Journal of Algebra 453 (2016) 377–399

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

W-graph ideals and duality ☆

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ARTICLE INFO

Article history:
Received 17 July 2015
Available online xxxx
Communicated by Gunter Malle

MSC:
20C08
20F55

Keywords:
Coxeter group
Hecke algebra
W-graph
Kazhdan–Lusztig basis
Kazhdan–Lusztig polynomial

ABSTRACT

This paper is the continuation of the work in [14]. In that paper we generalized the definition of W-graph ideal in the weighted Coxeter groups, and showed how to construct a W-graph from a given W-graph ideal in the case of unequal parameters.

In this paper we study the duality and the full W-graph for a given W-graph ideal. We show that there are two modules associated with a given W-graph ideal, they are connected by a duality map. The full W-graph includes all the W-graph data determined by the dual and contragredient representations. Our construction closely parallels that of Kazhdan and Lusztig [6,10,11], which can be regarded as the special case $J = \emptyset$. It also generalizes the work of Couillens [2], Deodhar [3,4], and Douglass [5], corresponding to the parabolic case.

Introduction

Let (W, S) be a Coxeter system and $\mathscr{H}(W)$ its Hecke algebra over $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in the indeterminate q. This is now called *the one parameter case*.

☆ The work is supported by China NSFC Grant 11271239.
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http://dx.doi.org/10.1016/j.jalgebra.2015.12.021
0021-8693/

In [9] Howlett and Nguyen introduced the concept of *a W-graph ideal* in $(W, \leqslant_L)$ with respect to a subset J of S, where $\leqslant_L$ is the left weak Bruhat order on W. They showed that a W-graph can be constructed from a given W-graph ideal, and a Kazhdan–Lusztig like algorithm was obtained.

In [14] we generalized the definition of W-graph ideal in the Coxeter groups with a weight function L, we showed that a W-graph can also be constructed from a given W-graph ideal.

In this paper we continue the work in [14], it grows out of our attempt to understand the "full W-graph" for a given W-graph ideal. We show that, if J is an arbitrary subset of S then there exist a pair of dual modules $M(\mathbf{E}_J, L)$ and $\widetilde{M}(\mathbf{E}_J, L)$ (denoted by M and $\widetilde{M}$) associated with a given W-graph ideal $\mathbf{E}_J$, they are connected by a duality map, this in turn leads to the construction of the dual W-graph bases. Generally D_J, the set of distinguished left coset representatives of W_J in W, is a W-graph ideal with respect to J and also with respect to $\emptyset$, and Couillens, Deodhar and Douglass's parabolic analogues of the Kazhdan–Lusztig construction are recovered (see Couillens [2], Deodhar [3,4] and Douglass [5]). In particular, W itself is a W-graph ideal with respect to $\emptyset$, and the "full" W-graph obtained is the Kazhdan–Lusztig W-graph for the regular representation of $\mathscr{H}(W)$ (as defined in [6]).

Inspired by Lusztig's work [11, Ch. 10], we can construct the W-graph bases for each of the $\mathscr{H}$-modules $Hom_A(M, A)$ and $Hom_A(\widetilde{M}, A)$ (denoted by $\hat{M}$ and $\hat{\widetilde{M}}$ respectively), where A is the ring of "generalized Laurent polynomials in q".

The paper is organized as follows. In Section 1 we present some basic concepts and facts concerning the weighted Coxeter groups, Hecke algebras and W-graphs. In Section 2, we recall the concept of W-graph ideal. In Section 3, we show a duality theorem for the W-graph ideals.

In Section 4 we study in general the W-graphs for the modules $\hat{M}$ and $\hat{\widetilde{M}}$.

In Section 5, we prove, in the case W is finite, an inversion formula that relates the two versions of the relative Kazhdan–Lusztig polynomials. In the last section we give some examples and remarks.

1. Preliminaries

Let W be a Coxeter group, with generating set S. In this section, we briefly recall some basic concepts concerning the general multi-parameter framework of Lusztig [10, 11], which introduces a weight function into Coxeter groups and their associated Hecke algebras on which all the subsequent constructions depend.

We denote by $\ell : W \to \mathbb{N} = \{0, 1, 2, \cdots\}$ the length function on W with respect to S. Let $\leqslant$ denote the Bruhat order on W.

In this section we follow the conventions in [1,8]. Let Γ be the totally ordered abelian group which will be denoted additively, the order on Γ will be denoted by $\leqslant$. Let $\{L(s) \mid s \in S\} \subseteq \Gamma$ be a collection of elements such that $L(s) = L(t)$ whenever $s, t \in S$ are conjugate in W. This gives rise to a weight function

$$L : W \longrightarrow \Gamma$$

in the sense of Lusztig [10,11]; we have $L(w) = L(s_1) + L(s_2) + \cdots + L(s_k)$ where $w = s_1 s_2 \cdots s_k (s_i \in S)$ is a reduced expression for $w \in W$. We assume throughout that

$$L(s) \geqslant 0$$

for all $s \in S$. (If $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$, then this is the original "equal parameter" setting of [6].)

Let $R \subseteq \mathbb{C}$ be a subring and $A = R[\Gamma]$ be a free R-module with basis $\{q^\gamma \mid \gamma \in \Gamma\}$ where q is an indeterminant. (The basic constructions in this section are independent of the choice of R and so we could just take $R = \mathbb{Z}$.) The flexibility of R will be useful once we consider the representations of W. There is a well-defined ring structure on A such that $q^\gamma q^{\gamma'} = q^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. We denote $1 = q^0 \in A$. If $a \in A$ we denote by a_γ the coefficient of a on q^γ so that $a = \sum_{\gamma\in\Gamma} a_\gamma q^\gamma$. If $a \neq 0$ we define the degree of a as the element of Γ equal to

$$\deg(a) = max\{\gamma \mid a_\gamma \neq 0\}$$

by convention (see [1]), we set $\deg 0 = -\infty$. So $\deg : A \to \Gamma \cup \{-\infty\}$ satisfies $\deg(ab) = \deg(a) + \deg(b)$.

Let $\mathscr{H} = \mathscr{H}(W, S, L)$ be the generic Hecke algebra corresponding to (W, S) with parameters $\{q^{L(s)} \mid s \in S\}$. Thus $\mathscr{H}$ has an A-basis $\{T_w \mid w \in W\}$ and the multiplication is given by the rules

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + (q^{L(s)} - q^{-L(s)})T_w & \text{if } \ell(sw) < \ell(w). \end{cases} \tag{1}$$

Let $\Gamma_{\geqslant\gamma_0} = \{\gamma \in \Gamma \mid \gamma \geqslant \gamma_0\}$ and denote by $A_{\geqslant\gamma_0}$ (or $R[\Gamma_{\geqslant\gamma_0}]$) the set of all R-linear combinations of terms q^γ where $\gamma \geqslant \gamma_0$. The notations $A_{\gamma>\gamma_0}, A_{\gamma\leqslant\gamma_0}, A_{\gamma<\gamma_0}$ have a similar meaning.

We denote by $A \mapsto A, a \mapsto \overline{a}$ the automorphism of A induced by the automorphism of Γ sending γ to $-\gamma$ for any $\gamma \in \Gamma$. This extends to a ring involution $\mathscr{H} \mapsto \mathscr{H}$, $h \mapsto \overline{h}$, where

$$\overline{\sum_{w\in W} a_w T_w} = \sum_{w\in W} \overline{a_w} T_{w^{-1}}^{-1},\ a_w \in A \text{ for all } w \in W,$$

and

$$\overline{T_s} = T_s^{-1} = T_s + (q^{-L(s)} - q^{L(s)}) \text{ for all } s \in S.$$

1.1. Definition of W-graph

Definition 1.1. (For equal parameter case see [6]; for general L see [7].) A W-graph for $\mathscr{H}$ consists of the following data:

(a) a base set Λ together with a map I which assigns to each $x \in \Lambda$ a subset $I(x) \subseteq S$;
(b) for each $s \in S$ with $L(s) > 0$, a collection of elements

$$\{\mu^s_{x,y} \mid x, y \in \Lambda \text{ such that } s \in I(x), s \notin I(y)\};$$

(c) for each $s \in S$ with $L(s) = 0$ a bijection $\Lambda \to \Lambda, x \to s.x$. These data are subject to the following requirements. First we require that, for any $x, y \in \Lambda$ and $s \in S$ where $\mu^s_{x,y}$ is defined, we have

$$q^{L(s)}\mu^s_{x,y} \in R[\Gamma_{>0}] \text{ and } \overline{\mu^s_{x,y}} = \mu^s_{x,y}.$$

Furthermore, let $[\Lambda]_A$ be a free A-module with basis $\{b_y \mid y \in \Lambda\}$. For $s \in S$, define an A-linear map

$$\rho_s(b_y) = \begin{cases} b_{s.y} & \text{if } L(s) = 0; \\ -q^{-L(y)}b_y & \text{if } L(s) > 0, s \in I(y); \\ q^{L(y)}b_y + \sum_{x\in\Lambda; s\in I(x)} \mu^s_{x,y}b_x & \text{if } L(s) > 0, s \notin I(y). \end{cases} \tag{2}$$

Then we require that the assignment $T_s \mapsto \rho_s$ defines a representation of $\mathscr{H}$.

