R_0 + yK_0[y]$, where $K_0 = qf(R_0)$ and y is an indeterminate over R_0 [15, Proposition 3.4]. This type of integral domains were first studied in [13] where the authors proved that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and GCD(d, X) exists in $D^{(S)}$ for all $0 \neq d \in D$. They also studied several ring-theoretic properties (for example, Bezout domain, Prüfer domain, v-domain, PvMD) of the ring $D + XK[X]$. Later, in [29], it was shown that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and S is a splitting set of D. Also, in [2], the authors proved that $D^{(S)}$ is a PvMD (resp., GGCD domain) if and only if D is a PvMD (resp., GGCD domain) and S is a t-splitting (resp., d-splitting) set of D.

Let $A \subseteq B$ be an extension of integral domains, X be an indeterminate over B, and $R = A + XB[X]$. It is known that if R is a PvMD, then B is an overring of A [6, Proposition 2.6(1)] and that R is a GCD domain if and only if A is a GCD domain and $B = A_S$ for S a splitting set of A [6, Theorem 2.10]. In this paper, we study when R is a PvMD or a GGCD domain; hence, by [6, Proposition 2.6(1)], we may assume that B is an overring of A. (An *overring* of A means a ring between A and the quotient field of A.) We begin with a study of a t-splitting set of ideals, in Section 1. Let $\mathfrak{S}$ be a multiplicative set of ideals of A. In Section 2, we show that if $\mathfrak{S}$ is a t-splitting set of ideals and A is a PvMD, then $A + XA_{\mathfrak{S}}[X]$ is a PvMD; moreover, if B is t-flat over A, then R a PvMD implies that A is a PvMD and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a t-splitting set of ideals of A. Finally, in Section 3, we first define the notion of d-splitting sets of ideals and give a nice characterization of d-splitting sets of ideals. We then prove that R is a GGCD domain if and only if A is a GGCD domain and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A. We use this result to recover Anderson and El Abidine's result [6, Theorem 2.10] that $R = A + XB[X]$ is a GCD domain if and only if A is a GCD domain and $B = A_S$ for some splitting set S of A.

0.1. Star operations and related notations

Let D be an integral domain with quotient field K. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D, i.e., $I \in \mathbf{F}(D)$ if I is a nonzero D-submodule of K with $dI \subseteq D$ for some $0 \neq d \in D$. For $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in \mathbf{F}(D) \text{ is finitely generated}\}$, and $I_d = I$. It is well known and easy to show that if $* = v, t$, or d, then $(aD)_* = aD$, $(aI)_* = aI_*$, $I \subseteq I_*$, $I \subseteq J$ implies $I_* \subseteq J_*$, and $(I_*)_* = I_*$ for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$.

More generally, a mapping $*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star-operation* on D if for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$, the following conditions are satisfied:

(1) $(aD)^* = aD$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given a star operation $*$ on D, one can construct a new star operation $*_f$ by setting $I^{*_f} = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in \mathbf{F}(D) \text{ is finitely generated}\}$ for all $I \in \mathbf{F}(D)$. A star operation $*$ on D is said to be of *finite type* if $*_f = *$. Obviously, $(*_f)_f = *_f$, and hence $*_f$ is of finite type. An $I \in \mathbf{F}(D)$ is called a $*$*-ideal* if $I^* = I$, and we say that a $*$-ideal is a *maximal* $*$*-ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) denote the set of maximal $*$-ideals of D. It may happen that $*$-Max$(D) = \emptyset$ even though D is not a field (for example, if D is a rank-one nondiscrete valuation domain, then v-Max$(D) = \emptyset$). However, it is well known that $*_f$-Max$(D) \neq \emptyset$ when D is not a field; a maximal $*_f$-ideal is a prime ideal; each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal; and $I^{*_f} = \bigcap_{P \in *_f\text{-Max}(D)} I^{*_f} D_P$ for all $I \in \mathbf{F}(D)$. We know that if $*$ is any star operation on D, then $I_d \subseteq I^{*_f} \subseteq I^* \subseteq I_v$ and $I^{*_f} \subseteq I_t$ for all $I \in \mathbf{F}(D)$. An $I \in \mathbf{F}(D)$ is said to be $*$*-invertible* if $(II^{-1})^* = D$. It is well known that I is $*_f$-invertible if and only if I^{*_f} is of finite type and ID_P is principal for all $P \in *_f$-Max(D) [21, Proposition 2.6]. We say that D is a *Prüfer* $*$*-multiplication domain* (P$*$MD) if every nonzero finitely generated ideal of D is $*_f$-invertible. Hence, PdMDs are just the Prüfer domains. An integral domain D is a GCD domain if $aD \cap bD$ is principal for all $0 \neq a, b \in D$, while D is a *generalized GCD domain* (GGCD domain) if $aD \cap bD$ is invertible for all $0 \neq a, b \in D$. Clearly,

$$\text{GCD domain} \;\Rightarrow\; \text{GGCD domain} \;\Rightarrow\; \text{P}v\text{MD}.$$

Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$, and let $Inv(D)$ (resp., $Prin(D)$) be its subgroup of invertible (resp., nonzero principal) fractional ideals of D. Then $Cl(D) = T(D)/Prin(D)$, called the *class group of D,* is an abelian group and $Pic(D) = Inv(D)/Prin(D)$, the *Picard group of D*, is a subgroup of $Cl(D)$. Clearly, if each maximal ideal of D is a t-ideal (e.g., in a Prüfer domain), then $Cl(D) = Pic(D)$. It is well known that D is a GCD domain (resp., GGCD domain) if and only if D is a PvMD and $Cl(D) = 0$ (resp., $Cl(D) = Pic(D)$). For basic properties of star operations, see [19, §32].

0.2. Multiplicative sets and multiplicative sets of ideals

Let S be a saturated multiplicative set of an integral domain D, and let $N(S) = \{0 \neq a \in D \mid (a, s)_v = D \text{ for all } s \in S\}$. We say that S is a *splitting set* if each nonzero $d \in D$ can be written as $d = st$ for some $s \in S$ and $t \in N(S)$. Let $* = t$ or d. Then S is called a $*$*-splitting set* if, for each $0 \neq d \in D$, we have $dD = (IJ)_*$, where I and J are ideals of D with $I_* \cap sD = sI_*$ for all $s \in S$ and $J_* \cap S \neq \emptyset$. The notions of $*$-splitting sets were introduced in [2] in order to study when $D + XD_S[X]$ is a PvMD or a GGCD domain.

Let $\mathfrak{S}$ be a multiplicative set of ideals of D, $sp(\mathfrak{S}) = \{I \mid I$ is an ideal of D and $J \subseteq I$ for some $J \in \mathfrak{S}\}$, and $\mathfrak{S}^\perp$ be the set of ideals I of D with $(I + A)_t = D$ for all $A \in \mathfrak{S}$. Then $D_\mathfrak{S} = \{x \in K \mid xI \subseteq D$ for some $I \in \mathfrak{S}\}$ is an overring of D called the *$\mathfrak{S}$-transform of D* or a *generalized transform of D*. Clearly, $\mathfrak{S}^\perp = sp(\mathfrak{S}^\perp) = sp(\mathfrak{S})^\perp$ and $D_\mathfrak{S} = D_{sp(\mathfrak{S})}$. For basic properties of generalized transforms of D, see [10]. As in [18], we say that $\mathfrak{S}$ is *v-finite* if for each $I \in \mathfrak{S}$, there is a nonzero finitely ideal J of D such that $J_v \in sp(\mathfrak{S})$ and $J_v \subseteq I_t$. Following [12], we say that $\mathfrak{S}$ is a *t-splitting set of ideals* if each nonzero $d \in D$ can be written as $dD = (IJ)_t$ for some $I \in sp(\mathfrak{S})$ and $J \in \mathfrak{S}^\perp$. Clearly, $\mathfrak{S}$ is t-splitting if and only if $sp(\mathfrak{S})$ is t-splitting, if and only if $\mathfrak{S}^\perp$ is t-splitting [12, Proposition 2]. Also, if S is a multiplicative set of D, then $\mathfrak{S} := \{aD \mid a \in S\}$ is a v-finite multiplicative set of ideals such that $D_S = D_\mathfrak{S}$, and S is a t-splitting set if and only if $\mathfrak{S}$ is a t-splitting set of ideals.

1. t-Splitting set of ideals and t-flatness

Let D be an integral domain, $K = qf(D)$, and $\mathfrak{S}$ be a multiplicative set of ideals of D. We begin this section by recalling a nice characterization of t-splitting sets of ideals.

Proposition 1.1. *(See [12, Proposition 5].) Let $\mathfrak{S}$ be a multiplicative set of ideals of D. Then $\mathfrak{S}$ is t-splitting if and only if $\mathfrak{S}$ is v-finite and $dD_\mathfrak{S} \cap D$ is t-invertible for all $0 \neq d \in D$.*

For an ideal I of D, let $I_\mathfrak{S} = \{x \in K \mid xJ \subseteq I$ for some $J \in \mathfrak{S}\}$. It is easy to see that $I_\mathfrak{S}$ is an ideal of $D_\mathfrak{S}$ and $ID_\mathfrak{S} \subseteq I_\mathfrak{S}$. Let T be a multiplicative set of D, and let $\mathfrak{F} = \{AD_T \mid A \in \mathfrak{S}\}$. Clearly, $\mathfrak{F}$ is a multiplicative set of ideals of D_T. We next show that if $\mathfrak{S}$ is v-finite, then $(D_T)_\mathfrak{F} = (D_\mathfrak{S})_T$.

