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On Lusztig-Dupont homology of flag complexes

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A B S T R A C T

Let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$. The spherical building X_V associated with $GL(V)$ is the order complex of the nontrivial linear subspaces of V. Let $\mathfrak{g}$ be the local coefficient system on X_V, whose value on the simplex $\sigma = [V_0 \subset \cdots \subset V_p] \in X_V$ is given by $\mathfrak{g}(\sigma) = V_0$. The homology module $\mathcal{D}^1(V) = \tilde{H}_{n-2}(X_V; \mathfrak{g})$ plays a key role in Lusztig's seminal work on the discrete series representations of $GL(V)$. Here, some further properties of $\mathfrak{g}$ and its exterior powers are established. These include a construction of an explicit basis of $\mathcal{D}^1(V)$, a computation of the dimension of $\mathcal{D}^k(V) = \tilde{H}_{n-k-1}(X_V; \wedge^k \mathfrak{g})$, and the following twisted analogue of a result of Smith and Yoshiara: For any $1 \leq k \leq n-1$, the minimal support size of a non-zero $(n-k-1)$-cycle in the twisted homology $\tilde{H}_{n-k-1}(X_V; \wedge^k \mathfrak{g})$ is $\frac{(n-k+2)!}{2}$.

1. Introduction

Let q be a prime power and let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$. The spherical building associated with $G = GL(V)$ is the order complex X_V

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of the nontrivial linear subspaces of V: The vertices of X_V are the linear subspaces $0 \neq U \subsetneq V$, and the k-simplices are families of subspaces of the form $\{U_0, \ldots, U_k\}$, where $U_0 \subsetneq \cdots \subsetneq U_k$. The homotopy type of X_V was determined by Solomon and Tits [9] (see also Theorem 4.73 in [1]).

Theorem 1.1 *(Solomon-Tits). X_V is homotopy equivalent to a wedge of $q^{\binom{n}{2}}$ $(n-2)$-spheres. In particular, the reduced homology of X_V with coefficients in a field $\mathbb{K}$ is given by*

$$\dim \tilde{H}_i(X_V; \mathbb{K}) = \begin{cases} 0 & i \neq n-2, \\ q^{\binom{n}{2}} & i = n-2. \end{cases}$$

The natural action of G on X_V induces a representation of G on $\tilde{H}_{n-2}(X_V; \mathbb{K})$. Viewed as a G-module, $\tilde{H}_{n-2}(X_V; \mathbb{K})$ is the *Steinberg module* of G over $\mathbb{K}$ (see e.g. section 6.4 in [6]). We recall some facts concerning X_V and the Steinberg module. For a subset $S \subset V$, let $\langle S \rangle = \text{span}\,(S)$ denote the linear span of S. Let $[n] = \{1, \ldots, n\}$. Let $B = \{v_1, \ldots, v_n\}$ be a basis of V and let $\tilde{B}$ be the set of vertices of X_V given by

$$\tilde{B} = \big\{ \langle v_i : i \in I \rangle \ : \ \emptyset \neq I \subsetneq [n] \big\}.$$

The induced subcomplex $X_V[\tilde{B}]$ is the *apartment* determined by B. Clearly, $X_V[\tilde{B}]$ is isomorphic to the barycentric subdivision of the boundary of a $(n-1)$-simplex, and thus

$$\tilde{H}_{n-2}(X_V[\tilde{B}]; \mathbb{K}) \cong \mathbb{K}.$$

We next exhibit a generator z_B of $\tilde{H}_{n-2}(X_V[\tilde{B}]; \mathbb{K})$. For a permutation π in the symmetric group $\mathbb{S}_n$ and for $1 \leq i \leq n$, let $V_\pi(i) = \langle v_{\pi(1)}, \ldots, v_{\pi(i)} \rangle$ and let σ_π be the ordered $(n-2)$-simplex

$$\sigma_\pi = [V_\pi(1) \subset \cdots \subset V_\pi(n-1)].$$

Then $z_B = \sum_{\pi \in \mathbb{S}_n} \text{sgn}(\pi) \sigma_\pi$ is a generator of $\tilde{H}_{n-2}(X_V[\tilde{B}]; \mathbb{K})$. The following explicit construction of a basis of $\tilde{H}_{n-2}(X_V[\tilde{B}]; \mathbb{K})$ is due to Solomon [9] (see also Theorem 4.127 in [1]).

Theorem 1.2 *(Solomon). Let σ be a fixed $(n-2)$-simplex of X_V. Then*

$$\big\{ z_B : B \textit{ is a basis of } V \textit{ such that } \sigma \in X_V[\tilde{B}] \big\}$$

is a basis of $\tilde{H}_{n-2}(X_V[\tilde{B}]; \mathbb{K})$.

The *support* of a $(n-2)$-chain $c = \sum_\sigma a_\sigma \sigma \in C_{n-2}(X_V; \mathbb{K})$ is

$$\text{supp}(c) = \{\sigma : a_\sigma \neq 0\}.$$

Clearly, $|\mathrm{supp}(z_B)| = n!$ for any basis B of V. Smith and Yoshiara [7] proved that the z_B's are in fact the nontrivial $(n-2)$-cycles of minimal support in X_V.

Theorem 1.3 *(Smith-Yoshiara).*

$$\min\left\{|\mathrm{supp}(z)| : 0 \neq z \in \tilde{H}_{n-2}(X_V;\mathbb{K})\right\} = n!.$$

In this paper we study analogues of Theorems 1.1, 1.2 and 1.3 for the homology of X_V with certain local coefficient systems introduced by Lusztig and Dupont. We first recall some definitions. Let X be a simplicial complex on a vertex set S. Let $\prec$ be an arbitrary fixed linear order on S. For $k \geq -1$ let $X(k)$ denote the set of k-dimensional simplices of X, and let $X^{(k)}$ denote the k-dimensional skeleton of X. A simplex $\sigma \in X(k)$ will be written as $\sigma = [s_1, \ldots, s_{k+1}]$ where $s_1 \prec \cdots \prec s_{k+1}$. The i-th face of σ as above is the $(k-1)$-simplex $\sigma_i = [s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{k+1}]$. For a 0-dimensional simplex $\sigma = [s_1]$, let $\sigma_1 = \emptyset$ be the empty simplex. A *local system* $\mathcal{F}$ on X is an assignment of an abelian group $\mathcal{F}(\sigma)$ to each simplex $\sigma \in X$, together with homomorphisms $\rho_\sigma^\tau : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$ for each $\sigma \subset \tau$ satisfying the usual compatibility conditions: $\rho_\sigma^\sigma =$ identity, and $\rho_\eta^\sigma \rho_\sigma^\tau = \rho_\eta^\tau$ if $\eta \subset \sigma \subset \tau$. A $\mathcal{F}$-twisted k-chain of X is a formal linear combination $c = \sum_{\sigma \in X(k)} c(\sigma)\sigma$, where $c(\sigma) \in \mathcal{F}(\sigma)$. Let $C_k(X;\mathcal{F})$ denote the group of $\mathcal{F}$-twisted k-chains of X. For $k \geq 0$ define the boundary map

$$\partial_k : C_k(X;\mathcal{F}) \to C_{k-1}(X;\mathcal{F})$$

by

$$\partial_k\left(\sum_{\sigma \in X(k)} c(\sigma)\sigma\right) = \sum_{\sigma \in X(k)} \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{\sigma_i}^\sigma\left(c(\sigma)\right)\sigma_i.$$

For $k = -1$ let ∂_{-1} denote the zero map $C_{-1}(X;\mathcal{F}) = \mathcal{F}(\emptyset) \to 0$. The homology of X with coefficients in $\mathcal{F}$, denoted by $H_*(X,\mathcal{F})$, is the homology of the complex $\oplus_{i \geq 0} C_i(X;\mathcal{F})$. The reduced homology $\tilde{H}_*(X,\mathcal{F})$ is the homology of $\oplus_{i \geq -1} C_i(X;\mathcal{F})$. Let X, Y be two simplicial complexes and let $f : X \to Y$ be a simplicial map such that $\dim f(\sigma) = \dim \sigma$ for all $\sigma \in X$. Let $\mathcal{G}$ be a local system on Y. The inverse image system $\mathcal{F} = f^{-1}\mathcal{G}$ given by $\mathcal{F}(\sigma) = \mathcal{G}(f(\sigma))$ is a local system on X. The induced mapping on homology is denoted by $f_* : \tilde{H}_k(X;\mathcal{F}) \to \tilde{H}_k(Y;\mathcal{G})$. For further discussion of local coefficient homology, see e.g. chapter 7 in [3] and chapter 10 in [6].

