Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Algebra 294 (2005) 569–589
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Constructing modules of finite projective dimension with prescribed intersection multiplicities

Greg Piepmeyer *, Paul Roberts [1]

Received 10 November 2004
Available online 3 May 2005
Communicated by Craig Huneke

Abstract

Recently Roberts and Srinivas proved the existence of large classes of modules of finite length and finite projective dimension with prescribed intersection multiplicities, giving many new counterexamples to earlier vanishing conjectures. We present an explicit method for constructing the examples whose existence is implied by their theorem.

Keywords: Intersection multiplicity; Finite projective dimension; Vanishing theorems; Perfect complexes

1. Introduction

The interest in intersection properties of modules of finite projective dimension goes back to Serre's definition of intersection multiplicities, where intersections for smooth schemes are defined using homological functors. Serre defined the intersection multiplicity for two finitely-generated modules M and N over a regular local ring A of dimension d as follows: if M and N satisfy the condition that $M \otimes_A N$ has finite length, then their intersection multiplicity is

$$\chi(M, N) = \sum_{i=0}^{d} (-1)^i \operatorname{length}\bigl(\operatorname{Tor}_i^A(M, N)\bigr).$$

* Corresponding author.
E-mail addresses: gpiepmey@math.unl.edu (G. Piepmeyer), roberts@math.utah.edu (P. Roberts).
[1] The second author was supported in part by a grant from the National Science Foundation.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.03.020

One of the conjectures concerning this multiplicity (proven by Serre [11] in the geometric case and later by Roberts [8,9] and Gillet and Soule [3,4] in the mixed characteristic case) was the vanishing conjecture, which stated that if

$$\dim(M) + \dim(N) < \dim(A),$$

then

$$\chi(M, N) = 0.$$

It was later asked to what extent this would hold if the condition that A be a regular local ring were dropped and replaced with the condition that M have finite projective dimension. The first counterexample to this generalized conjecture was constructed by Dutta, Hochster, and McLaughlin in [1]. More recently, in answering a question of multiplicities over Gorenstein rings, another example was constructed by Miller and Singh in [7]. A similar example had been suggested by Kurano [6], but he did not actually construct a module with the required property.

These examples answered the question on the vanishing conjecture but gave no idea why or where such examples existed. In Roberts and Srinivas [10], a general theorem was proven on the existence of examples of this type, examples that include the above and many more. However, while this in a certain sense explained why these examples exist, it gave no idea as to how to construct them. The construction of examples using this method is the topic of this paper.

2. The setup

In this section we describe the situation in which the theorem of Roberts and Srinivas applies, and modules of finite projective dimension with given intersection multiplicities are shown to exist.

Let R be a graded ring for which R_0 is a field and R is finitely generated over R_0 by R_1. All graded rings will be assumed to have these properties. In this situation one can define a projective scheme $X = \mathrm{Proj}(R)$. We assume that X is a smooth variety.

Before proceeding, we recall some facts about Chow groups and K-groups that we will need.

2.1. Chow groups

For any scheme X of finite type over a regular scheme, the Chow group $CH_*(X)$ is defined to be the group of cycles modulo rational equivalence. There are two cases of special interest here, and we describe these in more detail.

If A is a noetherian ring, then for each integer $i \geqslant 0$, we let $Z_i(A)$ be the free abelian group on the prime ideals $\mathfrak{p}$ of A such that the dimension of $A/\mathfrak{p}$ is equal to i. For each prime ideal $\mathfrak{q}$ with $\dim(A/\mathfrak{q}) = i + 1$ and for each $f \neq 0$ in $A/\mathfrak{q}$, define

$$\mathrm{div}(f, A/\mathfrak{q}) = \sum \mathrm{length}\big(A/(\mathfrak{q}, f)\big)_{\mathfrak{p}}[A/\mathfrak{p}],$$

where the sum is taken over all prime ideals $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}) = i$ (this sum is finite). The component of dimension i of the Chow group, $CH_i(A)$, is the quotient of $Z_i(A)$ by the subgroup generated by all $\operatorname{div}(f, A/\mathfrak{q})$ for all such $\mathfrak{q}$ and f.

If A has dimension d, then $CH_d(A)$ is the free abelian group on the components of $\operatorname{Spec}(A)$ of dimension d. If A is an integrally closed domain of dimension d, then $CH_{d-1}(A)$ is isomorphic to the divisor class group of A.

The other case of interest is where R is a graded ring over a field and X is the associated projective scheme. In this case the description is similar except for two major differences:

(1) $Z_i(X)$ is generated by graded prime ideals $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) = i + 1$ (so that the projective subscheme defined by $R/\mathfrak{p}$ has dimension i).
(2) The relation of rational equivalence is defined by setting $\operatorname{div}(f, R/\mathfrak{q}) = 0$, where $\mathfrak{q}$ is a graded prime ideal and f is a quotient of two homogeneous polynomials of the same degree. Thus $f = g/h$, and $\operatorname{div}(f, R/\mathfrak{q}) = \operatorname{div}(g, R/\mathfrak{q}) - \operatorname{div}(h, R/\mathfrak{q})$ is zero in $CH_*(X)$ (but neither $\operatorname{div}(g, R/\mathfrak{q})$ nor $\operatorname{div}(h, R/\mathfrak{q})$ is necessarily zero in $CH_*(X)$).

Note that in the case in which R is graded and $X = \operatorname{Proj}(R)$ there is a map from $CH_i(X)$ to $CH_{i+1}(A)$, where A is the localization of R at its graded maximal ideal, induced by the inclusion of the set of graded prime ideals into the set of all prime ideals of A.

For $X = \operatorname{Proj}(R)$ there is an important operator called the *hyperplane section* on $CH_*(X)$; we denote this operator h. It is defined as the map from $CH_i(X)$ to $CH_{i-1}(X)$ that sends a generator $[R/\mathfrak{p}]$ to $\operatorname{div}(R/\mathfrak{p}, x)$, where x is any homogeneous element of R of degree 1 that is not in $\mathfrak{p}$. It is easy to check that this definition is well defined up to rational equivalence.

If X is smooth, there is an intersection pairing defined on the Chow group of X, making the Chow group a ring. This pairing can be defined, for example, using Serre's definition given in Section 1. If d is the dimension of X and α and β are elements of $CH_i(X)$ and $CH_{d-i}(X)$, respectively, we let $\alpha \cdot \beta$ denote the degree of the intersection product of α and β.

