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F-singularities under generic linkage

Linquan Ma [a,*], Janet Page [b], Rebecca R.G. [c], William Taylor [d], Wenliang Zhang [b]

[a] *Department of Mathematics, University of Utah, Salt Lake City, UT 84112, United States*
[b] *Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60607, United States*
[c] *Department of Mathematics, Syracuse University, Syracuse, NY 13244, United States*
[d] *Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, United States*

A R T I C L E   I N F O

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A B S T R A C T

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a prefect field of positive characteristic. Let I be an equi-dimensional ideal in R and let J be a generic link of I in $S = R[u_{ij}]_{c\times r}$. We describe the parameter test submodule of S/J in terms of the test ideal of the pair (R, I) when a reduction of I is a complete intersection or almost complete intersection. As an application, we deduce a criterion for when S/J has F-rational singularities in these cases. We also compare the F-pure threshold of (R, I) and (S, J).

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1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field of positive characteristic. Let $I = (f_1, \ldots, f_r)$ be an equi-dimensional ideal in R of height c, where equi-dimensional

* Corresponding author.
E-mail addresses: lquanma@math.utah.edu (L. Ma), jpage8@uic.edu (J. Page), rirebhuh@syr.edu (R. R.G.), wdtaylor@uark.edu (W. Taylor), wlzhang@uic.edu (W. Zhang).

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means that all associated primes of I have the same height [15]. We can define a regular sequence $g_1, \ldots, g_c$ in $S = R[u_{ij}]_{c\times r}$ via $g_i := u_{i1}f_1 + \cdots + u_{ir}f_r$, where the u_{ij} are variables over S. Then $J = (g_1, \ldots, g_c) : I$ is called a generic link of I in $S = R[u_{ij}]$. The study of generic linkage has attracted considerable attention and has been developed widely from both algebraic and geometric points of view [11], [12], [4], [5], [16].

In contrast to the quick and deep development of singularity theories in the past decades, much less has been known about the behaviors of singularities under generic linkage. A special case is a result of Chardin and Ulrich [4] which says that if R/I is a complete intersection and has rational (resp. F-rational) singularities, then a generic link S/J also has rational (resp. F-rational singularities). This result in characteristic zero has been vastly extended in recent work of Niu [16], which is our main motivation for this research.

Theorem 1.1 *(Theorem 1.1 in [16]). Let J be a generic link of a reduced and equidimensional ideal I in $S = R[u_{ij}]$ and assume that the characteristic of k is 0. We have*

(1) $\omega^{GR}_{S/J} \cong \mathscr{J}(R, I^c) \cdot (S/J)$, *where $\omega^{GR}_{S/J}$ denotes the Grauert–Riemenschneider canonical sheaf of S/J and $\mathscr{J}(R, I^c)$ denotes the multiplier ideal of the pair (R, I^c),*
(2) $\mathrm{lct}(S, J) \geq \mathrm{lct}(R, I)$. *In particular, if the pair (R, I^c) is log canonical, then the pair (S, J^c) is also log canonical.*

This result gives a nice criterion for a generic link to have rational singularities in characteristic 0. It also has applications to bounding the Castelnuovo–Mumford regularity of projective varieties [16, Corollary 1.2]. Since test ideals and F-pure thresholds are characteristic p analogues of multiplier ideals and log canonical thresholds (*cf.* [2] and [13]), it is natural to ask whether analogues of Theorem 1.1 hold for test ideals and F-pure thresholds. Our main result is the following, which partially extends Theorem 1.1 to characteristic p and generalizes [4, Theorem 3.13] in characteristic p.

Theorem 1.2 *(Theorem 3.3, Corollary 4.4). Let J be a generic link of an equi-dimensional ideal I in $S = R[u_{ij}]$ and assume the characteristic of k is $p > 0$.*

(1) *Suppose I is reduced and that a reduction of I is a complete intersection or an almost complete intersection. Then $\tau(\omega_{S/J}) \cong \tau(R, I^c) \cdot (S/J)$, where $\tau(\omega_{S/J})$ denotes the parameter test submodule and $\tau(R, I^c)$ denotes the test ideal of the pair (R, I^c).*
(2) *Suppose that a reduction of I is a complete intersection. Then $\mathrm{fpt}_S(J) \geq \mathrm{fpt}_R(I)$. In particular, if the pair (R, I^c) is F-pure, then the pair (S, J^c) is also F-pure.*

This paper is organized as follows: in Section 2 we recall and prove some basic result for F-singularities and test ideals; in Section 3 we give a description of the parameter test submodule of S/J in terms of the test ideal of the pair (R, I), when a reduction of I is a complete intersection or an almost complete intersection. This generalizes earlier results in [4]. In Section 4 we compare the F-pure threshold of the pairs (S, J) and (R, I) when a reduction of I is a complete intersection.

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2. F-singularities and test ideals

In this section we collect some basic definitions of F-singularities and test ideals and prove a characteristic $p > 0$ analogue of Ein's Lemma in [16], which will be used in later sections.

Let R be a Noetherian commutative ring of characteristic $p > 0$. We will use $F^e_* R$ to denote the target of the e-th Frobenius endomorphism $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$, *i.e.* $F^e_* R$ is an R-bimodule, which is the same as R as an abelian group and as a right R-module, that acquires its left R-module structure via the e-th Frobenius endomorphism $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$. When R is reduced, we will use R^{1/p^e} to denote the ring whose elements are p^e-th roots of elements of R. Note that these notations (when R is reduced) $F^e_* R$ and R^{1/p^e} are used interchangeably in the literature; we will do so in this paper as well assuming no confusion will arise.

Remark 2.1. If R is a commutative ring essentially of finite type over a perfect field of characteristic $p > 0$, then R admits a canonical module denoted by ω_R. Applying $\operatorname{Hom}_R(-, \omega_R)$ to the e-th Frobenius $R \to F^e_* R$ produces an R-linear map

$$\operatorname{Hom}_R(F^e_* R, \omega_R) \to \operatorname{Hom}_R(R, \omega_R) = \omega_R.$$

Moreover, we have $F^e_* \omega_R \cong \operatorname{Hom}_R(F^e_* R, \omega_R)$ (see [2, Example 2.4] for more details). Hence we have a natural R-linear map:

$$\Phi^e_R : F^e_* \omega_R \cong \operatorname{Hom}_R(F^e_* R, \omega_R) \to \operatorname{Hom}_R(R, \omega_R) = \omega_R$$

called the trace map of the e-th Frobenius.