2. W-graph ideals

For each $J \subseteq S$, let $\hat{J} = S\backslash J$ (the complement of J) and define $W_J = \langle J \rangle$, the corresponding parabolic subgroup of W. Let $\mathscr{H}_J$ be the Hecke algebra associated with W_J. As is well known, $\mathscr{H}_J$ can be identified with a subalgebra of $\mathscr{H}$.

Let $D_J = \{\, w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\,\}$, the set of minimal coset representatives of W/W_J. The following lemma is well known.

Lemma 2.1 *(Modified). (See [3, Lemma 2.1(iii)].) Let $J \subseteq S$ and $s \in S$, and define*

$$D^-_{J,s} = \{\, w \in D_J \mid \ell(sw) < \ell(w)\,\},$$
$$D^+_{J,s} = \{\, w \in D_J \mid \ell(sw) > \ell(w) \text{ and } sw \in D_J\,\},$$
$$D^0_{J,s} = \{\, w \in D_J \mid \ell(sw) > \ell(w) \text{ and } sw \notin D_J\,\},$$

so that D_J is the disjoint union $D^-_{J,s} \cup D^+_{J,s} \cup D^0_{J,s}$. Then $sD^+_{J,s} = D^-_{J,s}$, and if $w \in D^0_{J,s}$ then $sw = wt$ for some $t \in J$.

In this section we shall recall [9, Section 5], with some modification.

Let $\leqslant_L$ denote the left weak (Bruhat) order on W. We say $x \leqslant_L y$ if and only if $y = zx$ for some $z \in W$ such that $\ell(y) = \ell(z) + \ell(x)$. We also say that x is a *suffix* of y. The following property of the Bruhat order is useful (see [11, Corollary 2.5], for example).

Lemma 2.2. *Let $y, z \in W$ and let $s \in S$.*

(i) *Assume that $sz < z$, then $y \leqslant z \Longleftrightarrow sy \leqslant z$.*
(ii) *Assume that $y < sy$, then $y \leqslant z \Longleftrightarrow y \leqslant sz$.*

Definition 2.3. If $X \subseteq W$, let $Pos(X) = \{s \in S \mid \ell(xs) > \ell(x) \text{for all} x \in X\}$.

Thus $Pos(X)$ is the largest subset J of S such that $X \subseteq D_J$. Let $\mathbf{E}$ be an ideal in the poset $(W, \leqslant_L)$; that is, $\mathbf{E}$ is a subset of W such that every $u \in W$ that is a suffix of an element of $\mathbf{E}$ is itself in $\mathbf{E}$. This condition implies that $Pos(\mathbf{E}) = S \backslash \mathbf{E} = \{s \in S \mid s \notin E\}$. Let J be a subset of $Pos(\mathbf{E})$, so that $\mathbf{E} \subseteq D_J$. In contexts we shall denote by $\mathbf{E}_J$ for the set $\mathbf{E}$, with reference to J, for each $s \in S$ we classify the elements in $\mathbf{E}_J$ as follows:

$$\mathbf{E}^-_{J,s} = \{\, w \in \mathbf{E}_J \mid \ell(sw) < \ell(w) \text{ and } sw \in \mathbf{E}_J \,\},$$

$$\mathbf{E}^+_{J,s} = \{\, w \in \mathbf{E}_J \mid \ell(sw) > \ell(w) \text{ and } sw \in \mathbf{E}_J \,\},$$

$$\mathbf{E}^{0,-}_{J,s} = \{\, w \in \mathbf{E}_J \mid \ell(sw) > \ell(w) \text{ and } sw \notin D_J \,\},$$

$$\mathbf{E}^{0,+}_{J,s} = \{\, w \in \mathbf{E}_J \mid \ell(sw) > \ell(w) \text{ and } sw \in D_J \backslash \mathbf{E}_J \,\}.$$

Since $\mathbf{E}_J \subseteq D_J$ it is clear that, for each $w \in \mathbf{E}_J$, each $s \in S$ appears in exactly one of the following four sets $SA(w) = \{s \in S \mid w \in \mathbf{E}^+_{J,s}\}, SD(w) = \{s \in S \mid w \in \mathbf{E}^-_{J,s}\}$, $WA_J = \{s \in S \mid w \in \mathbf{E}^{0,+}_{J,s}\}$ and $WD_J = \{s \in S \mid w \in \mathbf{E}^{0,-}_{J,s}\}$. We call the elements of these sets the strong ascents, strong descents, weak ascents and weak descents of w relative to $\mathbf{E}_J$ and J. In contexts where the ideal $\mathbf{E}_J$ and the set J is fixed we frequently omit reference to J, writing $WA(w)$ and $WD(w)$ rather than $WA_J(w)$ and $WD_J(w)$. We also define the sets of descents and ascents of w by $D(w) = SD(w) \cup WD(w)$ and $A(w) = SA(w) \cup WA(w)$.

Remark. It follows from Lemma 2.1 that

$$WA_J(w) = \{s \in S \mid sw \notin \mathbf{E}_J \text{ and } w^{-1}sw \notin J\,\},$$

$$WD_J(w) = \{s \in S \mid sw \notin \mathbf{E}_J \text{ and } w^{-1}sw \in J\,\}.$$

Since $sw \notin \mathbf{E}_J$ implies that $sw > w$ (given that $\mathbf{E}_J$ is an ideal in $(W, \leqslant L)$). Note also that $J = WD_J(1)$.

Definition 2.4 *(Modified).* (See [9, Definition 5.1].) Let (W, S) be a Coxeter group with weight function L such that $L(s) \geqslant 0$ for all $s \in S$, $\mathscr{H}$ be the corresponding Hecke algebra. The set $\mathbf{E}_J$ is said to be a W-graph ideal with respect to $J(\subseteq S)$ and L if the following hypotheses are satisfied.

(i) There exists an A-free $\mathscr{H}$-module $M(\mathbf{E}_J, L)$ possessing an A-basis

$$B = \{\Gamma_w | w \in \mathbf{E}_J\},$$

for any $s \in S$ and any $w \in \mathbf{E}_J$ we have

$$T_s\Gamma_w = \begin{cases} \Gamma_{sw} + (q^{L(s)} - q^{-L(s)})\Gamma_w & \text{if } w \in \mathbf{E}_{J,s}^{-}, \\ \Gamma_{sw} & \text{if } w \in \mathbf{E}_{J,s}^{+}, \\ -q^{-L(s)}\Gamma_w & \text{if } w \in \mathbf{E}_{J,s}^{0,-}, \\ q^{L(s)}\Gamma_w - \sum_{\substack{z \in \mathbf{E}_J \\ z<w}} r_{z,w}^s \Gamma_z & \text{if } w \in \mathbf{E}_{J,s}^{0,+}, \end{cases} \tag{3}$$

for some polynomials $r_{z,w}^s \in q^{L(s)}A_{>0}$.

(ii) The module $M(\mathbf{E}_J, L)$ admits an A-semilinear involution $\alpha \mapsto \overline{\alpha}$ satisfying $\overline{\Gamma_1} = \Gamma_1$ and $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathscr{H}$ and $\alpha \in M(\mathbf{E}_J, L)$.

An obvious induction on $\ell(w)$ shows that $\Gamma_w = T_w\Gamma_1$ for all $w \in \mathbf{E}_J$.

Definition 2.5. (See [9, Definition 5.2].) If $w \in W$ and $\mathbf{E}_J = \{u \in W \mid u \leqslant_L w\}$ is a W-graph ideal with respect to some $J \subseteq S$ then we call w a W-graph determining element.

Remark. It has been verified in [9, Section 5] that if W is finite then w_S, the maximal length element of W, is a W-graph determining element with respect to $\emptyset$ and d_J, the minimal length element of the left coset w_SW_J, is a W-graph determining element with respect to J and also with respect to $\emptyset$.

The W-graph for a given W-graph ideal $\mathbf{E}_J$, carries a representation. In this paper we show that the "dual" and "contragredient" representations are also determined by the W-graph data, which form the "the full W-graph" structures.

3. Duality theorem for W-graph ideals

Let (W, S) be a Coxeter group with weight function L such that $L(s) \geqslant 0$ for all $s \in S$, $\mathscr{H}$ be the corresponding Hecke algebra. There exists an algebra map $\Phi : \mathscr{H} \to \mathscr{H}$ given by $\Phi(q^{L(s)}) = q^{L(s)}$ for all $s \in S$, and $\Phi(T_w) = \epsilon_w\overline{T_w}$, where the bar is the standard involution in $\mathscr{H}$ and $\epsilon_w := (-1)^{\ell(w)}$. Further, $\Phi^2 = Id$ and Φ commutes with the bar involution.

3.1. Duality theorem

We now give an equivalent definition of a W-graph ideal, and the associated module is denoted by $\widetilde{M}(\mathbf{E}_J, L)$. The following theorem essentially provides the duality between the two set ups.

Theorem–Definition 3.1.