Proposition 1.2. *Let $\mathfrak{S}$ be a v-finite multiplicative set of ideals of D (e.g., $\mathfrak{S}$ is t-splitting), T be a multiplicative set of D, and $\mathfrak{F} = \{AD_T \mid A \in \mathfrak{S}\}$.*

1. *$(AD_\mathfrak{S})_t = D_\mathfrak{S}$ for all $A \in \mathfrak{S}$.*
2. *$D_\mathfrak{S}$ is t-flat over D, i.e., $(D_\mathfrak{S})_M = D_{M \cap D}$ for every $M \in t\text{-}Max(D_\mathfrak{S})$.*
3. *If J is a t-ideal of $D_\mathfrak{S}$, then $J = ((J \cap D)D_\mathfrak{S})_t = ((J \cap D)_t D_\mathfrak{S})_t$.*
4. *$(D_T)_\mathfrak{F} = (D_\mathfrak{S})_T$.*
5. *If $\mathfrak{S}$ is t-splitting, then $\mathfrak{F}$ is a t-splitting set of ideals of D_T.*

Proof. (1) Since $\mathfrak{S}$ is v-finite, there exists a nonzero finitely generated ideal B of D such that $B_v \subseteq A_t$ and $B_v \in sp(\mathfrak{S})$. Hence, $x \in (BD_\mathfrak{S})^{-1} \;\Leftrightarrow\; xB \subseteq D_\mathfrak{S}, \Rightarrow\; xBB_1 \subseteq D$ for some $B_1 \in \mathfrak{S}$ because B is finitely generated, $\Rightarrow\; xB_t(B_1)_t \subseteq (xBB_1)_t \subseteq D$, $\Rightarrow\; x \in D_\mathfrak{S}$. Thus, $(BD_\mathfrak{S})^{-1} = D_\mathfrak{S}$ or $(BD_\mathfrak{S})_t = (BD_\mathfrak{S})_v = D_\mathfrak{S}$, and since $B_t \subseteq A_t$, we have $D_\mathfrak{S} = (BD_\mathfrak{S})_t \subseteq (B_t D_\mathfrak{S})_t \subseteq (A_t D_\mathfrak{S})_t = (AD_\mathfrak{S})_t \subseteq D_\mathfrak{S}$ (see [21, Lemma 3.4] for the last equality). Therefore, $(AD_\mathfrak{S})_t = D_\mathfrak{S}$.

(2) Let Q be a maximal t-ideal of $D_{\mathfrak{S}}$, and put $Q \cap D = P$. Then $A \nsubseteq P$ for all $A \in sp(\mathfrak{S})$ by (1), and thus $Q = P_{\mathfrak{S}}$ and $(D_{\mathfrak{S}})_Q = D_P$ [10, Theorem 1.1].

(3) If $x \in J$, then $xA \subseteq J \cap D$ for some $A \in \mathfrak{S}$. Hence, $x \in xD_{\mathfrak{S}} = x(AD_{\mathfrak{S}})_t = (xAD_{\mathfrak{S}})_t \subseteq ((J \cap D)D_{\mathfrak{S}})_t$. The reverse containment is clear. The second equality is from [21, Lemma 3.4].

(4) ($\subseteq$) Let $0 \neq \beta \in (D_T)_{\mathfrak{F}}$. Then $\beta AD_T \subseteq D_T$ for some $A \in \mathfrak{S}$, and since $\mathfrak{S}$ is v-finite, there is a finitely generated ideal J of D such that $J_v \subseteq A_t$ and $J_v \in sp(\mathfrak{S})$. Note that $\beta J \subseteq \beta(JD_T) \subseteq \beta(J_vD_T)_t \subseteq \beta(A_tD_T)_t = \beta(AD_T)_t \subseteq D_T$. Since J is finitely generated, there exists an $s \in T$ such that $\beta sJ \subseteq D$, and so $\beta sJ_v \subseteq D$. Hence, $\beta s \in D_{\mathfrak{S}}$, and thus $\beta \in (D_{\mathfrak{S}})_T$. ($\supseteq$) Let $\alpha \in D_{\mathfrak{S}}$ and $s \in T$. Then $\alpha A \subseteq D$ for some $A \in \mathfrak{S}$, and hence $\frac{\alpha}{s}AD_T \subseteq D_T$ and $AD_T \in \mathfrak{F}$. Thus, $\frac{\alpha}{s} \in (D_T)_{\mathfrak{F}}$.

(5) Let $0 \neq \alpha \in D_T$. Then $\alpha D_T = aD_T$ for some $a \in D$, and since $\mathfrak{S}$ is t-splitting, $aD = (AB)_t$ for some $A \in sp(\mathfrak{S})$ and $B \in \mathfrak{S}^{\perp}$. Hence, $\alpha D_T = (AB)_tD_T = ((AB)D_T)_t = ((AD_T)(BD_T))_t$, where the second equality follows because AB is t-invertible. Note that $AD_T \in sp(\mathfrak{F})$. Also, if $C \in \mathfrak{F}$, then $C = C_1D_T$ for some $C_1 \in \mathfrak{S}$; hence $D_T \supseteq (C + BD_T)_t = (C_1D_T + BD_T)_t = ((C_1 + B)D_T)_t = ((C_1 + B)_tD_T)_t = D_T$ (cf. [21, Lemma 3.4] for the third equality). Thus, $BD_T \in \mathfrak{F}^{\perp}$. Therefore, $\mathfrak{F}$ is a t-splitting set of ideals of D_T. $\square$

Corollary 1.3. *Let $\mathfrak{S}$ be a t-splitting set of ideals of D, and let $\Lambda = \{P \in t\text{-}Max(D) \mid (PD_{\mathfrak{S}})_t \subsetneq D_{\mathfrak{S}}\}$.*

(1) *$t\text{-}Spec(D_{\mathfrak{S}}) = \{P_{\mathfrak{S}} \mid P \in t\text{-}Spec(D)$ and $A \nsubseteq P$ for all $A \in \mathfrak{S}\}$.*
(2) *$(PD_{\mathfrak{S}})_t = P_{\mathfrak{S}}$ and $P_{\mathfrak{S}} \in t\text{-}Max(D_{\mathfrak{S}})$ for all $P \in \Lambda$.*
(3) *$t\text{-}Max(D_{\mathfrak{S}}) = \{P_{\mathfrak{S}} \mid P \in \Lambda\}$.*

Proof. (1) ($\subseteq$) Let Q be a prime t-ideal of $D_{\mathfrak{S}}$, and set $P = Q \cap D$. Then $Q = (PD_{\mathfrak{S}})_t = (P_tD_{\mathfrak{S}})_t$ by Proposition 1.2(3), and hence $P = Q \cap D \supseteq P_tD_{\mathfrak{S}} \cap D \supseteq P_t$; so $P_t = P$. If $A \subseteq P$ for some $A \in \mathfrak{S}$, then $Q \supseteq (PD_{\mathfrak{S}})_t \supseteq (AD_{\mathfrak{S}})_t = D_{\mathfrak{S}}$ by Proposition 1.2(1), a contradiction. Thus, $A \nsubseteq P$ for all $A \in \mathfrak{S}$, and hence $Q = (PD_{\mathfrak{S}})_t = P_{\mathfrak{S}}$ [10, Theorem 1.1]. ($\supseteq$) Let P be a prime t-ideal of D such that $A \nsubseteq P$ for all $A \in \mathfrak{S}$. Then $P \in \mathfrak{S}^{\perp}$, and hence $P = (PD_{\mathfrak{S}})_t \cap (PD_{\mathfrak{S}^{\perp}})_t = (PD_{\mathfrak{S}})_t \cap D_{\mathfrak{S}^{\perp}} = (PD_{\mathfrak{S}})_t \cap D$ by Proposition 1.2(1) and [12, Proposition 8]. So $(PD_{\mathfrak{S}})_t = P_{\mathfrak{S}}$ [12, Lemma 11], and thus $P_{\mathfrak{S}}$ is a prime t-ideal of $D_{\mathfrak{S}}$.

(2) If $P \in \Lambda$, then $A \nsubseteq P$ for all $A \in \mathfrak{S}$ by Proposition 1.2(1), and thus, by the proof of ($\supseteq$) of (1) above, $(PD_{\mathfrak{S}})_t = P_{\mathfrak{S}}$. Also, by (1), $P_{\mathfrak{S}} \in t$-Max$(D_{\mathfrak{S}})$.

(3) Let Q be a maximal t-ideal of $D_{\mathfrak{S}}$. Then $Q = ((Q \cap D)D_{\mathfrak{S}})_t = ((Q \cap D)_tD_{\mathfrak{S}})_t$ by Proposition 1.2(3), and hence $Q \cap D$ is a t-ideal of D. Note that $A \nsubseteq Q \cap D$ for all $A \in \mathfrak{S}$; so $Q = (Q \cap D)_{\mathfrak{S}}$ [10, Theorem 1.1]. Let P be a maximal t-ideal of D with $Q \cap D \subseteq P$. Then $Q \cap D \in \mathfrak{S}^{\perp}$ implies $P \in \mathfrak{S}^{\perp}$, and thus $Q = (Q \cap Q)_{\mathfrak{S}} \subseteq P_{\mathfrak{S}}$ and $P_{\mathfrak{S}}$ is a t-ideal by (1). Therefore, $Q = P_{\mathfrak{S}}$ and $P \in \Lambda$. The reverse containment is from (2) above. $\square$

Corollary 1.4. *Let $\mathfrak{S}$ be a t-splitting set of ideals of D. If P is a prime t-ideal of D containing some $A \in \mathfrak{S}$, then $(D_{\mathfrak{S}})_{D\setminus P} = K$.*

Proof. Let $\mathfrak{F} = \{AD_P \mid A \in \mathfrak{S}\}$. Then $\mathfrak{F}$ is a t-splitting set of ideals of D_P and $(D_{\mathfrak{S}})_{D\setminus P} = (D_P)_{\mathfrak{F}}$ by Proposition 1.2(4) and (5). If $(D_P)_{\mathfrak{F}} \neq K$, then there is a nonzero prime ideal P_0 of D such that $((P_0D_P)((D_P)_{\mathfrak{F}}))_t$ is a maximal t-ideal of $(D_P)_{\mathfrak{F}}$ by Corollary 1.3. Clearly, $P \in sp(\mathfrak{S})$, and hence $P_0 \in sp(\mathfrak{S})$ [12, Proposition 10]; so $P_0D_P \in sp(\mathfrak{F})$, and by Proposition 1.2(1), $((P_0D_P)((D_P)_{\mathfrak{F}}))_t = (D_P)_{\mathfrak{F}}$, a contradiction. Therefore, $(D_{\mathfrak{S}})_{D\setminus P} = K$. $\square$

Corollary 1.5. *If $\mathfrak{S}$ is a t-splitting set of ideals of a valuation domain D, then $D_{\mathfrak{S}} = D$ or K.*

Proof. Let M be the maximal ideal of D. If $A \nsubseteq M$ for all $A \in \mathfrak{S}$, then $\mathfrak{S} = \{D\}$, and hence $D_{\mathfrak{S}} = D$. Next, if $A \subseteq M$ for some $A \in \mathfrak{S}$, then $D_{\mathfrak{S}} = (D_{\mathfrak{S}})_{D\setminus M} = K$ by Corollary 1.4. $\square$

Corollary 1.6. *(Cf. [29, Lemma 1.1].) Let D be a nontrivial valuation domain, V be an integral domain with $D \subsetneq V$, and X be an indeterminate over V. Then the following statements are equivalent.*

(1) *$D + XV[X]$ is a PvMD.*
(2) *$V = K$.*
(3) *$D + XV[X]$ is a Bezout domain.*
(4) *$D + XV[X]$ is a GCD domain.*

Proof. (1) $\Rightarrow$ (2) By [6, Proposition 2.6(i)], V is an overring of D, and since D is a valuation domain, $V = D_Q$ for some prime ideal Q of D. Let $S = D \setminus Q$. Then $D + XV[X] = D + XD_S[X]$, and hence S is a t-splitting set of D [2, Theorem 2.5]. Thus, $V = K$ by Corollary 1.5.