Lusztig, in his seminal work [5] on discrete series representations of $GL(V)$, defined and studied the local system $\mathfrak{g}$ on X_V given by $\mathfrak{g}(U_1 \subset \cdots \subset U_\ell) = U_1$ and $\mathfrak{g}(\emptyset) = V$, where the connecting homomorphisms ρ_σ^τ's are the natural inclusion maps. Dupont, in his study of homological approaches to scissors congruences [4], extended some of Lusztig's results to the higher exterior powers $\wedge^k \mathfrak{g}$ over flag complexes of Euclidean spaces. For $i \geq 0$ let $\tilde{H}_i(X_V;\wedge^k \mathfrak{g})$ denote the i-th homology $\mathbb{F}_q$-module of the chain complex of X_V

with $\wedge^k \mathfrak{g}$ coefficients. Note that $C_{-1}(X_V; \wedge^k \mathfrak{g}) = \wedge^k V$. The following result was proved by Lusztig (Theorem 1.12 in [5]) for $k = 1$, and extended by Dupont (Theorem 3.12 in [4]) to all $k \geq 1$.

Theorem 1.4 *(Lusztig, Dupont). Let $1 \leq k \leq n-1$. Then $\tilde{H}_i(X_V; \wedge^k \mathfrak{g}) = 0$ for $i \neq n-k-1$.*

Let $\mathcal{D}^k(V) = \tilde{H}_{n-k-1}(X_V; \wedge^k \mathfrak{g})$. Lusztig (Theorem 1.14 in [5]) proved that

$$\dim \mathcal{D}^1(V) = \prod_{i=1}^{n-1} (q^i - 1). \tag{1}$$

The proof of (1) in [5] is based on the case $k = 1$ of Theorem 1.4, combined with an Euler characteristic computation. In Section 2 we describe an explicit basis of $\mathcal{D}^1(V)$. This construction may be regarded as a twisted counterpart of Theorem 1.2. Concerning the dimension of $\mathcal{D}^k(V)$ for general k, we prove the following extension of Theorem 1.

Theorem 1.5.

$$\dim \mathcal{D}^k(V) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_{n-k} \leq n-1} \prod_{j=1}^{n-k} (q^{\alpha_j} - 1). \tag{2}$$

Our final result is an analogue of the Smith-Yoshiara Theorem 1.3 for the coefficient system $\wedge^k \mathfrak{g}$.

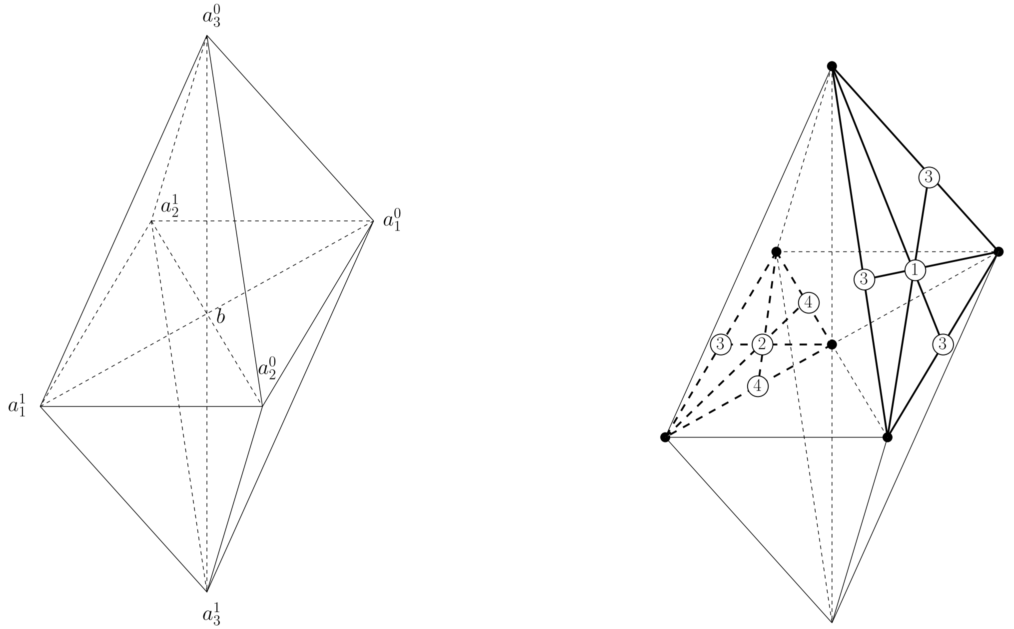
Theorem 1.6.

$$\min \left\{ |\mathrm{supp}(w)| : 0 \neq w \in \mathcal{D}^k(V) \right\} = \frac{(n-k+2)!}{2}.$$

The paper is organised as follows. In Section 2 we construct an explicit basis for $\mathcal{D}^1(V)$. In Section 3 we use an exact sequence due to Dupont to prove Theorem 1.5. In Section 4 we recall the Nerve lemma for homology with local coefficients, and obtain a vanishing result for a certain local system on the simplex. These results are used to prove Theorem 1.6. We conclude in Section 5 with some remarks and open problems.

2. A basis for $\mathcal{D}^1(V)$

In this section we construct an explicit basis for $\mathcal{D}^1(V) = \tilde{H}_{n-2}(X_V; \mathfrak{g})$. Let $V = \mathbb{F}_q^n$ and let $e_1, \ldots, e_n$ be the standard basis of V. For $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in V$ let $a \cdot b$ denote the standard bilinear form $\sum_{i=1}^n a_i b_i$. For a subset $S \subset V$, let $S^\perp = \{u \in V : u \cdot s = 0, \text{ for all } s \in S\}$. Let $\prec$ be any linear order on $X_V(0)$ such that $U \prec U'$ if $\dim U < \dim U'$. Then an $(n-2)$-simplex in X_V is of the form $[U_1, \ldots, U_{n-1}]$, where $0 \neq U_1 \subsetneq \cdots \subsetneq U_{n-1} \subsetneq V$.



(a) $(n-2)$-dimensional K for $n = 4$ (b) $(n-3)$-simplices in $\mathrm{sd}(K)$

Fig. 1. Four types of $(n-3)$-simplices in $\mathrm{sd}(K) = \mathrm{supp}(c_{\mathbf{v}})$.

For simplicial complexes Y, Z defined on disjoint vertex sets, let $Y * Z = \{\sigma \cup \tau : \sigma \in Y, \tau \in Z\}$ denote their simplicial join. Let $a_1^0, a_1^1, \ldots, a_{n-1}^0, a_{n-1}^1, b$ be $2n-1$ distinct elements. Let M denote the octahedral $(n-2)$-sphere

$$M = \{a_1^0, a_1^1\} * \cdots * \{a_{n-1}^0, a_{n-1}^1\},$$

and let $K = M \cup \big(\{b\} * M^{(n-3)}\big)$. See Fig. 1a for a depiction of the 2-dimensional complex K when $n = 4$.

Choose a linear order $\prec_1$ on the simplices of K such that $\sigma \prec_1 \tau$ if $\dim \sigma > \dim \tau$. The barycentric subdivision of K, denoted by $\mathrm{sd}(K)$, is the complex whose vertex set $\mathrm{sd}(K)(0)$ consists of all nonempty simplices of K, and whose k-simplices (ordered according to $\prec_1$) are $[\sigma_1, \ldots, \sigma_{k+1}]$ where $\sigma_1 \supsetneq \cdots \supsetneq \sigma_{k+1}$. For a sequence $\mathbf{x} = (x_1, \ldots, x_{n-1})$ of distinct vertices of K, such that $\{x_1, \ldots, x_{n-1}\} \in K$, let $S(\mathbf{x})$ denote the $(n-2)$-simplex of $\mathrm{sd}(K)$ given by

$$S(\mathbf{x}) = [\{x_1, \ldots, x_{n-1}\}, \{x_1, \ldots, x_{n-2}\}, \ldots, \{x_1\}].$$

For a permutation π in the symmetric group $\mathbb{S}_{n-1}$ let $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n-1)})$. Let $E = \{0,1\}^{n-1}$ and for $1 \le j \le n-1$ let

$$E_j = \{(\epsilon_1, \ldots, \epsilon_{n-1}) \in E : \epsilon_j = 0\}.$$

For $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1}) \in E$ and $1 \leq j \leq n-1$ let $\mathbf{a}^{\boldsymbol{\epsilon}} = (a_1^{\epsilon_1}, \ldots, a_{n-1}^{\epsilon_{n-1}})$ and let

$$\mathbf{a}^{\boldsymbol{\epsilon},j} = (a_1^{\epsilon_1}, \ldots, a_{j-1}^{\epsilon_{j-1}}, b, a_{j+1}^{\epsilon_{j+1}}, \ldots, a_{n-1}^{\epsilon_{n-1}}).$$

Let $T_{q,n}$ denote the set of all sequences $\mathbf{v} = (v_1, \ldots, v_{n-1}) \in V^{n-1}$ such that $v_i \in e_i + \langle e_{i+1}, \ldots, e_n \rangle$ and $v_i \neq e_i$ for all $1 \leq i \leq n-1$. Clearly $|T_{q,n}| = \prod_{i=1}^{n-1}(q^i - 1)$.