2.2. *K-groups*

We use K_0, the Grothendieck group of objects where relations are given by short exact sequences. There are two main cases. If X is a smooth scheme, we consider the group $K_0(X)$, which is the free abelian group on the set of coherent sheaves with relations given by short exact sequences. If X is not smooth, $K_0(X)$ will denote the K-group where the objects are perfect complexes. A *perfect complex* is a complex that is locally quasi-isomorphic to a bounded complex of free modules. In this case, the relations are of two types:

(1) if

$$0 \to F' \to F \to F'' \to 0$$

is a short exact sequence of complexes (that is, the sequence is exact in each degree), then $[F] = [F'] + [F'']$;
(2) if $f : F \to G$ is a quasi-isomorphism, then $[F] = [G]$.

If X is smooth, then this definition agrees with the other one, using the fact that every coherent sheaf has a finite, locally free resolution.

We now state a version of the existence theorem of Roberts and Srinivas.

Theorem 1. *Let R be a graded ring such that $X = \mathrm{Proj}(R)$ is smooth of dimension d. Let A be the localization of R at the graded maximal ideal; assume that A is Cohen–Macaulay. Let η be an element of $CH_i(X)$ in the kernel of the hyperplane section. Then for every graded prime ideal $\mathfrak{p}$ such that $W = \mathrm{Proj}(R/\mathfrak{p})$ has dimension $d - i$, there is an A-module M of finite length and finite projective dimension and a positive integer n such that*

$$\chi(M, A/\mathfrak{p}A) = n\bigl(\eta \cdot [W]\bigr).$$

In many cases the integer n can be taken to be 1, so that $\chi(M, R/\mathfrak{p})$ is exactly $\eta \cdot [W]$. If A is not Cohen–Macaulay, one can still define a perfect complex with these properties. In fact, in general the construction produces a complex, and in the Cohen–Macaulay case a module can be constructed from this complex.

In what follows, we will let R denote a graded ring and let A denote the localization of R at its graded maximal ideal.

To conclude this section we describe the rings and cycles under consideration in the examples mentioned in Section 1. We use the result of Kurano [6] that states that if R is a graded ring as above, the Chow group of A is isomorphic to $CH_*(X)/hCH_*(X)$, where h is the hyperplane section.

In most of the examples we consider, the cycles are defined by schemes and subschemes of the form $\mathbb{P}^m \times \mathbb{P}^n$ for various m and n, so we describe the Chow ring of these schemes in detail. The graded ring R corresponding to $\mathbb{P}^m \times \mathbb{P}^n$ by the Segre embedding is the quotient $k[X_{ij}]/I_2(X_{ij})$, where i runs from 0 to m, j runs from 0 to n, and $I_2(X_{ij})$ is the ideal of 2 by 2 minors of the $m+1$ by $n+1$ matrix (X_{ij}). The Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$ is isomorphic to $\mathbb{Z}[a,b]/(a^{m+1}, b^{n+1})$ (see Kurano [6]). Here a is the cycle of codimension 1 given by $H \times \mathbb{P}^n$ and b is the cycle given by $\mathbb{P}^m \times K$, where H and K are hyperplanes in $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. The corresponding ideals of R are defined by the entries in one column and one row of the matrix (X_{ij}). The hyperplane in the Chow group of $\mathrm{Proj}(R)$ is defined by one element X_{ij}, and the corresponding ideal is the intersection of the ideals defined by the ith row and the jth column, so under our identifications this gives the element $a+b$ in the Chow group. The class of a point is represented by the class $a^m b^n$.

In this situation it is very easy to compute the kernel of the operator given by intersection with the hyperplane. Since the hyperplane is $a+b$, an element in the Chow group will be in the kernel if and only if each homogeneous component of codimension i is an integer multiple of an element of the form

$$a^i - a^{i-1}b + a^{i-2}b^2 - \cdots + (-1)^{i+1}b^i,$$

where $i \geqslant m$ and $i \geqslant n$ (some of the terms in this sum may be zero).

We now describe the examples in detail.

2.3. *The example of Dutta, Hochster, and McLaughlin*

In this case $R = k[X, Y, Z, W]/(XW - YZ)$, so $X = \mathbb{P}^1 \times \mathbb{P}^1$. We choose the cycle η to $a - b$, so that the intersection with a is $a^2 - ab = -ab$, and the degree of the intersection is -1. In terms of the ring R, the cycle η can be chosen to be $[\mathrm{Proj}(R/(X, Y))] - [\mathrm{Proj}(R/(X, Z))]$ and the intersection with $[\mathrm{Proj}(R/(X, Y))]$ is -1. Thus, by the theorem, there is a module of finite length and finite projective dimension M with $\chi(M, R/(X, Z)) = -n$ for some positive integer n; in fact, n can be taken to be 1 in this case.

2.4. *The example of Miller and Singh*

In this case the ring R over which the example is constructed is $k[X, Y, Z, U, V, W]/(XU + YV + ZW)$. The projective scheme X is a quadric and the prime $\mathfrak{p}$ is the ideal defined by (X, Y, Z). The cycle η is $[\mathrm{Proj}(R/(U, V, W))] - [\mathrm{Proj}(R/(U, V, Z))]$. (The example of a Gorenstein ring for which Dutta multiplicity and ordinary multiplicity do not coincide is a finite extension of this ring R.)

2.5. *The example of a Gorenstein ring with nontrivial Todd class of Kurano*

In [6], Kurano showed that the ring obtained by dividing a polynomial ring in nine variables corresponding to the entries of a 3×3 matrix by the ideal of 2×2 minors of the matrix is a Gorenstein ring of dimension 5 such that the component of dimension 3 of the Todd class is nonzero. This gave a candidate for a Gorenstein ring where Dutta and ordinary multiplicity do not coincide. However, it was not known how to construct a module for which the two multiplicities are different. The technique of Roberts and Srinivas [10] shows that a module with these properties does exist in this case. Here $X = \mathbb{P}^2 \times \mathbb{P}^2$ and the Chow ring of X is $\mathbb{Z}[a, b]/(a^3, b^3)$. The cycle η is $a^2 - ab + b^2$. Kurano showed that if M is a module of finite length and finite projective dimension corresponding to η as in Theorem 1, the Dutta multiplicities and ordinary multiplicities of M are not equal.