(I) *With the above notations, let the set $\mathbf{E}_J$ be a W-graph ideal with respect to $J(\subseteq S)$ and L, then the following hypotheses are satisfied.*

(i) *There exists an A-free $\mathscr{H}$-module $\widetilde{M}(\mathbf{E}_J, L)$ possessing an A-basis*

$$\widetilde{B} = \{\widetilde{\Gamma}_w | w \in \mathbf{E}_J\},$$

for any $s \in S$ and any $w \in \mathbf{E}_J$ we have

$$T_s\widetilde{\Gamma}_w = \begin{cases} \widetilde{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\widetilde{\Gamma}_w & \text{if } w \in \mathbf{E}^-_{J,s}, \\ \widetilde{\Gamma}_{sw} & \text{if } w \in \mathbf{E}^+_{J,s}, \\ q^{L(s)}\widetilde{\Gamma}_w & \text{if } w \in \mathbf{E}^{0,-}_{J,s}, \\ -q^{-L(s)}\widetilde{\Gamma}_w + \sum_{\substack{z\in\mathbf{E}_J \\ z<w}} \widetilde{r}^s_{z,w}\widetilde{\Gamma}_z & \text{if } w \in \mathbf{E}^{0,+}_{J,s}, \end{cases} \tag{4}$$

where $\widetilde{r}^s_{z,w} = \epsilon_z\epsilon_w\overline{r^s_{z,w}} \in q^{-L(s)}A_{<0}$.

(ii) *The module $\widetilde{M}(\mathbf{E}_J, L)$ admits an A-semilinear involution $\widetilde{\alpha} \mapsto \overline{\widetilde{\alpha}}$ satisfying $\overline{\widetilde{\Gamma}_1} = \widetilde{\Gamma}_1$ and $\overline{h\widetilde{\alpha}} = \overline{h}\overline{\widetilde{\alpha}}$ for all $h \in \mathscr{H}$ and $\widetilde{\alpha} \in \widetilde{M}(\mathbf{E}_J, L)$.*

(II) *There exists a unique map $\eta : M(\mathbf{E}_J, L) \to \widetilde{M}(\mathbf{E}_J, L)$ such that*

$$(i)\ \eta(\Gamma_1) = \widetilde{\Gamma}_1;$$

$$(ii)\ \eta(h\Gamma) = \Phi(h)\eta(\Gamma), \text{for all } h \in \mathscr{H} \text{ and } \Gamma \in M(\mathbf{E}_J, L)$$

(i.e., η is Φ-linear). Further, it has the following properties:

(a) *η commutes with the involution on $M(\mathbf{E}_J, L)$ and $\widetilde{M}(\mathbf{E}_J, L)$.*

(b) *η is one-to-one onto and the inverse θ of η, satisfies properties (i) and (ii) of η.*

Proof. For $w \in \mathbf{E}_J$, define $\eta(\Gamma_w) = \epsilon_w\overline{\widetilde{\Gamma}_w}$. Extend η to the whole of $M(\mathbf{E}_J, L)$ by Φ-linearity. Let $s \in S$. Then we have,

$$\eta(T_s\Gamma_w) = \begin{cases} \eta[\Gamma_{sw} + (q^{L(s)} - q^{-L(s)})\Gamma_w] & \text{if } w \in \mathbf{E}^-_{J,s}, \\ \eta(\Gamma_{sw}) & \text{if } w \in \mathbf{E}^+_{J,s}, \\ \eta(-q^{-L(s)}\Gamma_w) & \text{if } w \in \mathbf{E}^{0,-}_{J,s}, \\ \eta(q^{L(s)}\Gamma_w - \sum_{\substack{z\in\mathbf{E}_J \\ z<w}} r^s_{z,w}\Gamma_z) & \text{if } w \in \mathbf{E}^{0,+}_{J,s}, \end{cases} \tag{5}$$

which equals to

$$\begin{cases} \epsilon_{sw}\overline{\widetilde{\Gamma}_{sw}} + (q^{L(s)} - q^{-L(s)})\epsilon_w\overline{\widetilde{\Gamma}_w} & \text{if } w \in \mathbf{E}^-_{J,s}, \\ \epsilon_{sw}\overline{\widetilde{\Gamma}_{sw}} & \text{if } w \in \mathbf{E}^+_{J,s}, \\ -q^{-L(s)}\epsilon_w\overline{\widetilde{\Gamma}_w} & \text{if } w \in \mathbf{E}^{0,-}_{J,s}\ , \\ q^{L(s)}\epsilon_w\overline{\widetilde{\Gamma}_w} - \sum_{\substack{z\in\mathbf{E}_J \\ z<w}} r^s_{z,w}\epsilon_z\overline{\widetilde{\Gamma}_z} & \text{if } w \in \mathbf{E}^{0,+}_{J,s}, \end{cases} \tag{6}$$

for some polynomials $r^s_{z,w} \in q^{L(s)}A_{>0}$. On the other hand

$$\begin{aligned} \Phi(T_s)\eta(\Gamma_w) &= -\overline{T_s}\epsilon_w\overline{\widetilde{\Gamma}_w} \\ &= (-1)^{\ell(w)+1}\overline{T_s\widetilde{\Gamma}_w} \\ &= (-1)^{\ell(w)+1}\begin{cases} \overline{\widetilde{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\widetilde{\Gamma}_w} & \text{if } w \in \mathbf{E}^-_{J,s}, \\ \overline{\widetilde{\Gamma}_{sw}} & \text{if } w \in \mathbf{E}^+_{J,s}, \\ \overline{q^{L(s)}\widetilde{\Gamma}_w} & \text{if } w \in \mathbf{E}^{0,-}_{J,s}\ , \\ \overline{-q^{-L(s)}\widetilde{\Gamma}_w - \sum_{\substack{z\in\mathbf{E}_J \\ z<w}} \widetilde{r}^s_{z,w}\widetilde{\Gamma}_z} & \text{if } w \in \mathbf{E}^{0,+}_{J,s}. \end{cases} \end{aligned}$$

It is easy to check that these two expressions give the same result, and this shows that $\eta(T_s\Gamma_w) = \Phi(T_s)\eta(\Gamma_w)$. It is also easy to see that $\eta(h\Gamma_w) = \Phi(h)\eta(\Gamma_w)$ for all $h \in \mathscr{H}$ and $\Gamma_w \in M(\mathbf{E}_J, L)$.

If η' is another map satisfying properties (i) and (ii), then

$$\eta'(\Gamma_w) = \eta'(T_w\Gamma_1) = \Phi(T_w)\widetilde{\Gamma}_1 = \epsilon_w\overline{T_w}\widetilde{\Gamma}_1 = \epsilon_w\overline{T_w\widetilde{\Gamma}_1} = \epsilon_w\overline{\widetilde{\Gamma}_w}.$$

It is now clear that $\eta' = \eta$.

To prove statement (a), observe that for any $\Gamma \in M(\mathbf{E}_J, L)$, there exists $h \in \mathscr{H}$ such that $\Gamma = h\Gamma_1$. Thus

$$\overline{\eta(\Gamma)} = \overline{\eta(h\Gamma_1)} = \overline{\Phi(h)\widetilde{\Gamma}_1} = \overline{\Phi(h)}\widetilde{\Gamma}_1 = \Phi(\overline{h})\widetilde{\Gamma}_1 = \eta(\overline{h}\Gamma_1) = \eta(\overline{\Gamma}).$$

This proves (a).

We interchange the roles of these two modules to obtain a map

$$\theta : \widetilde{M}(\mathbf{E}_J, L) \to M(\mathbf{E}_J, L)$$

such that $\theta(\widetilde{\Gamma}_w) = \epsilon_w\overline{\Gamma_w}$. It is easy to check that θ and η are inverses of each other. This proves (b). $\square$

Corollary 3.2. *If $R_{x,y}$ and $\widetilde{R}_{x,y}$ are the polynomials given by the formula*

$$\overline{\Gamma_y} = \sum_{x \in \mathbf{E}_J} R_{x,y} \Gamma_x, \overline{\widetilde{\Gamma}_y} = \sum_{x \in \mathbf{E}_J} \widetilde{R}_{x,y} \widetilde{\Gamma}_x$$

then

$$\overline{R_{x,y}} = \epsilon_x \epsilon_y \widetilde{R}_{x,y}.$$

Proof. Apply the function η to both sides of the formula for $\overline{\Gamma_y}$ and use the fact that η commutes with the involution and then use the formula for $\overline{\widetilde{\Gamma}}_y$. We omit the details. □

The above result can also be proved by the following recursive formulas.

