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HOMOMORPHISMS FROM AN ARBITRARY SPECHT MODULE TO ONE CORRESPONDING TO A HOOK

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ABSTRACT. We use the presentation of the Specht modules given by Kleshchev-Mathas-Ram to derive results about Specht modules. In particular, we determine all homomorphisms from an arbitrary Specht module to a fixed Specht module corresponding to any hook partition. Along the way, we give a complete description of the action of the standard KLR generators on the hook Specht module. This work generalizes a result of James.

1. INTRODUCTION

Let H_d be an Iwahori-Hecke algebra for the symmetric group $\mathfrak{S}_d$ with deformation parameter q, over a field F. Define e to be the smallest integer so that $1 + q + \cdots + q^{e-1} = 0$, setting $e = 0$ if no such value exists. To every partition λ of d, there is a corresponding *Specht module* S^λ over H_d. Brundan and Kleshchev show in [1] that H_d is isomorphic to a certain algebra $R_d^{\Lambda_0}$ known as a cyclotomic KLR (Khovanov-Lauda-Rouquier) algebra of type $A_{e-1}^{(1)}$ when $e \neq 0$, and type A_∞ when $e = 0$. The Specht modules are described in [2] as modules over the cyclotomic KLR algebras. This result is extended by Kleshchev, Mathas, and Ram in [12], in which Specht modules for the (full) KLR algebras R_d are explicitly defined in terms of generators and relations. The purpose of this paper is to use this presentation to completely determine $\mathrm{Hom}_{R_d}(S^\mu, S^\lambda)$ when μ is an arbitrary partition and λ is a hook. Of course, when $e = 0$ there are no nontrivial homomorphisms. Furthermore, our methods do not apply when $e = 2$, so we make the assumption starting in section 4 that $e \geq 3$.

To state the main theorem, we need some notation. The element $z^\mu \in S^\mu$ is the standard cyclic generator of weight $\boldsymbol{i}^\mu$, see Definition 3.2. When the $\boldsymbol{i}^\mu$-weight space of S^λ is nonempty, it will turn out that its top degree component is one-dimensional. In this case we take $[\sigma_\mu] \in S^\lambda$ to be any non-zero vector of that component. For any weakly decreasing sequence of positive integers $\boldsymbol{a} = (a_1, \ldots, a_N)$ we define its *Garnir content* $\mathrm{Gc}(\boldsymbol{a}) \in F$ by

$$\mathrm{Gc}(\boldsymbol{a}) = \gcd\left\{ \binom{a_i}{k} \mid 1 \leq k \leq a_{i+1} - 1, \ 1 \leq i \leq N - 1 \right\}$$

with the convention that $\gcd(\emptyset) = 0$.

Main Theorem. *Let μ be an arbitrary partition and $\lambda = (d - k, 1^k)$ for $k \geq 0$. Then $\mathrm{Hom}_{R_d}(S^\mu, S^\lambda)$ is at most one-dimensional, spanned by a map satisfying $z^\mu \mapsto [\sigma_\mu]$, and $\dim \mathrm{Hom}_{R_d}(S^\mu, S^\lambda) = 1$ if and only if one of the following conditions holds:*

(i) there exist $n \in \{1, \ldots, k+1\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$, and $0 \leq m < e$ such that $\mathrm{Gc}(\boldsymbol{a}) = 0$ and

$$\mu = (a_1 e, \ldots, a_{n-1} e, a_n e - m, 1^{k-n+1}),$$

(ii) e divides d, there exist $n \in \{1, \ldots, k\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$, and $0 \leq m < e$ such that $\mathrm{Gc}(\boldsymbol{a}) = 0$ and

$$\mu = (a_1 e, \ldots, a_{n-1} e, a_n e - m, 1^{k-n+2}), \text{ or}$$

(iii) there exist $n > k + 1$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$ and $0 \leq m < e$ such that $\mathrm{Gc}(\boldsymbol{a}) = 0$ and

$$\mu = (a_1 e, \ldots, a_k e, a_{k+1} e - 1, \ldots, a_{n-1} e - 1, a_n e - 1 - m).$$

This theorem generalizes James [**7**, Theorem 24.4] for the symmetric groups, and Lyle [**13**, Theorem 3.3] for Hecke algebras, corresponding to $k = 0$. Our Specht modules are the dual of the Specht modules defined in [**7**]. We note here that the main theorem also allows us to determine all homomorphisms between Specht modules when the source is a hook, as follows; see [**12**] for details. For any partition ν we define ν' to be the conjugate partition and S_ν to be the dual of S^ν. There is an automorphism $\mathtt{sgn}$ of R_d such that $S_\nu \cong (S^{\nu'})^{\mathtt{sgn}}$, and so

$$\mathrm{Hom}_{R_d}(S^\mu, S^\lambda) \cong \mathrm{Hom}_{R_d}(S_\lambda, S_\mu) \cong \mathrm{Hom}_{R_d}((S^{\lambda'})^{\mathtt{sgn}}, (S^{\mu'})^{\mathtt{sgn}})$$
$$\cong \mathrm{Hom}_{R_d}(S^{\lambda'}, S^{\mu'}).$$

Clearly, λ is a hook if and only if λ' is.

Section 2 introduces the notation to be used throughout the paper. In addition, we collect a few elementary facts which will be necessary in later sections. Section 3 reviews the definitions of the affine and cyclotomic KLR algebras, and their universal Specht modules as introduced in [**12**]. Section 4 provides a study of the structure of the Specht modules S^λ, where λ is a hook. Specifically, there is a basis for S^λ given by standard tableaux on λ, and in Theorem 4.3 we determine how the generators of the KLR algebra act on this basis. In section 5, we look at homomorphisms from an arbitrary Specht module S^μ to S^λ, where λ is a hook. In section 6, we consider some examples.

2. Preliminaries

2.1. **Lie theoretic notation.** We collect some notation that will be used in the sequel; the interested reader is refered to [**12**] for more details. Let $e \in \{0, 2, 3, 4, \ldots\}$ and $I := \mathbb{Z}/e\mathbb{Z}$. Let Γ be the quiver with vertex set I, with a directed edge from i to j if $j = i - 1$. Thus Γ is a quiver of type $A^{(1)}_{e-1}$.

We denote the simple roots by $\{\alpha_i \mid i \in I\}$, and define $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ to be the positive part of the root lattice. For $\alpha \in Q_+$ let $\mathrm{ht}(\alpha)$ be the *height of* α. That is, $\mathrm{ht}(\alpha)$ is the sum of the coefficients when α is expanded in terms of the α_i's.

Let $\mathfrak{S}_d$ be the symmetric group on d letters and let $s_r = (r, r+1)$, for $1 \leq r < d$, be the simple transpositions of $\mathfrak{S}_d$. Then $\mathfrak{S}_d$ acts on the left on the set I^d by place permutations. If $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d$ then its *weight* is $|\boldsymbol{i}| := \alpha_{i_1} + \cdots + \alpha_{i_d} \in Q_+$. The $\mathfrak{S}_d$-orbits on I^d are the sets

$$\langle I \rangle_\alpha := \{\boldsymbol{i} \in I^d \mid \alpha = |\boldsymbol{i}|\}$$

parametrized by all $\alpha \in Q_+$ of height d.

2.2. **Partitions.** Let $\mathscr{P}_d$ be the set of all partitions of d and put $\mathscr{P} := \bigsqcup_{d \geq 0} \mathscr{P}_d$. The *Young diagram* of the partition $\mu \in \mathscr{P}$ is

$$\{(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \leq b \leq \mu_a\}.$$

The elements of this set are the *nodes of* μ. More generally, a *node* is any element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. We identify a partition with its Young diagram.