Fix $\mathbf{v} = (v_1, \ldots, v_{n-1}) \in T_{q,n}$. For $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1}) \in E$, let $\mathbf{v}^{\boldsymbol{\epsilon}} = (u_1, \ldots, u_{n-1})$, where

$$u_i = \begin{cases} e_i & \epsilon_i = 0, \\ v_i & \epsilon_i = 1. \end{cases}$$

For $1 \leq j \leq n-1$ let $\mathbf{v}^{\boldsymbol{\epsilon},j} = (u_1, \ldots, u_{n-1})$, where

$$u_i = \begin{cases} e_n & i = j, \\ e_i & i \neq j \ \& \ \epsilon_i = 0, \\ v_i & i \neq j \ \& \ \epsilon_i = 1. \end{cases}$$

Define $\theta_{\mathbf{v}} : K(0) \to V$ by

$$\theta_{\mathbf{v}}(x) = \begin{cases} e_i & x = a_i^0, \\ v_i & x = a_i^1, \\ e_n & x = b, \end{cases}$$

and let $f_{\mathbf{v}} : \mathrm{sd}(K)(0) \to X_V(0)$ be the map given by

$$f_{\mathbf{v}}(\sigma) = \langle \theta_{\mathbf{v}}(x) : x \in \sigma \rangle^{\perp}.$$

Clearly, $f_{\mathbf{v}}$ extends to a simplicial map from $\mathrm{sd}(K)$ to X_V. The inverse of $\mathfrak{g}$ under $f_{\mathbf{v}}$ is the local system of $\mathrm{sd}(K)$ given by $\mathfrak{h}_{\mathbf{v}} = f_{\mathbf{v}}^{-1}\mathfrak{g}$. We next define an element

$$c_{\mathbf{v}} = \sum_{F \in \mathrm{sd}(K)(n-2)} c_{\mathbf{v}}(F)F \in C_{n-2}(\mathrm{sd}(K); \mathfrak{h}_{\mathbf{v}}).$$

For a sequence $\mathbf{u} = (u_1, \ldots, u_{n-1}) \in V^{n-1}$ of linearly independent vectors in V such that $e_n \notin \langle u_1, \ldots, u_{n-1} \rangle$, let $w(\mathbf{u})$ be the unique element $w \in \langle u_1, \ldots, u_{n-1} \rangle^{\perp}$ such that $w \cdot e_n = 1$. For $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1}) \in \{0,1\}^{n-1}$ and $\pi \in \mathbb{S}_{n-1}$ let $\chi(\boldsymbol{\epsilon}, \pi) = (-1)^{\sum_{j=1}^{n-1} \epsilon_j} \mathrm{sgn}(\pi)$. On an $(n-2)$-simplex $F \in \mathrm{sd}(K)(n-2)$ define

$$c_{\mathbf{v}}(F) = \begin{cases} \chi(\boldsymbol{\epsilon}, \pi) w(\mathbf{v}^{\boldsymbol{\epsilon}}) & \boldsymbol{\epsilon} \in E, F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}})), \\ \chi(\boldsymbol{\epsilon}, \pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) & \boldsymbol{\epsilon} \in E_j, F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j})), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Note that $c_{\mathbf{v}}(F) \in \mathfrak{h}_{\mathbf{v}}(F)$ for all $F \in \mathrm{sd}(K)(n-2)$. Indeed, if $F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}}))$ then

$$
\begin{aligned}
c_{\mathbf{v}}(F) &= \chi(\boldsymbol{\epsilon},\pi)w(\mathbf{v}^{\boldsymbol{\epsilon}}) \in \langle v_1^{\epsilon_1},\dots,v_{n-1}^{\epsilon_{n-1}}\rangle^{\perp} \\
&= \mathfrak{g}(f_{\mathbf{v}}(F)) = \mathfrak{h}_{\mathbf{v}}(F).
\end{aligned}
$$

If $F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))$ for $1 \le j \le n-1$ and $\boldsymbol{\epsilon} \in E_j$ then

$$
\begin{aligned}
c_{\mathbf{v}}(F) &= \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) \in \langle v_1^{\epsilon_1},\dots,v_{j-1}^{\epsilon_{j-1}}, e_n, v_{j+1}^{\epsilon_{j+1}},\dots,v_{n-1}^{\epsilon_{n-1}}\rangle^{\perp} \\
&= \mathfrak{g}(f_{\mathbf{v}}(F)) = \mathfrak{h}(F).
\end{aligned}
$$

Proposition 2.1. $c_{\mathbf{v}} \in \tilde{H}_{n-2}(\mathrm{sd}(K);\mathfrak{h}_{\mathbf{v}})$.

Proof. Let $G \in \mathrm{sd}(K)(n-3)$. We have to show that $\partial_{n-2}c_{\mathbf{v}}(G) = 0$. Let $\Gamma(G)$ denote the set of $(n-2)$-simplices in $\mathrm{sd}(K)$ that contain G. For $2 \le \ell \le n-1$ let $\eta_\ell \in \mathbb{S}_{n-1}$ denote the transposition $(n-\ell, n-\ell+1)$. We consider the following four cases according to the type of G. For $n = 4$ we depict the types of the 24 bold edges in Fig. 1b. The 6 edges incident with the vertex ① are of type 1, and the 6 edges incident with the vertex ② are of type 2 below. Of the remaining 12 edges, the 8 edges that are incident with vertices labelled ③ are of type 3, and the remaining 4 edges incident with vertices labelled by ④ are of type 4.

1. $G = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}}))_\ell$ for some $2 \le \ell \le n-1$, $\pi \in \mathbb{S}_{n-1}$ and $\boldsymbol{\epsilon} \in E$.
   Then

   $$\Gamma(G) = \{S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}})), S((\pi\eta_\ell)(\mathbf{a}^{\boldsymbol{\epsilon}}))\}$$

   As G is the ℓ-th face of both these simplices, it follows that

   $$
   \begin{aligned}
   (-1)^{\ell+1}\partial_{n-2}c_{\mathbf{v}}(G) &= c_{\mathbf{v}}(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}}))) + c_{\mathbf{v}}(S((\pi\eta_\ell)(\mathbf{a}^{\boldsymbol{\epsilon}}))) \\
   &= \chi(\boldsymbol{\epsilon},\pi)w(\mathbf{v}^{\boldsymbol{\epsilon}}) + \chi(\boldsymbol{\epsilon},\pi\eta_\ell)w(\mathbf{v}^{\boldsymbol{\epsilon}}) \\
   &= \chi(\boldsymbol{\epsilon},\pi)w(\mathbf{v}^{\boldsymbol{\epsilon}}) - \chi(\boldsymbol{\epsilon},\pi)w(\mathbf{v}^{\boldsymbol{\epsilon}}) = 0.
   \end{aligned}
   $$

2. $G = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))_\ell$ for some $2 \le \ell \le n-1$, $\pi \in \mathbb{S}_{n-1}$, $1 \le j \le n-1$ and $\boldsymbol{\epsilon} \in E_j$.
   Then

   $$\Gamma(G) = \{S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j})), S((\pi\eta_\ell)(\mathbf{a}^{\boldsymbol{\epsilon},j}))\}.$$

   As G is the ℓ-th face of both these simplices, it follows that

   $$
   \begin{aligned}
   (-1)^{\ell+1}\partial_{n-2}c_{\mathbf{v}}(G) &= c_{\mathbf{v}}(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))) + c_{\mathbf{v}}(S((\pi\eta_\ell)(\mathbf{a}^{\boldsymbol{\epsilon},j}))) \\
   &= \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) + \chi(\boldsymbol{\epsilon},\pi\eta_\ell)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) \\
   &= \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) - \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) = 0.
   \end{aligned}
   $$