2.6. *A cubic surface*

The component of the Chow group of a cubic surface of dimension 1 is known by classical results to have rank 7 (see, for instance, Hartshorne [5]), and it follows from this and from Theorem 1 that if R is the coordinate ring of a cubic surface, there are numerous examples of modules of finite length and finite projective dimension with different intersection multiplicities for various prime ideals $\mathfrak{p}$ for which $R/\mathfrak{p}$ has dimension 2. We do not pursue this topic here, but we do show that the existence of nontrivial examples in the case of $k[X, Y, Z, W]/(XZ - YW)$ implies at least that nontrivial examples exist in this case also. We assume that the field k is algebraically closed. Let $F(X, Y, Z, W)$ be a homogeneous cubic equation that defines a smooth surface S in $\mathbb{P}^3$. Then it is known that the surface contains a line (in fact, it contains 27 of them), so there are linear forms l_1

and l_2 such that the line defined by l_1 and l_2 is contained in S, which implies that there are quadratic forms q_1 and q_2 such that

$$F = l_1 q_1 + l_2 q_2.$$

If the four elements did not generate an ideal primary to the maximal ideal, by the Leibniz rule a nonmaximal prime would contain all the partial derivatives of F, and S would not be smooth. Thus we can map the generators of $k[X, Y, Z, W]/(XZ - YW)$ to $l_1, q_1, -l_2, q_2$ and taking the tensor product over this map will transform whatever example we had over $k[X, Y, Z, W]/(XZ - YW)$ to one over $k[X, Y, Z, W]/(F(X, Y, X, W))$. Note that we can do this for any line on the surface, and we get examples so that the intersection with any line is nonzero.

3. Outline of the construction

We outline the main steps in the construction. We recall that we are starting with a cycle η in the Chow group of X with zero intersection with the hyperplane and ending with a module of finite length and finite projective dimension with the same intersection with a given module of the form $R/\mathfrak{p}$.

There are four steps to the construction. They are:

(1) Finding an element in the K-group of X that corresponds to the element of the Chow group.
Since an element of the Chow group will be a linear combination of elements of the form $[\mathrm{Proj}(R/\mathfrak{p})]$, where $\mathfrak{p}$ is a graded prime ideal of R, it might look reasonable to take the same combination of the classes of the coherent sheaves defined by the $R/\mathfrak{p}$ in $K_0(X)$. However, this will not work, in general; the main problem is that the element of $K_0(X)$ defined in this manner need not be in the kernel of intersection with the hyperplane in $K_0(X)$ (we give an example below). The way to proceed in general is to use the inverse of the Riemann–Roch map, which defines an isomorphism $K_0(X)_{\mathbb{Q}} \to CH_*(X)_{\mathbb{Q}}$. Any denominators that appear under this map are the source of the integral multiple necessary in Theorem 1. In special cases there are also simpler methods that can be used; we discuss one of these below.

(2) Taking appropriate hyperplanes and representing the element as zero in the K-group.
In this step we find a concrete representation of the relations of the intersection of our class in the K-group of X with the hyperplane that show that its intersection is zero. This will consist of a set of short exact sequences such that when the corresponding relations in the K-group are taken all terms cancel.

(3) Lifting the short exact sequences to perfect complexes by taking partial resolutions and lifting.
This is the most difficult step and will be explained in full in a later section. The main idea is as follows. The relations from step 2 are in X, and involve modules that define coherent sheaves that have finite locally free resolutions (since X is smooth) but are not of finite projective dimension at the irrelevant maximal ideal. These are approximated

by maps of perfect complexes that agree with the original ones up to complexes with homology of finite length.

(4) Going from a perfect complex to a module.
This process was first developed by Foxby [2] and was also explained in detail in Roberts and Srinivas [10]. In this paper we do not carry out this step.

The remainder of the paper is devoted to explaining and working out these steps.

4. Constructing an element on the K-group of X

As outlined above, the first step of the construction is to start with a cycle on X that is in the kernel of the hyperplane in the Chow group of X and to use that the Riemann–Roch map is an isomorphism between $K(X)_{\mathbb{Q}}$ and $CH(X)_{\mathbb{Q}}$ to find a corresponding element in $K_0(X)$. As mentioned above, replacing a linear combination of integral subschemes by the corresponding combination of coherent sheaves does not work in general. One property that is necessary but is not necessarily satisfied by this element of $K_0(X)$ is that it must be in the kernel of the hyperplane. The inverse image under the Riemann–Roch map will be in the kernel of the hyperplane section and will agree with this element up to components of lower dimension.

We use the notation $[R/\mathfrak{p}]$ for the class of the coherent sheaf defined by $R/\mathfrak{p}$ in $K_0(X)$, and we denote the cycle determined by $\mathfrak{p}$ in $CH_*(X)$ by $[\mathrm{Proj}(R/\mathfrak{p})]$. For specific choices of R, the quotient ring $R/\mathfrak{p}$ may have a simple description, and in those cases, that description may be substituted in these expressions.

In general the Riemann–Roch map may be hard to compute, but in our cases all the cycles are products of the form $\mathbb{P}^i \times \mathbb{P}^j$, and the Riemann–Roch map is known in this case. In fact, if we denote the generators of the Chow group of $\mathbb{P}^m \times \mathbb{P}^n$ by a and b as above, then the image of the class of $\mathbb{P}^i \times \mathbb{P}^j$ under the Riemann–Roch maps is $Q(a)^{i+1}Q(b)^{j+1}a^{m-i}b^{n-j}$, where

$$Q(X) = \frac{X}{1 - e^{-X}}.$$

(For a proof of this equality see Kurano [6].) We use this formula in the examples.

4.1. The example of Dutta, Hochster, and McLaughlin

Here the class $[\mathrm{Proj}(R/(X, Y))] - [\mathrm{Proj}(R/(X, Z))]$ is in the kernel of the hyperplane and is in fact the image under the Riemann–Roch map of the class $[R/(X, Y)] - [R/(Y, Z)]$ of $K_0(X)$. The intersection of this element of $K_0(X)$ with the hyperplane can be computed by intersecting the first term with the element Z and the second with Y (since both of these elements have degree one), giving $[(R/(X, Y, Z))] - [(R/(X, Z, Y))]$, which is clearly zero.

4.2. *The Miller–Singh example*

In this example we can take the obvious element $[(R/(U,V,W))]-[(R/(U,V,Z))]$; intersection with Z in the first term and W in the second shows that the intersection with the hyperplane in $K_0(X)$ is zero.