To each node $A = (a, b)$ we associate its *residue*, which is the following element of $I = \mathbb{Z}/e\mathbb{Z}$:

$$\mathrm{res}\, A = (b - a) \pmod{e}. \tag{2.1}$$

An *i-node* is a node of residue i. Define the *residue content of* μ to be

$$\mathrm{cont}(\mu) := \sum_{A \in \mu} \alpha_{\mathrm{res}\, A} \in Q_+. \tag{2.2}$$

Denote

$$\mathscr{P}_\alpha := \{\mu \in \mathscr{P} \mid \mathrm{cont}(\mu) = \alpha\} \qquad (\alpha \in Q_+).$$

A node $A \in \mu$ is a *removable node (of μ)* if $\mu \setminus \{A\}$ is (the diagram of) a partition. A node $B \notin \mu$ is an *addable node (for μ)* if $\mu \cup \{B\}$ is a partition. We use the notation

$$\mu_A := \mu \setminus \{A\}, \qquad \mu^B := \mu \cup \{B\}.$$

2.3. **Tableaux.** Let $\mu \in \mathscr{P}_d$. A *μ-tableau* $\mathtt{T}$ is obtained from the diagram of μ by inserting the integers $1, \ldots, d$ into the nodes, allowing no repeats. If the node $A = (a, b) \in \mu$ is occupied by the integer r in $\mathtt{T}$ then we write $r = \mathtt{T}(a, b)$ and set $\mathrm{res}_\mathtt{T}(r) = \mathrm{res}\, A$. The *residue sequence* of $\mathtt{T}$ is

$$\boldsymbol{i}(\mathtt{T}) = (i_1, \ldots, i_d) \in I^d, \tag{2.3}$$

where $i_r = \mathrm{res}_\mathtt{T}(r)$ is the residue of the node occupied by r in $\mathtt{T}$ ($1 \leq r \leq d$).

A *μ-tableau* $\mathtt{T}$ is *row-strict* (resp. *column-strict*) if its entries increase from left to right (resp. from top to bottom) along the rows (resp. columns) of $\mathtt{T}$. A μ-tableau $\mathtt{T}$ is *standard* if it is row- and column-strict. Let $\mathrm{St}(\mu)$ be the set of standard μ-tableaux.

Let $\mu \in \mathscr{P}$, $i \in I$, and A be a removable i-node of μ. We set

$$d_A(\mu) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \mu \\ \text{strictly below } A\end{array}\right\} - \#\left\{\begin{array}{c}\text{removable } i\text{-nodes of } \mu \\ \text{strictly below } A\end{array}\right\}. \tag{2.4}$$

Given $\mu \in \mathscr{P}_d$ and $\mathtt{T} \in \mathrm{St}(\mu)$, the *degree* of $\mathtt{T}$ is defined in [**2**, section 3.5] inductively as follows. If $d = 0$, then $\mathtt{T}$ is the empty tableau $\emptyset$, and we set $\deg(\mathtt{T}) := 0$. Otherwise, let A be the node occupied by d in $\mathtt{T}$. Let $\mathtt{T}_{<d} \in \mathrm{St}(\mu_A)$ be the tableau obtained by removing this node and set

$$\deg(\mathtt{T}) := d_A(\mu) + \deg(\mathtt{T}_{<d}). \tag{2.5}$$

The group $\mathfrak{S}_d$ acts on the set of μ-tableaux from the left by acting on the entries of the tableaux. Let $\mathtt{T}^\mu$ be the μ-tableau in which the numbers $1, 2, \ldots, d$ appear in order from left to right along the successive rows, working from top row to bottom row.

Set

$$\boldsymbol{i}^\mu := \boldsymbol{i}(\mathtt{T}^\mu). \tag{2.6}$$

For each μ-tableau $\mathtt{T}$ define permutations $w^{\mathtt{T}} \in \mathfrak{S}_d$ by the equation

$$w^{\mathtt{T}} \mathtt{T}^\mu = \mathtt{T}. \tag{2.7}$$

2.4. **Binomial coefficients.** In this section, we state some elementary theorems about binomial coefficients that will be useful later. Let p be a fixed prime.

Definition 2.8. For $n \in \mathbb{Z}_{>0}$ we define
 (i) $\nu_p(n) = \max\{i \mid p^i \text{ divides } n\}$
 (ii) $\ell_p(n) = \min\{i \mid p^i > n\}$.
We also set $\ell_p(0) = -\infty$.

The following can be easily derived from [**7**, Lemma 22.4].

Lemma 2.9. *For any* $a, b \in \mathbb{Z}_{>0}$, *one has*

$$p \left| \binom{a}{k} \right. \text{ for } k = 1, \ldots, b \qquad \Leftrightarrow \qquad \nu_p(a) \geq \ell_p(b).$$

Recall the definition of $\mathrm{Gc}(a_1, \ldots, a_N)$ from the introduction. Then

Corollary 2.10. *We have* $p \mid \mathrm{Gc}(a_1, \ldots, a_N)$ *if and only if*

$$\nu_p(a_i) \geq \ell_p(a_{i+1} - 1) \text{ for } 1 \leq i \leq N - 1.$$

2.5. **Shuffles.** In this section, we fix $e \in \{0, 3, 4, 5, \ldots\}$ and $I := \mathbb{Z}/e\mathbb{Z}$. Given $a, b \in \mathbb{Z}_{\geq 0}$ we define

$$\mathrm{Sh}(a, b) := \{\sigma \in \mathfrak{S}_{a+b} \mid \sigma(1) < \cdots < \sigma(a) \text{ and } \sigma(a+1) < \cdots < \sigma(a+b)\}.$$

For $\boldsymbol{i} \in I^a, \boldsymbol{i}' \in I^b$ write $\boldsymbol{i}\boldsymbol{i}' \in I^{a+b}$ for their concatenation, and define

$$\mathrm{Shuf}(\boldsymbol{i}, \boldsymbol{i}') := \{\sigma \cdot \boldsymbol{i}\boldsymbol{i}' \mid \sigma \in \mathrm{Sh}(a, b)\},$$
$$\mathrm{Sh}(\boldsymbol{j}; \boldsymbol{i}, \boldsymbol{i}') := \{\sigma \in \mathrm{Sh}(a, b) \mid \sigma \cdot \boldsymbol{i}\boldsymbol{i}' = \boldsymbol{j}\}, \text{ and}$$
$$H(\boldsymbol{i}) := \langle s_m \mid s_m \boldsymbol{i} = \boldsymbol{i} \rangle < \mathfrak{S}_a.$$

For $i \in I$, define the elements of I^a

$$S^+(i, a) := (i, i+1, \ldots, i+a-1), \qquad (2.11)$$

$$S^-(i, a) := (i, i-1, \ldots, i-a+1). \qquad (2.12)$$

Fix $j, k \in I$, $a, b \in \mathbb{Z}_{\geq 0}$, and write $S^+ := S^+(j, a)$ and $S^- := S^-(k, b)$. If $i \in \mathrm{Shuf}(S^+, S^-)$ then i cannot contain three or more equal adjacent indices. In this case $H(i)$ is an elementary 2-subgroup of $\mathfrak{S}_{a+b}$.

Proposition 2.13. *For every $i \in \mathrm{Shuf}(S^+, S^-)$, there is a unique element $\sigma_i \in \mathrm{Sh}(i; S^+, S^-)$ of minimal length. Furthermore, $\mathrm{Sh}(i; S^+, S^-) = \{h\sigma_i \mid h \in H(i)\}$.*

Proof. Write $S^+ = (j_1, \ldots, j_a)$ and $S^- = (k_1, \ldots, k_b)$, and for every integer $d \geq 1$ introduce the following notation. For $\sigma \in \mathfrak{S}_d$, define $E\sigma \in \mathfrak{S}_{d-1}$ by

$$E\sigma(r) = \begin{cases} \sigma(r), & \text{for } r = 1, \ldots, \sigma^{-1}(d) - 1; \\ \sigma(r+1), & \text{for } r = \sigma^{-1}(d), \ldots, d-1. \end{cases}$$

Similarly, for $i = (i_1, \ldots, i_d) \in I^d$, define $Ei = (i_1, \ldots, i_{d-1})$.

The proposition will be proved by induction on $a + b$. If $a + b = 1$ or 2, or if $a = 0$ or $b = 0$ the claim follows immediately. Suppose $a + b \geq 3$, with $a, b \geq 1$, and assume the induction hypothesis. Fix $i \in \mathrm{Shuf}(S^+, S^-)$. We distinguish several cases based on the values of i_{a+b}, i_{a+b-1}, j_a, and k_b.

Case 1: $i_{a+b} = j_a$ and either $j_a \neq k_b$ or $i_{a+b-1} = i_{a+b} - 1$. Given $\sigma \in \mathrm{Sh}(i; S^+, S^-)$, we must have $\sigma(a) = a + b$. Then the map

$$\mathrm{Sh}(i; S^+, S^-) \to \mathrm{Sh}(Ei; ES^+, S^-), \ \sigma \mapsto E\sigma$$

is a bijection, and moreover $\ell(E\sigma) = \ell(\sigma) - b$ and $H(i) \cong H(Ei)$. We define σ_i by $E\sigma_i = \sigma_{Ei}$, which completes the induction in this case.