Lemma 3.3. *(See [14, Prop. 4.1].) Let x, $y \in \mathbf{E}_J$. If $s \in S$ is such that $y \in \mathbf{E}^-_{J,s}$ then*

$$R_{x,y} = \begin{cases} R_{sx,sy} & \text{if } x \in \mathbf{E}^-_{J,s}, \\ R_{sx,sy} + (q^{-L(s)} - q^{L(s)}) R_{x,sy} & \text{if } x \in \mathbf{E}^+_{J,s}, \\ -q^{L(s)} R_{x,sy} & \text{if } x \in \mathbf{E}^{0,-}_{J,s}\ , \\ q^{-L(s)} R_{x,sy} & \text{if } x \in \mathbf{E}^{0,+}_{J,s}. \end{cases}$$

Similarly we have

Lemma 3.4. *Let x, $y \in \mathbf{E}_J$. If $s \in S$ is such that $y \in \mathbf{E}^-_{J,s}$ then*

$$\widetilde{R}_{x,y} = \begin{cases} \widetilde{R}_{sx,sy} & \text{if } x \in \mathbf{E}^-_{J,s}, \\ \widetilde{R}_{sx,sy} + (q^{-L(s)} - q^{L(s)}) \widetilde{R}_{x,sy} & \text{if } x \in \mathbf{E}^+_{J,s}, \\ q^{-L(s)} \widetilde{R}_{x,sy} & \text{if } x \in \mathbf{E}^{0,-}_{J,s}\ , \\ -q^{L(s)} \widetilde{R}_{x,sy} & \text{if } x \in \mathbf{E}^{0,+}_{J,s}. \end{cases}$$

We have the further properties of $R_{x,y}$.

Lemma 3.5. *If $y \in \mathbf{E}^{0,-}_{J,s}$ then we have*

$$R_{x,y} = \begin{cases} -q^{-L(s)} R_{sx,y} & \text{if } x \in \mathbf{E}^-_{J,s}, \\ -q^{L(s)} R_{sx,y} & \text{if } x \in \mathbf{E}^+_{J,s}. \end{cases}$$

If $y \in \mathbf{E}^{0,+}_{J,s}$ then we have

$$R_{x,y} = \begin{cases} q^{L(s)} R_{sx,y} & \text{if } x \in \mathbf{E}^{-}_{J,s}, \\ q^{-L(s)} R_{sx,y} & \text{if } x \in \mathbf{E}^{+}_{J,s}. \end{cases}$$

Proof. If $y \in \mathbf{E}^{0,-}_{J,s}$ then

$$T_s\Gamma_y = -q^{-L(s)}\Gamma_y$$

Applying involution bar on both sides. On the left hand side we have

$$\overline{T_s\Gamma_y} = \overline{T_s}\overline{\Gamma_y} = [T_s + (q^{-L(s)} - q^{L(s)}] \sum_{x \in \mathbf{E}_J} R_{x,y}\Gamma_x$$

while the right hand side is $\overline{-q^{-L(s)}\Gamma_y} = -q^{L(s)} \sum_{x \in \mathbf{E}_J} R_{x,y}\Gamma_x$.

Comparing the coefficients of Γ_x in the two expressions, we get the result. The proof for the case $y \in \mathbf{E}^{0,+}_{J,s}$ is similar with the above. □

3.2. Dual bases for the modules $M(\mathbf{E}_J, L)$ and $\widetilde{M}(\mathbf{E}_J, L)$

Recall [14, Th. 4.4] that the invariants in $M(\mathbf{E}_J, L)$ (respectively $\widetilde{M}(\mathbf{E}_J, L)$) form a free A-module with a basis $\{ \mathbf{C}_w \mid w \in \mathbf{E}_J \}$ (respectively $\{ \widetilde{\mathbf{C}}_w \mid w \in \mathbf{E}_J \}$), where $\mathbf{C}_w = \sum_{y \in \mathbf{E}_J} P_{y,w}\Gamma_y$ and $\widetilde{\mathbf{C}}_w = \sum_{y \in \mathbf{E}_J} \widetilde{P}_{y,w}\widetilde{\Gamma}_y$.

Using the map θ, we obtain a dual basis $\{ \mathbf{C}'_w \mid w \in \mathbf{E}_J \}$ for the invariants in $M(\mathbf{E}_J, L)$. Analogously, using the map η we obtain the dual basis $\{ \widetilde{\mathbf{C}}'_w \mid w \in \mathbf{E}_J \}$ for the invariants in $\widetilde{M}(\mathbf{E}_J, L)$.

More precisely, we have:

Proposition 3.6. *Let $\mathbf{C}'_w = \theta(\widetilde{\mathbf{C}}_w)$, $\widetilde{\mathbf{C}}'_w = \eta(\mathbf{C}_w)$. Then*

(a) *The $\mathscr{H}$-module $M(\mathbf{E}_J, L)$ has a unique basis $\{ \mathbf{C}'_w \mid w \in \mathbf{E}_J \}$ such that $\overline{\mathbf{C}'_w} = \mathbf{C}'_w$ for all $w \in \mathbf{E}_J$, and $\mathbf{C}'_w = \sum_{y \in \mathbf{E}_J} \epsilon_y \overline{\widetilde{P}_{y,w}}\Gamma_y$, for some elements $\widetilde{P}_{y,w} \in A_{\geqslant 0}$ with the following properties*:
   - (a1) $\widetilde{P}_{y,w} = 0$ *if* $y \nleqslant w$;
   - (a2) $\widetilde{P}_{w,w} = 1$;
   - (a3) $\widetilde{P}_{y,w}$ *has zero constant term if* $y \neq w$ *and*

$$\overline{\widetilde{P}_{y,w}} - \widetilde{P}_{y,w} = \sum_{\substack{y < x \leqslant w \\ x \in \mathbf{E}_J}} \overline{\widetilde{R}_{y,x}} \widetilde{P}_{x,w} \text{ for any } y < w.$$

(b) *Analogously, the module* $\widetilde{M}(\mathbf{E}_J, L)$ *has another basis* $\{\, \widetilde{\mathbf{C}}'_w \mid w \in \mathbf{E}_J \,\}$, *where* $\widetilde{\mathbf{C}}'_w = \sum_{y \in \mathbf{E}_J} \epsilon_y \overline{P_{y,w}} \widetilde{\Gamma}_y$.

Proof.

$$\mathbf{C}'_w = \theta(\sum_{y \in \mathbf{E}_J} \widetilde{P}_{y,w} \widetilde{\Gamma}_y) = \sum_{y \in \mathbf{E}_J} \epsilon_y \widetilde{P}_{y,w} \overline{\Gamma_y}$$

Hence, $\overline{\mathbf{C}'_w} = \overline{\theta(\widetilde{\mathbf{C}}_w)} = \theta(\overline{\widetilde{\mathbf{C}}_w}) = \theta(\widetilde{\mathbf{C}_w}) = \mathbf{C}'_w$ and the result follows. $\square$

Theorem 3.7. *The bases* $\mathbf{C}$ *and* $\mathbf{C}'$, *give the module* $M(\mathbf{E}_J, L)$ *the structures of a* W*-graph module such that*

$$T_s \mathbf{C}_v = \begin{cases} q^{L(s)} \mathbf{C}_v + \mathbf{C}_{sv} + \sum_{z \in \mathbf{E}_J, sz < z < v} m^s_{z,v} \mathbf{C}_z & \text{if } s \in SA(v), \\ -q^{-L(s)} \mathbf{C}_v & \text{if } s \in D(v), \\ q^{L(s)} \mathbf{C}_v + \sum_{z \in \mathbf{E}_J, sz < z < v} m^s_{z,v} \mathbf{C}_z & \text{if } s \in WA(v), \end{cases} \tag{7}$$

$$T_s \mathbf{C}'_v = \begin{cases} -q^{-L(s)} \mathbf{C}'_v + \mathbf{C}'_{sv} + \sum_{z \in \mathbf{E}_J, sz < z < v} m^s_{z,v} \mathbf{C}'_z & \text{if } s \in SA(v), \\ q^{L(s)} \mathbf{C}'_v & \text{if } s \in D(v), \\ -q^{-L(s)} \mathbf{C}'_v + \sum_{z \in \mathbf{E}_J, sz < z < v} m^s_{z,v} \mathbf{C}'_z & \text{if } s \in WA(v). \end{cases} \tag{8}$$

The formulas for $T_s \mathbf{C}_v$, see [14, Th. 4.7]. The formulas for $T_s \mathbf{C}'_v$ are obtained by $\theta(T_s \widetilde{\mathbf{C}}_v)$.

3.3. Inversion

For $y, w \in \mathbf{E}_J$, we write the matrix $P = (P_{y,w})$, where $P_{y,w}$ are $\mathbf{E}_J$-relative Kazhdan–Lusztig polynomials. The formula for $\mathbf{C}_w$ in [14, Th. 4.4] may be written as

$$\mathbf{C}_w = \Gamma_w + \sum_{y \in \mathbf{E}_J} P_{y,w} \Gamma_y$$

and inverting this gives

$$\Gamma_w = \mathbf{C}_w + \sum_{y \in \mathbf{E}_J} Q_{y,w} \mathbf{C}_y$$

where the elements $Q_{y,w}$ (defined whenever $y < w$) are given recursively by

$$Q_{y,w} = -P_{y,w} - \sum_{z \in \mathbf{E}_J | y<z<w} Q_{y,z} P_{z,w} \tag{9}$$

An $\mathbf{E}_J$-chain is a sequence $\zeta : z_0 < z_1 < \cdots < z_n (n \geqslant 1)$ of elements in $\mathbf{E}_J$, we set $\ell(\zeta) = n$ and $P_\zeta = P_{z_0,z_1} P_{z_1,z_2} \cdots P_{z_{n-1},z_n}$. z_0 is called the initial element of ζ and z_n is called the final element of ζ. For $y < w$, let $\tau(y, w)$ denote the set of all $\mathbf{E}_J$-chains with y as the initial element and w as the final element.