3. $G = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}}))_1$ for some $\pi \in \mathbb{S}_{n-1}$ and $\boldsymbol{\epsilon} \in E_j$, where $j = \pi(n-1)$.
   Then

$$\Gamma(G) = \{S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}})), S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}+e_j})), S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))\}.$$

   As G is the 1-face of each of these simplices, it follows that

$$\begin{aligned}\partial_{n-2} c_{\mathbf{v}}(G) &= c_{\mathbf{v}}\left(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}}))\right) + c_{\mathbf{v}}\left(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}+e_j}))\right) + c_{\mathbf{v}}\left(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))\right)\\ &= \chi(\boldsymbol{\epsilon},\pi) w(\mathbf{v}^{\boldsymbol{\epsilon}}) + \chi(\boldsymbol{\epsilon}+e_j,\pi) w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) + \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big)\\ &= \chi(\boldsymbol{\epsilon},\pi)\left(w(\mathbf{v}^{\boldsymbol{\epsilon}}) - w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j})\right) + \chi(\boldsymbol{\epsilon},\pi)\big(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})\big) = 0.\end{aligned}$$

4. $G = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))_1$ for some $\pi \in \mathbb{S}_{n-1}$ and $\boldsymbol{\epsilon} \in E_j$, where $j \neq \pi(n-1)$.
   Let $j' = \pi(n-1)$ and let τ denote the transposition (j, j'). Since $S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))_1$ is independent of $\epsilon_{\pi(n-1)}$, we may assume that $\epsilon_{j'} = \epsilon_{\pi(n-1)} = 0$. Then:

$$\Gamma(G) = \{S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j})), S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}+e_{j'},j})), S((\tau\pi)(\mathbf{a}^{\boldsymbol{\epsilon},j'})), S((\tau\pi)(\mathbf{a}^{\boldsymbol{\epsilon}+e_j,j'}))\}.$$

   As G is the 1-face of each of these simplices, it follows that

$$\begin{aligned}\partial_{n-2} c_{\mathbf{v}}(G) &= c_{\mathbf{v}}(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j}))) + c_{\mathbf{v}}(S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}+e_{j'},j})))\\ &\qquad + c_{\mathbf{v}}(S((\tau\pi)(\mathbf{a}^{\boldsymbol{\epsilon},j'}))) + c_{\mathbf{v}}(S((\tau\pi)(\mathbf{a}^{\boldsymbol{\epsilon}+e_j,j'})))\\ &= \chi(\boldsymbol{\epsilon},\pi)(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})) + \chi(\boldsymbol{\epsilon}+e_{j'},\pi)(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}}))\\ &\qquad + \chi(\boldsymbol{\epsilon},\tau\pi)(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}}) - w(\mathbf{v}^{\boldsymbol{\epsilon}}))\\ &\qquad + \chi(\boldsymbol{\epsilon}+e_j,\tau\pi)(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j+e_{j'}}) - w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}))\\ &= \chi(\boldsymbol{\epsilon},\pi)[(w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})) - (w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}}))\\ &\qquad - (w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}}) - w(\mathbf{v}^{\boldsymbol{\epsilon}})) + (w(\mathbf{v}^{\boldsymbol{\epsilon}+e_{j'}+e_j}) - w(\mathbf{v}^{\boldsymbol{\epsilon}+e_j}))] = 0.\end{aligned}$$

We have thus shown that $c_{\mathbf{v}} \in \tilde{H}_{n-2}(\mathrm{sd}(K); \mathfrak{h})$. □

Proposition 2.1 implies that $\widetilde{c}_{\mathbf{v}} = (f_{\mathbf{v}})_* c_{\mathbf{v}} \in \tilde{H}_{n-2}(X_V; \mathfrak{g})$.

Theorem 2.2. *The family $\{\widetilde{c}_{\mathbf{v}} : \mathbf{v} \in T_{q,n}\}$ is a basis of $\mathcal{D}^1(V) = \tilde{H}_{n-2}(X_V; \mathfrak{g})$.*

Proof. Let $\mathbf{v} \in T_{q,n}$. Let $R(\mathbf{v}) \in X_V(n-2)$ be the $(n-2)$-simplex

$$R(\mathbf{v}) = [\langle v_1, \ldots, v_{n-1}\rangle^{\perp}, \langle v_1, \ldots, v_{n-2}\rangle^{\perp}, \ldots, \langle v_1, v_2\rangle^{\perp}, \langle v_1\rangle^{\perp}].$$

Let $\mathbf{1} = (1, \ldots, 1) \in E$. It is straightforward to check that $F = S(\mathbf{a}^{\mathbf{1}})$ is the unique $(n-2)$-simplex in $\mathrm{sd}(K)$ such that $f_{\mathbf{v}}(F) = R(\mathbf{v})$. It follows that

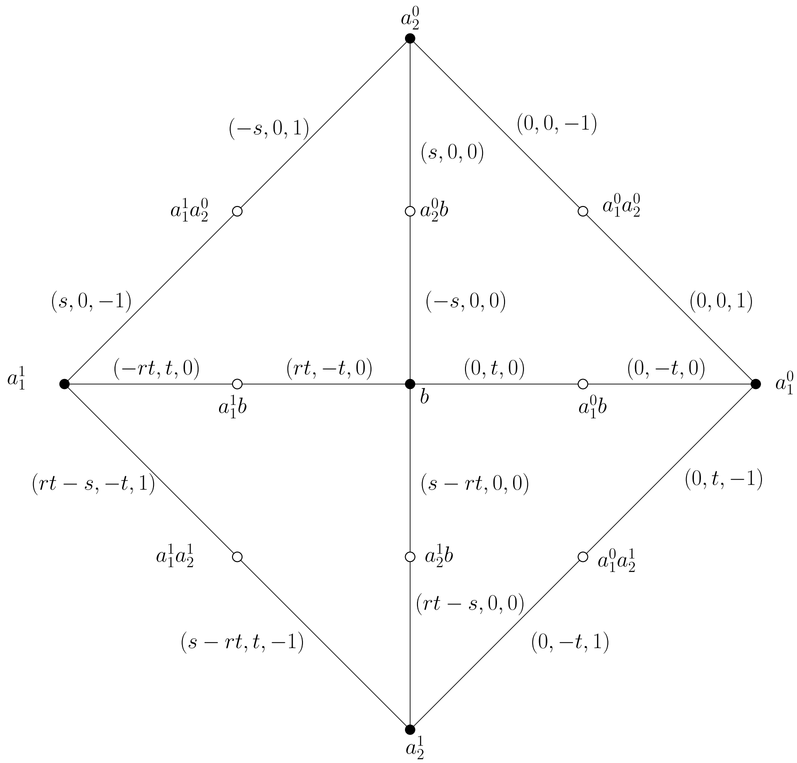


Fig. 2. The cycle $c_{\mathbf{v}}$ for $\mathbf{v} = (v_1, v_2) = ((1, r, s), (0, 1, t))$.

$$\widetilde{c}_{\mathbf{v}}(R(\mathbf{v})) = c_{\mathbf{v}}(S(\mathbf{a^1})) = (-1)^{n-1}w(\mathbf{v}).$$

On the other hand, if $\mathbf{v} \neq \mathbf{v}' \in T_{q,n}$, then $R(\mathbf{v}') \notin f_{\mathbf{v}}(\mathrm{sd}(K))$ and so $\widetilde{c}_{\mathbf{v}}(R(\mathbf{v}')) = 0$. It follows that the $(n-2)$-cycles $\{\widetilde{c}_{\mathbf{v}} : \mathbf{v} \in T_{q,n}\}$ are linearly independent in $\mathcal{D}^1(V)$. As $|T_{q,n}| = \prod_{i=1}^{n-1}(q^i - 1) = \dim \mathcal{D}^1(V)$, this completes the proof of Theorem 2.2. $\square$

Example. Let $n = 3$ and let

$$\mathbf{v} = (v_1, v_2) = ((1, r, s), (0, 1, t)) \in T_{q,3}.$$

Fig. 2 depicts the cycle $c_{\mathbf{v}} \in H_1(\mathrm{sd}(K); \mathfrak{h})$. Black vertices correspond to vertices of K and white vertices correspond to edges of K. The values of $c_{\mathbf{v}}$ are indicated on the edges of the diagram. For example, let $\boldsymbol{\epsilon} = (1, 1)$ and $\pi = (1, 2)$. Then $F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon}})) = [\{a_2^1, a_1^1\}, \{a_2^1\}]$, and

$$c_{\mathbf{v}}(F) = \chi(\boldsymbol{\epsilon}, \pi)w\big((v_1, v_2)\big) = -w\big((v_1, v_2)\big) = (s - rt, t, -1).$$