(2) $\Rightarrow$ (3) This follows directly from [13, Corollary 4.13] because a valuation domain is a Bezout domain.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) Clear. $\square$

For $0 \neq \alpha \in K$, let $(D : \alpha) = \{x \in D \mid x\alpha \in D\}$; so $(D : \alpha)$ is an ideal of D. Clearly, flat overrings of an integral domain D are t-flat over D. The next result is a t-flatness analogue of the fact that an overring T of D is flat over D if and only if there is a multiplicative set $\mathfrak{S}$ of ideals of D such that $T = D_{\mathfrak{S}}$ and $AT = T$ for all $A \in \mathfrak{S}$ [10, Theorem 1.3], if and only if $(D : \alpha)T = T$ for all $0 \neq \alpha \in T$ [27, Theorem 1]. Although this result is already known, we give a new proof because the proof is used in the proof of Theorem 3.5.

Theorem 1.7. *(See [24, Proposition 2.5].) If T is an overring of D, then the following statements are equivalent.*

(1) *T is t-flat over D.*
(2) *There is a multiplicative set $\mathfrak{S}$ of ideals of D such that $T = D_{\mathfrak{S}}$ and $(AT)_t = T$ for all $A \in \mathfrak{S}$.*
(3) *$((D : \alpha)T)_t = T$ for all $0 \neq \alpha \in T$.*

Proof. (1) ⇒ (3) Let M be a maximal t-ideal of T. Then $T_M = D_{M\cap D}$, and hence $(D : \alpha)T_M = (D : \alpha)D_{M\cap D} = (D_{M\cap D} : \alpha D_{M\cap D}) = D_{M\cap D} = T_M$. Thus, $T \supseteq ((D : \alpha)T)_t \supseteq \bigcap_{M\in t\text{-Max}(T)}(D : \alpha)T_M = \bigcap_{M\in t\text{-Max}(T)} T_M = T$ (cf. [21, Proposition 2.8(3)] for the second containment), and so $((D : \alpha)T)_t = T$.

(3) ⇒ (2) Let $\mathfrak{S}$ be the multiplicative set of ideals of D generated by $\{(D : \alpha) \mid 0 \neq \alpha \in T\}$. Clearly, if $A \in \mathfrak{S}$, then $(AT)_t = T$ by (3). Also, $T \subseteq D_{\mathfrak{S}}$. For the reverse containment, let $x \in D_{\mathfrak{S}}$. Then $xA \subseteq D$ for some $A \in \mathfrak{S}$, and thus $x \in xT = x(AT)_t = (xAT)_t \subseteq T_t = T$. Therefore, $T = D_{\mathfrak{S}}$.

(2) ⇒ (1) Let M be a maximal t-ideal of T, and put $P = M \cap D$. Then $A \nsubseteq P$ for all $A \in \mathfrak{S}$ because $(AT)_t = T$. Hence, by [10, Theorem 1.1], $M = P_{\mathfrak{S}}$ and $T_M = (D_{\mathfrak{S}})_{P_{\mathfrak{S}}} = D_P$. Thus, T is t-flat over D. □

The next result is a t-flat overring analogue of the fact that if T is a flat overring of D, then $(IT)^{-1} = I^{-1}T$ for every nonzero finitely generated ideal I of D.

Corollary 1.8. *Let T be a t-flat overring of D. If I is a nonzero finitely generated ideal of D, then $(IT)^{-1} = (I^{-1}T)_t$.*

Proof. Clearly, $(I^{-1}T)_t \subseteq (IT)^{-1}$. For the reverse containment, let $0 \neq \alpha \in (IT)^{-1}$. Then $\alpha I \subseteq \alpha IT \subseteq T = D_{\mathfrak{S}}$ for some multiplicative set $\mathfrak{S}$ of ideals of D by Theorem 1.7. Since I is finitely generated, $\alpha IA \subseteq D$ for some $A \in \mathfrak{S}$. Let Q be a maximal t-ideal of T. Then $T_Q = D_{Q\cap D}$, and since $(AT)_t = T$ by Theorem 1.7, $AD_{Q\cap D} = D_{Q\cap D}$. Hence, $\alpha ID_{Q\cap D} = \alpha IAD_{Q\cap D} \subseteq D_{Q\cap D}$, and so $\alpha \in (ID_{Q\cap D})^{-1} = I^{-1}D_{Q\cap D} = I^{-1}T_Q \subseteq (I^{-1}T)_tT_Q$. Hence, $\alpha \in \bigcap_{Q\in t\text{-Max}(T)}(I^{-1}T)_tT_Q = (I^{-1}T)_t$. Thus, $(IT)^{-1} \subseteq (I^{-1}T)_t$. □

An extension ring T of D is said to be *t-linked over D* if $I^{-1} = D$ for I a nonzero finitely generated ideal of D implies $(IT)^{-1} = T$. Clearly, t-flat overrings of D are t-linked over D by Corollary 1.8. Also, it is known that the integral closure of a Noetherian domain D is t-linked over D (cf. [17, Lemma 4.5]). The notion of t-linkedness was introduced in [16] in order to obtain a PvMD analogue of a characterization of Prüfer domains [14, Theorem 1] that D is a Prüfer domain if and only if each overring of D is integrally closed.

Another nice characterization of Prüfer domains is as follows: D is a Prüfer domain if and only if each overring of D is flat [27], if and only if each overring of D is an invertible generalized transform of D [10, Theorem 1.5]. (An overring T of D is an *invertible generalized transform* of D if $T = D_{\mathfrak{S}}$ for $\mathfrak{S}$ a multiplicative set of ideals consisting entirely of invertible ideals.) As a t-operation analogue, we will say that $\mathfrak{S}$ is a *t-invertible multiplicative set of ideals* of D if for each $A \in \mathfrak{S}$, there is a t-invertible ideal I of D such that $I_t \subseteq A_t$ and $I_t \in sp(\mathfrak{S})$. An overring D_1 of D is a *t-invertible generalized transform* of D if $D_1 = D_{\mathfrak{S}}$ for some t-invertible multiplicative set $\mathfrak{S}$ of ideals of D. Clearly, t-splitting sets of ideals are t-invertible [12, Proposition 2] and a t-invertible multiplicative set of ideals is v-finite.

Theorem 1.9. *The following statements are equivalent.*

(1) *D is a PvMD.*
(2) *Each t-linked overring of D is a PvMD.*
(3) *Each t-linked overring of D is integrally closed.*
(4) *Each t-linked overring of D is t-flat over D.*
(5) *Each t-linked overring of D is a t-invertible generalized transform of D.*
(6) *Each t-linked valuation overring of D is a t-invertible generalized transform of D.*
(7) *D_P is a valuation domain for each maximal t-ideal P of D.*

Proof. (1) ⇒ (2) Let R be a t-linked overring of D, $K = qf(D)$, and $c_D(h)$ (resp., $c_R(h)$) be the fractional ideal of D (resp., R) generated by the coefficients of a polynomial $h \in K[X]$. Note that if $0 \neq f \in D[X]$ with $c_D(f)_v = D$, then $c_R(f)_v = (c_D(f)R)_v = R$ because R is t-linked over D. Hence, there exists a set Δ of prime t-ideals of D such that $R = \bigcap_{P\in\Delta} D_P$, i.e., R is a subintersection of D [21, Theorem 3.8]. Thus, R is a PvMD [25, Proposition 5.1].

(2) ⇒ (3) Clear.

(3) ⇔ (1) [16, Theorem 2.10].

(1) ⇔ (4) [24, Proposition 2.10].

(1) ⇒ (5) Let D_1 be a t-linked overring of D. Then $D_1 = \bigcap D_{P_\alpha}$, where $\{P_\alpha\}$ is a set of prime t-ideals of D [21, Theorem 3.8]. Note that if $0 \neq x \in K$ and P is a prime ideal of D, then $x \in D_P$ if and only if $(D : x)D_P = D_P$. Hence, $x \in D_1$ if and only if $x \in D_{P_\alpha}$ for all α, if and only if $(D : x) \nsubseteq P_\alpha$ for all α.

Let $\mathfrak{S}$ be the multiplicative set of ideals of D generated by $\{(D : x) \mid 0 \neq x \in D_1\}$. Clearly, $D_1 \subseteq D_{\mathfrak{S}}$. For the reverse containment, let $x \in D_{\mathfrak{S}}$. Then $xA \subseteq D$ for some $A \in \mathfrak{S}$, and since $(D : x) \nsubseteq P_\alpha$ for all α, we have $A \nsubseteq P_\alpha$. Hence $x \in \bigcap xD_{P_\alpha} = \bigcap xAD_{P_\alpha} \subseteq \bigcap D_{P_\alpha} = D_1$. Thus, $D_1 = D_{\mathfrak{S}}$. Also, since D is a PvMD, $(D : x)$ is t-invertible, and thus each ideal in $\mathfrak{S}$ is t-invertible.

(5) ⇒ (6) Clear.

(6) ⇒ (7) Let P be a maximal t-ideal of D. Then there is a valuation overring V of D with maximal ideal M such that $M \cap D = P$. So $D_P \subseteq V_M = V$, and, in particular, V is t-linked over D. Hence, $V = D_{\mathfrak{S}}$ for some t-invertible multiplicative set $\mathfrak{S}$ of

ideals of D. Clearly, $\mathfrak{S}$ is v-finite, and so V is t-flat over D by Proposition 1.2(2). Thus, $V = V_M = D_{M\cap D} = D_P$.