The following results are inspired by Lusztig [11, Ch. 10] and [12]. For the sake of completeness we attach the proof.

Proposition 3.8. *For any $y, w \in \mathbf{E}_J$ we have*

$$Q_{y,w} = \sum_{\zeta \in \tau(y,w)} (-1)^{\ell(\zeta)} P_\zeta$$

We have $Q_{y,w} \in A_{\geqslant 0}$ with the following properties:

(a1) $Q_{y,w} = 0$ *if* $y \nleqslant w$;
(a2) $Q_{w,w} = 1$.

Proof. If $\ell(w) - \ell(y) = 1$, by Eq. (9) we have $Q_{y,w} = -P_{y,w}$. The statement is true. Applying induction on $\ell(w) - \ell(y) \geqslant 1$. For any $z \in \mathbf{E}_J$, $y < z < w$, in the sum of Eq. (9) we use the induction hypothesis.

$$Q_{y,z} = \sum_{\zeta' \in \tau(y,z)} (-1)^{\ell(\zeta')} P_{\zeta'}$$

We have

$$\begin{aligned} Q_{y,w} &= -P_{y,w} - \sum_{\zeta' \in \tau(y,z)} (-1)^{\ell(\zeta')} P_{\zeta'} P_{z,w} \\ &= \sum_{\zeta \in \tau(y,w)} (-1)^{\ell(\zeta)} P_\zeta \end{aligned}$$

where the sequence $\zeta = (y, w)(\in \tau(y, w))$ is with $\ell(\zeta) = 1$ and $(\zeta', w)(\in \tau(y, w))$ with the length $\ell(\zeta') + 1$. The listed properties of $Q's$ are by Eq. (9). The result is proved. □

We define

$$Q'_{y,w} = \epsilon_y \epsilon_w Q_{y,w}$$

If P is a property we set $\delta_P = 1$ if P is true and $\delta_P = 0$ if P is false. We write $\delta_{x,y}$ instead of $\delta_{x=y}$.

Proposition 3.9. *For any $y, w \in \mathbf{E}_J$ we have* $\overline{Q'_{y,w}} = \sum_{z;y \leqslant_L z \leqslant_L w} Q'_{y,z}\overline{\widetilde{R}_{z,w}}$.

Proof. The triangular matrices $Q = (Q_{y,w}), P = (P_{y,w}), R = (R_{y,w})$ are related by

$$PQ = QP = 1, \overline{P} = \overline{R}P, \overline{R}R = R\overline{R} = 1$$

where the bar involution over a matrix is the matrix obtained by applying $^-$ to each entry. We deduce that

$$QP = 1 = \overline{QP} = \overline{QR}P$$

Multiplying on the right by Q and using the fact $PQ = 1$ we deduce $Q = \overline{QR}$. This gives

$$\overline{Q} = QR$$

Let S be the matrix whose (y, w)-entry is $\epsilon_y \delta_{y,w}$. We have $S^2 = 1$. Note that $Q' = SQS$. By Corollary 3.2 we have $\overline{R} = S\widetilde{R}S$. Hence

$$\overline{Q'} = \overline{SQS} = S(QR)S = SQS \cdot SRS = Q'\overline{\widetilde{R}}$$

The result follows. □

4. *W*-graphs for the modules $\hat{M}$ and $\hat{\widetilde{M}}$

Denote by $M := M(\mathbf{E}_J, L)$ and $\widetilde{M} := \widetilde{M}(\mathbf{E}_J, L)$. Let $\hat{M} := Hom_A(M, A)$ and $\hat{\widetilde{M}} := Hom_A(\widetilde{M}, A)$.

Define an left $\mathscr{H}$-module structure on $\hat{M}$ by

$$hf(m) = f(hm) \text{ (with } f \in \hat{M}, m \in M, h \in \mathscr{H}).$$

We define a bar operator $\hat{M} \mapsto \hat{M}$ by $\overline{f}(m) = \overline{f(\overline{m})}$ (with $f \in \hat{M}, m \in M$); in $\overline{f(\overline{m})}$ the lower bar is that of M and the upper bar is that of A.

$$\overline{h \cdot f}(m) = \overline{hf(\overline{m})} = \overline{f(h\overline{m})} = \overline{f}(\overline{h\overline{m}}) = \overline{f}(\overline{h}m) = \overline{h} \cdot \overline{f}(m).$$

Hence we have $\overline{h \cdot f} = \overline{h} \cdot \overline{f}$ for $f \in \hat{M}$, $h \in \mathscr{H}$.

In the following contexts we focus on the module $\hat{M}$, and usually omit the analogous details for $\hat{\widetilde{M}}$.

4.1. The basis of $\hat{M}$

We firstly introduce two bases for the module $\hat{M}$. For any $z \in \mathbf{E}_J$ we define $\hat{\Gamma}_z \in \hat{M}$ by $\hat{\Gamma}_z(\Gamma_w) = \delta_{z,w}$ for any $w \in \mathbf{E}_J$. Then $\hat{B} =: \{\hat{\Gamma}_z; z \in \mathbf{E}_J\}$ is an A-basis of $\hat{M}$.

Further, for any $z \in \mathbf{E}_J$ we define $D_z \in \hat{M}$ by $D_z(\mathbf{C}_w) = \delta_{z,w}$ for any $w \in \mathbf{E}_J$. Then $D := \{D_z; z \in \mathbf{E}_J\}$ is an A-basis of $\hat{M}$.

Obviously we have

$$D_z = \sum_{y \in \mathbf{E}_J, z<y} Q_{z,y} \hat{\Gamma}_y.$$

An equivalent definition of the basis element $D_w \in \hat{M}$ is

$$D_z(\Gamma_y) = Q_{z,y}$$

for all $y \in \mathbf{E}_J$. In fact, we have

$$D_z(\mathbf{C}_w) = D_z \sum_{y \in \mathbf{E}_J} P_{y,w}\Gamma_y) = \sum_{y \in \mathbf{E}_J} Q_{z,y}P_{y,w} = \delta_{z,w}$$

Lemma 4.1. *For any $y \in \mathbf{E}_J$ we have*

$$\overline{\hat{\Gamma}_y} = \sum_{w \in \mathbf{E}_J, y \leqslant w} \overline{R_{y,w}} \hat{\Gamma}_w.$$

Proof. For any $x \in \mathbf{E}_J$ we have

$$\begin{aligned}
\overline{\hat{\Gamma}_y}(\Gamma_x) &= \overline{\hat{\Gamma}_y(\overline{\Gamma_x})} \\
&= \overline{\hat{\Gamma}_y(\sum_{x' \in \mathbf{E}_J, x' \leqslant x} R_{x',x}\Gamma_{x'})} = \overline{\delta_{y \leqslant x} R_{y,x}} = \delta_{y \leqslant x} \overline{R_{y,x}} \\
&= \sum_{w \in \mathbf{E}_J, y \leqslant w} \overline{R_{y,w}} \hat{\Gamma}_w(\Gamma_x) \qquad \square
\end{aligned}$$

Theorem 4.2. *The $\mathscr{H}$-module $\hat{M}(\mathbf{E}_J, L)$ has a unique basis $\{ D_z \mid z \in \mathbf{E}_J \}$ such that $\overline{D_z} = D_z$ for all $z \in \mathbf{E}_J$, and $D_z = \sum_{y \in \mathbf{E}_J} Q_{z,y}\hat{\Gamma}_y$, for some elements $Q_{z,y} \in A_{\geqslant 0}$ with the following properties*:

(a1) $Q_{z,y} = 0$ *if* $z \nleqslant y$;
(a2) $Q_{z,z} = 1$*;*
(a3) $Q_{z,y}$ *has zero constant term if* $z \neq y$ *and*

$$Q_{z,y} - \overline{Q_{z,y}} = \sum_{\substack{z \leqslant x < y \\ x \in \mathbf{E}_J}} \overline{Q_{z,x}R_{x,y}} \textit{for any } z < y.$$

The proof is very similar with that of [11, Th. 5.2]. It uses induction on $\ell(w) - \ell(y)$, the equation $\overline{Q} = QR$ in Proposition 3.9 and Lemma 4.1, and the fact:

If $f = \sum_{\substack{z \leqslant x < y \\ y \in \mathbf{E}_J}} Q_{z,x} R_{x,y}$ then $\overline{f} = -f$. We omit further details of the proof.