Similarly, if $j = 1$, $\boldsymbol{\epsilon} = (0, 1) \in E_1$ and $\pi = (1, 2)$, then $F = S(\pi(\mathbf{a}^{\boldsymbol{\epsilon},j})) = [\{a_2^1, b\}, \{a_2^1\}]$ and

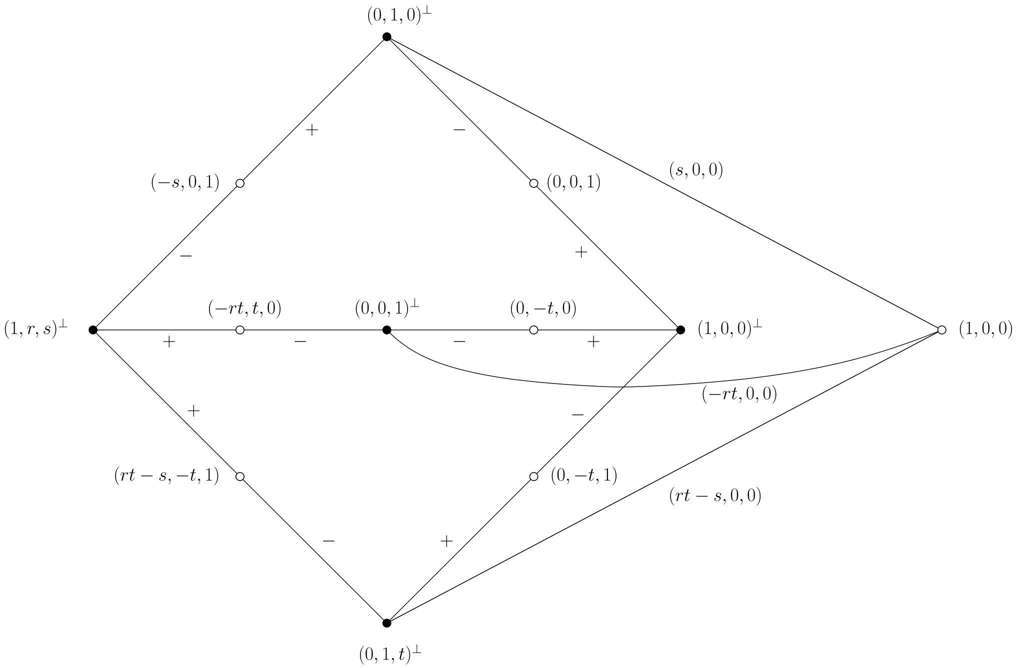


Fig. 3. The cycle $\widetilde{c}_{\mathbf{v}}$ for a generic $\mathbf{v} = ((1, r, s), (0, 1, t))$.

$$\begin{aligned} c_{\mathbf{v}}(F) &= \chi(\boldsymbol{\epsilon}, \pi)\left(w\big((v_1, v_2)\big) - w\big((e_1, v_2)\big)\right) \\ &= (rt - s, -t, 1) - (0, -t, 1) = (rt - s, 0, 0). \end{aligned}$$

Figs. 3 and 4 depict the 1-cycle $\widetilde{c}_{\mathbf{v}} \in H_1(X_V; \mathfrak{g})$. Here, the black vertices correspond to 2-dimensional subspaces of V. The white vertices and their labels correspond to 1-dimensional subspaces and their generating vectors. Fig. 3 depicts the generic case when $rst(rt - s) \neq 0$. The labels of the left most 6 white points together with the $\pm$ signs, indicate the values of $\widetilde{c}_{\mathbf{v}}$ on the incident edges. The remaining three values of $\widetilde{c}_{\mathbf{v}}$ are indicated on the edges incident with the vertex corresponding to the line spanned by $(1, 0, 0)$. Fig. 4 similarly depicts the case $s = 0$. Note that in both cases, the simplicial map $f_{\mathbf{v}} : \mathrm{sd}(K) \to X_V$ is not injective.

3. The dimension of $\mathcal{D}^k(V)$

Proof of Theorem 1.5. For an $\mathbb{F}_q$-space W let $\mathrm{St}(W) = \tilde{H}_{\dim W - 2}(X_W; \mathbb{F}_q)$ denote the Steinberg module of W over $\mathbb{F}_q$. Recall that $\dim \mathrm{St}(W) = q^{\binom{\dim W}{2}}$ by Theorem 1.1. Let $G_j(V)$ denote the family of all j-dimensional linear subspaces of V. The following result is due to Dupont (Proposition 5.38 in [4]).

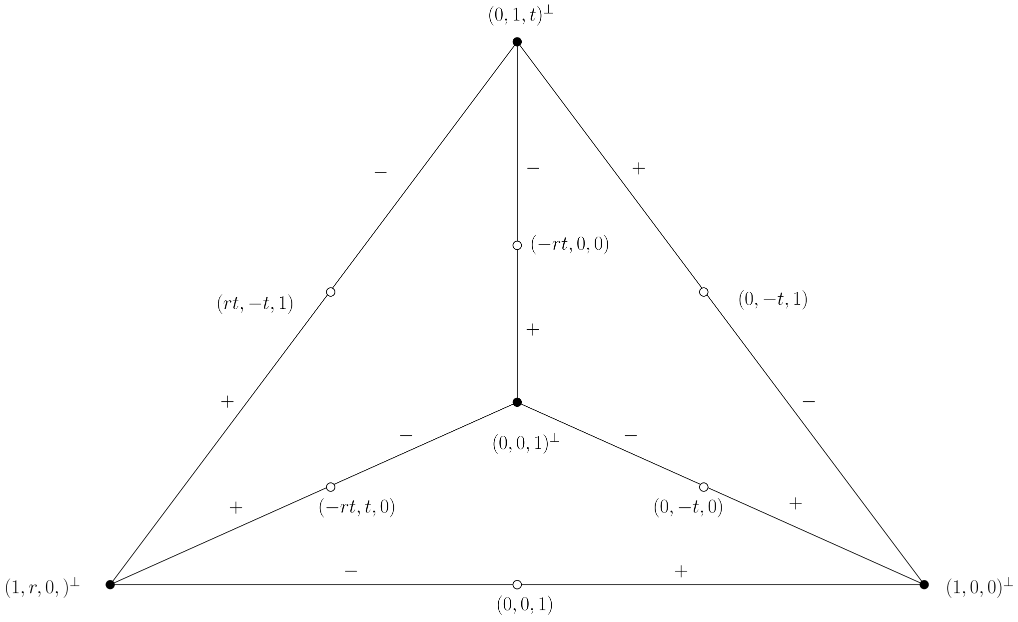


Fig. 4. The cycle $\tilde{c}_{\mathbf{v}}$ for $\mathbf{v} = (v_1, v_2) = ((1,r,0),(0,1,t))$.

Theorem 3.1 *(Dupont). There is an exact sequence*

$$\begin{aligned} 0 \to \mathcal{D}^k(V) \to \bigoplus_{U_k \in G_k(V)} \wedge^k U_k \otimes \mathrm{St}(V/U_k) \to \bigoplus_{U_{k+1} \in G_{k+1}(V)} \wedge^k U_{k+1} \otimes \mathrm{St}(V/U_k) \to \\ \ldots \to \bigoplus_{U_{n-2} \in G_{n-2}(V)} \wedge^k U_{n-2} \otimes \mathrm{St}(V/U_{n-2}) \\ \to \bigoplus_{U_{n-1} \in G_{n-1}(V)} \wedge^k U_{n-1} \to \wedge^k V \to 0. \end{aligned}$$

Writing $\begin{bmatrix} n \\ j \end{bmatrix}_q$ for the q-binomial coefficient, Theorem 3.1 implies that

$$\dim \mathcal{D}^k(V) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{4}$$

By the q-binomial theorem (see e.g. (1.87) in [8])

$$\prod_{j=0}^{n-1} (1 + q^j \lambda) = \sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \lambda^j . \tag{5}$$

Substituting $\lambda = -t^{-1}$ in (5) and multiplying by t^n it follows that

$$\prod_{j=0}^{n-1}(t-q^j)=\sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} n \\ j\end{bmatrix}_q t^{n-j}. \tag{6}$$