(7) ⇒ (1) [21, Theorem 3.2]. □

We end this section with a PvMD analogue of the fact that an integrally closed domain D is a Prüfer domain if and only if $(IT)^{-1} = I^{-1}T$ for every overring T of D and a nonzero finitely generated ideal I of D [9, Corollary 4.3].

Corollary 1.10. *An integrally closed domain D is a PvMD if and only if $(IT)^{-1} = (I^{-1}T)_t$ for every t-linked overring T of D and a nonzero finitely generated ideal I of D.*

Proof. (⇒) This follows directly from Corollary 1.8 because a t-linked overring of a PvMD is t-flat by Theorem 1.9. (⇐) By Theorem 1.9, it suffices to show that T is integrally closed. Let $K = qf(D)$, and let $0 \neq f, g \in K[X]$. Then $c_D(fg)_v = (c_D(f)c_D(g))_v$ [19, Proposition 34.8] because D is integrally closed. Hence, by assumption, $c_T(fg)^{-1} = (c_D(fg)^{-1}T)_t = ((c_D(f)c_D(g))^{-1}T)_t = ((c_D(f)c_D(g))T)^{-1} = (c_T(f)c_T(g))^{-1}$, and thus $c_T(fg)_v = (c_T(f)c_T(g))_v$. Thus, T is integrally closed [26, Lemme 1]. □

2. Prüfer v-multiplication domains

Let $A \subseteq B$ be an extension of integral domains, X be an indeterminate over B, and $R = A + XB[X]$. Let

- $\Lambda = \{P \in t\text{-Max}(A) \mid (PB)_t \subsetneq B\}$,
- $\Lambda' = \{P \in t\text{-Max}(A) \mid (PB)_t = B\}$.

Clearly, $\Lambda \cap \Lambda' = \emptyset$ and $\Lambda \cup \Lambda' = t\text{-Max}(A)$. In this section, we study the PvMD property of R when $B = A_{\mathfrak{S}}$, where $\mathfrak{S}$ is a multiplicative set of ideals of A. (We usually use D instead of A when $B = A_{\mathfrak{S}}$.)

Lemma 2.1. *Let $R = A + XB[X]$ and I be a nonzero ideal of A.*

(1) $(IR)^{-1} = I^{-1} + X(IB)^{-1}[X]$.

(2) *Let $\mathfrak{S}$ be a multiplicative set of ideals of A, and suppose that $B = A_{\mathfrak{S}}$.*

  (a) *If I is finitely generated or a v-ideal of finite type, then $(IR)^{-1} = I^{-1} + X(I^{-1})_{\mathfrak{S}}[X]$.*

  (b) *If I is t-invertible, then $(IR)_v = I_v + X(I_v)_{\mathfrak{S}}[X] = I_v + X(IA_{\mathfrak{S}})_v[X]$.*

Proof. (1) By [7, Lemma 2.1], $(IR)^{-1} = I^{-1} \cap (IB)^{-1} + X(IB)^{-1}[X]$, and since $I^{-1} \subseteq (IB)^{-1}$, we have $(IR)^{-1} = I^{-1} + X(IB)^{-1}[X]$.

(2) Note that $(I_{\mathfrak{S}})^{-1} = (IA_{\mathfrak{S}})^{-1} = (I^{-1})_{\mathfrak{S}}$ (cf. [21, Lemma 3.4]). Hence, (a) follows directly from (1). For (b), note that $(IR)^{-1} = I^{-1} + X(I^{-1})_{\mathfrak{S}}[X]$ by (1), $(IA_{\mathfrak{S}})_v =$

$((I^{-1})_{\mathfrak{S}})^{-1} = (I_v)_{\mathfrak{S}}$ because I is t-invertible, $I^{-1} \subseteq (I^{-1})_{\mathfrak{S}}$, and $I_v \subseteq (I_v)_{\mathfrak{S}}$. Thus, $(IR)_v = (I^{-1} + X(I^{-1})_{\mathfrak{S}}[X])^{-1} = I_v + X(I_v)_{\mathfrak{S}}[X] = I_v + X(IA_{\mathfrak{S}})_v[X]$. $\square$

Theorem 2.2. *If D is a PvMD and $\mathfrak{S}$ is a t-splitting set of ideals of D, then $R = D + XD_{\mathfrak{S}}[X]$ is a PvMD.*

Proof. Let Q be a maximal t-ideal of R. If $Q \cap D = (0)$, then $R_{D\setminus\{0\}} = K[X]$, and thus $R_Q = (R_{D\setminus\{0\}})_{Q_{D\setminus\{0\}}}$ is a rank-one discrete valuation domain. Next, assume that $Q \cap D \neq (0)$, and put $P = Q \cap D$. If I is a nonzero finitely generated subideal of P, then I is t-invertible, and so by Lemma 2.1, $I_v + X(I_v)_{\mathfrak{S}}[X] = (IR)_v \subseteq Q_t = Q$. Hence, $I_v \subseteq Q \cap D = P$. Thus, $P_t = P$, and by assumption, D_P is a valuation domain. Note that $R_{D\setminus P} = D_P + X(D_{\mathfrak{S}})_{D\setminus P}[X]$, and because $\mathfrak{S}$ is a t-splitting set of ideals, by Proposition 1.2(5) and Corollary 1.5, $(D_{\mathfrak{S}})_{D\setminus P} = D_P$ or K. Thus, $R_{D\setminus P} = D_P[X]$ or $D_P + XK[X]$.

Case 1. If $R_{D\setminus P} = D_P + XK[X]$, then $R_{D\setminus P}$ is a Bezout domain by Corollary 1.6, and thus $R_Q = (R_{D\setminus P})_{Q_{D\setminus P}}$ is a valuation domain.

Case 2. Assume $R_{D\setminus P} = D_P[X]$; equivalently, $(D_{\mathfrak{S}})_{D\setminus P} = D_P$. If $X \in Q$, then $(XD_{\mathfrak{S}}[X])^2 = X(XD_{\mathfrak{S}}[X]) \subseteq Q$, and since Q is a prime ideal, $XD_{\mathfrak{S}}[X] \subseteq Q$. Hence, $Q = P + XD_{\mathfrak{S}}[X]$. Since $(D_{\mathfrak{S}})_{D\setminus P} = D_P$, by Corollary 1.4 $J' \nsubseteq P$ for all $J' \in \mathfrak{S}$; hence there is a finitely generated ideal I of D such that $I \in \mathfrak{S}^{\perp}$ and $I \subseteq P$ because $\mathfrak{S}$ is t-splitting. Let $u \in (I, X)^{-1}$. Then $uI \subseteq R$ and $uX \in R$; so $u \in D_{\mathfrak{S}}[X]$. Hence, there is a $J \in \mathfrak{S}$ such that $u(0)J \subseteq D$. If $u(0) = 0$, then $u \in R$. If $u(0) \neq 0$, then $u(0)I \subseteq D$, and so $u(0)(I + J) = u(0)I + u(0)J \subseteq D \Rightarrow u(0) \in u(0)D = u(0)(I + J)_t = (u(0)(I + J))_t \subseteq D_t = D \Rightarrow u \in R$. Thus, $(I, X)^{-1} = R$, and hence $R = (I, X)_v \subseteq Q \subsetneq R$, a contradiction. Thus, $X \notin Q$, and since Q is a maximal t-ideal, $(Q, X)_t = R$. Let $T = \{X^n \mid n \geq 0\}$. We claim that Q_T is a t-ideal. If not, there are some $f_1, \ldots, f_m \in Q$ such that $(f_1, \ldots, f_m, X)_v = R$ and $((f_1, \ldots, f_m)^{-1})_T = ((f_1, \ldots, f_m)_T)^{-1} = R_T$. Hence, if $z \in (f_1, \ldots, f_m)^{-1}$, then $z \in R_T \Rightarrow zX^k \in R$ for some $k \geq 1$, $\Rightarrow z \in (f_1, \ldots, f_m, X^k)^{-1} = R$ (the equality follows because $(f_1, \ldots, f_m, X)_v = R$). Thus, $(f_1, \ldots, f_m)^{-1} = R$, and so $R = (f_1, \ldots, f_m)_v \subseteq Q$, a contradiction. Hence, Q_T is a t-ideal of R_T. Note that $D_{\mathfrak{S}}$ is a PvMD; so $R_T = D_{\mathfrak{S}}[X, X^{-1}]$ is a PvMD. Thus, $R_Q = (R_T)_{Q_T}$ is a valuation domain, and eventually $Q_{D\setminus P} = PD_P[X]$. $\square$

In [6, Proposition 2.6(ii)], the authors gave a necessary condition for $R = A + XB[X]$ to be a PvMD when B is flat over A. We next give in Theorem 2.4 a necessary and sufficient condition for $R = A + XB[X]$ to be a PvMD when B is t-flat over A.

Lemma 2.3. *Let I be a nonzero finitely generated ideal of A and $R = A + XB[X]$.*

(1) *If $P \in \Lambda'$, then $(PR)_t = P + XB[X]$ and $(PR)_t \in t$-Max(R).*
(2) *IR is t-invertible if and only if IB is t-invertible and there exists $F \subseteq II^{-1}$, a nonzero finitely generated ideal of A such that $F^{-1} \cap B = A$.*

Proof. (1) Let $\sum$ be the set of (F,G) such that $F \subseteq P$ (resp., $G \subseteq PB$) is a nonzero finitely generated ideal of A (resp., B) with $F \subseteq G$. Then, by [8, Lemma 2.8],

$$\begin{aligned}(PR)_t &= (\bigcup_{(F,G)\in\sum}(F^{-1}\cap G^{-1})^{-1})\cap(PB)_t + X(PB)_t[X] \\ &= (\bigcup_{(F,G)\in\sum}(F^{-1}\cap G^{-1})^{-1})\cap B + XB[X] \\ &= (\bigcup_{(F,G)\in\sum}(F^{-1}\cap G^{-1})^{-1}) + XB[X],\end{aligned}$$

where the last equality follows because $(F^{-1}\cap G^{-1})^{-1} \subseteq A$ for all $(F,G) \in \sum$. Let $(F,G) \in \sum$. Since $(PB)_t = B$, there is a nonzero finitely generated ideal G' of B such that $G \subseteq G' \subseteq PB$ and $(G')^{-1} = B$. Hence, $(F^{-1}\cap G^{-1})^{-1} \subseteq (F^{-1}\cap (G')^{-1})^{-1} = (F^{-1}\cap B)^{-1}$, and as $(F,G') \in \sum$, we have

$$\bigcup_{(F,G)\in\sum}(F^{-1}\cap G^{-1})^{-1} = \bigcup_{(F,G)\in\sum}(F^{-1}\cap B)^{-1}.$$

Note that if $(FB)^{-1} = B$, then $F^{-1} \subseteq B$ because $x \in F^{-1} \Rightarrow xF \subseteq A \Rightarrow xFB \subseteq B \Rightarrow x \in xB = x(FB)_v = (xFB)_v \subseteq B_v = B$; hence if F' is a nonzero finitely generated ideal of A with $F' \subseteq P$ and $(F'B)_v = B$, then $(F^{-1}\cap B)^{-1} \subseteq ((F+F')^{-1}\cap B)^{-1} = (F+F')_v \subseteq P_t = P$ for any nonzero finitely generated ideal F of A with $F \subseteq P$. So $P = P_t = \bigcup_{\sum} F_v \subseteq \bigcup_{\sum}(F^{-1}\cap B)^{-1} \subseteq P$. Therefore, $(PR)_t = P + XB[X]$.