Example 2.2. When $R = k[x_1, \dots, x_n]$ is a polynomial ring over a perfect field k of characteristic $p > 0$, we can identify ω_R with R, and Φ^e_R can be identified with the usual trace Tr^e_R, that is:

$$\begin{aligned}&\mathrm{Tr}^e_R(F^e_*(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}))\\ &= \begin{cases} x_1^{\frac{i_1-(p^e-1)}{p^e}} x_2^{\frac{i_2-(p^e-1)}{p^e}} \cdots x_n^{\frac{i_n-(p^e-1)}{p^e}}, & \text{if } \frac{i_t-(p^e-1)}{p^e} \in \mathbb{Z} \text{ for each } t \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

In this case $\mathrm{Hom}_R(F^e_*R, R)$ is a cyclic F^e_*R-module generated by Tr^e_R. Furthermore, if $f_1, \dots, f_c$ is a regular sequence in R and $T = R/(f_1, \dots, f_c)$, then we have ([6, Corollary on page 465][1])

$$\Phi^e_T(F^e_*(-)) = \text{image of } \mathrm{Tr}^e_R(F^e_*(f_1^{p^e-1} \cdots f_c^{p^e-1} \cdot -)) \text{ in } T.$$

Remark 2.3. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$ and $A = R[y_1, \dots, y_m]$ be a polynomial ring over R. For each ideal I in R, it is well known and straightforward to check that

$$\mathrm{Tr}^e_R(F^e_*(I))A = \mathrm{Tr}^e_A(F^e_*(IA)).$$

Lemma 2.4. *Let $S \to R$ be a surjection of Noetherian commutative rings of characteristic p. Assume that both S and R admit canonical module ω_S and ω_R respectively and* $\dim S = \dim R$. *Then*

$$\Phi^e_R = \Phi^e_S|_{\omega_R}.$$

Proof. Under our assumptions, we have $\omega_R = \mathrm{Hom}_S(R, \omega_S)$ and the surjection $S \to R$ induces an inclusion $\omega_R = \mathrm{Hom}_S(R, \omega_S) \hookrightarrow \omega_S$. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(F^e_*R, \mathrm{Hom}_S(R, \omega_S)) \arrow[r] \arrow[d] & \mathrm{Hom}_R(R, \mathrm{Hom}_S(R, \omega_S)) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_S(R, \omega_S) \arrow[d] \\
\mathrm{Hom}_S(F^e_*S, \omega_S) \arrow[r] & \mathrm{Hom}_S(S, \omega_S) \arrow[r, "\sim"] & \omega_S
\end{tikzcd}$$

Note that the top row (resp. the bottom row) induces the trace map Φ^e_R (resp. Φ^e_S). To prove our lemma, it suffices to prove

[1] Fedder's result [6, Corollary on page 465] assumes that the ring R is a Gorenstein local ring only to ensure that $\mathrm{Hom}_R(F_*R, R) \cong F_*R$. In our case, $R = k[x_1, \dots, x_n]$ is a polynomial ring, so $\mathrm{Hom}_R(F_*R, R)$ is clearly isomorphic to F_*R. Hence Fedder's result applies in our case.

(a) the vertical map on the left is an inclusion, and
(b) the diagram commutes.

To prove (a), note that the vertical map on the left can be refined further as

$$\begin{aligned}\mathrm{Hom}_R(F^e_*R, \mathrm{Hom}_S(R,\omega_S)) &= \mathrm{Hom}_S(F^e_*R, \mathrm{Hom}_S(R,\omega_S))\\ &\hookrightarrow \mathrm{Hom}_S(F^e_*S, \mathrm{Hom}_S(R,\omega_S)) \text{ since } F^e_*S \twoheadrightarrow F^e_*R\\ &\hookrightarrow \mathrm{Hom}_S(F^e_*S, \omega_S) \text{ since } \mathrm{Hom}_S(R,\omega_S) \hookrightarrow \omega_S\end{aligned}$$

To prove (b), note that the commutativity follows directly from the commutativity of

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F^e_*S \arrow[d] \\ R \arrow[r] & F^e_*R \end{tikzcd} \qquad \square$$

Definition 2.5 *(cf. Definition 3.1 in [8] and Definition 2.33 in [2]).* Let R be an F-finite Noetherian integral domain of characteristic p. The *parameter test submodule* $\tau(\omega_R)$ is the unique smallest nonzero submodule M of ω_R such that $\Phi_R(F_*M) \subseteq M$. R is called *F-rational* if R is Cohen–Macaulay and $\tau(\omega_R) = \omega_R$. Note that this is not the original definition of F-rationality, but is known to be equivalent [18].

Definition 2.6 *(cf. Definition 3.16 and Theorem 3.18 in [17]).* Let R be an F-finite Noetherian integral domain of characteristic p. Let $I \subseteq R$ be a nonzero ideal and $t \in \mathbb{Q}_{\geq 0}$. We define the *test ideal* $\tau(R, I^t)$, abbreviated $\tau(I^t)$, to be the unique smallest nonzero ideal $J \subseteq R$ such that $\phi(F^e_*(I^{\lceil t(p^e-1)\rceil}J)) \subseteq J$ for all $e > 0$ and all $\phi \in \mathrm{Hom}_R(F^e_*R, R)$.

Definition 2.7 *(cf. Definitions 1.3 and 2.1 and Proposition 2.2 in [21]).* Let R be an F-finite, local, Noetherian, integral domain of characteristic p. Let $I \subset R$ be an ideal and $t \geq 0$ be a real number.

(1) The pair (R, I^t) is *F-pure* if for all large $e \gg 0$, there exists an element $d \in I^{\lfloor t(p^e-1)\rfloor}$ such that $(F^e_*d)R \hookrightarrow F^e_*R$ splits as an R-module homomorphism.
(2) The pair (R, I^t) is *strongly F-regular* if for every $c \neq 0$ there exists $e \geq 0$ and $d \in I^{\lceil tp^e\rceil}$ such that $F^e_*(cd)R \hookrightarrow F^e_*R$ splits as an R-module homomorphism.
(3) The *F-pure threshold* $\mathrm{fpt}_R(I)$ of (R, I) is $\sup\{s \in \mathbb{R}_{\geq 0} |$ the pair (R, I^s) is F-pure$\}$, and when R is strongly F-regular, we also have $\mathrm{fpt}_R(I) = \sup\{s \in \mathbb{R}_{\geq 0} |$ the pair (R, I^s) is strongly F-regular$\}$.