Case 2: $i_{a+b} = k_b$ and either $j_a \neq k_b$ or $i_{a+b-1} = i_{a+b} + 1$. Here the map

$$\mathrm{Sh}(i; S^+, S^-) \to \mathrm{Sh}(Ei; S^+, ES^-), \ \sigma \to E\sigma$$

gives a bijection as above.

Case 3: $i_{a+b} = i_{a+b-1} = j_a = k_b$. We must either have $\sigma(a) = a + b$ and $\sigma(a + b) = a + b - 1$ or $\sigma(a) = a + b - 1$ and $\sigma(a + b) = a + b$. Thus, we get a well-defined map

$$\mathrm{Sh}(i; S^+, S^-) \to \mathrm{Sh}(EEi; ES^+, ES^-), \ \sigma \mapsto EE\sigma$$

which is two-to-one, with $s_{a+b-1}\sigma$ and σ having the same image. This proves the claim, since $H(i) \cong H(EEi) \times \langle s_{a+b-1} \rangle$. $\qquad\square$

3. KLR algebras and universal Specht modules

3.1. **KLR algebras.** Let $\mathcal{O}$ be a commutative ring with identity and $\alpha \in Q_+$. There is a unital $\mathcal{O}$-algebra $R_\alpha = R_\alpha(\mathcal{O})$ called a *Khovanov-Lauda-Rouquier (KLR) algebra*, first defined in [**9**, **10**, **16**]. We follow the notations and conventions of [**12**, Section 3.1], so R_α is generated by elements

$$\{e(i) \mid i \in \langle I \rangle_\alpha\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}, \qquad (3.1)$$

subject to some explicit relations.

The algebras R_α have $\mathbb{Z}$-gradings determined by setting $e(\boldsymbol{i})$ to be of degree 0, y_r of degree 2, and $\psi_r e(\boldsymbol{i})$ of degree $-a_{i_r,i_{r+1}}$ for all r and $\boldsymbol{i} \in \langle I \rangle_\alpha$.

For a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, with finite dimensional graded components its *graded dimension* is $\dim_q V := \sum_{n\in\mathbb{Z}} (\dim V_n) q^n \in \mathbb{Z}[[q, q^{-1}]]$.

For $x \in R_\alpha$ such that $x = e(\boldsymbol{i})x$, define the *weight* of x to be $\operatorname{wt}(x) = \boldsymbol{i}$.

3.2. Universal Specht modules.

Fix $\mu \in \mathscr{P}_\alpha$. In this section we define the Specht module S^μ over R_α, mostly following [12]. A node $A = (x, y) \in \mu$ is called a *Garnir node* of μ if also $(x + 1, y) \in \mu$. We define the *Garnir belt* of A to be the set $\mathbf{B}^A$ of nodes of μ containing A and all nodes directly to the right of A, along with the node directly below A and all nodes directly to the left of this node. Explicitly,

$$\mathbf{B}^A = \{(x, z) \in \mu \mid y \le z \le \mu_x\} \cup \{(x+1, z) \in \mu \mid 1 \le z \le y\}.$$

We define an *A-brick* to be a set of e successive nodes in the same row

$$\{(w, z), (w, z+1), \ldots, (w, z+e-1)\} \subseteq \mathbf{B}^A$$

such that $\operatorname{res}(w, z) = \operatorname{res}(A)$. Let $k \ge 0$ be the number of bricks in $\mathbf{B}^A$. We label the bricks

$$B_1^A, B_2^A, \ldots, B_k^A$$

going from left to right along row x, and then from left to right along row $x + 1$. Define C^A to be the set of nodes in row x of $\mathbf{B}^A$ not contained in any A-brick, and D^A to be the set of nodes in row $x + 1$ of $\mathbf{B}^A$ not contained in any A-brick. Define f to be the number of A-bricks in row x of $\mathbf{B}^A$.

Let $u, u+1, \ldots, v$ be the values in $\mathbf{B}^A$ in the standard tableau T^μ. We obtain a new tableau T^A by placing the numbers $u, u+1 \ldots, v$ from left to right in the following order: D^A, B_1^A, B_2^A, $\ldots$, B_k^A, C^A. The rest of the numbers are placed in the same positions as in T^μ. Define $\boldsymbol{i}^A := \boldsymbol{i}(\mathrm{T}^A)$.

Assume that $k > 0$, and let $n = \mathrm{T}^A(A)$. Define

$$w_r^A = \prod_{z=n+re-e}^{n+re-1} (z, z+e) \in \mathfrak{S}_d \qquad (1 \le r < k).$$

Informally, w_r^A swaps the bricks B_r^A and B_{r+1}^A. The elements $w_1^A, w_2^A, \ldots, w_{k-1}^A$ are Coxeter generators of the group

$$\mathfrak{S}^A := \langle w_1^A, w_2^A, \ldots, w_{k-1}^A \rangle \cong \mathfrak{S}_k.$$

By convention, if $k = 0$ we define $\mathfrak{S}^A$ to be the trivial group.

Recall that f is the number of A-bricks in row x of $\mathbf{B}^A$. Let $\mathscr{D}^A$ be the set of minimal length left coset representatives of $\mathfrak{S}_f \times \mathfrak{S}_{k-f}$ in $\mathfrak{S}^A \cong \mathfrak{S}_k$. Define the elements of R_α

$$\sigma_r^A := \psi_{w_r^A} e(\boldsymbol{i}^A) \quad \text{and} \quad \tau_r^A := (\sigma_r^A + 1) e(\boldsymbol{i}^A).$$

Let $u \in \mathscr{D}^A$ with reduced expression $u = w_{r_1}^A \ldots w_{r_a}^A$. Since every element of $\mathscr{D}^A$ is fully commutative, the element $\tau_u^A := \tau_{r_1}^A \ldots \tau_{r_a}^A$ does not depend upon this reduced expression. We now define the *Garnir element* to be

$$g^A := \sum_{u \in \mathscr{D}^A} \tau_u^A \psi^{T^A} \in R_\alpha.$$

Definition 3.2. Let $\alpha \in Q_+$, $d = \mathrm{ht}(\alpha)$, and $\mu \in \mathscr{P}_\alpha$. Define the following left ideals of R_α.

   (i) $J_1^\mu = \langle e(\boldsymbol{j}) - \delta_{\boldsymbol{j}, \boldsymbol{i}^\mu} \mid \boldsymbol{j} \in \langle I \rangle_\alpha \rangle$;
   (ii) $J_2^\mu = \langle y_r \mid r = 1, \ldots, d \rangle$;
   (iii) $J_3^\mu = \langle \psi_r \mid r$ and $r+1$ appear in the same row of $T^\mu \rangle$;
   (iv) $J_4^\mu = \langle g^A \mid$ Garnir nodes $A \in \mu \rangle$.

Let $J^\mu = J_1^\mu + J_2^\mu + J_3^\mu + J_4^\mu$ and define the *universal graded Specht module* $S^\mu := R_\alpha / J^\mu \langle \deg(T^\mu) \rangle$. Define $z^\mu := 1 + J^\mu \in S^\mu$.

4. Specht modules corresponding to hooks

For the rest of the paper, we assume that $e \geq 3$. In this section we fix two integers $d \geq k \geq 0$. We set $\lambda := (d - k, 1^k)$ and $\alpha := \mathrm{cont}(\lambda)$. We write Sh^λ for the image of $\mathrm{Sh}(d - k - 1, k)$ under the embedding $\mathfrak{S}_{d-1} \to \mathfrak{S}_d$ determined by $s_i \mapsto s_{i+1}$. For $\sigma \in \mathrm{Sh}^\lambda$, we define $[\sigma] := \psi_\sigma z^\lambda \in S^\lambda$ of weight $\boldsymbol{i}_\sigma := \sigma \boldsymbol{i}^\lambda$. Note that $\mathrm{Sh}^\lambda = \{\sigma \in \mathfrak{S}_d \mid \sigma T^\lambda$ is standard$\}$. Therefore, by [**12**, Corollary 6.24], the set $\{[\sigma] \mid \sigma \in \mathrm{Sh}^\lambda\}$ is a basis of S^λ.

Remark 4.1. Recall that in order to define the element $\psi_\sigma \in R_\alpha$ we needed to fix a reduced decomposition for σ. However, every $\sigma \in \mathrm{Sh}^\lambda$ is fully commutative, and so the element ψ_σ is independent of this choice. Thus $[\sigma] \in S^\lambda$ only depends on σ.