Next, let Q be a maximal t-ideal of R with $(PR)_t \subseteq Q$. Clearly, $P \subseteq Q \cap A$. If $P \neq Q \cap A$, then $(Q\cap A)_t = A$ and $((Q\cap A)B)_t = B$. Hence, there is a nonzero finitely generated ideal I of A such that $I \subseteq Q \cap A$, $I_v = A$, and $(IB)_v = B$. Note that $(IR)^{-1} = I^{-1} + X(IB)^{-1}[X]$ by Lemma 2.1(1); so $(IR)^{-1} = A + XB[X]$, and thus $R = (IR)_v \subseteq Q_t = Q$, a contradiction. Hence, $Q \cap A = P$. Let $f = a + Xg \in Q$ where $a \in A$ and $g \in B[X]$. Since $P + XB[X] \subseteq Q$, we have $a \in Q \cap A = P$, and hence $f = a + Xg \in P + XB[X]$. Thus, $Q = P + XB[X] = (PR)_t$.

(2) This is an immediate consequence of [8, Lemma 3.8]. $\square$

Theorem 2.4. *Let $R = A + XB[X]$, and assume that B is t-flat over A. Then R is a PvMD if and only if A is a PvMD and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a t-splitting set of ideals of A.*

Proof. ($\Leftarrow$) Theorem 2.2.

($\Rightarrow$) *Claim* 1. A is a PvMD, and hence B is also a PvMD by Theorem 1.9 because B is t-flat (hence t-linked) over A. (For this, let I be a nonzero finitely generated ideal of A. Then IB is t-invertible by Lemma 2.3(2); hence $B = ((IB)(IB)^{-1})_t = ((IB)(I^{-1}B)_t)_t = ((IB)(I^{-1}B))_t = ((II^{-1})B)_t$ (see Corollary 1.8 for the second equality). This implies that $II^{-1} \nsubseteq P$ for all $P \in \Lambda$. Also, $(IR)^{-1} = I^{-1} + X(IB)^{-1}[X]$ by Lemma 2.1(1), and hence $(IR)(IR)^{-1} = II^{-1} + XI(IB)^{-1}[X] + XI^{-1}(IB)[X] + X^2(IB)(IB)^{-1}[X] \subseteq$

$II^{-1} + XB[X] \subseteq R$. Thus, $(II^{-1} + XB[X])_t = R$, and so if $P \in \Lambda'$, then $II^{-1} \nsubseteq P$ because $(PR)_t = P + XB[X] \subsetneq R$ by Lemma 2.3(1). Therefore, $II^{-1} \nsubseteq P$ for all $P \in t\text{-Max}(A)$, and so $(II^{-1})_t = A$.)

Let $\mathfrak{S}$ be the multiplicative set of ideals of A generated by $\{(A : \alpha) \mid 0 \neq \alpha \in B\}$.

Claim 2. $B = A_{\mathfrak{S}}$. (Clearly, $B \subseteq A_{\mathfrak{S}}$ because $\alpha(A : \alpha) \subseteq A$ for each $0 \neq \alpha \in B$. For the reverse containment, we first show that $(IB)_t = B$ for all $I \in \mathfrak{S}$. To do this, it suffices to show that $((A : \alpha)B)_t = B$ for all $0 \neq \alpha \in B$. If M is a maximal t-ideal of B, then $B_M = A_{M\cap A}$ because B is t-flat over A; so $\alpha \in B \subseteq A_{M\cap A}$ implies $(A : \alpha)B_M = (A : \alpha)A_{M\cap A} = (A_{M\cap A} : \alpha A_{M\cap A}) = A_{M\cap A} = B_M$. Thus, $((A : \alpha)B)_t = \bigcap_{M\in t\text{-Max}(B)}(A : \alpha)B_M = \bigcap_{M\in t\text{-Max}(B)} B_M = B$ [21, Theorem 3.5] because B is a PvMD by Claim 1. Hence, if $0 \neq \beta \in A_{\mathfrak{S}}$, then $\beta I \subseteq A$ for some $I \in \mathfrak{S}$, and thus $\beta \in \beta B = \beta(IB)_t = (\beta IB)_t \subseteq B$.)

Next, we show that $\mathfrak{S}$ is t-splitting. By Proposition 1.1, we only have to show that $\mathfrak{S}$ is v-finite and $dA_{\mathfrak{S}} \cap A$ is t-invertible for each $0 \neq d \in A$.

Claim 3. $\mathfrak{S}$ is v-finite. (Note that A is a PvMD by Claim 1; so $(A : \alpha)$ is a t-invertible t-ideal for all $0 \neq \alpha \in B$, and thus $(A : \alpha)$ is of finite type. Hence, $\mathfrak{S}$ is v-finite because $\mathfrak{S}$ is generated by $\{(A : \alpha) \mid 0 \neq \alpha \in B\}$.)

Claim 4. $dA_{\mathfrak{S}} \cap A$ is t-invertible for each $0 \neq d \in A$. (Note that $((d, X)R)^{-1} = (d^{-1}A \cap B) + XB[X]$; so if we let $I = d^{-1}A \cap B$, then I is a fractional ideal of A, $((d, X)R)^{-1} = I + XB[X]$, and $((d, X)R)_v = (I^{-1} \cap B) + XB[X]$ [7, Lemma 2.1]. Note that $A \subseteq I$, and so $I^{-1} \subseteq A$ and $((d, X)R)_v = I^{-1} + XB[X]$. Hence, $R = (((d, X)R)^{-1}((d, X)R)_v)_t \subseteq (II^{-1} + XB[X])_t \subseteq R$, and so $(II^{-1} + XB[X])_t = R$. By Lemma 2.3(1), $II^{-1} \nsubseteq P$ for all $P \in \Lambda'$. Next, if $P \in \Lambda$, then A_P is a valuation domain and $R_{A\setminus P} = A_P + XB_{A\setminus P}[X]$ is a PvMD, and since $(PB)_t \subsetneq B$ implies $B_{A\setminus P} \neq K$, we have $R_{A\setminus P} = A_P + XA_P[X] = A_P[X]$ by Corollary 1.6. Hence, $A_P[X] = R_{A\setminus P} = ((d, X)R_{A\setminus P})_v = ((d, X)R)_v R_{A\setminus P} = (I^{-1} + XB[X])R_{A\setminus P}$ (the third equality follows because $(d, X)R$ is t-invertible), and thus $I^{-1}A_P = A_P$. Also, $IA_P = A_P$ because $A \subseteq I \subseteq B$ and $B_{A\setminus P} = A_P$. Thus, $(II^{-1})A_P = (IA_P)(I^{-1}A_P) = A_P$ which means $II^{-1} \nsubseteq P$. Therefore, $(II^{-1})_t = A$. Thus, $I = d^{-1}A \cap B = d^{-1}A \cap A_{\mathfrak{S}}$, and so $dI = dA_{\mathfrak{S}} \cap A$ is t-invertible.) $\square$

It is known that if $\mathfrak{S}$ is a multiplicative set of ideals of D, then $D_{\mathfrak{S}}$ is t-linked over D [16, Proposition 2.2], and since a t-linked overring of a PvMD is t-flat, by Theorems 2.2 and 2.4, we have

Corollary 2.5. *Let $\mathfrak{S}$ be a multiplicative set of ideals of a PvMD D, and let $R = D + XD_{\mathfrak{S}}[X]$. Then R is a PvMD if and only if $\mathfrak{S}$ is a t-splitting set of ideals.*

Let $A \subseteq B$ be an extension of integral domains. It is known that $I_v \subseteq (IB)_v$ for any nonzero finitely generated ideal I of A if and only if $I_t \subseteq (IB)_t$ for any nonzero ideal I of A [11, Proposition 1.1].

Lemma 2.6. *Let $\mathfrak{S}$ be a multiplicative set of ideals of D, $R = D + XD_{\mathfrak{S}}[X]$, and $K = qf(D)$.*

(1) *$I_v \subseteq (IR)_v$ for any nonzero finitely generated ideal I of D.*
(2) *$(IR)_v \cap K = I_v$ for any nonzero finitely generated fractional ideal I of D.*
(3) *$(IR)_t \cap K = I_t$ for any nonzero fractional ideal I of D.*

Proof. (1) By Lemma 2.1(1), $(IR)^{-1} = I^{-1} + X(ID_{\mathfrak{S}})^{-1}[X]$, and hence $I_v(IR)^{-1} = I_vI^{-1} + XI_v(ID_{\mathfrak{S}})^{-1}[X]$. Note that $(I_vD_{\mathfrak{S}})_t = (ID_{\mathfrak{S}})_t$ [21, Lemma 3.4(3)] since I is finitely generated; so

$$\begin{aligned} I_v(ID_{\mathfrak{S}})^{-1} &= (I_vD_{\mathfrak{S}})(ID_{\mathfrak{S}})^{-1} \subseteq ((I_vD_{\mathfrak{S}})(ID_{\mathfrak{S}})^{-1})_t \\ &= ((I_vD_{\mathfrak{S}})_t(ID_{\mathfrak{S}})^{-1})_t = ((ID_{\mathfrak{S}})_t(ID_{\mathfrak{S}})^{-1})_t \\ &= ((ID_{\mathfrak{S}})(ID_{\mathfrak{S}})^{-1})_t \subseteq D_{\mathfrak{S}}. \end{aligned}$$

Hence, $I_v(IR)^{-1} \subseteq D + XD_{\mathfrak{S}}[X] = R$, and thus $I_v \subseteq (IR)_v$.