Remark 2.8. Note that when R is local, (R, I^t) is strongly F-regular if and only if $\tau(I^t) = R$. Indeed, suppose (R, I^t) is strongly F-regular. Pick a nonzero element $c \in J$

and take $e \gg 0$ and $d \in I^{\lceil tp^e \rceil}$ satisfying the conditions of strong F-regularity for c, and let $\phi : F^e_* R \to R$ be a map such that $\phi(F^e_*(cd)) = 1$. Then

$$\phi(F^e_*(I^{\lceil t(p^e-1) \rceil} J)) \supseteq \phi(F^e_*(I^{\lceil tp^e \rceil} J)) = R,$$

and so $\tau(I^t) = R$.

On the other hand, if $\tau(I^t) = R$, $0 \neq c \in R$, and $a \in I^{\lceil t \rceil}$, then there exists $e \geq 0$ and $\phi : F^e_* R \to R$ such that $\phi(F^e_*(I^{\lceil t(p^e-1) \rceil} acR)) = R$. Let $b \in I^{\lceil t(p^e-1) \rceil}$ and $f \in R$ such that $\phi(F^e_*(c(abf))) = 1$. Then we are done once we note that $abf \in I^{\lceil t \rceil} I^{\lceil t(p^e-1) \rceil} \subseteq I^{\lceil tp^e \rceil}$.

We will need the following important description of test ideals:

Theorem 2.9 *(cf. Proof of Theorem 3.18 in [17]). With the notations as in Definition 2.6, for any nonzero $a \in \tau(I^t)$, we have:*

$$\tau(I^t) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_*(aI^{\lceil t(p^e-1) \rceil}))$$

where the inner sum runs over all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$.

Remark 2.10. With the notations as in Definition 2.6, the following holds ([3, 3.3])

$$\tau(I^t) = \sum_{e \geq 0} \sum_{\phi \in \mathrm{Hom}_R(F^e_* R, R)} \phi(F^e_*(dI^{\lceil tp^e \rceil})) \tag{2.10.1}$$

where d is a big test element (which is just a nonzero element in $\tau(R) = \tau(R, I^0)$).

If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a perfect field k of characteristic $p > 0$, then one can set $d = 1$ in (2.10.1) and $\mathrm{Hom}_R(F^e_* R, R)$ is a cyclic $F^e_* R$-module generated by Tr^e_R as discussed earlier. Hence by (2.10.1),

$$\begin{aligned}
\tau(I^t) &= \sum_{e \geq 0} \sum_{\phi \in \mathrm{Hom}_R(F^e_* R, R)} \phi(F^e_*(aI^{\lceil t(p^e-1) \rceil})) \\
&= \sum_{e \geq 0} \mathrm{Tr}^e_R(F^e_*(aI^{\lceil t(p^e-1) \rceil})), \text{ for any } \ a \in \tau(I^t) \\
&= \sum_{e \geq 0} \sum_{\phi \in \mathrm{Hom}_R(F^e_* R, R)} \phi(F^e_*(I^{\lceil tp^e \rceil})) \\
&= \sum_{e \geq 0} \mathrm{Tr}^e_R(F^e_*(I^{\lceil tp^e \rceil}))
\end{aligned}$$

Remark 2.11. With the notations as in Definition 2.5, one can show that if $R_{a'}$ is regular, then for every sufficiently large power a of a', $\tau(\omega_R) = \sum_e \Phi^e_R(F^e_*(a \cdot \omega_R))$. This can be proved by a similar argument as [19, Lemma 3.6, Lemma 3.8] so we omit the details.

The following result from [19] will also be used. These results were originally proved in [13] and [9], and they hold as long as R is F-finite. We will only state the version of these results that we need.

Lemma 2.12 *(cf. Theorem 6.9 in [19]). Let R be an integral domain essentially of finite type over a perfect field of characteristic $p > 0$ and let $I, J \subseteq R$ be nonzero ideals and $t \in \mathbb{R}_{\geq 0}$.*

(1) *If J is a reduction of I, then $\tau(I^t) = \tau(J^t)$.*
(2) *If J is generated by r elements, then $\tau(J^r) = J\tau(J^{r-1})$.*

We are ready to prove the characteristic $p > 0$ analogue of Ein's Lemma in [16]:

Lemma 2.13 *(Ein's Lemma in characteristic $p > 0$). Let R be an integral domain essentially of finite type over a perfect field of characteristic $p > 0$ and let $I \subseteq R$ be an equi-dimensional and unmixed ideal of codimension c. If $\tau(I^{c-1}) = R$, then $\tau(I^c) = I$. In particular, if R is strongly F-regular and (R, I^c) is F-pure, then $\tau(I^c) = I$.*

Proof. The lemma will follow from the following two inclusions:

$$\tau(I^c) \subseteq I. \tag{2.13.1}$$

$$I\tau(I^{t-1}) \subseteq \tau(I^t) \text{ for all } t \geq 1. \tag{2.13.2}$$

Indeed, if $\tau(I^{c-1}) = R$, then $I = I\tau(I^{c-1}) \subseteq \tau(I^c) \subseteq I$, and so we have equality throughout.

Proof of (2.13.1). Since inclusion is a local condition, we may assume that R is local with maximal ideal $\mathfrak{m}$. By replacing R by $R[x]_{\mathfrak{m}R[x]}$, we may assume that R has infinite residue field: it is straightforward to check that $\tau(I^c)R[x]_{\mathfrak{m}R[x]} = \tau((IR[x]_{\mathfrak{m}R[x]})^c)$. Now let $\mathfrak{p}$ be a minimal prime of I. Since I is equi-dimensional, $\dim R_{\mathfrak{p}} = c$. Hence $IR_{\mathfrak{p}}$ has a reduction $J \subseteq IR_{\mathfrak{p}}$ generated by c elements. Therefore, since test ideals localize,

$$\tau(I^c)R_{\mathfrak{p}} = \tau((IR_{\mathfrak{p}})^c) = \tau(J^c) = J\tau(J^{c-1}) \subseteq J \subseteq IR_{\mathfrak{p}}.$$

Since every associated prime of I is minimal, this inclusion holds for all associated primes of I, hence it holds globally, *i.e.* $\tau(I^c) \subseteq I$. $\square$