Definition 4.2. Given $\sigma \in \mathrm{Sh}^\lambda$, the nodes in positions $(1, y)$ of σT^λ for $y = 2, 3, \ldots, d-k$ are called *arm nodes*, and we define $\mathrm{Arm}(\sigma) = \{\sigma(2), \ldots, \sigma(d-k)\}$. Similarly, the nodes in positions $(x, 1)$ of σT^λ for $x = 2, 3, \ldots, d$ are called *leg nodes*, and $\mathrm{Leg}(\sigma) = \{\sigma(d - k + 1), \ldots, \sigma(d)\}$.

The following theorem tells us how the KLR generators act on the standard basis of S^λ.

Theorem 4.3. *Let $\sigma \in \mathrm{Sh}^\lambda$ and write $\boldsymbol{i}_\sigma = (i_1, \ldots, i_d)$. Then*
   (i) For $\boldsymbol{j} \in \langle I \rangle_\alpha$, $e(\boldsymbol{j})[\sigma] = \delta_{\boldsymbol{i}_\sigma, \boldsymbol{j}}[\sigma]$.
   (ii) For $1 \leq r \leq d$,

$$y_r[\sigma] = \begin{cases} -[s_r \sigma], & \text{if } i_r = i_{r+1}, \ r \in \mathrm{Leg}(\sigma), \text{ and } r + 1 \in \mathrm{Arm}(\sigma) \\ [s_{r-1}\sigma], & \text{if } i_{r-1} = i_r, \ r - 1 \in \mathrm{Leg}(\sigma), \text{ and } r \in \mathrm{Arm}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

(iii) We have $\psi_1[\sigma] = 0$, and for $2 \le r \le d-1$,

$$\psi_r[\sigma] = \begin{cases} [s_r\sigma], & \text{if } r+1 \in \mathrm{Leg}(\sigma) \text{ and } r \in \mathrm{Arm}(\sigma), \text{ or if } i_r \ne i_{r+1} \\ [s_{r+1}s_r\sigma], & \text{if } r \in \mathrm{Leg}(\sigma), r+1, r+2 \in \mathrm{Arm}(\sigma), \text{ and } i_r \to i_{r+1} \\ -[s_{r-1}s_r\sigma], & \text{if } r-1, r \in \mathrm{Leg}(\sigma), r+1 \in \mathrm{Arm}(\sigma), \text{ and } i_r \leftarrow i_{r+1} \\ [s_r s_{r-1}\sigma], & \text{if } r-1 \in \mathrm{Leg}(\sigma), r, r+1 \in \mathrm{Arm}(\sigma), \text{ and } i_r = i_{r-1} \\ -[s_r s_{r+1}\sigma], & \text{if } r, r+1 \in \mathrm{Leg}(\sigma), r+2 \in \mathrm{Arm}(\sigma), \text{ and } i_{r+1} = i_{r+2} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) Immediate from the definitions.

(ii) Recall the definition of wt from Section 3.1. We define $\mathrm{Sh}^\lambda(\boldsymbol{i}) = \{\tau \in \mathrm{Sh}^\lambda \mid \mathrm{wt}([\tau]) = \boldsymbol{i}\}$. By Proposition 2.13 there is a unique element $\sigma_{\boldsymbol{i}_\sigma} \in \mathrm{Sh}^\lambda(\boldsymbol{i}_\sigma)$ of minimal length, and $\sigma = h\sigma_{\boldsymbol{i}_\sigma}$ for some $h \in H(\boldsymbol{i}_\sigma)$. Choosing any reduced decomposition $h = s_{r_1} \ldots s_{r_a}$, it follows that $[\sigma] = \psi_{r_1} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}]$. Moreover $\deg([\sigma_{\boldsymbol{i}_\sigma}]) - \deg([\sigma]) = 2a$, and so $[\sigma_{\boldsymbol{i}_\sigma}]$ is the unique vector of weight $\boldsymbol{i}_\sigma$ with largest degree. In particular, $y_s[\sigma_{\boldsymbol{i}_\sigma}] = 0$ for $1 \le s \le d$. Since $H(\boldsymbol{i}_\sigma)$ is commutative, at most one of $r-1$ and r is an element of $\{r_1, \ldots, r_a\}$.

Suppose that $r \in \{r_1, \ldots, r_a\}$. This is equivalent to $i_r = i_{r+1}$, $r \in \mathrm{Leg}(\sigma)$ and $r+1 \in \mathrm{Arm}(\sigma)$. Since the ψ_{r_i} commute with each other, we may assume that $r = r_1$. We compute

$$\begin{aligned} y_r[\sigma] &= y_r \psi_r \psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}] \\ &= (\psi_r y_{r+1} - 1)\psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}] \\ &= \psi_r \psi_{r_2} \ldots \psi_{r_a} y_{r+1}[\sigma_{\boldsymbol{i}_\sigma}] - \psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}] \\ &= -\psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}] = -[s_r\sigma]. \end{aligned}$$

Similarly, $r-1 \in \{r_1, \ldots, r_a\}$ is equivalent to $i_{r-1} = i_r$, $r-1 \in \mathrm{Leg}(\sigma)$, and $r \in \mathrm{Arm}(\sigma)$. An argument similar to the above shows that $y_r[\sigma] = [s_{r-1}\sigma]$. If neither r nor $r-1$ is among $\{r_1, \ldots, r_a\}$, then y_r commutes with each ψ_{s_r}, and thus $y_r \psi_{s_{r_1}} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}_\sigma}] = \psi_{r_1} \ldots \psi_{r_a} y_r[\sigma_{\boldsymbol{i}_\sigma}] = 0$.

(iii) Observe that $\mathrm{wt}([\sigma])$ begins with either $(0, 1, \ldots)$ or $(0, e-1, \ldots)$, and thus $\mathrm{wt}(\psi_1[\sigma])$ does not occur in R_α. Thus $\psi_1[\sigma] = 0$. We break down the rest of the proof into many cases.

Case 1: Assume $r \in \mathrm{Arm}(\sigma)$ and $r+1 \in \mathrm{Leg}(\sigma)$. Clearly $\psi_r[\sigma] = [s_r\sigma]$.

Case 2: Assume $r \in \mathrm{Leg}(\sigma)$, $r+1 \in \mathrm{Arm}(\sigma)$. Then $[\sigma] = \psi_r[s_r\sigma]$, and so $\psi_r[\sigma] = \psi_r^2[s_r\sigma]$. If $i_r \ne i_{r+1}$ then $\psi_r^2[s_r\sigma] = [s_r\sigma]$, and if $i_r = i_{r+1}$ then $\psi_r^2[s_r\sigma] = 0$.

If $i_r \to i_{r+1}$, then $\psi_r^2[s_r\sigma] = (y_r - y_{r+1})[s_r\sigma]$. Since $r \in \mathrm{Arm}(s_r\sigma)$, part (ii) of this theorem shows that if $y_r[s_r\sigma] \ne 0$ then $r-1 \in \mathrm{Leg}(s_r\sigma)$ and $i_{r-1} = i_{r+1}$. This is seen to be impossible by considering weights, and so $y_r[s_r\sigma] = 0$. Similarly, $r+1 \in \mathrm{Leg}(s_r\sigma)$, so part (ii) says that $y_{r+1}[s_r\sigma] = 0$ unless $r+2 \in \mathrm{Arm}(s_r\sigma)$ and $i_r = i_{r+2}$. But $r+2 \in \mathrm{Arm}(s_r\sigma)$ if and only if $r+2 \in \mathrm{Arm}(\sigma)$, in which case $i_{r+2} = i_{r+1} + 1 = i_r$. We then conclude that

$y_{r+1}[s_r\sigma] = -[s_{r+1}s_r\sigma]$. To summarize, if $i_r \to i_{r+1}$, then

$$\psi_r[\sigma] = \begin{cases} [s_{r+1}s_r\sigma], & \text{if } r+2 \in \text{Arm}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $i_r \leftarrow i_{r+1}$. An argument similar to the one above shows that

$$\psi_r[\sigma] = \begin{cases} -[s_{r-1}s_r\sigma], & \text{if } r-1 \in \text{Leg}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

Case 3: Assume $r, r+1 \in \text{Arm}(\sigma)$. We prove the following statement by induction on t: If $\tau \in \text{Sh}^\lambda$ satisfies $t, t+1 \in \text{Arm}(\tau)$ we have

$$\psi_t[\tau] = \begin{cases} [s_ts_{t+1}\sigma], & \text{if } t-1 \in \text{Leg}(\tau) \text{ and } i_{t-1} = i_t \\ 0, & \text{otherwise.} \end{cases}$$

The base case of $t = 1$ has been shown above.