4.3. *The Kurano example*

Here $X=\mathbb{P}^2\times\mathbb{P}^2$ and the cycle is a^2-ab+b^2. In this case the obvious choice of cycle in $K_0(X)$ does not work, and we work out this example in detail, showing why the Riemann–Roch map is important for this construction.

Let $R=k[X_{ij}]/I_2$ where the X_{ij} are the entries of the matrix

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

and I_2 is the ideal generated by the 2×2 minors of this matrix. The element a^2 of the Chow group corresponds to the ideal generated by the entries of two columns of the matrix, which we take to be the first two. The subscheme corresponding to this quotient is $\operatorname{Proj}(k[X_{13},X_{23},X_{33}])$, which is $\mathbb{P}^2$. The element b^2 is similar and is defined by the ideal generated by the first two rows. The element ab is defined by the ideal of entries in the first row and first column; the subscheme is $\operatorname{Proj}(k[X_{22},X_{23},X_{32},X_{33}]/(X_{22}X_{33}-X_{32}X_{23}))$, which is $\mathbb{P}^1\times\mathbb{P}^1$. We denote the prime ideals that define the cycles a^2, ab, and b^2 by $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$, respectively.

If we attempt to intersect the element of the K-group defined by the corresponding combination of coherent sheaves, the best strategy is to take the hyperplanes defined by X_{13} for $\mathfrak{p}_1$, X_{22} for $\mathfrak{p}_2$, and X_{31} for $\mathfrak{p}_3$. This gives

$$\big[k[X_{23},X_{33}]\big]-\big[k[X_{23},X_{32},X_{33}]/(X_{32}X_{23})\big]+\big[k[X_{32},X_{33}]\big].$$

This expression is not zero in the K-group; there is a short exact sequence

$$0\to k[X_{23},X_{33}](-1)\xrightarrow{X_{23}} k[X_{23},X_{32},X_{33}]/(X_{32}X_{23})\longrightarrow k[X_{32},X_{33}]\to 0,$$

but the difference $[k[X_{23},X_{33}](-1)]-[k[X_{23},X_{33}]]=[k[X_{33}]]$ is not zero in $K_0(X)$, so the above expression is not zero.

One way around this is to use the element

$$\big[R/\mathfrak{p}_1(-1)\big]-[R/\mathfrak{p}_2]+[R/\mathfrak{p}_3].$$

This has all the necessary properties; in particular, the above short exact sequence shows that the intersection with the hyperplane is zero.

In general, the solution to this problem is more complicated, and we briefly outline how the general procedure works in this case. To keep notation simple, we denote the coherent sheaf corresponding to the cycle $a^i b^j$ by $A^i B^j$. We denote the Riemann–Roch map by τ.

We recall that the Riemann–Roch map sends the class of $\mathbb{P}^i \times \mathbb{P}^j$ to $Q(a)^{i+1} Q(b)^{j+1} \times a^{m-i} b^{n-j}$. Using that A^2 is $\mathbb{P}^2$, AB is $\mathbb{P}^1 \times \mathbb{P}^1$, and so on, we obtain

$$\tau\big(A^2\big) = a^2\left(1 + \frac{3}{2}b + b^2\right), \qquad \tau(AB) = ab(1+a)(1+b),$$
$$\tau\big(B^2\big) = b^2\left(1 + \frac{3}{2}a + a^2\right), \qquad \tau\big(A^2B\big) = a^2b(1+b),$$
$$\tau\big(AB^2\big) = ab^2(1+a), \quad \text{and} \quad \tau\big(A^2B^2\big) = a^2b^2.$$

Using these expressions, it is not difficult to compute that the element

$$A^2 - AB + B^2 - \frac{1}{2}\big(A^2B + AB^2\big)$$

maps to $a^2 - ab + b^2$ and can be used in the construction.

We note that in this notation the element derived previously is

$$A^2(-1) - AB + B^2 = A^2 - AB + B^2 - A^2B.$$

This gives a different element of the K group, but both have the correct components in dimension 2 and are in the kernel of the hyperplane, so either one can be used.

5. Transforming the intersection with the hyperplane into a perfect complex

We assume now that we have an element β of $K_0(X)$ whose intersection with the hyperplane is zero. We can write β in the form

$$\beta = \sum a_i \big[R/\mathfrak{p}_i(n_i)\big],$$

for some graded prime ideals $\mathfrak{p}_i$ and some rational numbers a_i and integers n_i.

The intersection of β with the hyperplane in $K_0(X)$ is taken by choosing an element y_i of degree 1 that is not in $\mathfrak{p}_i$ for each i and taking the element $\sum a_i[R/(\mathfrak{p}_i, y_i)(n_i)]$. In this section we show how to replace $R/(\mathfrak{p}_i, y_i)$ by a perfect complex that reduces to $R/(\mathfrak{p}_i, y_i)$ in $U = \operatorname{Spec}(R) - \{\mathfrak{m}\}$. To simplify notation, we drop the subscripts and denote $\mathfrak{p}_i$ by $\mathfrak{p}$ and y_i by y.

The reason this is not a simple process is that $R/\mathfrak{p}$ will usually not have finite projective dimension. The main technique is suggested by a construction of Thomason and Trobaugh [12] and involves taking a partial resolution of $R/\mathfrak{p}$ and using maps defined on the tensor product of this resolution with a truncated Koszul complex. We now explain this construction in detail.

Let $x_1, \dots, x_m$ be a sequence of homogeneous elements in $\mathfrak{m}$ and let $K = K(x_1, \dots, x_m)$ be the Koszul complex on these elements. We define the complex $K^+ = K^+(x_1, \dots, x_m)$ by letting $K_i^+ = K_{i+1}$ if $i \geqslant 0$ and $K_i^+ = 0$ if $i < 0$. In other words, we remove K_0 and shift the other degrees by one. Let E be a graded truncated resolution of $R/\mathfrak{p}$; that is, we have a resolution F of $R/\mathfrak{p}$ by graded free R-modules and a positive integer t so that $E_i = F_i$ for $i \leqslant t$ and $E_i = 0$ for $i > t$. The boundary maps in K^+ and E are induced by those of K and F, respectively.