Differentiating (6) k times and multiplying by $\frac{(-1)^{n-k}}{k!}$ we obtain

$$\begin{aligned}\prod_{j=0}^{n-1}(q^j-t) & \sum_{0\le\alpha_0<\cdots<\alpha_{k-1}\le n-1}\prod_{\ell=0}^{k-1}\frac{1}{q^{\alpha_\ell}-t} \\ &=\sum_{j=0}^{n}(-1)^{n-k+j}\binom{n-j}{k}q^{\binom{j}{2}}\begin{bmatrix} n \\ j\end{bmatrix}_q t^{n-j-k} \\ &=\sum_{j=0}^{n}(-1)^{j-k}\binom{j}{k}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j\end{bmatrix}_q t^{j-k}.\end{aligned} \tag{7}$$

Substituting $t=1$ in (7) and using (4) we obtain (2). $\square$

3.1. A basis for $\mathcal{D}^{n-1}(V)$

In this subsection we describe an explicit basis for $\mathcal{D}^{n-1}(V)=\tilde{H}_0(X_V;\wedge^{n-1}\mathfrak{g})$. We first recall some facts concerning the exterior algebra $\wedge V$. Let $V=\mathbb{F}_q^n$. Using the notation of Section 2, recall that $e_1,\ldots,e_n$ are the unit vectors in V, and $a\cdot b$ denotes the standard symmetric bilinear form on V. Let $\mathbf{e}=e_1\wedge\cdots\wedge e_n\in\wedge^n V$. The induced bilinear form on $\wedge^p V$ is given by

$$(u_1\wedge\cdots\wedge u_p)\cdot(v_1\wedge\cdots\wedge v_p)=\det\left(u_i\cdot v_j\right)_{i,j=1}^{p}.$$

The *star operator* $*:\wedge^{n-k}V\to\wedge^k V$ is the unique linear map that satisfies

$$(*\alpha)\cdot\beta=\mathbf{e}\cdot(\alpha\wedge\beta)$$

for any $\alpha\in\wedge^{n-k}V,\beta\in\wedge^k V$.

Claim 3.2. *Let $v_1,\ldots,v_{n-k}$ be linearly independent vectors in V and let $M=\langle v_1,\ldots,v_{n-k}\rangle^\perp$. Then*

$$0\neq *(v_1\wedge\cdots\wedge v_{n-k})\in\wedge^k M.$$

Proof. Extend $\{v_i\}_{i=1}^{n-k}$ to a basis $\{v_i\}_{i=1}^{n}$ of V, and let $\{w_j\}_{j=1}^{n}$ be the dual basis, i.e. $v_i\cdot w_j=\delta_{i,j}$. Then $M=\langle w_{n-k+1},\ldots,w_n\rangle$. For a subset $L=\{i_1,\ldots,i_\ell\}\in\binom{[n]}{\ell}$ such that $1\le i_1<\cdots<i_\ell\le n$ let $v_L=v_{i_1}\wedge\cdots\wedge v_{i_\ell}$ and $w_L=w_{i_1}\wedge\cdots\wedge w_{i_\ell}$. If $L,L'\in\binom{[n]}{\ell}$ then $v_L\cdot w_{L'}=\delta_{L,L'}$.

Let $I_0=\{1,\ldots,n-k\}$, $J_0=\{n-k+1,\ldots,n\}$, and let $*v_{I_0}=\sum_{|J|=k}\lambda_J w_J$. Then for any $J'\in\binom{[n]}{k}$

$$*v_{I_0} \cdot v_{J'} = \sum_{|J|=k} \lambda_J w_J \cdot v_{J'} = \lambda_{J'}. \tag{8}$$

On the other hand

$$\begin{aligned} *v_{I_0} \cdot v_{J'} &= \mathbf{e} \cdot (v_{I_0} \wedge v_{J'}) \\ &= \begin{cases} \det(v_1, \ldots, v_n) & J' = J_0, \\ 0 & J' \neq J_0. \end{cases} \end{aligned} \tag{9}$$

Combining (8) and (9), it follows that $0 \neq *v_{I_0} = \det(v_1, \ldots, v_n) w_{J_0} \in \wedge^k M$. $\square$

We proceed to construct a basis of $\mathcal{D}^{n-1}(V) = \tilde{H}_0(X_V; \wedge^{n-1}\mathfrak{g})$. Note that if $u \in V$, then by Claim 3.2, $(*u)u^\perp \in C_0(X_V; \wedge^{n-1}\mathfrak{g})$. For any $1 \leq i \leq n$ let

$$z_{u,i} = (*e_i)e_i^\perp + (*u)u^\perp - (*(u+e_i))(u+e_i)^\perp \in C_0(X_V; \wedge^{n-1}\mathfrak{g}).$$

Then

$$\partial_0(z_{u,i}) = *e_i + *u - *(u+e_i) = *(e_i + u - (u+e_i)) = 0$$

and therefore $z_{u,i} \in \mathcal{D}^{n-1}(V)$. For $2 \leq i \leq n$ let $R_i = \left(\mathbb{F}_q^{i-1} \setminus \{0\}\right) \times \{0\}^{n-i+1}$.

Claim 3.3.

$$\mathcal{B} = \left\{z_{u,i} : 2 \leq i \leq n \ , \ u \in R_i\right\} \tag{10}$$

is a basis of $\mathcal{D}^{n-1}(V)$.

Proof. By Theorem 1.5

$$\dim \mathcal{D}^{n-1}(V) = \sum_{i=2}^{n}(q^{i-1} - 1) = \sum_{i=2}^{n} |R_i| = |\mathcal{B}|.$$

It therefore suffices to show that the elements of $\mathcal{B}$ are linearly independent. This in turn follows from the fact that for any $2 \leq j \leq n$ and $v \in R_j$, it holds that $(v + e_j)^\perp \in \mathrm{supp}(z_{v,j})$, but $(v + e_j)^\perp \notin \mathrm{supp}(z_{u,i})$ for any $(u, i) \neq (v, j)$ such that $2 \leq i \leq j$ and $u \in R_i$. $\square$

4. Minimal cycles in $\mathcal{D}^k(V)$

In this section we prove Theorem 1.6. The upper bound follows from a construction of certain explicit $(n - k - 1)$-cycles of $\mathcal{D}^k(V)$ given in Subsection 4.1. The lower bound is established in Subsection 4.2.

4.1. The upper bound

Let $1 \leq k \leq n-1$ and let $m = n-k+2$. Let $\mathbf{u} = (u_1, \ldots, u_m) \in V^m$ be an ordered m-tuple of vectors in V whose only linear dependence is $\sum_{i=1}^m u_i = 0$. Let $\mathbb{I}_{m-2,m}$ denote the family of injective functions $\pi : [n-k] = [m-2] \to [m]$. For $\pi \in \mathbb{I}_{m-2,m}$ let $T(\mathbf{u}, \pi)$ be the $(n-k-1)$-simplex given by

$$T(\mathbf{u}, \pi) = [\langle u_{\pi(1)}, \ldots, u_{\pi(n-k)} \rangle^{\perp} \subset \cdots \subset \langle u_{\pi(1)} \rangle^{\perp}].$$

Let $\gamma_{\mathbf{u}} \in C_{n-k-1}(X_V; \wedge^k \mathfrak{g})$ be the chain whose value on an $(n-k-1)$-simplex F is given by

$$\gamma_{\mathbf{u}}(F) = \begin{cases} * \left(u_{\pi(1)} \wedge \cdots \wedge u_{\pi(n-k)} \right) & F = T(\mathbf{u}, \pi), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Proposition 4.1. $\gamma_{\mathbf{u}} \in \mathcal{D}^k(V)$.