We proceed with the induction, fixing $t > 1$, $\tau \in \text{Sh}^\lambda$ with $t, t+1 \in \text{Arm}(\tau)$ and assuming the claim holds for all smaller values of t. Let u be the largest value of $\text{Leg}(\tau)$ which is less than t. If no such value exists, then $\psi_t[\tau] = \psi_t\psi_\tau z^\lambda = \psi_\tau\psi_t z^\lambda = 0$. Otherwise, there is a reduced expression for τ beginning with $s_u \ldots s_{t-1}s_t$. Write $\tau' = s_ts_{t-1}\ldots s_u\tau$. Then

$$\begin{aligned} \psi_t[\tau] &= \psi_t(\psi_u \ldots \psi_{t-2})\psi_{t-1}\psi_t[\tau'] = (\psi_u \ldots \psi_{t-2})\psi_t\psi_{t-1}\psi_t[\tau'] \\ &= (\psi_u \ldots \psi_{t-2})(\psi_{t-1}\psi_t\psi_{t-1} + \delta_{i_t,i_u})[\tau'] \\ &= (\psi_u \ldots \psi_t)\psi_{t-1}[\tau'] + \delta_{i_t,i_u}(\psi_u \ldots \psi_{t-2})[\tau']. \end{aligned}$$

Assume that $t - 1 \in \text{Arm}(\tau)$. Then the first term is zero by induction. Furthermore, the second term is zero unless $i_t = i_u$ and (by induction) $u = t - 2$, $t - 3 \in \text{Leg}(\tau)$, and $i_{t-3} = i_{t-1}$. But since $i_{t-1} = i_t - 1$ and $i_{t-3} = i_u + 1$, this is impossible. Therefore the second term is zero as well.

Now suppose that $u = t - 1$. We can see that the first term is zero using an argument similar to the above. If the second term appears, it is visibly nonzero. We have thus proved the induction step.

Applying this to case 3, we have

$$\psi_r[\sigma] = \begin{cases} [s_rs_{r-1}\sigma], & \text{if } r-1 \in \text{Leg}(\sigma), \text{ and } i_r = i_{r-1} \\ 0, & \text{otherwise.} \end{cases}$$

Case 4: Assume $r, r+1 \in \text{Leg}(\sigma)$. This situation is entirely analogous to case 3, except now induction runs backwards from $t = d - 1$ to $t = 2$. We find that

$$\psi_r[\sigma] = \begin{cases} -[s_rs_{r+1}\sigma], & \text{if } r+2 \in \text{Arm}(\sigma), \text{ and } i_{r+1} = i_{r+2} \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Corollary 4.4. *Suppose that* $e(\boldsymbol{i})S^\lambda \neq 0$. *Then*

$$\{v \in e(\boldsymbol{i})S^\lambda \mid y_r v = 0 \ \text{ for } r = 1, \ldots, d\} = \mathcal{O}[\sigma_{\boldsymbol{i}}].$$

Proof. By Theorem 4.3, we have $y_r[\sigma_{\boldsymbol{i}}] = 0$ for $r = 1, \ldots, d$. Suppose given

$$v = \sum_{\sigma \in \mathrm{Sh}^\lambda} c_\sigma[\sigma] \in e(\boldsymbol{i})S^\lambda,$$

and let $h = s_{r_1} \ldots s_{r_m} \in H(\boldsymbol{i})$ be a reduced expression for an element of maximal length for which $c_{h\sigma_{\boldsymbol{i}}} \neq 0$. Apply Theorem 4.3 to obtain $y_{r_1} \ldots y_{r_m} v = (-1)^m c_{h\sigma_{\boldsymbol{i}}}[\sigma_{\boldsymbol{i}}] \neq 0$. Therefore v is killed by every y_r if and only if $v \in \mathcal{O}[\sigma_{\boldsymbol{i}}]$. $\qquad\square$

5. Homomorphisms into S^λ

We again fix two integers $d > k \geq 0$, $\lambda := (d - k, 1^k)$ and $\alpha := \mathrm{cont}(\lambda)$. We also fix $\mu = (\mu_1, \ldots, \mu_N)$ a partition of d. The goal of this section is to determine $\mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda)$, and the answer is given in Theorem 5.1. In order for a nonzero homomorphism to exist $\boldsymbol{i}^\mu$ must be a weight of S^λ. We assume this, so in particular we are able to define $\sigma_\mu := \sigma_{\boldsymbol{i}^\mu}$. This determines a mapping from μ to λ in which the node containing r in $\mathtt{T}^\mu$ is sent to the node containing r in $\sigma_\mu \mathtt{T}^\lambda$. We define *arm nodes* and *leg nodes* of μ to be the nodes which under this map are sent to the arm or the leg of λ, respectively.

Theorem 5.1. *There is a homomorphism* $S^\mu \to S^\lambda$ *satisfying* $z^\mu \mapsto v$ *for* $v \neq 0$ *if and only if*

(i) *there exist* $c \in \{1, \ldots, k+1\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^c$, *and* $0 \leq m < e$ *such that* $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ *and*

$$\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+1}),$$

(ii) e *divides* d, *there exist* $c \in \{1, \ldots, k\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^c$, *and* $0 \leq m < e$ *such that* $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ *and*

$$\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+2}), \ or$$

(iii) $N > k + 1$, *there exist* $\boldsymbol{a} \in (\mathbb{Z}_{>0})^N$ *and* $0 \leq m < e$ *such that* $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ *and*

$$\mu = (a_1 e, \ldots, a_k e, a_{k+1} e - 1, \ldots, a_{N-1} e - 1, a_N e - 1 - m).$$

The proof of this will follow from several lemmas. Recall the ideals J_i^μ of Definition 3.2.

Lemma 5.2. *A nonzero* $v \in S^\lambda$ *satisfies* $(J_1^\mu + J_2^\mu + J_3^\mu)v = 0$ *if and only if* $v \in \mathcal{O}[\sigma_\mu]$, *and all leg nodes of* μ *are in the first column of its Young diagram.*

Proof. Assume that $(J_1^\mu + J_2^\mu + J_3^\mu)v = 0$. Corollary 4.4 implies that $v = \gamma[\sigma_\mu]$ for some $\gamma \in \mathcal{O}$. By considering weights we see that $(1, 2)$ cannot be a leg node. Suppose that $(a, b) \neq (1, 2)$ is a leg node of μ with $b \geq 2$, and define $r = \mathtt{T}^\mu(a, b)$. Then since $J_3^\mu v = 0$ we must have $\psi_{r-1} v = 0$. Furthermore

$(a, b-1)$ must be an arm node, for in $\boldsymbol{i}^\mu = (i_1, \ldots, i_d)$ we have $i_r = i_{r-1} + 1$, whereas if they were both leg nodes we would have $i_r = i_{r-1} - 1$. By Theorem 4.3 we have $\psi_{r-1}\gamma[\sigma_\mu] = \gamma[s_{r-1}\sigma_\mu] \neq 0$, a contradiction.

Conversely, assume that $v = \gamma[\sigma_\mu]$ and all leg nodes of μ are in the first column. Suppose that r and $r+1$ are in the same row of $\mathtt{T}^\mu$, and so in particular $r+1$ is an arm node. If the node of μ containing r in $\mathtt{T}^\mu$ is a leg node then by considering Theorem 4.3 we see that $\psi_r v = 0$ unless the node containing $r-1$ is also a leg node. But then the nodes containing $r-1$ and r are both in the first column of $\mathtt{T}^\mu$ forcing $r+1$ (and every subsequent node) to also be in the first column. This contradicts the assumption that r and $r+1$ appear in the same row of T^μ, and so necessarily $\psi_r v = 0$.