Lemma 1. *Let E and F be complexes. Suppose we have, for each ordered subset $I = \{i_1 < i_2 < \dots < i_r\}$ of $\{1 < 2 < \dots < m\}$, a map $\phi_I : E \to F[r-1]$ such that for each I we have*

$$d_F \phi_I = \sum_{j=1}^{r} (-1)^j x_{i_j} \phi_{I-\{i_j\}} + (-1)^{r-1} \phi_I d_E.$$

Assume also that $\phi_\emptyset = 0$. Then the ϕ_I define a map of complexes $\Phi : K^+ \otimes E \to F$. More precisely, for each I let b_I be the standard basis element of K_{r-1}^+. If $e \in E_n$, then

$$\Phi_{n+r-1}(b_I \otimes e) = \phi_I(e).$$

Proof. The proof is exactly a matter of checking the condition for a map to be a map of complexes. Let $b_I \otimes e$ be as above. We have

$$\begin{aligned} \Phi_{n+r-2} d_{K^+ \otimes E}(b_I \otimes e) &= \Phi_{n+r-2}\big(d_{K^+}(b_I) \otimes e + (-1)^{r-1} b_I \otimes d_E(e)\big) \\ &= \Phi_{n+r-2}\Big(\sum (-1)^j x_{i_j} b_{I-\{x_{i_j}\}} \otimes e + (-1)^{r-1} b_I \otimes d_E(e)\Big) \\ &= \sum (-1)^j x_{i_j} \phi_{I-\{x_{i_j}\}}(e) + (-1)^{r-1} \phi_I\big(d_E(e)\big) \\ &= d_F\big(\phi_I(e)\big) = d_F\big(\Phi_{n+r-1}(b_I \otimes e)\big). \qquad \square \end{aligned}$$

We note that if $I = \{i\}$ has one element, the condition is $d_F \phi_i = \phi_i d_E$, which states that ϕ_i is a map of complexes.

To motivate the construction, we note first that if $x_1, \dots, x_m$ generate an $\mathfrak{m}$-primary ideal, then the Koszul complex on $x_1, \dots, x_m$ is exact on $U = \operatorname{Spec}(R) - \{\mathfrak{m}\}$, and this implies that the map from K^+ to R defined by the boundary map from K_1 to $K_0 = R$ is a quasi-isomorphism on U. Thus we can tensor this map with E and obtain a map $\sigma : K^+ \otimes E \to E$ that is a quasi-isomorphism on U. (We could also define this map using the above lemma by letting $\phi_i =$ multiplication by x_i and letting $\phi_I = 0$ for $|I| > 1$.)

The aim of this section is to represent the quotient $R/(\mathfrak{p}, y)$ by a perfect complex. The first approximation to $R/\mathfrak{p}$ is the complex E, its graded resolution truncated in degree t. This complex has nonzero homology in two degrees, 0 and t. If we could split E into a direct sum of two complexes, each with homology in one degree, we could take the map y in degree 0 and the identity in degree t, and the associated mapping cone would solve the problem. Usually, however, that will not be possible. What we do instead is to split the above map from $K^+ \otimes E$ to E; that is, we show that it is a sum of two maps, one of which is zero in low degrees and the other in high degrees and use this decomposition to construct the desired perfect complex. We denote the natural map from $K^+ \otimes E$ to E by Φ.

Assume that $x_1, \ldots, x_m$ is a sequence of homogeneous elements of R of positive degree that generate an $\mathfrak{m}$-primary ideal of R. We note that since $\mathrm{Proj}(R)$ is smooth, $(R/\mathfrak{p})_{(x)}$ has finite projective dimension over the ring $R_{(x)}$. (Here, following standard notation, $R_{(x)}$ denotes the ring of homogeneous elements of degree zero in the localization R_x.) We denote $E(-n)$ the complex E with the grading on each module in the complex shifted by $-n$.

Lemma 2. *Let E be a projective resolution of $R/\mathfrak{p}$ truncated at t for some integer t. Let x be a homogeneous element of R of positive degree, and let s be an integer greater than or equal to the projective dimension of $(R/\mathfrak{p})_{(x)}$ and less than t. Then there exists an integer n such that multiplication by x^n from $E(-n)$ to E can be written as a sum $\phi' + \phi''$, where*

(1) *ϕ' and ϕ'' are maps of complexes that are maps of graded modules in each degree*;
(2) *$\phi'_j = 0$ for $j < s$ and $\phi'_j =$ multiplication by x^n for $j > s$;*
(3) *$\phi''_j = 0$ for $j > s$ and $\phi''_j =$ multiplication by x^n for $j < s$.*

Proof. Since s is greater than or equal to the projective dimension of $(R/\mathfrak{p})_{(x)}$, and $E_{(x)}$ is a projective resolution of $(R/\mathfrak{p})_{(x)}$ up to a degree t greater than s, $(E_{(x)})_s$ splits into a direct sum $\mathrm{Im}(d_{s+1})_{(x)} \oplus M$, where d denotes the boundary map on E and M is a submodule that maps injectively into $(E_{(x)})_{s-1}$. Taking the projections onto $\mathrm{Im}(d_{s+1})_{(x)}$ and M, respectively, and clearing denominators, we obtain a decomposition of the map given by multiplication by x^n on E_s into a sum $f' + f''$. Note that $d_s f' = 0$ since f' maps into the image of d_{s+1}. We now define ϕ' by letting $\phi'_j = x^n$ for $j > s$, f' for $j = s$, and 0 for $j < s$. Since $d_s f' = 0$, ϕ' is a chain map. Let $\phi''_j = 0$ for $j > s$, f'' for $j = s$, and x^n for $j < s$; then $\phi' + \phi'' = x^n$, and so ϕ'' is also a chain map. This entails $f'' d_{s+1} = 0$. □

We next combine the maps given by Lemma 2 to split the natural map Φ from $K^+ \otimes E$ to E.

Proposition 1. *Let F be a graded resolution of a module M. Let $x_1, \ldots, x_m$ be a set of homogeneous elements of R of positive degree, and let s be an integer greater or equal to the maximum of the projective dimensions of the $M_{(x_j)}$, and let $t > s + m$ be an integer. Let E be the truncation of F in degree t. Then there is a positive integer n such that the natural map $\Phi : K^+(x_1^n, \ldots, x_m^n) \otimes E \to E$ splits into a sum $\Phi = \Phi' + \Phi''$, where*

(1) *$\Phi'_j = 0$ for $j < s$ and $\Phi'_j = \Phi_j$ for $j \geqslant s + m$;*
(2) *$\Phi''_j = 0$ for $j \geqslant s + m$ and $\Phi''_j = \Phi_j$ for $j < s$.*

Proof. We construct Φ' and Φ'' by constructing maps ϕ'_I and ϕ''_I as in Lemma 1 using induction on the number $|I|$ of elements of I. At each step in the induction we may change the integer n. We begin with multiples of the maps ϕ' and ϕ'' given by Lemma 2. Begin by setting n to be the maximum of the n_i's from Lemma 2. The multiples are $x_i^{n-n_i}\phi'$ and $x_i^{n-n_i}\phi''$. Rather than carry around a power, we rename $x_i^{n_i}$ as simply x_i. Then, since $\phi'_{\{i\}}$

is either 0 or x_i except in degree s, we have that $x_l\phi'_{\{j\}} - x_j\phi'_{\{l\}}$ has a nonzero component only in degree s, and similarly for ϕ''.