Proof. Let G be an $(n-k-2)$-simplex in X_V. Let $\Gamma_{\mathbf{u}}(G)$ denote the set of $(n-k-1)$-simplices in $\mathrm{supp}(\gamma_{\mathbf{u}})$ that contain G. For $2 \leq \ell \leq n-k$ let $\eta_\ell \in \mathbb{S}_{n-k-2}$ denote the transposition $(n-k-\ell+1, n-k-\ell+2)$. We consider the following two cases:

1. $G = T(\mathbf{u}, \pi)_\ell$ for some $2 \leq \ell \leq n-k$ and $\pi \in \mathbb{I}_{m-2,m}$.
   Then

   $$\Gamma_{\mathbf{u}}(G) = \{T(\mathbf{u}, \pi), T(\mathbf{u}, \pi\eta_\ell)\}.$$

   As G is the ℓ-th face of both these simplices, it follows that

   $$\begin{aligned} (-1)^{\ell+1} \partial_{n-k-1} \gamma_{\mathbf{u}}(G) &= \gamma_{\mathbf{u}}(T(\mathbf{u}, \pi)) + \gamma_{\mathbf{u}}(T(\mathbf{u}, \pi\eta_\ell)) \\ &= * \left(u_{\pi(1)} \wedge \cdots \wedge u_{\pi(n-k-\ell+1)} \wedge u_{\pi(n-k-\ell+2)} \wedge \cdots \wedge u_{\pi(n-k)} \right) \\ &\quad + * \left(u_{\pi(1)} \wedge \cdots \wedge u_{\pi(n-k-\ell+2)} \wedge u_{\pi(n-k-\ell+1)} \wedge \cdots \wedge u_{\pi(n-k)} \right) = 0. \end{aligned}$$

2. $G = T(\mathbf{u}, \pi)_1$ for some $\pi \in \mathbb{I}_{m-2,m}$.
   Let $[m] \setminus \pi([m-3]) = \{\alpha_1, \alpha_2, \alpha_3\}$. For $i = 1, 2, 3$ define $\pi_i \in \mathbb{I}_{m-2,m}$ by

   $$\pi_i(j) = \begin{cases} \pi(j) & 1 \leq j \leq n-k-1, \\ \alpha_i & j = n-k. \end{cases}$$

   Then

   $$\Gamma_{\mathbf{u}}(G) = \{T(\mathbf{u}, \pi_1), T(\mathbf{u}, \pi_2), T(\mathbf{u}, \pi_3)\}.$$

   As G is the 1-th face of these three simplices, it follows that

$$\begin{aligned}\partial_{n-k-1}\gamma_{\mathbf{u}}(G) &= \sum_{i=1}^{3}\gamma_{\mathbf{u}}(T(\mathbf{u},\pi_i)) \\ &= \sum_{i=1}^{3} *\big(u_{\pi(1)}\wedge\cdots\wedge u_{\pi(n-k-1)}\wedge u_{\alpha_i}\big) \\ &= *\big(u_{\pi(1)}\wedge\cdots\wedge u_{\pi(n-k-1)}\wedge (\sum_{i=1}^{3} u_{\alpha_i})\big) \\ &= *\big(u_{\pi(1)}\wedge\cdots\wedge u_{\pi(n-k-1)}\wedge (\sum_{j=1}^{m} u_j)\big) = 0.\end{aligned}$$

We have thus shown that $\gamma_{\mathbf{u}} \in \mathcal{D}^k(V)$. $\square$

Corollary 4.2.

$$\begin{aligned}\min\big\{|\mathrm{supp}(w)| : 0 \neq w \in \mathcal{D}^k(V)\big\} &\leq |\mathrm{supp}(\gamma_{\mathbf{u}})| \\ = |\mathbb{I}_{m-2,m}| &= \frac{(n-k+2)!}{2}.\end{aligned}$$

Example. Let $n = 3, k = 1$. A minimal twisted 1-cycle in $\mathcal{D}^1(X_V)$ is depicted in Fig. 4.

4.2. The lower bound

In preparation for the proof of the lower bound in Theorem 1.6, we first recall a twisted version of the nerve lemma. Let $\mathcal{F}$ be a local system on a finite simplicial complex Y, and let $\mathcal{Y} = \{Y_i\}_{i=1}^m$ be a family of subcomplexes of Y such that $Y = \bigcup_{i=1}^m Y_i$. The *nerve* of the cover $\mathcal{Y}$ is the simplicial complex $N = N(\mathcal{Y})$ on the vertex $[m] = \{1, \ldots, m\}$, whose simplices are the subsets $\tau \subset [m]$ such that $Y_\tau := \bigcap_{i\in\tau} Y_i \neq \emptyset$. For $j \geq 1$ let $N_j(\mathcal{F})$ be the local system on N given by $N_j(\mathcal{F})(\tau) = H_j(Y_\tau; \mathcal{F})$. The following result is twisted version of the Mayer-Vietoris spectral sequence (see e.g. [5]).

Proposition 4.3. *There exists a spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(Y;\mathcal{F})$ such that $E^1_{p,q} = \bigoplus_{\sigma\in N(p)} H_q(Y_\sigma;\mathcal{F})$ and $E^2_{p,q} = H_p(N; N_q(\mathcal{F}))$.*

The Nerve Lemma is the following

Corollary 4.4. *Suppose that $H_q(Y_\sigma;\mathcal{F}) = 0$ for all $q \geq 1$ and $\sigma \in N(p)$ such that $p+q \leq t$. Then $H_p(Y;\mathcal{F}) \cong H_p(N; N_0(\mathcal{F}))$ for all $0 \leq p \leq t$.*

We will also need a simple observation concerning a certain twisted homology of the simplex. Let $r \geq 2$ and let $W_1, \ldots, W_r$ be arbitrary linear subspaces of a finite dimensional vector space W over a field $\mathbb{K}$. Let Δ_{r-1} denote the simplex on the vertex set $[r]$, and let $\mathcal{G}$ be the local system on Δ_{r-1} given by

$$\mathcal{G}(\sigma) = \begin{cases} \bigcap_{i\in\sigma} W_i & \emptyset \neq \sigma \in \Delta_{r-1}, \\ W & \sigma = \emptyset, \end{cases}$$

with the natural inclusion maps.

Proposition 4.5. *$\tilde{H}_k(\Delta_{r-1};\mathcal{G}) = 0$ for $k \geq r-2$.*

Proof. Using the natural order on $\{1,\ldots,r\}$, the top dimensional simplex in Δ_{r-1} is $\tau = [1,2,\cdots,r]$, and its i-th face is $\tau_i = [1,\ldots,i-1,i+1,\ldots,r]$. For $1 \leq i < j \leq r$ let

$$\tau_{i,j} = [1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,r].$$

Then

$$C_{r-1}(\Delta_{r-1},\mathcal{G}) = \left\{ w\tau : w \in \bigcap_{i=1}^{r} W_i \right\}$$

and

$$C_{r-2}(\Delta_{r-1},\mathcal{G}) = \left\{ \sum_{i=1}^{r} w_i\tau_i : w_i \in \bigcap_{\ell\in\tau_i} W_\ell \right\}.$$

The boundary map $\partial_{r-1} : C_{r-1}(\Delta_{r-1};\mathcal{G}) \to C_{r-2}(\Delta_{r-1};\mathcal{G})$ is given by

$$\partial_{r-1}(w\tau) = \sum_{i=1}^{r} (-1)^{i+1} w\tau_i. \tag{12}$$

Note that for $1 \leq i \leq r$ and $1 \leq j \leq r-1$, the j-th face of τ_i is

$$(\tau_i)_j = \begin{cases} \tau_{j,i} & 1 \leq j < i \leq r, \\ \tau_{i,j+1} & 1 \leq i \leq j \leq r-1. \end{cases}$$

It follows that the boundary map $\partial_{r-2} : C_{r-2}(\Delta_{r-1};\mathcal{G}) \to C_{r-3}(\Delta_{r-1};\mathcal{G})$ is given by

$$\begin{aligned} \partial_{r-2}\big(\sum_{i=1}^{r} w_i\tau_i\big) &= \sum_{i=1}^{r}\sum_{j=1}^{r-1} (-1)^{j+1} w_i(\tau_i)_j \\ &= \sum_{i=1}^{r}\sum_{j=1}^{i-1} (-1)^{j+1} w_i\tau_{j,i} + \sum_{i=1}^{r}\sum_{j=i}^{r-1} (-1)^{j+1} w_i\tau_{i,j+1} \\ &= \sum_{j=1}^{r}\sum_{i=1}^{j-1} (-1)^{i+1} w_j\tau_{i,j} + \sum_{i=1}^{r}\sum_{j=i+1}^{r} (-1)^{j} w_i\tau_{i,j} \\ &= \sum_{1\leq i<j\leq r} \left((-1)^{i+1}w_j + (-1)^j w_i\right)\tau_{i,j}. \end{aligned} \tag{13}$$

Eq. (12) implies that $\tilde{H}_{r-1}(\Delta_{r-1};\mathcal{G}) = 0$. Next let $c = \sum_{i=1}^{r} w_i\tau_i \in \ker\partial_{r-2}$ be a $\mathcal{G}$-twisted $(r-2)$-cycle of Δ_{r-1}. It follows by (13) that $w_j = (-1)^{j+1}w_1$ for all $1 \le j \le r$. Therefore $w_1 \in \bigcap_{i=1}^{r} W_i$ and hence $w_1\tau \in C_{r-1}(X;\mathcal{G})$. Eq. (12) then implies that $\partial_{r-1}(w_1\tau) = c$. Thus $\tilde{H}_{r-2}(\Delta_{r-1};\mathcal{G}) = 0$. $\square$