Proof of (2.13.2). This should be well known to experts in the field; we opt to provide a proof here since we could not locate a proper reference. Let $t \in \mathbb{R}_{\geq 1}$, and pick $0 \neq a \in \tau(I^t)$. Then

$$I\tau(I^{t-1}) = I\sum_{e\geq 0}\sum_{\phi} \phi\left(F^e_*\left(aI^{\lceil (t-1)(p^e-1)\rceil}\right)\right)$$

$$
\begin{aligned}
&= \sum_{e\geq 0}\sum_{\phi}\phi\left(F^e_*\left(aI^{[p^e]}I^{\lceil (t-1)(p^e-1)\rceil}\right)\right)\\
&\subseteq \sum_{e\geq 0}\sum_{\phi}\phi\left(F^e_*\left(aI^{p^e}I^{\lceil (t-1)(p^e-1)\rceil}\right)\right)\\
&= \sum_{e\geq 0}\sum_{\phi}\phi\left(F^e_*\left(aI^{p^e+\lceil (t-1)(p^e-1)\rceil}\right)\right)\\
&\subseteq \sum_{e\geq 0}\sum_{\phi}\phi\left(F^e_*\left(aI^{\lceil t(p^e-1)\rceil}\right)\right)\\
&= \tau(I^t),
\end{aligned}
$$

where the inner sum runs over all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and the last inclusion following from the fact that

$$p^e + \lceil (t-1)(p^e-1)\rceil = \lceil p^e + (t-1)(p^e-1)\rceil = \lceil t(p^e-1)+1\rceil > \lceil t(p^e-1)\rceil. \quad \square$$

For the last statement, if (R, I^c) is F-pure, then the F-pure threshold of I is at least c. Since the F-pure threshold is the supremum of those values t for which (R, I^t) is strongly F-regular when R is strongly F-regular [21, Proposition 2.2], we have that (R, I^{c-1}) is strongly F-regular. This means that $\tau(I^{c-1}) = R$ by Remark 2.8, and hence the first statement of the lemma tells us $\tau(I^c) = I$. $\square$

3. *F*-rationality under generic linkage

In this section, we investigate how F-singularities (*e.g.* F-purity, F-rationality, etc) behave under a generic linkage. To this end, we will also consider the behaviors of test ideals under a generic linkage. We begin with recalling the definition of a generic link.

Definition 3.1. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field of positive characteristic. Let I be an equi-dimensional and unmixed ideal of R of height c. Fix a generating set $\{f_1, \ldots, f_r\}$ of I. Let (u_{ij}), $1 \leq i \leq c$, $1 \leq j \leq r$, be a $c \times r$ matrix of variables. Consider c elements $g_1, \ldots, g_c$ in $S = R[u_{ij}]$ defined by

$$g_i := u_{i1}f_1 + u_{i2}f_2 + \cdots + u_{ir}f_r$$

for $1 \leq i \leq c$. Then $J = (g_1, \ldots, g_c) : (IS)$ is called the *first generic link* of I with respect to $\{f_1, \ldots, f_r\}$ (we also call S/J the generic link of R/I with respect to $\{f_1, \ldots, f_r\}$).

Remark 3.2. It is well known that under the above assumptions, if I is reduced, then IS and J are *geometrically linked*, *i.e.*, $IS = (g_1, \ldots, g_c) : J$ and $IS \cap J = (g_1, \ldots, g_c)$. Moreover, J is actually a prime ideal of height c [10, 2.6].

The following theorem is our main technical result in this section.

Theorem 3.3. *With the notation as in Definition 3.1, assuming I is reduced, we have*

(1) $\tau(\omega_{S/J}) \subseteq \tau(I^c) \cdot (S/J)$;
(2) *If I has a minimal reduction generated by at most $c+1$ elements, then* $\tau(\omega_{S/J}) \supseteq \tau(I^c) \cdot (S/J)$; *hence* $\tau(\omega_{S/J}) = \tau(I^c) \cdot (S/J)$ *in this case.*

Our proof of Theorem 3.3(2) requires considering different sets of generators of I. A priori, a generic link (S, J) depends on the choice of generators. The following lemma guarantees that the statement in Theorem 3.3(2) is independent of the choice of generators of I. Its proof follows the same line as the proof of [11, Proposition 2.11].

Lemma 3.4. *Let Λ_1 and Λ_2 be two sets of generators of I and let (S_1, J_1) and (S_2, J_2) be generic links of I with respect to Λ_1 and Λ_2 respectively. Then $\tau(\omega_{S_1/J_1}) \supseteq \tau(I^c)\cdot(S_1/J_1)$ iff $\tau(\omega_{S_2/J_2}) \supseteq \tau(I^c) \cdot (S_2/J_2)$.*

Proof. By considering $\Lambda_1 \cup \Lambda_2$, we can assume that $\Lambda_1 \subseteq \Lambda_2$. By induction on the difference between the cardinality of Λ_1 and Λ_2, we may assume that Λ_2 has one more element than Λ_1, *i.e.* we may assume that $\Lambda_1 = \{f_1, \dots, f_r\}$ and $\Lambda_2 = \Lambda_1 \cup \{f_{r+1}\}$.

Denote the height of I by c. Let $\{u_{ij} \mid 1 \le i \le c, 1 \le j \le r+1\}$ be indeterminates over R. Set $S_1 = R[u_{ij}]_{1\le i\le c, 1\le j\le r}$ and $S_2 = R[u_{ij}]_{1\le i\le c,1\le j\le r+1}$. For $i = 1, \dots, c$, set

$$g_i := u_{i1}f_1 + \cdots + u_{ir}f_r$$

and

$$h_i := u_{i1}f_1 + \cdots u_{i,r+1}f_{r+1}.$$

Then $J_1 = ((g_1, \dots, g_c) :_S IS)$ is the first generic link of I with respect to Λ_1 and $J_2 = ((h_1, \dots, h_c) :_{S_2} IS_2)$ is the first generic link of I with respect to Λ_2.

It is clear that $S_2 = S_1[u_{1,r+1}, \dots, u_{c,r+1}]$. Since $f_{r+1} \in I$, we must have that $f_{r+1} = \sum_{j=1}^r a_j f_j$ for some $a_j \in R$. Let $\varphi : S_2 \to S_2$ be the automorphism given by the linear change of variables

$$u_{ij} \mapsto u_{ij} + u_{i,r+1}a_j$$

for $1 \le i \le c$ and $1 \le j \le r$ and

$$u_{i,r+1} \mapsto u_{i,r+1}$$

for $1 \le i \le c$.