Assume $r \geqslant 2$. We now construct, for each ordered subset $I = \{i_1 < \cdots < i_r\}$ of $\{1 < \cdots < m\}$, maps ϕ'_I and ϕ''_I from E to $E[r-1]$ such that each map is zero except in degree s and, denoting the components of ϕ'_I and ϕ''_I in degree s by f'_I and f''_I, respectively, we have

$$f'_I d_E = 0 \quad \text{and} \quad d_E f'_I = \sum (-1)^j x_{i_j} f'_{I-\{i_j\}},$$

and similarly for f''_I. Fix r, and assume that we have such maps for all smaller values of r. We then have

$$d_E\Big(\sum (-1)^j x_{i_j} f'_{I-\{i_j\}}\Big) = \sum (-1)^j x_{i_j} d_E f'_{I-\{i_j\}} = \sum\sum (-1)^{j+l} x_{i_j} x_{i_l} f'_{I-\{i_j,i_l\}} = 0;$$

each term appears twice, but with opposite signs since the punctured subsets of I are also ordered. Since E is exact in degree $s+r-2$, we can lift the map $\sum(-1)^j x_{i_j} f'_{I-\{i_j\}}$ and find a map satisfying the second of the required conditions. To satisfy the first condition, we localize by inverting $x_{i_1}x_{i_2}\ldots x_{i_r}$ and consider the part of degree zero in the graded localization. Since $s+r-1$ is greater than the projective dimension of $M_{(x_{i_1}x_{i_2}\ldots x_{i_r})}$ and $s+r$ is smaller than t, the degree where the complex is truncated, the complex is split exact at this point. Therefore we can find a map satisfying also $f'_I d_E = 0$ by projecting the lift onto a complement of the image of $d_{E_{(x_{i_1}\ldots x_{i_r})}}$ and then clearing denominators. We assume that we have done this for each set I with $|I| = r$, and that n_r is sufficiently large so that it suffices to multiply by $(x_{i_1}\ldots x_{i_r})^{n_r}$ to define the map f'_I for all I. If $n_r > 1$ we replace the maps $\phi'_{i_1,\ldots,i_s}$ by $x_{i_1}^{n_r-1}\ldots x_{i_s}^{n_r-1}\phi'_{i_1,\ldots,i_s}$ for $s < r$.

We let $\phi''_I = -\phi'_I$ for each I with $|I| \geqslant 2$; this ensures that the Φ' and Φ'' from Lemma 1 sum to give the natural Φ. □

We can now define the complex we want.

Definition 1. We let $C(\mathfrak{p}; y)$ denote the mapping cone of the map $\Phi' + y\Phi''$ from $K^+ \otimes E$ to E defined above.

It is clear that $C(\mathfrak{p}; y)$ is a perfect complex.

Lemma 3. *The natural map from $C(\mathfrak{p}; y)$ to $R/(\mathfrak{p}, y)$ is a quasi-isomorphism on U.*

Proof. The complex $C(\mathfrak{p}; y)$ in degree 0 is just E_0, and its homology is the cokernel under the sum of the images of the map from $E_1 \to E_0$ and that from $K^+ \otimes E_0 \to E_0$. The cokernel of the first map is, of course, $R/\mathfrak{p}$ and the image of the second map in this cokernel is the ideal generated by the yx_j, which is equal to the ideal generated by y on U.

The homology in the rest of $C(\mathfrak{p}; y)$ is determined by that in degree t in E. Since the natural map is a quasi-isomorphism on U and the maps in homology in degrees $\geqslant s+m$

are the same as those induced by the natural map, the homology of the mapping cone is zero in these degrees on U. Hence our map is a quasi-isomorphism on U. □

6. Building the complex from the pieces

As we have discussed, the fact that the intersection of our element of $K_0(X)$ with the hyperplane is zero implies that the modules $R/(\mathfrak{p}_i, y_i)$ fit into short exact sequences such that the corresponding alternating sum of terms that occur in these sequences is zero. The final step in the construction of the complex is to replace these maps by maps from complexes tensored with truncated Koszul complexes in a manner similar to that in the previous section and replace the exact sequences by appropriate mapping cones. We do not work out this formalism in general but describe two examples in detail.

6.1. *The example on $k[X, Y, Z, W]/(XW - YZ)$*

In this case we have two ideals $\mathfrak{p}_1 = (X, Y)$ and $\mathfrak{p}_2 = (X, Z)$ and homogeneous elements $y_1 = Z$ and $y_2 = Y$ such that

$$R/(\mathfrak{p}_1, y_1) = R/(X, Y, Z) = R/(X, Z, Y) = R/(\mathfrak{p}_2, y_2).$$

Thus $C(\mathfrak{p}_1; y_1)$ and $C(\mathfrak{p}_2; y_2)$ are isomorphic on U, and we can find a map from $K^+ \otimes C(\mathfrak{p}_1; y_1)$ to $C(\mathfrak{p}_2; y_2)$ that is a quasi-isomorphism on U for K^+ the truncated Koszul complex on some sequence of homogeneous elements of $\mathfrak{m}$ that generate an $\mathfrak{m}$-primary ideal. (A proof of this fact can be found in Thomason and Trobaugh [12].) The mapping cone of this map will have homology of finite length and will satisfy the desired conditions.

In the last section of the paper we describe a much more efficient method to construct the complex in this case.