If the node of μ containing r in $\mathtt{T}^\mu$ is an arm node, then write $\boldsymbol{i}^\mu = (i_1, \ldots, i_d)$. Referring again to Theorem 4.3 we see that $\psi_r v = 0$ unless the node containing $r-1$ is a leg node, and furthermore $i_{r-1} = i_r$. But if the the node containing $r-1$ is a leg node, then by assumption it is in the first column. Once again the assumption that r and $r+1$ are in the same row of T^μ forces the node containing $r-1$ to be directly to the left of the node containing r. But in this case $i_{r-1} = i_r - 1$ and once again we see that $\psi_r v = 0$. $\qquad\square$

Proposition 5.3. *If there is a nonzero element $v \in S^\lambda$ with $J^\mu v = 0$, then the leg nodes of μ are precisely $(2,1), (3,1), \ldots, (k+1, 1)$.*

Proof. Let $v = \gamma[\sigma_\mu]$ be such an element. By Lemma 5.2, the leg nodes appear in the first column of μ. Suppose that $(a, 1)$ is a leg node. Then $a \neq 1$, and so we may consider the Garnir node $A = (a-1, 1)$. Define $r = \mathtt{T}^\mu(a-1, 1)$ and $s = \mathtt{T}^\mu(a, 1)$. It is clear that $g^A = \psi^{T^A} = \psi_{(r, r+1, \ldots, s)}$ where $(r, r+1, \ldots, s)$ is a cycle of the entire Garnir belt. In particular, if $a \geq 3$ and $(a-1, 1)$ is not a leg node, then $g^A[\sigma_\mu] = [(r, r+1, \ldots, s)\sigma_\mu]$. This contradicts $J_4^\mu v = 0$. We see that for every leg node $(a, 1)$ with $a \geq 3$, the node $(a-1, 1)$ is also a leg node, which is equivalent to the statement of the proposition. $\qquad\square$

As a consequence of this proposition, we see that the leg nodes of μ are mapped to exactly the same nodes of λ, and therefore that $\sigma_\mu \mathtt{T}^\lambda$ is the tableau obtained by moving all of the arm nodes of $\mathtt{T}^\mu$ to the first row. This suggests the following definition.

Definition 5.4. Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of d with $N \geq k+1$. Define $\mathtt{T}_\mu^\lambda$ to be the unique standard λ-tableau such that $\mathtt{T}_\mu^\lambda(A) = \mathtt{T}^\mu(A)$ for all $A \in \lambda \cap \mu$. Furthermore, define $\sigma_\mu^\lambda := w^{\mathtt{T}_\mu^\lambda} \in \mathfrak{S}_n$.

If we have a nonzero homomorphism $S^\mu \to S^\lambda$, then Corollary 4.4 tells us it must send z^μ to a multiple of $[\sigma_\mu]$. On the other hand, Proposition 5.3 requires the image of z^μ to be a multiple of $[\sigma_\mu^\lambda]$. Since $[\sigma_\mu]$ and $[\sigma_\mu^\lambda]$ are elements of an $\mathcal{O}$-basis of S^λ, a necessary condition for a homomorphism is that $[\sigma_\mu] = [\sigma_\mu^\lambda]$. The next two results tell us when this happens.

Lemma 5.5. *Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of d such that $N \geq k+1$. Then $[\sigma_\mu^\lambda]$ has weight $\boldsymbol{i}^\mu$ if and only if μ is of one of the following three forms.*

- *(i) $N = k+1$, and there exist $c \in \{1, \ldots, k+1\}$, $a_1, \ldots, a_c \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ such that $\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+1})$.*
- *(ii) $N > k+1$ and $\mu_{k+1} = 1$, and there exist $c \in \{1, \ldots, k\}$, $a_1, \ldots, a_c \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ satisfying $m + c \equiv k + 2 \pmod{e}$ such that $\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+2})$.*
- *(iii) $N > k+1$ and $\mu_{k+1} > 1$, and there exist $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ such that $\mu = (a_1 e, \ldots, a_k e, a_{k+1} e - 1, \ldots, a_{N-1} e - 1, a_N e - 1 - m)$.*

Proof. Define $(i_1, \ldots, i_d) := \text{wt}([\sigma_\mu^\lambda])$ and $(j_1, \ldots, j_d) := \boldsymbol{i}^\mu$. Define $r_x = \mathtt{T}^\mu(x+1, 1)$ for $x = 1, 2, \ldots, k$. Since $r_x = \mathtt{T}^\mu(x+1, 1) = \mathtt{T}_\mu^\lambda(x+1, 1)$, it is immediate that $i_{r_x} = j_{r_x}$ for $x = 1, \ldots, k$. Letting $S = \{1, 2, \ldots, d\} \setminus \{r_1, \ldots, r_k\}$, we therefore have that $\text{wt}([\sigma_\mu^\lambda]) = \boldsymbol{i}^\mu$ if and only if $i_s = j_s$ for all $s \in S$.

For $p = 1, 2, \ldots, d - k$, let s_p be the pth element of the set S (under the usual ordering). It is clear that $s_p = \mathtt{T}_\mu^\lambda(1, p)$. In particular, we see that $i_{s_{p+1}} = i_{s_p} + 1$ for $p = 1, \ldots, d - k - 1$. Since $j_{s_1} = i_{s_1}$, we have $\text{wt}([\sigma_\mu^\lambda]) = \boldsymbol{i}^\mu$ if and only if $j_{s_{p+1}} = j_{s_p} + 1$ for $p = 1, \ldots, d - k - 1$.

Set $r_0 = 0$. Note that the condition $r_1 \neq r_0 + 1$ is automatically satisfied, so $x_0 = \max\{x \mid r_x \neq r_{x-1} + 1 \text{ where } 1 \leq x \leq k\}$ is well defined. We divide S as follows

$$
\begin{aligned}
(s_1, s_2, \ldots, s_{n-k}) = (1, & 2, \ldots, r_1 - 1, \\
& r_1 + 1, \ldots, r_2 - 1, \\
& \ldots, \\
& r_{x_0 - 1} + 1, \ldots, r_{x_0} - 1, \\
& r_k + 1, \ldots, d).
\end{aligned}
$$

Now, $(r_{x-1} + 1, \ldots, r_x - 1) = (\mathtt{T}^\mu(x, 2), \ldots, \mathtt{T}^\mu(x, \mu_x))$ for $x \leq x_0$, so it is clear that $j_{p+1} = j_p + 1$ when $r_{x-1} + 1 \leq p \leq r_x - 2$. Furthermore, for $x < x_0$ we have $r_x - 1 = \mathtt{T}^\mu(x, \mu_x)$ while $r_x + 1 = \mathtt{T}^\mu(x+1, 2)$. If we let $i = \text{res}(x, 1)$, then $j_{r_x - 1} = \text{res}(x, \mu_x) \equiv i + \mu_x - 1 \pmod{e}$, and $j_{r_x + 1} = \text{res}(x+1, 2) = i$. Thus $j_{r_x + 1} = j_{r_x - 1} + 1$ if and only if $\mu_x = a_x e$ for some $a_x \in \mathbb{Z}_{>0}$.

At this point we have shown that $i_p = j_p$ for $p = 1, \ldots, r_k$ if and only if for every $x < x_0$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$.

Suppose that $N = k + 1$. If $r_k = d$, then we set $c = x_0$ choose a_c, m so that $\mu_c = a_c e - m$. This clearly is of the form in part (i) of the Lemma. Otherwise, $x_0 = k$ and $r_k + 1, \ldots, d$ are all on row $k + 1$ of $\mathtt{T}^\mu$. The same analysis as above shows that $[\sigma_\mu^\lambda]$ has weight $\boldsymbol{i}^\mu$ if and only if for every $x \leq k$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$. In this case we set $c = k + 1$ and choose a_c, m as required. We have established part (i) of the Lemma.

Suppose that $N > k + 1$ and $\mu_{k+1} = 1$. Set $c = x_0$, and let a_c, m be the unique integers satisfying $\mu_c = a_c e - m$ and $0 \leq m < e$. Then $(r_k + 1, \ldots, d) = (\mathtt{T}^\mu(k+2, 1), \ldots, \mathtt{T}^\mu(N, 1))$. Let $i = \mathrm{res}(c, 1)$. Then $j_{r_c - 1} = \mathrm{res}(c, \mu_c) \equiv i + \mu_c - 1 \pmod{e}$, and $j_{r_k + 1} \equiv i + c - k - 2 \pmod{e}$. Thus $j_{r_k + 1} = j_{r_c - 1} + 1$ if and only if $\mu_c \equiv c - k - 2 \pmod{e}$. But $\mu_c = a_c e - m$, so this is equivalent to $m + c \equiv k + 2 \pmod{e}$. This is exactly the requirement in part (ii) of the Lemma. Finally, suppose that $N \geq k + 3$. Then $r_k + 1 = \mathtt{T}^\mu(k+2, 1)$ and $r_k + 2 = \mathtt{T}^\mu(k+3, 1)$. In this case we see that $j_{r_k + 2} = j_{r_k + 1} - 1 \neq j_{r_k + 1} + 1$. Thus part (ii) of the Lemma is finished.