We claim that $\varphi(J_1S_2) = J_2$ and we reason as follows. For $i = 1, \dots, c$, we have that

$$\begin{aligned}\varphi(g_i) &= \sum_{j=1}^{r}(u_{ij} + u_{i,r+1}a_j)f_j = \sum_{j=1}^{r} u_{ij}f_j + u_{i,r+1}\sum_{j=1}^{r} a_jf_j \\ &= \sum_{j=1}^{r} u_{ij}f_j + u_{i,r+1}f_{r+1} = h_i.\end{aligned}$$

Now since $S_1 \hookrightarrow S_2$ is a faithfully flat extension, we have that

$$J_1S_2 = ((g_1, \dots, g_c) :_{S_1} IS_1)S_2 = ((g_1, \dots, g_c)S_2 :_{S_2} IS_2),$$

and hence

$$\begin{aligned}\varphi(J_1S_2) &= \varphi((g_1, \dots, g_c)S_2 :_{S_2} IS_2) = (\varphi(g_1, \dots, g_c)S_2 :_{S_2} \varphi(IS_2)) \\ &= ((h_1, \dots, h_c) :_{S_2} IS_2) = J_2.\end{aligned}$$

Let S_2^{φ} denote the S_1-algebra that is the same as S_2 as a ring and whose S_1-module structure is induced by $S_1 \hookrightarrow S_2 \xrightarrow{\varphi} S_2$. Then we have shown that $J_1 \otimes_{S_1} S_2^{\varphi} = J_2$ and hence $S_1/J_1 \otimes_{S_1} S_2^{\varphi} = S_2/J_2$. Combining Remarks 2.3 and 2.11, one can check that

$$\tau(\omega_{S_1/J_1}) \otimes_{S_1} S_2^{\varphi} = \tau(\omega_{S_1/J_1} \otimes_{S_1} S_2^{\varphi})$$

where the right-hand side is precisely $\tau(\omega_{S_2/J_2})$. Our lemma follows immediately since S_2^{φ} is faithfully flat over S_1. $\square$

The following lemma is also needed in the proof of Theorem 3.3.

Lemma 3.5. *Let c, r be positive integers such that $c = r$ or $c = r-1$. Let $\beta = (\beta_1, \dots, \beta_r)$ be an element of $\mathbb{N}^r$, where $\mathbb{N}$ is the set of non-negative integers. Assume $\sum_i \beta_i = c(p^e - 1)$. Then there exist c elements $\alpha_1, ..., \alpha_c$ in $\mathbb{N}^r$ such that:*

(1) *each α_i has at most two nonzero entries;*
(2) *the sum of the entries of each α_i is $p^e - 1$;*
(3) *$\beta_j = \sum_i \alpha_{ij}$, where $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ir})$.*

Proof. We will induce on r. If $c = r = 1$, then $\beta = (p^e - 1)$ and we let $\alpha_1 = \beta$. If $c = 1, r = 2$, we have $\beta = (\beta_1, \beta_2)$ where $\beta_1 + \beta_2 = p^e - 1$ and we can let $\alpha_1 = (\beta_1, \beta_2)$ and again (1)–(3) hold.

If $c = r$ and $\beta_1 = \cdots = \beta_c = p^e - 1$, then we can set α_i to be the vector with $p^e - 1$ at i-th spot and 0 elsewhere. Otherwise, there must be a $\beta_i < p^e - 1$. Without loss of generality, we assume that $\beta_r < p^e - 1$.

We claim that $\beta_j \geq p^e - 1 - \beta_r$ for some j between 1 and $r-1$, and we reason as follows. If $c = r$, then there must be a j such that $\beta_j > p^e - 1$, and hence $\beta_j \geq p^e - 1 - \beta_r$. Now assume that $c = r - 1$. Suppose $\beta_i < p^e - 1 - \beta_r$ for all $i \leq r-1$, as then we would have:

$$\sum_{i=1}^{r} \beta_i < (r-1)(p^e - 1 - \beta_r) + \beta_r \leq (r-2)(p^e - 1 - \beta_r) + (p^e - 1) \leq (r-1)(p^e - 1)$$
$$= c(p^e - 1)$$

which contradicts the assumption that $\sum_{i=1}^{r} \beta_i = c(p^e - 1)$. So, there is a j between 1 and $r-1$ such that $\beta_j \geq p^e - 1 - \beta_r$.

Set $\alpha_c := (0, \ldots, 0, p^e - 1 - \beta_r, 0, \ldots, \beta_r)$ where $p^e - 1 - \beta_r$ appears in the j-th spot. Consider

$$(\beta_1, \ldots, \beta_{j-1}, \beta_j - (p^e - 1 - \beta_r), \beta_{j+1}, \ldots, \beta_{r-1}).$$

This is an element of $\mathbb{N}^{r-1}$ such that the sum of its entries is $(c-1)(p^e - 1)$. By our induction hypotheses, there are $\gamma_1, \ldots, \gamma_{c-1} \in \mathbb{N}^{r-1}$ that satisfy (1), (2), and (3). For $1 \leq i \leq c-1$, setting α_i be γ_i with a 0 added to the end completes the proof of our lemma. $\square$

Proof of Theorem 3.3. By Remark 3.2, J is a minimal prime of $(g_1, \ldots, g_c)$. Hence once we identify

$$\omega_{S/J} = \mathrm{Hom}_{S/(g_1,\ldots,g_c)}(S/J, S/(g_1, \ldots, g_c)) = ((g_1, \ldots, g_c) : J) \cdot (S/J) = I \cdot (S/J),$$

we know from Lemma 2.4 that

$$\Phi^e_{S/J}(F^e_*(-)) = \Phi^e_{S/(g_1,\ldots,g_c)}(F^e_*(-))|_{\omega_{S/J}} = \mathrm{Tr}^e_S(F^e_*(g_1^{p^e-1} \cdots g_c^{p^e-1} \cdot -))|_{I\cdot(S/J)}.$$

Next we notice that for every $1 \leq k \leq r$, $(S/J)_{f_k} \cong R_{f_k}[u_{ij} | j \neq k]$ is regular. Hence for $N \gg 0$, f_k^N is a test element for S/J. Thus by Remark 2.11, we have:

$$\tau(\omega_{S/J}) = \sum_{e \geq 0} \Phi^e_{S/J}(F^e_*(f_k^N \cdot \omega_{S/J})) = \sum_{e \geq 0} \mathrm{Tr}^e_S(F^e_*(g_1^{p^e-1} \cdots g_c^{p^e-1} \cdot f_k^N \cdot IS)) \cdot (S/J) \tag{3.5.1}$$

Since $f_k \in I$ and R is regular, by Remark 2.10, for $N \gg 0$ we also have:

$$\tau(I^c) \cdot (S/J) = \sum_{e \geq 0} \mathrm{Tr}^e_R(F^e_*((f_1, \ldots, f_r)^{c(p^e-1)} \cdot f_k^N \cdot R)) \cdot (S/J) \tag{3.5.2}$$