The (left) ascent set of $z \in \mathbf{E}_J$ is

$$A(z) = \{ s \in S \mid z \in \mathbf{E}^{+}_{J,s} \cup \mathbf{E}^{0,+}_{J,s} \}$$

Theorem 4.3. *Let $s \in S$ and assume that $L(s) > 0$. The basis elements*

$$\{ D_z \mid z \in \mathbf{E}_J \}$$

give $\hat{M}$ the structure of a W-graph module such that

$$T_s D_z = \begin{cases} -q^{-L(s)} D_z + D_{sz} + \sum\limits_{z<u, s\in A(u)} m^s_{z,u} D_u & \text{if } s \in SD(z), \\ q^{L(s)} D_z & \text{if } s \in A(z), \\ -q^{-L(s)} D_z + \sum\limits_{z<u, s \in A(u)} m^s_{z,u} D_u & \text{if } s \in WD(z). \end{cases} \tag{10}$$

Proof. In the case $s \in SD(z)$, $T_s D_z(\mathbf{C}_w) = D_z(T_s \mathbf{C}_w)$ gives

$$T_s D_z(\mathbf{C}_w) = \begin{cases} D_z(q^{L(s)} \mathbf{C}_w + \mathbf{C}_{sw} + \sum\limits_{x \in \mathbf{E}_J, sx<x<w} m^s_{x,w} \mathbf{C}_x) & \text{if } s \in SA(w), \\ D_z(-q^{-L(s)} \mathbf{C}_w) & \text{if } s \in D(w), \\ D_z(q^{L(s)} \mathbf{C}_w + \sum\limits_{x \in \mathbf{E}_J, sx<x<w} m^s_{x,w} \mathbf{C}_x) & \text{if } s \in WA(w), \end{cases}$$

$$= \begin{cases} \delta_{z,sw} + \sum\limits_{x \in \mathbf{E}_J, sx<x<w} m^s_{x,w} \delta_{z,x} & \text{if } s \in SA(w), \\ -q^{-L(s)} \delta_{z,w} & \text{if } s \in SD(w), \\ 0 & \text{if } s \in WD(w), \\ \sum\limits_{x \in \mathbf{E}_J, sx<x<w} m^s_{x,w} \delta_{z,x} & \text{if } s \in WA(w), \end{cases}$$

$$= \begin{cases} \delta_{z,sw} + m^s_{z,w} \delta_{z<w} & \text{if } s \in SA(w), \\ -q^{-L(s)} \delta_{z,w} & \text{if } s \in SD(w), \\ 0 & \text{if } s \in WD(w), \\ m^s_{z,w} \delta_{z<w} & \text{if } s \in WA(w), \end{cases}$$

$$= \begin{cases} (D_{sz} + \sum\limits_{z<u, u \in \mathbf{E}^{+}_{J,s}} m^s_{z,u} D_u)(\mathbf{C}_w) & \text{if } s \in SA(w), \\ -q^{-L(s)} D_z(\mathbf{C}_w) & \text{if } s \in SD(w), \\ 0 & \text{if } s \in WD(w), \\ \sum\limits_{z<u, u \in \mathbf{E}^{0,+}_{J,s}} m^s_{z,u} D_u(\mathbf{C}_w) & \text{if } s \in WA(w). \end{cases}$$

Hence, we obtain

$$T_sD_z(\mathbf{C}_w) = (-q^{-L(s)}D_z + D_{sz} + \sum_{z<u, s\in A(u)} m^s_{z,u}D_u)(\mathbf{C}_w)$$

for all $w \in \mathbf{E}_J$. The desired formula follows.

In other cases the computation is similar with the above, we omit the details. $\square$

The following is by [14, Prop. 4.8].

Corollary 4.4. *For $s \in S$ with $L(s) = 0$, $z \in \mathbf{E}_J$, we have*

$$T_sD_z = \begin{cases} D_{sz} & \text{if } s \in SD(z) \text{ or } s \in SA(z), \\ -D_z & \text{if } s \in WD(z), \\ D_z & \text{if } s \in WA(z). \end{cases}$$

4.2. The D'-basis for $\hat{M}$

Theorem 4.5. *The $\mathscr{H}$-module $\hat{M}(\mathbf{E}_J, L)$ has a unique basis $\{ D'_z \mid z \in \mathbf{E}_J \}$ such that $\overline{D'_z} = D'_z$ for all $z \in \mathbf{E}_J$, and $D'_z = \sum_{y\in\mathbf{E}_J} \epsilon_y \overline{\widetilde{Q}_{z,y}} \hat{\Gamma}_y$, where $\widetilde{Q}_{z,y} \in A_{\geqslant 0}$, are the analogous elements in the case of $\widetilde{M}$.*

$$T_sD'_z = \begin{cases} q^{L(s)}D'_z + D'_{sz} + \sum_{z<u, s\in A(u)} m^s_{z,u}D'_u & \text{if } s \in SD(z), \\ -q^{-L(s)}D'_z & \text{if } s \in A(z), \\ q^{L(s)}D'_z + \sum_{z<u, s\in A(u)} m^s_{z,u}D'_u & \text{if } s \in WD(z). \end{cases} \tag{11}$$

For a given W-graph ideal $\mathbf{E}_J$, two pairs of dual bases $\mathbf{C}, \mathbf{C}'$ and D, D' provide the "full W-graph" bases.

4.3. The module $\hat{M}(D_J, L)$

Set $\mathbf{E}_J := D_J$. If D_J is regarded as a W-graph ideal with respect to $\emptyset$ (see the remark on Deodhar's construction, in Section 6), we have

Lemma 4.6. *The modules $\hat{M}(D_J, L)$ and $M(D_J, L)$ are identical.*

Proof. For any basis element $\hat{\Gamma}_w$ of $\hat{M}(D_J, L)$ and element Γ_y of $M(D_J, L)$, we have

$$\begin{aligned} T_s\hat{\Gamma}_w(\Gamma_y) &= \hat{\Gamma}_w(T_s\Gamma_y) \\ &= \delta_{y\in D^-_{J,s}}\delta_{w,sy} + (q^{L(s)} - q^{-L(s)})\delta_{y\in D^-_{J,s}}\delta_{w,y} + \delta_{y\in D^+_{J,s}}\delta_{w,sy} \\ &\quad + q^{L(s)}\delta_{y\in D^0_{J,s}}\delta_{w,y} \end{aligned}$$

$$
\begin{aligned}
&= \delta_{w\in D^+_{J,s}}\delta_{sw,y} + (q^{L(s)} - q^{-L(s)})\delta_{w\in D^-_{J,s}}\delta_{w,y} + \delta_{w\in D^-_{J,s}}\delta_{sw,y} \\
&\qquad + q^{L(s)}\delta_{w\in D^0_{J,s}}\delta_{w,y} \\
&= (\delta_{w\in D^+_{J,s}}\hat{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\delta_{w\in D^-_{J,s}}\hat{\Gamma}_w + \delta_{w\in D^-_{J,s}}\hat{\Gamma}_{sw} \\
&\qquad + q^{L(s)}\delta_{w\in D^0_{J,s}}\hat{\Gamma}_w)(\Gamma_y)
\end{aligned}
$$

hence we have

$$
T_s\hat{\Gamma}_w = \begin{cases} \hat{\Gamma}_{sw} & \text{if } w \in D^+_{J,s}, \\ \hat{\Gamma}_{sw} + (q^{L(s)} - q^{-L(s)})\hat{\Gamma}_w & \text{if } w \in D^-_{J,s}, \\ q^{L(s)}\hat{\Gamma}_w & \text{if } w \in D^0_{J,s}. \end{cases}
$$

The result follows. $\square$

Corollary 4.7. *The $\mathscr{H}$-module $M(D_J, L)$ has basis $\{D_z \mid z \in D_J\}$, where $D_z = \sum_{y\in D_J, z<y} Q_{z,y}\Gamma_y$. This basis gives the structure of W-graph module such that*

$$
T_sD_z = \begin{cases} -q^{-L(s)}D_z + D_{sz} + \sum\limits_{z<u, u\in D^+_{J,s}\cup D^0_{J,s}} m^s_{z,u}D_u & \textit{if } z \in D^-_{J.s}, \\ q^{L(s)}D_z & \textit{if } z \in D^+_{J,s} \cup D^0_{J,s}. \end{cases}
$$

5. In the case W is finite

Let (W, S) be a finite Coxeter system and w_0 be the longest element in W. Define the function $\pi\colon W \to W$ by $\pi(w) = w_0ww_0$, it satisfies $\pi(S) = S$ and it extends to a $\mathbb{C}$-algebra isomorphism $\pi\colon \mathbb{C}[W] \longmapsto \mathbb{C}[W]$. We denote by $s_0 = \pi(s)$. For $s \in S$ we have $\ell(w_0) = \ell(w_0s) + \ell(s) = \ell(\pi(s)) + \ell(\pi(s)w_0)$, hence

$$
L(w_0) = L(w_0s) + L(s) = L(\pi(s)) + L(\pi(s)w_0) = L(\pi(s)) + L(w_0s)
$$

so that $L(\pi(s)) = L(s)$. It follows that $L(\pi(w)) = L(w)$ for all $w \in W$ and that we have an A-algebra automorphism $\pi : \mathscr{H} \longmapsto \mathscr{H}$ where $\pi(T_w) = T_{\pi(w)}$ for any $w \in W$.