Finally, suppose that $N > k + 1$ and $\mu_{k+1} > 1$. Then $x_0 = k$. Once again the analysis of the $N = k + 1$ case shows that $i_p = j_p$ for $p = 1, \ldots, r_k + 1$ if and only if for every $x \leq k$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$. Furthermore, a similar analysis shows that $i_p = j_p$ for $p = 1, \ldots, d$ if and only if additionally for every $k + 1 \leq x < N$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e - 1$. This is exactly the situation of part (iii). $\qquad\square$

Corollary 5.6. *If μ is as in Lemma 5.5, then $(J_2^\mu + J_3^\mu)[\sigma_\mu^\lambda] = 0$. In particular, $[\sigma_\mu^\lambda] = [\sigma_\mu]$.*

Proof. Let $(i_1, \ldots, i_d) = \mathrm{wt}([\sigma_\mu^\lambda])$. It is clear that in each of the cases above if $i_r = i_{r+1}$ then $r \in \mathrm{Arm}(\sigma_\mu^\lambda)$ and $r + 1 \in \mathrm{Leg}(\sigma_\mu^\lambda)$. This implies that $J_2^\mu[\sigma_\mu^\lambda] = 0$. In turn, this implies that $[\sigma_\mu^\lambda] = [\sigma_\mu]$, by Corollary 4.4. Lemma 5.2 now shows us that $J_3^\mu[\sigma_\mu^\lambda] = 0$. $\qquad\square$

As expected, the Garnir relations are the most difficult to verify.

Lemma 5.7. *Suppose that μ has one of the forms in Lemma 5.5 and let $A = (x, y) \in \mu$ be a Garnir node. Define $r, s,$ and t to be the values in $\mathtt{T}^\mu$ of the nodes $(x, y), (x + 1, 1),$ and $(x + 1, y)$ respectively so that the Garnir belt is as in the following picture*
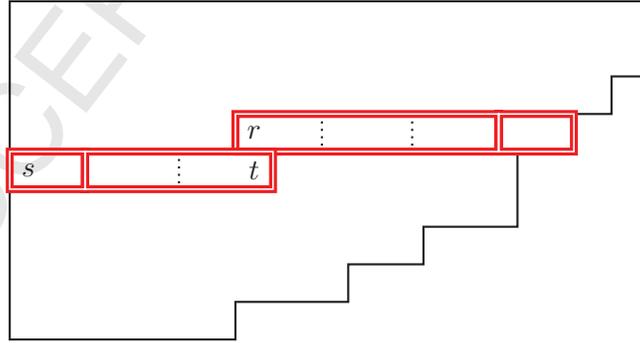
Then

$$\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \begin{cases} [(t+1, t, \ldots, s)\sigma_\mu^\lambda], & \text{if } x \leq k, \ y \equiv 0 \pmod{e}, \text{ and } y < \mu_{x+1} \\ [(r, r+1, \ldots, s)\sigma_\mu^\lambda], & \text{if } x \leq k, \ y \equiv 1 \pmod{e}, \text{ and } y > 1 \\ [\sigma_\mu^\lambda], & \text{if } x > k \text{ and } y \equiv 0 \pmod{e} \\ 0, & \text{otherwise} \end{cases}$$

Proof. We will use braid diagrams; see for example [**12**]. We label the top of the diagram by weight, and the bottom of the diagram by position. We compute $\psi^{\mathrm{T}^A}[\sigma_\mu^\lambda]$ in each case. Multiplying two elements left to right corresponds to concatenating two diagrams bottom to top.

We begin with the case $x \leq k$ and $y \equiv 0 \pmod{e}$. Then $\mu_x \equiv 0 \pmod{e}$ and, recalling the notation from Section 3.2, C^A consists of a single node and D^A is empty. Define $s_p = s + pe$ for $p \in \mathbb{N}$, and let i be the residue of A.

$$(5.8)$$

The second term is zero by Theorem 4.3(iii), and the first is equal to

using the defining relations of R_α and Theorem 4.3(ii). We note that this is of the same form as (5.8) above. An easy induction argument allows us to conclude that this is equal to

Given our assumptions, we see that the condition $t + 1 \in \mathrm{Arm}(\sigma_\mu^\lambda)$ is equivalent to $y < \mu_{x+1}$. If this holds, then by Theorem 4.3 the above element is

equal to

$$= [(t+1, t, \ldots, s)\sigma_\mu^\lambda].$$

as claimed. Otherwise, we get zero.

We next consider the case that $x \leq k$ and $y = 1$. If $x = 1$, then $\psi^{\mathrm{T}^A}[\sigma_\mu^\lambda] = 0$ by the Garnir relation for the node $(1, 1) \in \lambda$. Otherwise A is a leg node, and so is $(x + 1, 1)$. Therefore, $\psi^{\mathrm{T}^A}[\sigma_\mu^\lambda]$ takes the following form.

which is zero by Theorem 4.3(iii).

Suppose now that $A = (x, y)$ with $x \leq k$, $y \equiv 1 \pmod{e}$, and $y > 1$. Then C^A is empty and D^A consists of a single node, and $\psi^{\mathrm{T}^A}[\sigma_\mu^\lambda]$ takes the following form.


which is $[(r, r+1, \ldots, s)\sigma_\mu^\lambda]$.

Consider now the case $x \leq k$, and $y \not\equiv 0, 1 \pmod{e}$. Since A is a Garnir node and $y > 1$ we have $\mu_x \equiv 0 \pmod{e}$. This implies that C^A and D^A each contain at least two nodes. The portion of $\psi^{\mathrm{T}^A}[\sigma_\mu^\lambda]$ showing the crossing between C^A and D^A is depicted in the following picture


which is zero by Theorem 4.3(iii).

We next check the special case that $A = (k+1, 1)$. If μ is as in Lemma 5.5(ii), then the picture we have is as follows.
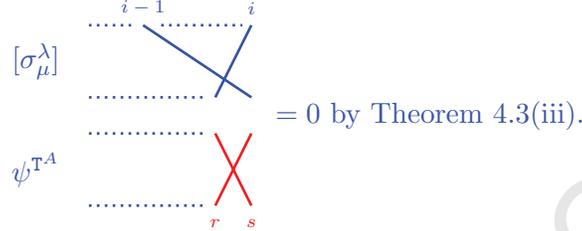
If μ is as in Lemma 5.5(iii), then the picture is as follows
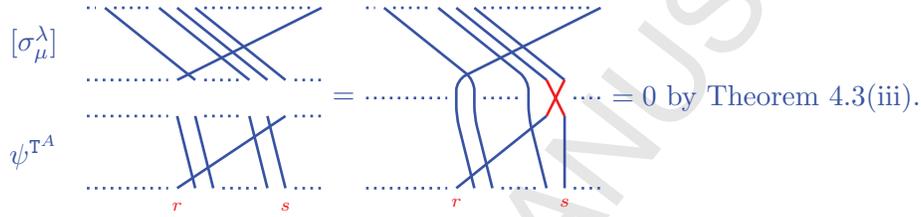
Suppose now that $x > k$, $y \not\equiv 0 \pmod{e}$, and $A \neq (k+1, 1)$. Then C^A is nonempty, and so $\psi^{\mathtt{T}^A} = X\psi_{s-1}$ for some $X \in R_d$. Thus $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = X\psi_{s-1}[\sigma_\mu^\lambda] = 0$ by Theorem 4.3(iii).

Finally, if $x > k$ and $y \equiv 0 \pmod{e}$ then μ must be as in Lemma 5.5(iii), so that $\mathtt{T}^A = \mathtt{T}^\mu$. Then $\psi^{\mathtt{T}^A} = 1$, and so $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = [\sigma_\mu^\lambda]$. $\qquad\square$

Lemma 5.9. *Suppose that μ is as in Lemma 5.5 and let $A \in \mu$ be a Garnir node. Let r, s, t be as in Lemma 5.7. Then for $1 \leq f \leq a_{x+1} - 1$ we have*

$$g^A[\sigma_\mu^\lambda] = \begin{cases} \binom{a_x}{f}[(t+1, t, \ldots, s)\sigma_\mu^\lambda], & \text{if } A = (x, fe) \text{ with } x \leq k \\ \binom{a_x}{f}[(r, r+1, \ldots, s)\sigma_\mu^\lambda], & \text{if } A = (x, fe+1) \text{ with } x \leq k \\ \binom{a_x}{f}[\sigma_\mu^\lambda], & \text{if } A = (x, fe) \text{ with } x > k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = 0$ there is nothing to show.