6.2. *The example of Kurano*

We also describe the case suggested by Kurano. In this case the fact that the intersection with the hyperplane is zero is expressed by a short exact sequence involving three terms. As described earlier, we let $\mathfrak{p}_1$ be the ideal generated by the first two columns of the matrix (X_{ij}), $\mathfrak{p}_2$ the ideal generated by the first row and the first column, and $\mathfrak{p}_3$ the ideal generated by the first two rows. The element of $K_0(X)$ is $[R/\mathfrak{p}_1(-1)] - [R/\mathfrak{p}_2] + [R/\mathfrak{p}_3]$. We let $y_1 = X_{13}$, $y_2 = X_{22}$, and $y_3 = X_{31}$. We then have a short exact sequence

$$0 \to R/(\mathfrak{p}_1, y_1)(-1) \xrightarrow{\alpha} R/(\mathfrak{p}_2, y_2) \longrightarrow R/(\mathfrak{p}_3, y_3) \to 0.$$

This means that there is a quasi-isomorphism β from the mapping cone of α to $R/(\mathfrak{p}_3, y_3)$. We let $\tilde{\alpha}$ be a map from $K^+ \otimes C(\mathfrak{p}_1; y_1)$ to $C(\mathfrak{p}_2; y_2)$ that restricts to α on U. We then let $\tilde{\beta}$ be a map from the cone on $\tilde{\alpha}$ tensored with an appropriate truncated Koszul complex to $C(\mathfrak{p}_3; y_3)$ that restricts to β on U. The mapping cone of $\tilde{\beta}$ will then have the right intersection properties.

7. Proof that the construction gives the correct result

In this section we show that the above construction gives a complex with the correct intersection properties. First we recall the process used in the proof in Roberts and Srinivas [10].

Let α be an element of $K_0(X)$ in the kernel of the hyperplane section. The first step in the proof in [10] is to push this element forward into $K_0(Y)$, where $Y = \text{Proj}(R[T])$ with T an indeterminate of degree 1 and where X is embedded into Y as $\text{Proj}(R[T]/(T))$. A computation shows that this element of $K_0(Y)$ goes to zero on $Y - \{p\}$, where p is the point defined by the maximal ideal of R in $R[T]$, and the theorem of Thomason and Trobaugh [12] then implies that it is equal in $K_0(Y)$ to the class of a complex supported at p. The final step is to restrict this element to $K_0(A)$, where A is the localization of R at the graded maximal ideal. To prove that the complex we have constructed is correct, we have to show that it is the restriction to $\text{Spec}(A)$ of a complex that is supported at p and whose class in the K-group of Y is the same as the pushforward of the original element of $K_0(X)$.

We prove this result with one further assumption, that in $x_1, \dots, x_m$ we have m greater than the dimension of R. We also assume that R is Cohen–Macaulay. We then have the following result.

Lemma 4. *Let E be a perfect complex on Y. Then, if $K^+ = K^+(x_1, \dots, x_m)$, where the x_i are homogeneous elements of R and $m > \dim(R)$, we have the equality $[K^+ \otimes_R E] = [E]$ in the K-group of Y.*

Proof. We show that if K is the whole Koszul complex on $x_1, \dots, x_m$, then $K \otimes E$ is the class of zero, which is equivalent to the statement of the lemma. It suffices to show that the class of $K \otimes_R R[T]$ itself is zero. To see this, we note that $K \otimes_R R[T]$ is built up out of copies of $\mathcal{O}_Y(n)$ for various n, and its class in the K-group of Y depends only on the degrees of the elements x_i. Since $m > \dim R$, we can choose m homogeneous elements of $R[T]$ of the same degrees that generate an ideal primary to the irrelevant maximal ideal of $R[T]$. The support of this complex in Y is empty, so its class is clearly zero. □

The element we begin with in the general construction is a combination of classes of the form $[R/\mathfrak{p}_i(n_i)]$. The image of this class in the K-group of Y is the class of $R[T]/(\mathfrak{p}_i, T)(n_i)$. We assume for simplicity of notation that $n_i = 0$; this does not affect the proof. The main part of the proof here is to show that there is a complex with homology supported at p except in degree 0 that defines the same element in the K-group of Y as $R/(\mathfrak{p}_i, T)$ and restricts in the K-group of A to the complex $C(\mathfrak{p}_i; y_i)$ defined in the previous sections. We note (a fact already used) that any complex with support contained in (T) is perfect on Y, since $\text{Proj}(R[T]/T) = \text{Proj}(R)$ is smooth and the embedding of X in Y is a regular embedding.

In the remainder of this section we use the notation E and K^+ to denote the extensions $E \otimes_R R[T]$ and $K^+ \otimes_R R[T]$.

The procedure we use involves defining complexes on Y that reduce to the given complex when T is set equal to 1 (note that we can obtain the restriction of the complex to

$Y - X = \mathrm{Spec}(R)$ by setting $T = 1$ using the isomorphism $R \cong R[T]_{(T)}$, where $R[T]_{(T)}$ denotes the graded part of the localization $R[T]_T$ of degree zero). The class of mapping cone of the map T times the natural map from $K^+ \otimes E$ to E is equal to the class of the mapping cone of the map $T\Phi' + y_i\Phi''$ in the K-group, since they are mapping cones on maps between the same two complexes (with the same gradings). We denote these complexes $C(\mathfrak{p}_i; T, T)$ and $C(\mathfrak{p}_i; T, y_i)$, respectively. Note that $C(\mathfrak{p}_i; T, y_i)$ becomes $C(\mathfrak{p}_i; y_i)$ if T is set equal to 1.

Lemma 5. *For each i we can find a perfect complex Q_i and maps $\gamma_i : Q_i \to C(\mathfrak{p}_i; T, T)$ and $\delta_i : Q_i \to C(\mathfrak{p}_i; T, y_i)$ such that*:

(1) *Q_i is supported on $X = \mathrm{Proj}(R/TR[T])$*;
(2) *the mapping cones on γ_i and δ_i have homology supported at p except in degree zero*;
(3) *we have $[R[T]/(\mathfrak{p}_i, T)] = [C(\mathfrak{p}_i; T, T)] - [Q_i]$ in $K_0(Y)$*;
(4) *there is a map from the mapping cone on δ_i to $R[T]/(\mathfrak{p}_i, y_i)$ that is an isomorphism up to homology supported at $\{p\}$.*

Proof. We first examine the homology of the mapping cones more closely. The complex E has homology $H_0 = R[T]/(\mathfrak{p}_i, T)$ in degree 0 and H_t in degree t; the rest of the homology is zero. Let F be the mapping cone of T times the natural map from K^+ to $R[T]$. Let ψ be the embedding of $H_t[t]$ into E. Then, since F is a complex of free graded modules, ψ induces maps from $F \otimes H_t[t]$ to $C(\mathfrak{p}_i; T, T)$ and to $C(\mathfrak{p}_i; T, y_i)$ that induce isomorphisms in homology in degrees $\geqslant t$.