Lemma 5.1. *The $\mathscr{H}$-modules M and $\widetilde{M}$ have basis $\Gamma^\pi = \{T_{w_0}\overline{\Gamma_w} \mid w \in \mathbf{E}_J\}$ and $\widetilde{\Gamma}^\pi = \{T_{w_0}\overline{\widetilde{\Gamma}_w} \mid w \in \mathbf{E}_J\}$ respectively. Furthermore, $\eta(T_{w_0}\overline{\Gamma_w}) = \epsilon_{w_0w}\overline{T_{w_0}\overline{\widetilde{\Gamma}_w}}$.*

Proof. Since the involution is square 1 and T_{w_0} is invertible in $\mathscr{H}$, the statement follows.
Moreover

$$
\eta(T_{w_0}\overline{\Gamma_w}) = \Phi(T_{w_0})\eta(\overline{\Gamma_w}) = \epsilon_{w_0}\overline{T_{w_0}}\epsilon_w\widetilde{\Gamma}_w = \epsilon_{w_0w}\overline{T_{w_0}\overline{\widetilde{\Gamma}_w}}. \qquad \square
$$

Still, we focus primarily on the module M and omit the analogous details for $\widetilde{M}$, unless it is needed. For any $w \in \mathbf{E}_J$ we denote by $w' := w_0 w$ and $\Gamma^\pi_{w'} := T_{w_0}\overline{\Gamma_w}$.

Remark. Note that, generally $w_0\mathbf{E}_J \neq \mathbf{E}_J$. In the following contexts, the set $w_0\mathbf{E}_J$ will be just used as the index set for the objects involved.

Direct computation gives the following multiplication rules for the basis Γ^π.

$$T_{s_0}\Gamma^\pi_{w'} = \begin{cases} \Gamma^\pi_{s_0w'} + (q^{L(s)} - q^{-L(s)})\Gamma^\pi_{w'} & \text{if } w \in \mathbf{E}^+_{J,s}, \\ \Gamma^\pi_{s_0w'} & \text{if } w \in \mathbf{E}^-_{J,s}, \\ -q^{-L(s)}\Gamma^\pi_{w'} & \text{if } w \in \mathbf{E}^{0,-}_{J,s}\ , \\ q^{L(s)}\Gamma^\pi_{w'} - \sum_{\substack{z' \in w_0\mathbf{E}_J \\ w'<z'}} r^{s_0}_{w',z'}\Gamma^\pi_{z'} & \text{if } w \in \mathbf{E}^{0,+}_{J,s}, \end{cases}$$

where $r^{s_0}_{w',z'} = \overline{r^s_{z,w}} \in q^{-L(s)}A_{<0}$.

Lemma 5.2. *For any $y' \in w_0\mathbf{E}_J$ there exist coefficients $R^\pi_{x',y'} \in A$, defined for $x' \in w_0\mathbf{E}_J$, such that $\overline{\Gamma^\pi_{y'}} = \sum_{x' \in w_0\mathbf{E}_J} R^\pi_{x',y'}\Gamma^\pi_{x'}$. If $R^\pi_{x',y'} \neq 0$ then $x' \leqslant y'$; particularly $R^\pi_{y',y'} = 1$.*

The proof is trivial.

We have further properties of $R^\pi_{x',y'}$.

Lemma 5.3. *If $y' \in w_0\mathbf{E}^{0,-}_{J,s}$ then we have*

$$R^\pi_{x',y'} = \begin{cases} -q^{L(s_0)}R^\pi_{s_0x',y'} & \text{if } x' \in w_0\mathbf{E}^-_{J,s}, \\ -q^{-L(s_0)}R^\pi_{s_0x',y'} & \text{if } x' \in w_0\mathbf{E}^+_{J,s}. \end{cases}$$

If $y' \in w_0\mathbf{E}^{0,+}_{J,s}$ then we have

$$R^\pi_{x',y'} = \begin{cases} q^{-L(s_0)}R^\pi_{s_0x',y'} & \text{if } x' \in w_0\mathbf{E}^-_{J,s}, \\ q^{L(s_0)}R^\pi_{s_0x',y'} & \text{if } x' \in w_0\mathbf{E}^+_{J,s}. \end{cases}$$

Proof. The proof is similar with that of Lemma 3.5. ◻

5.1. The basis $\mathbf{C}^\pi$ for M

The elements $R^\pi_{w',y'}$, where $w', y' \in w_0\mathbf{E}_J$, lead to the construction of another set of elements $P^\pi_{w',y'}$ and the following basis of $M(\mathbf{E}_J, L)$.

Theorem 5.4.

(1) *The $\mathscr{H}$-module $M(\mathbf{E}_J, L)$ has a unique basis $\{ \mathbf{C}^\pi_{y'} \mid y' \in w_0\mathbf{E}_J \}$ such that $\overline{\mathbf{C}^\pi_{y'}} = \mathbf{C}^\pi_{y'}$ for all $y' \in w_0\mathbf{E}_J$, and $\mathbf{C}^\pi_{y'} = \sum_{w' \in w_0\mathbf{E}_J} P^\pi_{w',y'}\Gamma^\pi_{w'}$, for some elements $P^\pi_{w',y'} \in A_{\geqslant 0}$ with the following properties:*
 (a1) $P^\pi_{w',y'} = 0$ *if* $w' \nleqslant y'$;
 (a2) $P^\pi_{y',y'} = 1$;
 (a3) $P^\pi_{w',y'}$ *has zero constant term if* $y' \neq w'$ *and*

$$\overline{P^\pi_{w',y'}} - P^\pi_{w',y'} = \sum_{\substack{w'<x'\leqslant y' \\ x' \in w_0\mathbf{E}_J}} \overline{R^\pi_{w',x'}} P^\pi_{x',y'} \text{for any } w' < y'.$$

(2) *We have the analogous version for the $\mathscr{H}$-module $\widetilde{M}(\mathbf{E}_J, L)$.*

The proof is very similar to that of [10, Section 2].

Lemma 5.5. *For $y, w \in \mathbf{E}_J$. We have*

(i) $y \leqslant_L w \Longleftrightarrow w' \leqslant_L y'$;
(ii) $R^\pi_{w',y'} = R_{y,w}$; $\widetilde{R}^\pi_{w',y'} = \widetilde{R}_{y,w}$;
(iii) *for any $w', y' \in w_0\mathbf{E}_J$ and $w' < y'$ we have*

$$\overline{P^\pi_{w',y'}} = \sum_{\substack{w'\leqslant x'\leqslant y' \\ x' \in w_0\mathbf{E}_J}} P^\pi_{x',y'} \overline{R_{x,w}},$$

$$\overline{\widetilde{P}^\pi_{w',y'}} = \sum_{\substack{w'\leqslant x'\leqslant y' \\ x' \in w_0\mathbf{E}_J}} \widetilde{P}^\pi_{x',y'} \overline{\widetilde{R}_{x,w}}$$

Proof. (i) is obvious. We prove (ii) by induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$. We have $R_{y,1} = \delta_{y,1}$. Now $R^\pi_{w_0,w_0y} = 0$ unless $w_0 \leqslant_L w_0y$. On the other hand we have $w_0y \leqslant_L w_0$. Hence $R^\pi_{w_0,w_0y} = 0$ unless $w_0y = w_0$, that is $y = 1$ in which case it is 1. The desired equality holds when $\ell(w) = 0$. Assume that $\ell(w) \geqslant 1$. We can find $s \in S$ such that $sw < w$.

In the case (a) $y \in \mathbf{E}^-_{J,s}$. By the induction hypothesis we have

$$R_{y,w} = R_{sy,sw} = R^\pi_{w_0sw,w_0sy} = R^\pi_{s_0w_0w,s_0w_0y} = R^\pi_{w_0w,w_0y}$$

In the case (b) $y \in \mathbf{E}^+_{J,s}$. Using Lemma 3.3, by the induction hypothesis we have

$$\begin{aligned} R_{y,w} &= R_{sy,sw} + (q^{-L(s)} - q^{L(s)})R_{y,sw} \\ &= R^\pi_{w_0sw,w_0sy} + (q^{-L(s_0)} - q^{L(s_0)})R^\pi_{w_0sw,w_0y} \end{aligned}$$

$$\begin{aligned}&= R^{\pi}_{s_0 w', s_0 y'} + (q^{-L(s_0)} - q^{L(s_0)}) R^{\pi}_{s_0 w', y'}\\&= R^{\pi}_{s_0 w', s_0 y'} + (q^{-L(s_0)} - q^{L(s_0)}) R^{\pi}_{w', s_0 y'}\\&= R^{\pi}_{w', y'}\end{aligned}$$

In the case (c) $y \in \mathbf{E}^{0,-}_{J,s}$. Using Lemma 3.5 and Lemma 5.3, by the induction hypothesis we have

$$\begin{aligned}R_{y,w} = -q^{L(s)} R_{y,sw} &= -q^{L(s_0)} R^{\pi}_{w_0(sw), w_0 y} = -q^{L(s_0)} R^{\pi}_{s_0 w', y'}\\&= -q^{L(s_0)}(-q^{-L(s_0)} R^{\pi}_{w', y'}) = R^{\pi}_{w', y'}.\end{aligned}$$

Case (d) $y \in \mathbf{E}^{0,+}_{J,s}$. Using Lemmas 3.5 and 5.3, by the induction hypothesis we have

$$R_{y,w} = q^{-L(s)} R_{y,sw} = q^{-L(s_0)} R^{\pi}_{s_0 w', y'} = R^{\pi}_{w', y'}.$$