(2) and (3) These follow directly from (1) and [11, Lemma 1.3] because $R \cap K = D$. $\square$

An integral domain D is said to be of *finite t-character* if each nonzero nonunit of D is contained in at most a finite number of maximal t-ideals. As in [20], we say that D is a *ring of Krull type* if D is a locally finite intersection of essential valuation overrings of D; equivalently, D is a PvMD of finite t-character. A ring of Krull type is called an *independent ring of Krull type* if no two distinct maximal t-ideals contain a nonzero prime ideal.

In [3, Section 2], the authors studied when $D^{(S)} = D + XD_S[X]$ is a ring (resp., an independent ring) of Krull type. We next give in Corollary 2.8 a ring of Krull type property of $R = D + XD_{\mathfrak{S}}[X]$. For this, we first study the set of maximal t-ideals of R.

Lemma 2.7. *Let $\mathfrak{S}$ be a multiplicative set of ideals of D and $R = D + XD_{\mathfrak{S}}[X]$. Assume that D and R are both PvMDs. Then $t\text{-}Max(R) = \{Q \in t\text{-}Max(R) \mid Q \cap D = (0)\} \cup \{P + XD_{\mathfrak{S}}[X] \mid P \in \Lambda'\} \cup \{P + X(PD_{\mathfrak{S}})_t[X] \mid P \in \Lambda\}$.*

Proof. ($\supseteq$) Let P be a maximal t-ideal of D. If $(PD_{\mathfrak{S}})_t = D_{\mathfrak{S}}$, then $(PR)_t = P + XD_{\mathfrak{S}}[X]$ is a maximal t-ideal of R by Lemma 2.3. Next, assume $(PD_{\mathfrak{S}})_t \subsetneq D_{\mathfrak{S}}$. Note that each nonzero finitely generated subideal of PR is contained in IR for some finitely generated ideal $I \subseteq P$ of D. So if we let $f(P)$ be the set of nonzero finitely generated subideals of P, then

$$\begin{aligned} (PR)_t &= \bigcup\{(IR)_v \mid I \in f(P)\} \\ &= \bigcup\{I_v + X(ID_{\mathfrak{S}})_v[X] \mid I \in f(P)\} \\ &= P + X(PD_{\mathfrak{S}})_t[X] \subsetneq R, \end{aligned}$$

where the second equality follows from Lemma 2.1(2). Hence, there is a maximal t-ideal Q of R with $(PR)_t \subseteq Q$. By Lemma 2.6, $Q \cap D = P$ because $P \subseteq Q$ and $P \in t\text{-Max}(D)$. Note that $(PD_{\mathfrak{S}})_t \subsetneq D_{\mathfrak{S}}$ implies $I \nsubseteq P$ for all $I \in \mathfrak{S}$ by Proposition 1.2; so $D_P = (D_{\mathfrak{S}})_{P_{\mathfrak{S}}}$ [10, Theorem 1.1(4)], and since D_P is a valuation domain, $(D_{\mathfrak{S}})_{D \setminus P} = D_P$ and $R_{D \setminus P} = D_P + X(D_{\mathfrak{S}})_{D \setminus P}[X] = D_P[X]$. Since R is a PvMD, $Q_{D \setminus P}$ is a maximal t-ideal of $R_{D \setminus P}$. Clearly, $Q_{D \setminus P} \cap D_P = PD_P$, and hence $Q_{D \setminus P} = PD_P[X]$ [21, Lemma 4.1] and $X \notin Q$. Let $T = \{X^k \mid k \geq 0\}$. Then Q_T is a maximal t-ideal of $R_T = D_{\mathfrak{S}}[X, X^{-1}]$, because R is a PvMD and $X \notin Q$. Note that $\mathfrak{S}$ is t-splitting by Corollary 2.5 and P is a maximal t-ideal of D; hence $(PD_{\mathfrak{S}})_t$ is a maximal t-ideal of $D_{\mathfrak{S}}$ by Corollary 1.3, and so $(PD_{\mathfrak{S}})_t[X, X^{-1}]$ is a maximal t-ideal of R_T (cf. [21, Proposition 2.2, Lemmas 3.17 and 4.1]). Also, since $X(PD_{\mathfrak{S}})_t[X] \subseteq Q$, we have $Q_T = (PD_{\mathfrak{S}})_t[X, X^{-1}]$. Thus, $Q = Q_T \cap R = (PD_{\mathfrak{S}})_t[X, X^{-1}] \cap R = P + X(PD_{\mathfrak{S}})_t[X] = (PR)_t$.

($\subseteq$) Let Q be a maximal t-ideal of R with $Q \cap D \neq (0)$. Put $Q \cap D = P$. Since $(PR)_t \subseteq Q$, we have $P_t \subsetneq D$ by Lemma 2.6, and D being a PvMD implies $P_t = P$. Also, since Q is homogeneous [5, Theorem 1.2], $Q \subseteq P + XD_{\mathfrak{S}}[X]$. Let P_0 be a maximal t-ideal of D with $P \subseteq P_0$. If $(P_0 D_{\mathfrak{S}})_t = D_{\mathfrak{S}}$, then $(P_0 R)_t = P_0 + XD_{\mathfrak{S}}[X]$ by Lemma 2.3(1), and since Q is a maximal t-ideal, $Q = P_0 + XD_{\mathfrak{S}}[X]$ and $P = P_0$. Next, assume that $(P_0 D_{\mathfrak{S}})_t \subsetneq D_{\mathfrak{S}}$. Then $R_{D \setminus P_0} = D_{P_0}[X]$ (see the proof of ($\supseteq$) above) and $Q_{D \setminus P_0}$ is a maximal t-ideal of $R_{D \setminus P_0}$. Note that $P_0 D_{P_0}[X]$ is a unique maximal t-ideal of $R_{D \setminus P_0}$ that does not contract to zero; hence $Q_{D \setminus P_0} = P_0 D_{P_0}[X]$. This means that $P_0 = P$ and $Q = (PR)_t = P + X(PD_{\mathfrak{S}})_t[X]$. $\square$

Corollary 2.8. *Let $\mathfrak{S}$ be a multiplicative set of ideals of D and $R = D + XD_{\mathfrak{S}}[X]$. If D is a PvMD, then*

(1) *R is a ring of Krull type if and only if D is a ring of Krull type, $\mathfrak{S}$ is a t-splitting set of ideals, and $|\Lambda'| < \infty$.*
(2) *R is an independent ring of Krull type if and only if D is an independent ring of Krull type, $\mathfrak{S}$ is a t-splitting set of ideals, and $|\Lambda'| \leq 1$.*

Proof. (1) Let $K = qf(D)$. Then $R_{D \setminus \{0\}} = K[X]$ is a principal ideal domain, and hence the finite t-characterness of R is completely determined by $\{Q \in t\text{-Max}(R) \mid Q \cap D \neq (0)\}$. Note that $XD_{\mathfrak{S}}[X]$ is a prime ideal of R. Thus, by Lemma 2.7, R is of finite t-character if and only if D is of finite t-character and $|\Lambda'| < \infty$. Hence, the result follows directly from Corollary 2.5.

(2) This can be proved by an argument similar to the proof of (1) above. $\square$

Let $X^1(D)$ be the set of height-one prime ideals of an integral domain D. It is well known that if D is a Krull domain, then $X^1(D) = t\text{-Max}(D)$, and hence $D = \bigcap_{P \in X^1(D)} D_P$. Also, D is a Krull domain if and only if, for each $0 \neq d \in D$, $dD = (P_1^{e_1} \cdots P_k^{e_k})_t$, where each P_i is a height-one prime ideal of D and $e_i \geq 1$ is an integer [22, Theorem 3.9].

Lemma 2.9. *If $\mathfrak{S}$ is a multiplicative set of ideals of a Krull domain D, then $\mathfrak{S}$ is a t-splitting set of ideals.*

Proof. Let $\mathbb{X}$ be the set of height-one prime ideals of D that are contained in $sp(\mathfrak{S})$. So if $0 \neq d \in D$, then $dD = ((P_1^{e_1} \cdots P_k^{e_k})_t(Q_1^{k_1} \cdots Q_n^{k_n})_t)_t$ for some $P_i \in \mathbb{X}$, $Q_j \in X^1(D) \setminus \mathbb{X}$, and positive integers e_i and k_j, because D is a Krull domain. Clearly, $(P_1^{e_1} \cdots P_k^{e_k})_t \in sp(\mathfrak{S})$ and $(Q_1^{k_1} \cdots Q_n^{k_n})_t \in \mathfrak{S}^\perp$. Thus, $\mathfrak{S}$ is a t-splitting set of ideals. $\square$

Let $\mathfrak{S}$ be a multiplicative set of ideals of a Krull domain D. It is clear that if we let $\mathfrak{S}'$ be the multiplicative set of ideals generated by $X^1(D) \cap sp(\mathfrak{S})$, then $sp(\mathfrak{S}) = sp(\mathfrak{S}')$, and hence $D_{\mathfrak{S}} = D_{\mathfrak{S}'}$, $\Lambda' = X^1(D) \cap sp(\mathfrak{S})$, and $\Lambda = X^1(D) \setminus sp(\mathfrak{S})$.