When we expand $g_1^{p^e-1} \cdots g_c^{p^e-1}$, it is easy to see from (3.5.1) that $\tau(\omega_{S/J})$ can be generated by elements of the form

$$\operatorname{Tr}_S^e\left(F_*^e\left(\binom{p^e-1}{\alpha_{11},\ldots,\alpha_{1r}}\cdots\binom{p^e-1}{\alpha_{c1},\ldots,\alpha_{cr}}f_1^{\beta_1}f_2^{\beta_2}\cdots f_r^{\beta_r}\prod_{\substack{1\le i\le c\\1\le j\le r}}u_{ij}^{\alpha_{ij}}\cdot f_k^N\cdot s\right.\right.$$
$$\left.\left.\cdot\prod_{\substack{1\le i\le c\\1\le j\le r}}u_{ij}^{\gamma_{ij}}\right)\right) \tag{3.5.3}$$

where $0\le\alpha_{ij}\le p^e-1$, $\beta_j=\sum_{i=1}^c\alpha_{ij}$, $\sum_{j=1}^r\beta_j=c(p^e-1)$ and $s\in I$. By definition of the trace map, this is equal to

$$\prod_{\substack{1\le i\le c\\1\le j\le r}}u_{ij}^{\frac{\alpha_{ij}+\gamma_{ij}-(p^e-1)}{p^e}}$$
$$\cdot\operatorname{Tr}_R^e\left(F_*^e\left(\binom{p^e-1}{\alpha_{11},\ldots,\alpha_{1r}}\cdots\binom{p^e-1}{\alpha_{c1},\ldots,\alpha_{cr}}f_1^{\beta_1}f_2^{\beta_2}\cdots f_r^{\beta_r}\cdot f_k^N\cdot s\right)\right)$$

where $\frac{\alpha_{ij}+\gamma_{ij}-(p^e-1)}{p^e}$ denotes 0 if $\alpha_{ij}+\gamma_{ij}\not\equiv-1 \bmod p^e$. But it is clear that this element is in $\tau(I^c)\cdot S$ by expression (3.5.2). This proves (1).

Next we prove (2). By Lemma 3.4 we can assume that $\tilde{I}=(f_1,\ldots,f_{c+1})$ is a reduction of I (the case that I has a reduction generated by c elements is similar). Hence by the arguments above, we have that, for $1\le k\le c$ and $N\gg0$,

$$\tau(I^c)\cdot(S/J)=\tau(\tilde{I}^c)\cdot(S/J)=\sum_{e\ge0}\operatorname{Tr}_R^e(F_*^e((f_1,\ldots,f_{c+1})^{c(p^e-1)}\cdot f_k^{N+1}\cdot R))\cdot(S/J)$$

Given a generator $f_1^{\beta_1}\cdots f_{c+1}^{\beta_{c+1}}$ of $(f_1,\ldots,f_{c+1})^{c(p^e-1)}$, we can find $\alpha_1,\ldots,\alpha_c\in\mathbb{N}^{c+1}$ satisfying the conclusion of Lemma 3.5. Then

$$\prod_{\substack{1\le i\le c\\1\le j\le c+1}}(u_{ij}f_j)^{\alpha_{ij}}=\prod_{\substack{1\le i\le c\\1\le j\le c+1}}u_{ij}^{\alpha_{ij}}\prod_{1\le j\le c+1}f_j^{\beta_j}$$

appears with coefficient $\binom{p^e-1}{\alpha_{11},\ldots,\alpha_{1,c+1}}\cdots\binom{p^e-1}{\alpha_{c1},\ldots,\alpha_{c,c+1}}$ in the product $g_1^{p^e-1}\cdots g_c^{p^e-1}$. Because each multinomial coefficient $\binom{p^e-1}{\alpha_{i1},\ldots,\alpha_{i,c+1}}=\binom{p^e-1}{\alpha_{ij_i}}$ for some j_i by Lemma 3.5 (1)–(2), they are nonzero in k.

Each α_{ij} is less than p^e, so let

$$s'=\left(\prod_{\substack{1\le i\le c\\1\le j\le c+1}}u_{ij}^{p^e-1-\alpha_{ij}}\right)\left(\prod_{\substack{1\le i\le c\\c+2\le j\le r}}u_{ij}^{p^e-1}\right).$$

Then $\operatorname{Tr}_S^e(F_*^e(-\cdot s'))$ sends $\prod_{1\le l\le n}x_l^{p^e-1}\prod_{\substack{1\le i\le c\\1\le j\le c+1}}u_{ij}^{\alpha_{ij}}$ to 1 and all other basis elements to 0. Hence,

$$\begin{aligned}
&\operatorname{Tr}^e_S(F^e_*(g_1^{p^e-1}\cdots g_c^{p^e-1}\cdot f_k^{N+1}\cdot s'\cdot R))\\
&=\operatorname{Tr}^e_S\left(F^e_*\left(\binom{p^e-1}{\alpha_{11},\dots,\alpha_{1,c+1}}\cdots\binom{p^e-1}{\alpha_{c1},\dots,\alpha_{c,c+1}}\cdot\prod_{\substack{1\le i\le c\\1\le j\le r}}u_{ij}^{p^e-1}\prod_{j=1}^{c+1}f_j^{\beta_j}\cdot f_k^{N+1}R\right)\right)\\
&=\operatorname{Tr}^e_R\left(F^e_*\left(\prod_{j=1}^{c+1}f_j^{\beta_j}\cdot f_k^{N+1}\cdot R\right)\right).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\operatorname{Tr}^e_R\left(F^e_*\left(\prod_{j=1}^{c+1}f_j^{\beta_j}\cdot f_k^{N+1}\cdot R\right)\right)\cdot(S/J)\\
&\quad=\operatorname{Tr}^e_S(F^e_*(g_1^{p^e-1}\cdots g_c^{p^e-1}\cdot f_k^N\cdot f_k s'R))\cdot(S/J)\subseteq\tau(\omega_{S/J})
\end{aligned}$$

for every generator $\prod_{j=1}^{c+1}f_j^{\beta_j}$ of $(f_1,\dots,f_{c+1})^{c(p^e-1)}$, where the second inclusion follows from expression (3.5.1). Therefore we have

$$\begin{aligned}
\tau(I^c)\cdot(S/J)&=\tau(\tilde{I}^c)\cdot(S/J)\\
&=\sum_{e\ge 0}\operatorname{Tr}^e_R(F^e_*((f_1,\dots,f_{c+1})^{c(p^e-1)}\cdot f_k^{N+1}\cdot R))\cdot(S/J)\\
&\subseteq\tau(\omega_{S/J}).\quad\square
\end{aligned}$$

Remark 3.6. The proof of Theorem 3.3 (2) requires the minimal reduction be generated by at most $c+1$ elements. If not, then we are not in the case of Lemma 3.5 and it may be the case that there are always at least three nonzero entries in some α_i. Consequently, multinomial coefficients must be taken into consideration.