Proof of the lower bound in Theorem 1.6. We argue by induction on $n-k$. For the induction basis $k = n-1$, we have to show that if $0 \ne z \in \mathcal{D}^{n-1}(V) = \tilde{H}_0(X_V;\wedge^{n-1}\mathfrak{g})$, then $|\mathrm{supp}(z)| \ge \frac{(n-k+2)!}{2} = 3$. Suppose for contradiction that $|\mathrm{supp}(z)| < 3$. Then $z = (*u)u^{\perp} + (*v)v^{\perp}$ for some $u, v \in V$. As

$$0 = \partial_0 z = (*u) + (*v) = *(u+v),$$

it follows that $u+v = 0$ and hence $z = 0$, a contradiction. For the induction step, assume that $n-k \ge 2$ and let

$$0 \ne z = \sum_{z \in X_V(n-k-1)} z(\tau)\tau \in H_{n-k-1}(X_V;\wedge^k\mathfrak{g}) = Z_{n-k-1}(X_V;\wedge^k\mathfrak{g}).$$

Let $\mathrm{supp}(z) = \{\tau_1,\ldots,\tau_s\} \in X_V(n-k-1)$ and write

$$\tau_i = [V_k(i),\ldots,V_{n-1}(i)],$$

where $\dim V_j(i) = j$ for all $1 \le i \le s$ and $k \le j \le n-1$. Let

$$\{V_{n-1}(i) : 1 \le i \le s\} = \{U_1,\ldots,U_r\},$$

where the U_i's are distinct $(n-1)$-dimensional subspaces. Let $\mathcal{U}_i = \{U : 0 \ne U \subset U_i\}$ and let $Y_i = X_V[\mathcal{U}_i]$. Let $Y = \cup_{i=1}^{r} Y_i$ then clearly $z \in Z_{n-k-1}(Y;\wedge^k\mathfrak{g})$. Let N be the nerve of the cover $\{Y_i\}_{i=1}^{r}$ of Y. For $\sigma \subset [r]$ let $U_\sigma = \cap_{i\in\sigma}U_i$ and $Y_\sigma = \cap_{i\in\sigma}Y_i$. If $\sigma \in N$ then Y_σ is the order complex of the poset $P_\sigma = \{W : 0 \ne W \subset U_\sigma\}$. As P_σ has a unique maximal element U_σ it follows (see e.g. Lemma 1.4 in [5]) that

$$N_q(\wedge^k\mathfrak{g})(\sigma) = H_q(Y_\sigma;\wedge^k\mathfrak{g}) = \begin{cases} \wedge^k U_\sigma & q = 0, \\ 0 & q > 0. \end{cases} \tag{14}$$

Write

$$\mathcal{F}(\sigma) = N_0(\wedge^k\mathfrak{g})(\sigma) = \wedge^k U_\sigma.$$

Eq. (14) and Corollary 4.4 imply that for all $p \ge 0$

$$H_p(Y;\wedge^k\mathfrak{g}) \cong H_p(N;\mathcal{F}). \tag{15}$$

Proposition 4.6. $r \ge n-k+2$.

Proof. Suppose to the contrary that $r \leq n-k+1$. Then $\Delta_{r-1}^{(r-2)} \subset N \subset \Delta_{r-1}$. For $1 \leq i \leq r$ let $W_i = \wedge^k U_i \subset \wedge^k V$. Let $\mathcal{G}$ be the local system on Δ_{r-1} given by $\mathcal{G}(\sigma) = \cap_{i\in\sigma} W_i = \wedge^k U_\sigma$. Then $\mathcal{G}(\sigma) = \mathcal{F}(\sigma)$ if $\sigma \in N$ and $\mathcal{G}(\sigma) = 0$ otherwise. Hence $H_*(\Delta_{r-1};\mathcal{G}) = H_*(N;\mathcal{F})$. As $n-k-1 \geq r-2$, it follows by combining (15) and Proposition 4.5 that

$$H_{n-k-1}(Y;\wedge^k \mathfrak{g}) \cong H_{n-k-1}(N;\mathcal{F}) = H_{n-k-1}(\Delta_{r-1};\mathcal{G}) = 0,$$

in contradiction with the assumption that z is a nonzero element of $H_{n-k-1}(Y;\wedge^k\mathfrak{g})$. □

We now conclude the proof of Theorem 1.6. For $1 \leq j \leq r$ define $z_j \in C_{n-k-2}(X_{U_j};\wedge^k\mathfrak{g})$ as follows. For an $(n-k-2)$-simplex $F = [V_k,\ldots,V_{n-2}] \in X_{U_j}(n-k-2)$ let $z_j(F) = z([V_k,\ldots,V_{n-2},U_j])$. Then $\partial_{n-k-2} z_j = 0$. Indeed, suppose that

$$[V_k,\ldots,V_{i-1},V_{i+1},\ldots,V_{n-2}] \in X_{U_j}(n-k-3),$$

where $\dim V_\ell = \ell$ for $i \neq \ell \in \{k,\ldots,n-2\}$. Then:

$$\begin{aligned}
&\partial_{n-k-2} z_j\left([V_k,\ldots,V_{i-1},V_{i+1},\ldots,V_{n-2}]\right)\\
&= (-1)^{i+k} \sum_{V_{i-1}\subset V_i \subset V_{i+1}} z_j\left([V_k,\ldots,V_{i-1},V_i,V_{i+1},\ldots,V_{n-2}]\right)\\
&= (-1)^{i+k} \sum_{V_{i-1}\subset V_i \subset V_{i+1}} z\left([V_k,\ldots,V_{i-1},V_i,V_{i+1},\ldots,V_{n-2},U_j]\right)\\
&= \partial_{n-k-1} z\left([V_k,\ldots,V_{i-1},V_{i+1},\ldots,V_{n-2},U_j]\right) = 0.
\end{aligned}$$

As $0 \neq z_j \in H_{n-k-2}(X_{U_j};\wedge^k\mathfrak{g})$, it follows by induction that $|\mathrm{supp}(z_j)| \geq \frac{(n-k+1)!}{2}$. Therefore by Proposition 4.6

$$|\mathrm{supp}(z)| = \sum_{j=1}^{r} |\mathrm{supp}(z_j)| \geq (n-k+2)\frac{(n-k+1)!}{2} = \frac{(n-k+2)!}{2}. \quad \square$$

5. Concluding remarks

In this paper we studied some aspects of the twisted homology modules $\mathcal{D}^k(V) = \tilde{H}_{n-k-1}(X_V;\wedge^k\mathfrak{g})$. Our results suggest several problems and directions for further research:

- In Sections 2 and 3.1 we described explicit bases for $\mathcal{D}^1(V) = \tilde{H}_{n-2}(X_V;\mathfrak{g})$ and for $\mathcal{D}^{n-1}(V) = \tilde{H}_0(X_V;\wedge^{n-1}\mathfrak{g})$. It would be interesting to obtain analogous constructions for other $\mathcal{D}^k(V)$'s.

- The Nerve Lemma argument used in the proof of Theorem 1.6 can be adapted to give a simple alternative proof of the Smith-Yoshiara Theorem 1.3. We hope that this approach can also be useful for the study of minimal cycles of local systems over other highly symmetric complexes.
- The Smith-Yoshiara Theorem 1.3 and its counterpart for the local system $\wedge^k \mathfrak{g}$, Theorem 1.6, show that the linear codes that arise from (twisted) homology of X_V have small distance relative to their length, and are therefore far from good codes. On the other hand, it is known (see [2]) that for fixed integers $n \geq 2$ and $K > 0$ there is a constant $\lambda = \lambda(n, K) > 0$, such that for sufficiently large N there exists a complex $X_N \subset \Delta_{N-1}^{(n)}$ whose number of n-faces satisfies $f_n(X_N) = K\binom{N}{n}$, and such that $|\mathrm{supp}(z)| \geq \lambda\binom{N}{n}$ for all $0 \neq z \in C = H_n(X_N; \mathbb{F}_2)$. In particular, the rate $r(C)$ and relative distance $\delta(C)$ of C satisfy

$$r(C) = \frac{\dim C}{f_n(X_N)} \geq \frac{K-1}{K}$$

and

$$\delta(C) = \frac{\min\{|\mathrm{supp}(c)| : 0 \neq c \in C\}}{f_n(X_N)} \geq \frac{\lambda}{K}.$$

It would be interesting to give explicit constructions of simplicial complexes that give rise to homological codes with similar parameters.

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