Corollary 2.10. *Let D be a Krull domain.*

(1) *$R = D + XD_{\mathfrak{S}}[X]$ is a PvMD.*
(2) *R is a ring of Krull type if and only if $|\Lambda'| < \infty$.*
(3) *R is an independent ring of Krull type if and only if $|\Lambda'| \leq 1$.*

Proof. Since D is a Krull domain, D is an independent ring of Krull type. Thus, the result follows directly from Theorem 2.2, Lemma 2.9, and Corollary 2.8. $\square$

3. Generalized GCD domains

Let D be an integral domain, $K = qf(D)$, and X be an indeterminate over D. In [2, Theorem 3.3], it was shown that if S is a multiplicative set of D, then $D^{(S)} = D + XD_S[X]$ is a GGCD domain if and only if D is a GGCD domain and S is a d-splitting set. The purpose of this section is to generalize the result of [2, Theorem 3.3] to the ring $R = A + XB[X]$ where $A \subseteq B$ is an extension of integral domains. For this, let $\mathfrak{S}$ be a multiplicative set of ideals of D. We will say that $\mathfrak{S}$ is a *d-splitting set of ideals* if, for each $0 \neq d \in D$, we have $dD = IJ$ for some $I \in sp(\mathfrak{S})$ and $J \in \mathfrak{S}^\perp$. Clearly, d-splitting sets of ideals are t-splitting. Also, if we set $\mathfrak{S} = \{sD \mid s \in S\}$, then S is a d-splitting set if and only if $\mathfrak{S}$ is a d-splitting set of ideals.

We begin this section with a nice characterization of d-splitting sets of ideals (cf. Proposition 1.1 for t-splitting sets of ideals).

Proposition 3.1. *Let $\mathfrak{S}$ be a multiplicative set of ideals of D. Then $\mathfrak{S}$ is d-splitting if and only if $\mathfrak{S}$ is v-finite and $dD_{\mathfrak{S}} \cap D$ is invertible for all $0 \neq d \in D$.*

Proof. ($\Rightarrow$) Let $0 \neq d \in D$. Then $dD = IJ$ for some $I \in sp(\mathfrak{S})$ and $J \in \mathfrak{S}^\perp$. We note that $dD_{\mathfrak{S}} \cap D = J$. (For if $x \in dD_{\mathfrak{S}} \cap D$, then $d^{-1}xI' \subseteq D$ for some $I' \in \mathfrak{S}$. So $xI' \subseteq dD \subseteq J$, and since $(I' + J)_t = D$, we have $x \in xD = x(I' + J)_t = (xI' + xJ)_t \subseteq J_t = J$. For the reverse containment, note that $dD_{\mathfrak{S}} = (IJ)D_{\mathfrak{S}} = (ID_{\mathfrak{S}})(JD_{\mathfrak{S}}) = JD_{\mathfrak{S}}$ because

$I \in sp(\mathfrak{S})$ is invertible. Thus, $J \subseteq dD_{\mathfrak{S}} \cap D$.) Thus, $dD_{\mathfrak{S}} \cap D$ is invertible. Next, for $I_1 \in \mathfrak{S}$, choose $0 \neq d \in I_1$, and let the notation be as in the previous paragraph. Then I is invertible, and hence $I = I_t = (I(J + I_1)_t)_t = (I(J + I_1))_t \subseteq (I_1)_t$. Thus, $\mathfrak{S}$ is v-finite.

($\Leftarrow$) Let $0 \neq d \in D$. Then $J := dD_{\mathfrak{S}} \cap D$ is invertible and $dD \subseteq J$; hence $dD = IJ$, where $I = dJ^{-1}$, so I is invertible.

Claim 1. $I \in sp(\mathfrak{S})$. (Note that $dD_{\mathfrak{S}} = (IJ)D_{\mathfrak{S}} = (ID_{\mathfrak{S}})(JD_{\mathfrak{S}}) = (ID_{\mathfrak{S}})(dD_{\mathfrak{S}})$. Hence $ID_{\mathfrak{S}} = D_{\mathfrak{S}}$, and thus $I \in sp(\mathfrak{S})$.)

Claim 2. $J \in \mathfrak{S}^{\perp}$, i.e., $(I' + J)_t = D$ for all $I' \in \mathfrak{S}$. (Since $\mathfrak{S}$ is v-finite and $(I' + J)_t = ((I')_t + J)_t$, we may assume that I' is a v-ideal of finite type. If $x \in J^{-1} \cap (I')^{-1}$, then $x \in D_{\mathfrak{S}}$. Hence, $xJ \subseteq JD_{\mathfrak{S}} \cap D = dD_{\mathfrak{S}} \cap D = J$, and since J is invertible, $x \in D$. Thus, $(I' + J)^{-1} = J^{-1} \cap (I')^{-1} = D$, and since I' is of finite type, we have $(I' + J)_t = (I' + J)_v = D$.) $\square$

It is known that $R = A + XB[X]$ is flat over A if and only if B is flat over A [7, Lemma 3.6]. While we don't know if the t-flatness analogue holds, we next give the t-linkedness analogue.

Lemma 3.2. *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. Then the following statements are equivalent.*

(1) *B is t-linked over A.*
(2) $B = \bigcap_{P \in t\text{-}Max(A)} B_{A \setminus P}$.
(3) *R is t-linked over A.*
(4) $R = \bigcap_{P \in t\text{-}Max(A)} R_{A \setminus P}$.

Proof. (1) $\Rightarrow$ (2) For $0 \neq x \in \bigcap_{P \in t\text{-Max}(A)} B_{A \setminus P}$, let $I = (B : x) \cap A$. Then $I \nsubseteq P$ for all $P \in t$-Max(A), and hence $I_t = A$. Since B is t-linked over A by (1), $B = (IB)_t \subseteq ((B : x)B)_t = (B : x) \subseteq B$ (see [4, Proposition 2.1] for the first equality), and so $(B : x) = B$. Thus, $x \in B$. The reverse containment is clear.

(2) $\Rightarrow$ (1) Let $P \in t$-Max(A). If I is a nonzero finitely generated ideal of A such that $I^{-1} = A$, then $I \nsubseteq P$, and hence $IB_{A \setminus P} = B_{A \setminus P}$; so $(IB_{A \setminus P})^{-1} = B_{A \setminus P}$. Thus, $B_{A \setminus P}$ is t-linked over A. Since $B = \bigcap_{P \in t\text{-Max}(A)} B_{A \setminus P}$ by (2), B is t-linked over A [4, Proposition 2.3(2)].

(1) $\Leftrightarrow$ (3) Let I be a nonzero finitely generated ideal of A such that $I^{-1} = A$. Then $(IR)^{-1} = A + X(IB)^{-1}[X]$ by Lemma 2.1, and thus $(IR)^{-1} = R$ if and only if $(IB)^{-1} = B$.

(3) $\Leftrightarrow$ (4) This follows directly from the equivalence of (1) and (2) above. $\square$

Lemma 3.3. *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. If R is a GGCD domain, then*

(1) *A is a GGCD domain,*

(2) *B is t-linked over A, and*
(3) *$B = A_{\mathfrak{S}}$ for some multiplicative set $\mathfrak{S}$ of ideals of A.*

Proof. If R is a GGCD domain, then R is a PvMD, and hence B is an overring of A [6, Proposition 2.1(1)].

(1) and (2) Let I be a nonzero finitely generated ideal of A. Then $(IR)_v$ is invertible and $(IR)_v = (I^{-1} + X(IB)^{-1}[X])^{-1} = I_v \cap (IB)_v + X(IB)_v[X]$ by Lemma 2.1 and [7, Lemma 2.1]. Hence, $R = (IR)_v(IR)^{-1} = (I_v \cap (IB)_v + X(IB)_v[X])(I^{-1} + X(IB)^{-1}[X])$, and so $A = (I_v \cap (IB)_v)I^{-1} \subseteq I_v I^{-1} \subseteq A$. Thus, $I_v I^{-1} = A$. Therefore, A is a GGCD domain. Moreover, if $I^{-1} = A$, then $A = (I_v \cap (IB)_v)I^{-1} = A \cap (IB)_v \subseteq (IB)_v \subseteq B$, and so $(IB)_v = B$. Thus, B is t-linked over A.

(3) Let P be a maximal t-ideal of A. Then $R_{A\setminus P} = A_P + XB_{A\setminus P}[X]$ is a PvMD and A_P is a valuation domain. Hence, $B_{A\setminus P} = A_P$ or $qf(A)$ by Corollary 1.6. Let $T = \{P \in t\text{-Max}(A) \mid B_{A\setminus P} = A_P\}$, $A_1 = \bigcap_{P\in T} A_P$, and $\mathfrak{S} = \{I \mid I \nsubseteq P \text{ for all } P \in T\}$. Then, by Lemma 3.2, $B = A_1$ since B is t-linked over A by (2). Also, note that $A_1 = A_{\mathfrak{S}}$. (For $0 \neq \alpha \in A_1$, let $I = (A : \alpha)$. Then $\alpha I \subseteq A$ and $I \nsubseteq P$ for all $P \in T$, and hence $\alpha \in A_{\mathfrak{S}}$. Thus, $A_1 \subseteq A_{\mathfrak{S}}$. For the reverse containment, let $0 \neq \beta \in A_{\mathfrak{S}}$. Then $\beta J \subseteq A$ for some $J \in \mathfrak{S}$, and hence $\beta \in \bigcap_{P\in T} \beta A_P = \bigcap_{P\in T} \beta J A_P \subseteq \bigcap_{P\in T} A_P = A_1$. Thus, $A_{\mathfrak{S}} \subseteq A_1$.) Thus, $B = A_{\mathfrak{S}}$. □

Let S be a t-splitting saturated multiplicative set of D. It is known that if $Cl(D) = 0$, then S is a splitting set. We next give a multiplicative set of ideals analogue.

Lemma 3.4. *Let $\mathfrak{S}$ be a t-splitting set of ideals of D and $S = \{a \in D \mid aD = I_v$ for some $I \in sp(\mathfrak{S})\}$. If $Cl(D) = 0$, then S is a splitting set of D and $D_S = D_{\mathfrak{S}}$.*

Proof. Let $0 \neq d \in D$. Then $dD = (IJ)_t$ for some $I \in sp(\mathfrak{S})$ and $J \in \mathfrak{S}^{\perp}$. Clearly, I and J are t-invertible, and hence $I_t = aD$ and $J_t = bD$ for some $a, b \in D$ because $Cl(D) = 0$. Hence, $dD = (I_t J_t)_t = abD$, and so $d = uab = (ua)b$ for some unit u of D. Clearly, $ua \in S$ and $b \in N(S)$. Thus, S is a splitting set of D.