Corollary 3.7. *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2), $\tau(\omega_{S/J})=\omega_{S/J}$ if and only if $\tau(I^c)=I$. In particular, S/J has F-rational singularities if and only if S/J is Cohen–Macaulay and $\tau(I^c)=I$.*

Proof. If $\tau(I^c)=I$, then Theorem 3.3 immediately implies $\tau(\omega_{S/J})=\omega_{S/J}$.

Conversely, assume that $\tau(I^c)\neq I$ and $\tau(\omega_{S/J})=\omega_{S/J}$. Since $\tau(I^c)$ is always contained in I by (2.13.1), at least one of the generators of I is not in $\tau(I^c)$, say f_1. From Theorem 3.3, we can see that $\tau(I^c)S+J=IS+J$; hence $f_1\in\tau(I^c)S+J$. Thus, there are elements $a\in\tau(I^c)S$ and $b\in J$ such that $f_1=a+b$. (Note that $b\neq 0$.) Then we have $f_1-a\in J$ which implies that $(f_1-a)f_1\in(g_1,\dots,g_c)$. However, this is impossible because of the degrees in the u_{ij}. This is a contradiction.

The last assertion is clear because S/J is F-rational if and only if S/J is Cohen–Macaulay and $\tau(\omega_{S/J})=\omega_{S/J}$. $\square$

Corollary 3.8. *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2), if the pair (R, I^c) is F-pure and R/I is Cohen–Macaulay, then S/J is F-rational. In particular, if R/I is an F-pure complete intersection, then S/J is F-rational.*

Proof. By Lemma 2.13, (R, I^c) is F-pure implies $\tau(I^c) = I$. The first statement thus follows from Corollary 3.7. Finally, it is well known that when R/I is an F-pure complete intersection, the pair (R, I^c) is F-pure. This follows from a Fedder type criterion ([20, Lemma 3.9] and others). $\square$

We can recover [16, Corollary 3.4] in the complete intersection and almost complete intersection cases.

Corollary 3.9. *Let $I = (f_1, \ldots, f_r)$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$ and let c be the codimension of I. Let S and J be in Definition 3.1. Assume that $r \leq c+1$. Then S/J has rational singularities if and only if S/J is Cohen–Macaulay and $\mathscr{I}(I^c) = I$, where $\mathscr{I}(I^c)$ is the multiplier ideal of I^c.*

Proof. By [18] and [7], S/J has rational singularities if and only if its reduction $(S/J)_p$ is F-rational for all $p \gg 0$. It is easy to see that, for $p \gg 0$, the reduction J_p of J is a generic link of the reduction I_p of I. Hence, S/J has rational singularities if and only if $(S/J)_p$ is Cohen–Macaulay and $\tau(I_p^c) = I_p$ for $p \gg 0$ by Corollary 3.7. On the other hand, it was proved in [13] that $\mathscr{I}(I^c)_p = \tau(I_p^c)$ for all $p \gg 0$. Therefore, we have S/J has rational singularities if and only if $(S/J)_p$ is Cohen–Macaulay and $\mathscr{I}(I^c)_p = I_p$ for $p \gg 0$. This completes the proof. $\square$

4. Behavior of F-pure threshold under generic linkage

In this section we investigate behaviors of F-pure thresholds under generic linkages. We begin with an easy lemma.

Lemma 4.1. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of R. Let Λ_1 and Λ_2 be 2 sets of generators of I and let (S_i, J_i) be the generic link with respect to Λ_i ($i = 1, 2$). Then*

$$\mathrm{fpt}_{S_1}(J_1) = \mathrm{fpt}_{S_2}(J_2).$$

Proof. As in the proof of Lemma 3.4, we can assume that $\Lambda_1 = \{f_1, \ldots, f_r\}$ and $\Lambda_2 = \{f_1, \ldots, f_r, f_{r+1}\}$. Let φ and S_2^φ be the same as in the proof of Lemma 3.4. It is straightforward to check that

$$\tau(J_1^t) \otimes_{S_1} S_2^\varphi = \tau(J_2^t)$$

for each nonnegative real number t. Our lemma follows immediately. $\square$

Remark 4.2. Let $k \subseteq K$ be an extension of perfect fields and let $R = k[x_1, \ldots, x_n]$ and $T = K[x_1, \ldots, x_n]$. Since $\mathrm{Hom}_R(R^{1/p^e}, R)$ and $\mathrm{Hom}_T(T^{1/p^e}, T)$ are generated by the same projection, we have $\tau_R(I^t) = \tau_T((IT)^t)$ (*cf.* [1, Remark 2.18]).

Theorem 4.3. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of height c in R. Assume that $I = (f_1, \ldots, f_s)$ and that I has a reduction $\tilde{I}$ generated by r elements. Let $S = R[u_{ij}]_{1 \le i \le c, 1 \le j \le s}$ be a polynomial ring over R. For $1 \le i \le c$, let*

$$g_i = u_{i1} f_1 + u_{i2} f_2 + \ldots + u_{is} f_s.$$

Then $\mathrm{fpt}_S(g_1, \ldots, g_c) \ge \frac{c}{r} \mathrm{fpt}_R(I)$.

Proof. By Lemma 4.1, we can add the generators of $\tilde{I}$ to those of I and then assume that $\tilde{I} = (f_1, \ldots, f_r)$. Since $\tilde{I}$ is a reduction of I, it follows from [21, Proposition 2.2(6)] that $\mathrm{fpt}_R(I) = \mathrm{fpt}_R(\tilde{I})$. Hence it suffices to show that $\tau_R(\tilde{I}^t) = R$ implies $\tau_S((g_1, \ldots, g_c)^{\frac{ct}{r}}) = S$ for a positive real number t. To this end, assume that $\tau_R(\tilde{I}^t) = R$. By Remark 4.2, we may assume that k is algebraically closed.

We wish to show that $\tau_S((g_1, \ldots, g_c)^{\frac{ct}{r}}) = S$. Suppose otherwise and we seek a contradiction. There is a maximal ideal $\mathfrak{m}$ of S such that $\tau_S((g_1, \ldots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$. Since k is algebraically closed, we can write $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n, u_{11} - b_{11}, \ldots, u_{cr} - b_{cr})$ for some $a_i, b_{ij} \in k$. Set $\mathfrak{n} = (x_1 - a_1, \ldots, x_n - a_n)$. Since $\tau_R(\tilde{I}^t) = R$, there exist an integer e, an R-linear map $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, and nonnegative integers $\alpha_1, \ldots, \alpha_r$ with $\sum_i \alpha_i = \lceil tp^e \rceil$ such that $\phi(f_1^{\alpha_1/p^e} \cdots f_r^{\alpha_r/p^e}) \notin \mathfrak{n}$.