Recall the definitions of σ_p^A and τ_p^A from Section 3.2. In each of the three cases in Lemma 5.7 for which $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] \neq 0$, multiplying σ_p^A with $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ has the effect of shuffling e arm strands past e arm strands. For instance, if $x \leq k$, $y \equiv 0 \pmod{e}$, and $y < \mu_{x+1}$ then $\sigma_p^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \sigma_p^A[(t+1, t, \ldots, s)\sigma_\mu^\lambda]$. By Theorem 4.3, we have that $\sigma_p^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = 0$. Thus $\tau_p^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ for all p. This furthermore implies that $\tau_w^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ for all $w \in \mathfrak{S}^A$. Therefore

$$g^A[\sigma_\mu^\lambda] = \sum_{u \in \mathscr{D}^A} \tau_u^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \sum_{u \in \mathscr{D}^A} \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \left|\mathscr{D}^A\right| \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda].$$

The problem is reduced to calculating $\left|\mathscr{D}^A\right|$ in each of the three cases above, which follows immediately from the definitions. $\qquad\square$

Proof of Theorem 5.1 By the discussion preceding Lemma 5.5, any homomorphism $S^\mu \to S^\lambda$ is of the form $z^\mu \mapsto \gamma[\sigma_\mu^\lambda]$ for some $\gamma \in \mathcal{O}$. By the discussion preceding Lemma 5.5 if there is a nonzero homomorphism, then $[\sigma_\mu] = [\sigma_\mu^\lambda]$. Lemma 5.5 tells us for which partitions this happens.

Suppose now that μ is as in Lemma 5.5. Corollary 5.6 ensures that $(J_1^\mu + J_2^\mu + J_3^\mu)\gamma[\sigma_\mu^\lambda] = 0$. By Lemma 5.9, $J_4^\mu \gamma[\sigma_\mu^\lambda] = 0$ if and only if for each $x = 1, \ldots, N-1$, we have $\binom{a_x}{f}\gamma = 0$ for $f = 1, \ldots, a_{x+1} - 1$. But $\mathrm{Gc}(\boldsymbol{a})$ is the greatest common divisor of these binomial coefficients. Therefore $\gamma[\sigma_\mu^\lambda]$ satisfies the Garnir relations if and only if $\mathrm{Gc}(\boldsymbol{a})\gamma = 0$. This is exactly the claim of the theorem. $\square$

6. EXAMPLES

We now present some examples of Theorem 5.1. We will restrict our attention to the case that $\mathcal{O}$ is a torsion-free algebra over either $\mathbb{Z}$ or $\mathbb{F}_p$ for some prime $p \geq 3$. In this case for any $r \in \mathbb{Z}$ the submodule $\mathrm{Ann}_\mathcal{O}(r)$ is either (0) or $\mathcal{O}$, and so in Theorem 5.1 all of our homomorphism spaces will be free $\mathcal{O}$-modules. In particular, we can talk about their dimensions.

In characteristic 0 the situation simplifies. The following corollary covers in particular Hecke algebras over $\mathbb{C}$ at a root of unity.

Corollary 6.1. *Let μ be an arbitrary partition, and $\lambda = (d-k, 1^k)$. Then* $\mathrm{Hom}(S^\mu, S^\lambda)$ *is at most one-dimensional, and* $\dim \mathrm{Hom}(S^\mu, S^\lambda) = 1$ *if and only if one of the following holds:*

(i) $\mu = \lambda$;

(ii) e *divides* d *and* $\mu = (d-k-1, 1^{k+1})$;

(iii) $k = 0$ *and there exist* $n \geq 0$, $a > 0$, *and* $0 \leq m < e$ *such that*
$$\mu = (ae - 1, (e-1)^n, m);$$

(iv) $k \geq 1$, *there exist* $0 \leq n < k$, $a \in \mathbb{Z}_{>0}$, *and* $1 \leq m \leq e$ *such that*
$$\mu = (ae, e^n, m, 1^{k-n-1});$$

(v) $k \geq 1$, *there exist* $n > k+1$, $a \in \mathbb{Z}_{>0}$ *and* $1 \leq m < e$ *such that*
$$\mu = (ae, e^{k-1}, (e-1)^{n-k-1}, m);$$

(vi) $k \geq 2$, e *divides* d, *there exist* $0 \leq n \leq k-2$, $a \in \mathbb{Z}_{>0}$, *and* $1 \leq m \leq e$ *such that*
$$\mu = (ae, e^n, m, 1^{k-n}).$$

For the rest of the section we assume that $\mathcal{O}$ is an $\mathbb{F}_p$-algebra, with $p \geq 3$, and we furthermore assume that $e = p$. This applies in the case of symmetric groups in positive characteristic.

6.1. **The trivial module.** Let $\lambda = (d)$. The module S^λ is referred to as the *trivial* module. It is one-dimensional over $\mathcal{O}$ having basis z^λ. We recover the following graded version of the result of James [**7**, Theorem 24.4].

Corollary 6.2. *Let* $\mu = (\mu_1, \ldots, \mu_N)$ *be a partition of* d. *Then* $\mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = 0$ *unless there exist* $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ *and* $m \in \{0, \ldots, p-1\}$ *such that* $\mu = (a_1 p - 1, \ldots, a_{N-1}p - 1, a_N p - 1 - m)$, *and furthermore* $\nu_p(a_x) \geq \ell_p(a_{x+1} - 1)$ *for* $1 \leq x < N$. *In this case,*

$$\dim_q \mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q^r \text{ where } r = N - \left\lceil \frac{N+m}{p} \right\rceil.$$

Proof. Theorem 5.1(i) can only occur if $\mu = \lambda$, and Theorem 5.1(ii) never occurs because there are no valid choices for c. Therefore Theorem 5.1(iii) and Corollary 2.10 tell us the form that μ must take.

To calculate the degree of the homomorphisms, we calculate the degrees of T^λ and T^μ. For any partition ν with M parts, we have $\deg(\mathsf{T}^\nu) = \sum_{x=1}^{M} \lfloor \frac{\nu_x}{p} \rfloor$. In particular we get $\deg(\mathsf{T}^\mu) = \sum_{x=1}^{N} a_x - N$ and $\deg(\mathsf{T}^\lambda) = \sum_{x=1}^{N} a_x - \lceil \frac{N+m}{p} \rceil$, and

$$\deg(T^\lambda) - \deg(\mathsf{T}^\mu) = N - \left\lceil \frac{N+m}{p} \right\rceil.$$

$\square$

6.2. **The standard module.** Let $\lambda = (d-1, 1)$. Then S^λ is referred to as the *standard* or *reflection* module. We have the following Corollary.

Corollary 6.3. *For any partition* $\mu \neq \lambda$, $\dim \mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = 0$ *with the following exceptions:*

   (i) if there are $a_1, a_2 \in \mathbb{Z}_{>0}$ *and* $0 \leq m < p$ *such that*

$$\mu = (a_1 p, a_2 p - m),$$

   and futhermore $\nu_p(a_1) \geq \ell_p(a_2 - 1)$, *then*

$$\dim_q \mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = \begin{cases} 1, & \text{if } p \mid d - 1 \\ q, & \text{otherwise;} \end{cases}$$

   (ii) if p *divides* d *and* $\mu = (d-2, 1, 1)$, *then* $\dim_q \mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q$;
   (iii) if there are $0 \leq m < p$, $N \geq 3$, *and* $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ *such that*

$$\mu = (a_1 p, a_2 p - 1, a_3 p - 1, \ldots, a_{N-1}p - 1, a_N p - 1 - m),$$

   and furthermore $\nu_p(a_i) \geq \ell_p(a_{i+1} - 1)$ *for* $i = 1, \ldots, N-1$, *then* $\dim_q \mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q^r$ *where*

$$r = N - \left\lceil \frac{N+m}{p} \right\rceil + \begin{cases} -1, & \text{if } p \mid d-1 \\ 1, & \text{if } p \mid d \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Everything follows directly from Theorem 5.1 and Corollary 2.10 except the degree calculations. The degree in cases (i) and (ii) is straightforward to calculate, and for case (iii) we can make the following calculations.

$$\deg(\mathtt{T}^\mu) = \sum_{x=1}^{N} a_x - N + 1$$

$$\deg(\mathtt{T}^\lambda) = \sum_{x=1}^{N} a_x - \left\lceil \frac{N+m}{p} \right\rceil$$

$$\deg([\sigma_\mu]) = \deg(\mathtt{T}^\lambda) + \begin{cases} 0, & \text{if } p \mid d-1 \\ 2, & \text{if } p \mid d \\ 1, & \text{otherwise.} \end{cases}$$

$\square$

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