Next, obviously, $D_S \subseteq D_{\mathfrak{S}}$. For the reverse containment, let $0 \neq \alpha \in D_{\mathfrak{S}}$. Then $\alpha I' \subseteq D$ for some $I' \in \mathfrak{S}$. Since $\mathfrak{S}$ is t-splitting, there is a t-invertible ideal J' of D such that $(J')_v \in sp(\mathfrak{S})$ and $(J')_v \subseteq (I')_t$ [12, Proposition 2], and since $Cl(D) = 0$, we have $(J')_v = sD$ for some $s \in D$. Clearly, $s \in S$ and $\alpha s \in \alpha sD = \alpha(J')_v \subseteq \alpha(I')_t \subseteq D$. Thus, $\alpha \in D_S$. □

We next give the main result of this section.

Theorem 3.5. *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. Then the following statements are equivalent.*

(1) *R is a GGCD domain.*

(2) *A is a GGCD domain and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A.*
(3) *A is a GGCD domain and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a t-splitting set of ideals of A.*

Proof. (1) ⇒ (2) Note that a GGCD domain is a PvMD; so B is an overring of A [6, Proposition 2.6(1)]. Hence, by Lemma 3.3, A is a GGCD domain and $B = A_{\mathfrak{S}}$ for some multiplicative set $\mathfrak{S}$ of ideals of A. Next, to show that $\mathfrak{S}$ is d-splitting, it suffices to show that $dA_{\mathfrak{S}} \cap A$ is invertible for each $0 \neq d \in A$ and $\mathfrak{S}$ is v-finite by Proposition 3.1.

Let $0 \neq d \in A$. Then $((d, X)R)^{-1} = d^{-1}A \cap A_{\mathfrak{S}} + XA_{\mathfrak{S}}[X]$, and hence $((d, X)R)_v = (d^{-1}A \cap A_{\mathfrak{S}})^{-1} \cap A_{\mathfrak{S}} + XA_{\mathfrak{S}}[X] = (d^{-1}A \cap A_{\mathfrak{S}})^{-1} + XA_{\mathfrak{S}}[X]$ [7, Lemma 2.1]. Put $I = d^{-1}A \cap A_{\mathfrak{S}}$. Since R is a GGCD domain,

$$\begin{aligned} R &= ((d, X)R)_v((d, X)R)^{-1} \\ &= (I^{-1} + XA_{\mathfrak{S}}[X])(I + XA_{\mathfrak{S}}[X]) \\ &= II^{-1} + XI^{-1}A_{\mathfrak{S}}[X] + XIA_{\mathfrak{S}}[X] + X^2A_{\mathfrak{S}}[X]. \end{aligned}$$

Hence, $II^{-1} = A$, and since $dA_{\mathfrak{S}} \cap A = d^{-1}I$, $dA_{\mathfrak{S}} \cap A$ is invertible.

Next, note that $A_{\mathfrak{S}}$ is flat (hence t-flat) over A [1, Theorem 5] because A is a GGCD domain. So if we let $\mathfrak{F}$ be the multiplicative set of ideals generated by $\{(A : \alpha) \mid 0 \neq \alpha \in A_{\mathfrak{S}}\}$, then $A_{\mathfrak{F}} = A_{\mathfrak{S}}$ by the proof of Theorem 1.7. Hence, $sp(\mathfrak{F}) = sp(\mathfrak{S})$. Since A is a GGCD domain, $(A : \alpha)$ is invertible for all $0 \neq \alpha \in A_{\mathfrak{S}}$, and thus for each $I \in \mathfrak{S}$, I_t contains an invertible ideal in $sp(\mathfrak{S})$.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Since A is a GGCD domain, A is a PvMD, and hence R is a PvMD by Theorem 2.2. Recall that R is a GGCD domain if and only if R is both a PvMD and a locally GCD domain, i.e., R_M is a GCD domain for all maximal ideals M of R [28, Corollary 3.4]; hence it suffices to show that R is a locally GCD domain.

Let M be a maximal ideal of R, and set $M \cap A = P$. If $P = (0)$, then $R_M = K[X]_{MK[X]}$ is a valuation domain, and hence a GCD domain. Next, assume that $P \neq (0)$. Then $R_{A\setminus P} = A_P + X(A_{\mathfrak{S}})_{A\setminus P}[X]$. Note that if we let $\mathfrak{F} = \{IA_P \mid I \in \mathfrak{S}\}$, then $(A_{\mathfrak{S}})_{A\setminus P} = (A_P)_{\mathfrak{F}}$ and $\mathfrak{F}$ is a t-splitting set of ideals of A_P by Proposition 1.2(4) and (5). Let $T = \{\alpha \in A_P \mid \alpha A_P = (IA_P)_t$ for some $IA_P \in sp(\mathfrak{F})\}$. Then $(A_P)_{\mathfrak{F}} = (A_P)_T$ and T is a splitting set of A_P by Lemma 3.4 because A_P is a GCD domain and $\mathfrak{F}$ is t-splitting. Thus, $R_{A\setminus P}$ is a GCD domain [29, Corollary 1.5]. Hence, $R_M = (R_{A\setminus P})_{M_{A\setminus P}}$ is a GCD domain. □

Clearly, a multiplicative set S of D is d-splitting if and only if $\{sD \mid s \in S\}$ is a d-splitting set of ideals. Thus, by Theorem 3.5, we have

Corollary 3.6. *(See [2, Theorem 3.3].) Let S be a multiplicative set of D. Then $D^{(S)} = D + XD_S[X]$ is a GGCD domain if and only if D is a GGCD domain and S is a d-splitting set.*

Corollary 3.7. *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. If A is a Prüfer domain, then the following statements are equivalent.*

(1) *R is a PvMD.*
(2) *R is a GGCD domain.*
(3) *$B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A.*

Proof. Clearly, Prüfer domain $\Rightarrow$ GGCD domain $\Rightarrow$ PvMD. Thus, the result follows directly from Theorems 2.4 and 3.5 because each overring of a Prüfer domain is flat (hence t-flat). $\square$

A π-*domain* is a Krull domain in which each height-one prime ideal is invertible. Hence, D is a π-domain if and only if, for each $0 \neq d \in D$, $dD = P_1^{e_1} \cdots P_k^{e_k}$ for some height-one prime ideals P_i of D and integers $e_i \geq 1$. It is well known that a Krull domain D is a π-domain if and only if D is a GGCD domain; a Dedekind domain is a π-domain; and the polynomial ring over a π-domain is a π-domain. The next result is a d-splitting set of ideals analogue of Lemma 2.9.

Lemma 3.8. *If $\mathfrak{S}$ is a multiplicative set of ideals of a π-domain D, then $\mathfrak{S}$ is a d-splitting set of ideals.*

Proof. Let $\mathbb{X} = X^1(D) \cap sp(\mathfrak{S})$. So if $0 \neq d \in D$, then $dD = (P_1^{e_1} \cdots P_k^{e_k})(Q_1^{k_1} \cdots Q_n^{k_n})$ for some $P_i \in \mathbb{X}$, $Q_j \in X^1(D) \setminus \mathbb{X}$, and positive integers e_i and k_j, because D is a π-domain. Clearly, $P_1^{e_1} \cdots P_k^{e_k} \in sp(\mathfrak{S})$ and $Q_1^{k_1} \cdots Q_n^{k_n} \in \mathfrak{S}^{\perp}$. Thus, $\mathfrak{S}$ is a d-splitting set of ideals. $\square$

Corollary 3.9. *Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. If A is a π-domain, then R is a GGCD domain if and only if $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a multiplicative set of ideals of D.*

Proof. This is an immediate consequence of Theorem 3.5 and Lemma 3.8. $\square$

Let S be a splitting set of D, and let $\mathfrak{S} = \{sD \mid s \in S\}$. Note that if $0 \neq d \in D$, then $d = st$ for some $s \in S$ and $t \in N(S)$; hence $dD = stD = (sD)(tD)$. Clearly, $sD \in \mathfrak{S}$ and $tD \in \mathfrak{S}^{\perp}$. Thus, $\mathfrak{S}$ is a d-splitting set of ideals of D. Conversely, if $\mathfrak{S}$ is a d-splitting set of ideals of D with $Cl(D) = 0$, then $S := \{a \in D \mid aD = I_v \text{ for some } I \in sp(\mathfrak{S})\}$ is a splitting set of D by Lemma 3.4.

Corollary 3.10. *(See [6, Theorem 2.10].) Let $A \subseteq B$ be an extension of integral domains and $R = A + XB[X]$. Then R is a GCD domain if and only if A is a GCD domain and $B = A_S$ for S a splitting set of A.*

Proof. (⇐) [29, Corollary 1.5]. (⇒) Clearly, A is a GCD domain. Also, since a GCD domain is a GGCD domain, by Lemma 3.3, $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A. Hence, if we let $S = \{a \in D \mid aD = I_v \text{ for some } I \in sp(\mathfrak{S})\}$, then, by Lemma 3.4, S is a splitting set of D with $D_S = D_{\mathfrak{S}}$ because $Cl(A) = 0$. □

Remark 3.11. After this article was submitted for publication, the author was told that Kim studied when the ring $A + XB[X]$ is a GGCD domain from a different perspective. Let $\mathfrak{S}$ be a multiplicative set of ideals of D. In [23], Kim called $\mathfrak{S}$ a *d-splitting set of ideals* if for each $0 \neq d \in D$, there are integral ideals I, I' of D such that $dD = II'$, $I \cap J = IJ$ for all $J \in \mathfrak{S}$ and $I' \supseteq J'$ for some $J' \in \mathfrak{S}$. He also noted that if $D_{\mathfrak{S}}$ is an invertible generalized transform of D, then $\mathfrak{S}$ is d-splitting if and only if $dD_{\mathfrak{S}} \cap D$ is invertible for all $0 \neq d \in D$ [23, Lemma 3.12], and he proved that if $A \subseteq B$ is an extension of integral domains, then $R = A + XB[X]$ is a GGCD domain if and only if A is a GGCD domain and $B = A_{\mathfrak{S}}$ for $\mathfrak{S}$ a d-splitting set of ideals of A [23, Theorem 3.13].

Let $D_{\mathfrak{S}}$ be an invertible generalized transform of D. Clearly, $\mathfrak{S}$ is v-finite, and hence by Proposition 3.1 and [23, Lemma 3.12], the notion of d-splitting sets of this paper is the same as that of Kim's d-splitting sets. (However, we don't know if the two notions of d-splitting sets are the same in general.) Note that an overring of a GGCD-domain is a generalized transform if and only if it is an invertible generalized transform [1, Theorem 5]. Hence, the equivalence of (1) and (2) in Theorem 3.5 is the same as Kim's result [23, Theorem 3.13].

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