At this point we show that each $f_j \in \mathfrak{n}$, and therefore $\alpha_j \le p^e - 1$ for all j. Indeed, let $e \ge 1$ such that $p^e \ge c/(c-t)$ and let $\psi : S^{1/p^e} \to S$ send the basis element $u_{1j}^{(p^e-1)/p^e} u_{2j}^{(p^e-1)/p^e} \cdots u_{cj}^{(p^e-1)/p^e}$ to 1 and all other basis elements $x_\ell^{a_\ell/p^e} u_{ij}^{b_{ij}/p^e}$ to 0. Now $f_j^c g_1^{p^e-1} g_2^{p^e-1} \cdots g_c^{p^e-1} \in (g_1, \cdots, g_c)^{\lceil (ct/r)p^e \rceil}$, because $(ct/r)p^e \le tp^e \le c(p^e - 1)$. Therefore

$$\begin{aligned} f_j^c &= \psi(f_j^{c/p^e} u_{1j}^{(p^e-1)/p^e} u_{2j}^{(p^e-1)/p^e} \cdots u_{cj}^{(p^e-1)/p^e} f_j^{c(p^e-1)/p^e}) \\ &= \psi(f_j^{c/p^e} g_1^{(p^e-1)/p^e} g_2^{(p^e-1)/p^e} \cdots g_c^{(p^e-1)/p^e}) \\ &\subseteq \psi(((g_1, \cdots, g_c)^{\lceil (ct/r)p^e \rceil})^{1/p^e}) \subseteq \mathfrak{m} \end{aligned}$$

by our choice of ψ and the assumption that $\tau_S((g_1, \ldots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$. It follows that $f_j \in \mathfrak{m} \cap R = \mathfrak{n}$.

Without loss of generality, we may assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_r$. Consequently,

$$\alpha_1 + \cdots + \alpha_c \ge \left\lceil \frac{c}{r}(\alpha_1 + \cdots + \alpha_r) \right\rceil = \left\lceil \frac{c}{r} \lceil tp^e \rceil \right\rceil \ge \left\lceil \frac{c}{r} tp^e \right\rceil$$

Let $\phi_{\underline{\alpha}} = \phi(f_{c+1}^{\alpha_{c+1}/p^e} \cdots f_r^{\alpha_r/p^e} \cdot -)$, *i.e.* pre-multiplication by $f_{c+1}^{\alpha_{c+1}/p^e} \cdots f_r^{\alpha_r/p^e}$ followed by the application of ϕ. It is clear that $\phi_{\underline{\alpha}} : R^{1/p^e} \to R$ is an R-linear map and that $\phi_{\underline{\alpha}}(f_1^{\alpha_1/p^e}\cdots f_c^{\alpha_c/p^e}) \notin \mathfrak{n}$. We can extend $\phi_{\underline{\alpha}}$ to an S-linear map $\psi_{\underline{\alpha}} : R^{1/p^e}[u_{ij}] \to S = R[u_{ij}]$ that sends each u_{ij} to itself and restricts to $\phi_{\underline{\alpha}}$ on R^{1/p^e}.

It is clear that $S^{1/p^e} = R^{1/p^e}[u_{ij}^{1/p^e}]$ is a free $R^{1/p^e}[u_{ij}]$-module with a basis $\{\prod_{0\le b_{ij}\le p^e-1} u_{ij}^{b_{ij}/p^e}\}$. Let $\pi_{\underline{\alpha}} : R^{1/p^e}[u_{ij}^{1/p^e}] \to R^{1/p^e}[u_{ij}]$ be the projection that sends $u_{11}^{\alpha_1/p^e}\cdots u_{cc}^{\alpha_c/p^e}$ to 1 and all other basis element to 0.

Let $\theta_{\underline{\alpha}}$ be the composition of $S^{1/p^e} \xrightarrow{\pi_{\underline{\alpha}}} R^{1/p^e}[u_{ij}] \xrightarrow{\psi_{z_{\underline{\alpha}}}} S$. It is clear that $\theta_{\underline{\alpha}}$ is S-linear. By the construction of $\pi_{\underline{\alpha}}$, it is straightforward to check that

$$\theta_{\underline{a}}(g_1^{\alpha_1/p^e}\cdots g_c^{\alpha_c/p^e}) = \theta_{\underline{\alpha}}((u_{11}f_1)^{\alpha_1/p^e}\cdots (u_{cc}f_c)^{\alpha_c/p^e}) = \phi(f_1^{\alpha_1/p^e}\cdots f_r^{\alpha_r/p^e}).$$

Since $\phi(f_1^{\alpha_1/p^e}\cdots f_r^{\alpha_r/p^e})$ in R but not in $\mathfrak{n} = (x_1-a_1,\dots,x_n-a_n)$, we must have

$$\phi(f_1^{\alpha_1/p^e}\cdots f_r^{\alpha_r/p^e}) \notin \mathfrak{m} = (x_1-a_1,\dots,x_n-a_n,u_{11}-b_{11},\dots,u_{cr}-b_{cr}),$$

a contradiction to the assumption that $\tau_S((g_1,\dots,g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$ (note that $g_1^{\alpha_1}\cdots g_c^{\alpha_c} \in (g_1,\dots,g_c)^{\lceil \frac{ct}{r}p^e\rceil}$). $\square$

We have some immediate corollaries.

Corollary 4.4. *Let $R = k[x_1,\dots,x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of height c in R. Let J be a generic link of I in $S = R[u_{ij}]$. The following hold:*

(1) *If I has a reduction generated by r elements, then* $\operatorname{fpt}_S(J) \ge \frac{c}{r}\operatorname{fpt}_R(I)$.
(2) *If I has a reduction generated by c elements, in particular if I is a complete intersection, then* $\operatorname{fpt}_S(J) \ge \operatorname{fpt}_R(I)$.
(3) $\operatorname{fpt}_S(J) \ge \frac{c}{n}\operatorname{fpt}_R(I)$ *(note $n = \dim(R)$).*

Proof. To prove (1), note that since $(g_1,\dots,g_c) \subseteq J$, we have $\operatorname{fpt}_S(J) \ge \operatorname{fpt}_S(g_1,\dots,g_c)$. Theorem 4.3 then completes the proof.

(2) is an immediate consequence of (1).

(3) By Remark 4.2, passing to the algebraic closure of k doesn't affect $\operatorname{fpt}_R(I)$ and $\operatorname{fpt}_S(J)$. Hence we can assume that k is algebraically closed and hence is infinite. [14, Theorem] asserts that each ideal I admits a reduction generated by n elements. We are done by (1). $\square$

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