(iii) follows (ii). $\square$

Proposition 5.6. *For any $y, w \in \mathbf{E}_J$ we have $Q_{y,w} = \epsilon_y \epsilon_w \widetilde{P}^{\pi}_{w', y'}$. (Analogously $\widetilde{Q}_{y,w} = \epsilon_y \epsilon_w P^{\pi}_{w', y'}$.)*

Proof. We argue by induction on $\ell(w) - \ell(y) \geqslant 0$. If $\ell(w) - \ell(y) = 0$ we have $y = w$ and both sides are 1. Assume that $\ell(w) - \ell(y) > 0$. Subtracting the identity in Lemma 5.5 (iii) from that in Proposition 3.8 and using induction hypothesis, we obtain

$$\overline{\epsilon_y \epsilon_w Q_{y,w}} - \overline{\widetilde{P}^{\pi}_{w', y'}} = \epsilon_y \epsilon_w Q_{y,w} - \widetilde{P}^{\pi}_{w', y'}$$

The right hand side is in $A_{>0}$; since it is fixed by the involution bar, it is 0. $\square$

More precisely, we have the following inversion formulas

Corollary 5.7. *In the above situation,*

$$\sum_{z \in \mathbf{E}_J, x \leqslant z \leqslant w} \varepsilon_w \varepsilon_z P_{x,z} \widetilde{P}^{\pi}_{w', z'} = \delta_{x,w};$$

$$\sum_{z \in \mathbf{E}_J, x \leqslant z \leqslant w} \varepsilon_w \varepsilon_z \widetilde{P}_{x,z} P^{\pi}_{w', z'} = \delta_{x,w}$$

for all $x, w \in \mathbf{E}_J$.

Corollary 5.8. *Assume that W is finite. We set $\mathbf{E}_J := D_J$ and use the above notations. Let D and C^{π} be the bases of $M(D_J, L)$, let $\widetilde{D}$ and $\widetilde{C}^{\pi}$ be the analogous bases for $\widetilde{M}(D_J, L)$. For any $z \in D_J$ we have*

$$T_{w_0}D_z = \epsilon_{w_0z}\theta(\widetilde{C}^{\pi}_{w_0z}) \text{ and } T_{w_0}\widetilde{D}_z = \epsilon_{w_0z}\eta(C^{\pi}_{w_0z}),$$

where η and θ are the maps described in Theorem–Definition 3.1 (replacing E_J by D_J).

Proof. By the proposition we have

$$D_z = \sum_{y\in D_J} Q_{z,y}\Gamma_y = \sum_{y\in D_J} \epsilon_z\epsilon_y \widetilde{P}^{\pi}_{w_0y,w_0z}\Gamma_y.$$

The equality $D_z = \overline{D_z}$ gives $D_z = \epsilon_z \sum_{y\in D_J} \epsilon_y \overline{\widetilde{P}^{\pi}_{w_0y,w_0z}}\,\overline{\Gamma_y}$. Hence

$$\begin{aligned} T_{w_0}D_z &= \epsilon_z \sum_{y\in D_J} \epsilon_y \overline{\widetilde{P}^{\pi}_{w_0y,w_0z}} T_{w_0}\overline{\Gamma_y} \\ &= \epsilon_{w_0z} \sum_{y\in D_J} \epsilon_{w_0y} \overline{\widetilde{P}^{\pi}_{w_0y,w_0z}} \Gamma^{\pi}_{w_0y} \\ &= \epsilon_{w_0z}\theta(\widetilde{C}^{\pi}_{w_0z}) \qquad \square \end{aligned}$$

6. Some remarks

6.1. An example: the dual Solomon modules

In this subsection, let (W, S) be a finite Coxeter group system. Assume that $L(s) > 0$ for all $s \in S$. In [14] we introduced the A-free $\mathscr{H}$-module $\mathscr{H}C_{w_J}C'_{w_{\hat{J}}}$, which is called the **Solomon module** (see [13]) with respect to J and L, and where

$$\begin{aligned} C_{w_J} &= \epsilon_{w_J} \sum_{w\in W_J} \epsilon_w q^{L(ww_J)}T_w = \epsilon_{w_J} q^{L(w_J)} \sum_{w\in W_J} \epsilon_w q^{-L(w)}T_w; \\ C'_{w_{\hat{J}}} &= \sum_{w\in W_{\hat{J}}} q^{-L(ww_{\hat{J}})}T_w = q^{-L(w_{\hat{J}})} \sum_{w\in W_{\hat{J}}} q^{L(w)}T_w, \end{aligned}$$

that is, $C'_{w_{\hat{J}}}$ is the C'-basis element corresponding to $w_{\hat{J}}$, the maximal length element of $W_{\hat{J}}$, or c-basis element corresponding to $w_{\hat{J}}$ (see [11, Corollary 12.2]). C_{w_J} is the C-basis element corresponding to w_J.

In [14] we showed that $\mathscr{H}C_{w_J}C'_{w_{\hat{J}}}$ has basis $\{\, T_xC_{w_J}C'_{w_{\hat{J}}} \mid x \in F_J \,\}$, where $F_J = E_Jw_J$ and $E_J = \overline{D_J} \cap D_{\hat{J}}$. This basis admits the multiplication rules listed in the Definition 2.4, and F_J is a W-graph ideal with respect to J and weight function L.

Similarly, the $\mathscr{H}$-module $\mathscr{H}C'_{w_J}C_{w_{\hat{J}}}$ has basis $\{\, T_xC'_{w_J}C_{w_{\hat{J}}} \mid x \in F_J \,\}$. We can easily prove that this basis admits the multiplication rules listed in the Theorem–Definition 3.1. We call this the **dual module** of $\mathscr{H}C_{w_J}C'_{w_{\hat{J}}}$.

6.2. The Kazhdan–Lusztig construction

Assume that $J = \emptyset$. Then $D_J = W$ and the sets $WD_J(w)$ and $WA_J(w)$ are empty for all $w \in W$.

If $L(s) > 0$ (for all $s \in S$), both modules $M(\mathbf{E}_J, L)$ and $\widetilde{M}(\mathbf{E}_J, L)$ have A-basis $(X_w \mid w \in \mathbf{E}_J)$ such that,

$$T_s X_w = \begin{cases} X_{sw} & \text{if } \ell(sw) > \ell(w) \\ X_{sw} + (q^{L(s)} - q^{-L(s)})X_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where the elements X_w stand for Γ_w or $\widetilde{\Gamma}_w$. If we let $X_w = T_w$ for all $w \in W$, then both modules are the regular module $\mathscr{H}$ with weight function L. Thus we can recover some of Lusztig's results (for example, see [11, Chs. 5–6, Chs. 10–11]) for the regular case.

6.3. Deodhar's construction: the parabolic case

Let J be an arbitrary subset of S and $L(s) = 1$ for all $s \in S$, we can now turning to Deodhar's construction.

Set $\mathbf{E}_J := D_J$, then D_J is a W-graph ideal with respect to J, and also it is a W-graph ideal with respect with $\emptyset$.

In the latter case we have $D_\emptyset = W$, if $w \in \mathbf{E}_J$ then

$$\begin{aligned} SA(w) &= \{s \in S | sw > w \text{ and } sw \in D_J\}, \\ SD(w) &= \{s \in S | sw < w\}, \\ WD_\emptyset(w) &= \{s \in S | sw \notin D_\emptyset\} = \emptyset, \\ WA_\emptyset(w) &= \{s \in S | sw \in D_\emptyset \setminus D_J\} = \{s \in S | sw = wt \text{ for some } t \in J\}. \end{aligned}$$

Let $\mathscr{H}_J$ be the Hecke algebra associated with the Coxeter system (W_J, J). Let $M_\psi = \mathscr{H} \otimes_{\mathscr{H}_J} A_\psi$, where A_ψ is A made into an $\mathscr{H}_J$-module via the homomorphism $\psi : \mathscr{H}_J \to A$ that satisfies $\psi(T_v) = q^{\ell(v)}$ for all $v \in W_J$, it is an A-free with basis $B = \{b_w \mid w \in D_J\}$ defined by $b_w = T_w \otimes 1$. This corresponds to M^J in [3] in the case $u = q$ (we note that this is denoted by $\widetilde{M}^J$ in [4]).

Let $M_\phi = \mathscr{H} \otimes_{\mathscr{H}_J} A_\phi$, where A_ϕ is A made into an $\mathscr{H}_J$-module via the homomorphism $\phi : \mathscr{H}_J \to A$ that satisfies $\phi(T_v) = (-q)^{-\ell(v)}$ for all $v \in W_J$, again it is an A-free with basis $B = \{b_w \mid w \in D_J\}$ defined by $b_w = T_w \otimes 1$. This corresponds to M^J in [3] in the case $u = -1$ (this is denoted by M^J in [4]).

Our module $M(\mathbf{E}_J, L)$ is now essentially reduced to be the module M_ψ, while $\widetilde{M}(\mathbf{E}_J, L)$ is reduced to be the module M_ϕ, the only differences being due to our non-traditional definition of $\mathscr{H}$.

In the case D_J is a W-graph ideal with respect to J, the discussion is similar with the above. For more details see [9, Section 8].

Acknowledgments

The author wishes to thank the referee for the comments.

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