We recall that we are trying to find a complex Q_i supported on X and maps from Q_i to $C(\mathfrak{p}_i; T, T)$ and $C(\mathfrak{p}_i; T, y_i)$ such that their mapping cones have homology supported at p except in degree zero. Since the maps defined above from $F \otimes H_t[t]$ to these mapping cones induce isomorphisms in homology in degrees $\geqslant t$ it suffices to find a map to $F \otimes H_t[t]$ with these properties. The homology of $F \otimes H_t[t]$ is supported at p except in degree t, where it is isomorphic to $H_t/(x_1T, \dots, x_mT)H_t$.

Let d be the dimension of R, and let $z_1, \dots, z_d$ be a homogeneous system of parameters of R such that the ideal $(z_1, \dots, z_d)$ is contained in $(x_1, \dots, x_m)$. Let $F(z_j)$ be the complex defined in the same way as F with $x_1, \dots, x_m$ replaced by $z_1, \dots, z_d$. Since $(z_1, \dots, z_d) \subseteq (x_1, \dots, x_m)$, there is a map from the Koszul complex on $z_1, \dots, z_d$ to the Koszul complex on $x_1, \dots, x_m$, and the same maps define a map from $F(z_j)$ to F, and hence a map from $F(z_j) \otimes H_t[t]$ to $F \otimes H_t[t]$. In degree t this map induces the surjection from $R[T]/(Tz_j) \otimes H_t$ to $R[T]/(Tx_j) \otimes H_t$.

Since R is assumed to be Cohen–Macaulay and t is greater than the dimension of R, the module H_t is a Cohen–Macaulay module of dimension d. Hence, since the z_j form a system of parameters on R, $F(z_j) \otimes H_t[t]$ is exact except in degree t.

The homology of $F(z_j) \otimes H_t[t]$ in degree t is $H_t/(z_1T, \dots, z_dT)H_t$. Let N be the submodule $(z_1, \dots, z_d)H_t/(z_1T, \dots, z_dT)H_t$. Over $Y - \{p\}$ the submodule N is isomorphic to the degree t homology of $F(z_j) \otimes H_t$. Set Q_i as the mapping cone of T times the natural map from $K^+(z_1, \dots, z_d) \otimes H_t[t]$ to $(z_1, \dots, z_d) \otimes H_t[t]$. Then Q_i is perfect on Y. In fact, by the regularity of the the z_j on H_t, the complex is acyclic in degrees $> t$. In degree t its homology is $(z_1, \dots, z_d)H_t/(z_1T, \dots, z_dT)H_t$. Over points of Y where T is invertible, this quotient is zero. Over points of Y where some z_j is invertible, the quotient agrees with H_t/TH_t, but more importantly, the localization is regular, so that every finitely-generated module has a finite free resolution. Thus everywhere on Y, the complex Q_i is locally quasi-isomorphic to bounded complexes of finitely-generated free modules. Hence Q_i is perfect on Y.

The inclusion of $(z_1, \ldots, z_d)$ into $R[T]$ induces a map from Q_i into $F(z_j) \otimes H_t[t]$ which is a quasi-isomorphism since the degree t homology of Q_i is N and that agrees with the degree t homology of $F(z_j) \otimes H_t[t]$ over $Y - \{p\}$, while in higher degrees, the chain map consists of identity maps.

Let γ_i and δ_i be the induced maps from Q_i to $C(\mathfrak{p}_i; T, T)$ and $C(\mathfrak{p}_i; T, y_i)$, respectively. It is clear that Q_i is supported on X. The map induced on homology from $H_t(Q_i)$ to $H_t(F(x_j) \otimes H_t[t])$ is the inclusion of $(z_j)H_t/(Tz_j)H_t$ into $(x_j)H_t/(Tx_j)H_t$, and the quotient of this map is supported at p (in fact its support is empty). It thus follows that Q_i, γ_i, and δ_i satisfy statements (1) and (2) of the lemma. The map in statement (4) is the same as that defined in Lemma 3. It remains to be shown that statement (3) is satisfied.

Let Φ denote the natural map from $K^+(x_1, \ldots, x_m) \otimes E$ to E. Then $C(\mathfrak{p}_i; T, T)$ is the mapping cone of $T\Phi$. We have a commutative square

$$\begin{array}{ccc} K^+ \otimes E & \xrightarrow{\;\Phi\;} & E \\ \| & & \downarrow{\scriptstyle T} \\ K^+ \otimes E & \xrightarrow{\;T\Phi\;} & E \end{array}$$

It follows, there is a map Ψ from the mapping cone on Φ, which is $K(x_1, \ldots, x_m) \otimes E$, to $C(\mathfrak{p}_i; T, T)$, and that the mapping cone of Ψ is quasi-isomorphic to the mapping cone of multiplication by T on E. The homology of the mapping cone of multiplication by T on E is $R/(\mathfrak{p}_i, T)$ in degree 0 and H_t/TH_t in degree t. Combining these facts, we obtain

$$\left[C(\mathfrak{p}_i; T, T)\right] - \left[K(x_j) \otimes E\right] = \left[R/(\mathfrak{p}_i, T)\right] + \left[H_t/TH_t[t]\right].$$

On the other hand, we have $[Q_i] = [(z_j)H_t/(Tz_j)H_t[t]]$. The cokernel of the injection of $(z_j)H_t/(Tz_j)H_t[t]$ into $H_t/TH_t[t]$ is supported at the irrelevant ideal of $R[T]$, so we thus have

$$[H_t/TH_t] = \left[(z_j)H_t/(Tz_j)H_t[t]\right] = [Q_i].$$

Finally, Lemma 4 implies that $[K(x_j) \otimes E] = 0$. Combining these equalities, we obtain

$$\left[R/(\mathfrak{p}_i, T)\right] = \left[C(\mathfrak{p}_i; T, T)\right] - [Q_i].$$

This completes the proof of Lemma 5. □

We now complete the proof that the complex we constructed has the right intersection properties. Let $C(\delta_i)$ denote the mapping cone of δ_i. From the above lemma, each $C(\delta_i)$ is a resolution of $R[T]/(\mathfrak{p}_i, y_i)$ on $Y - \{p\}$. Hence, as we did for the $C(\mathfrak{p}_i; y_i)$, we can find maps from truncated Koszul complexes tensored with the $C(\delta_i)$ so that the result of taking the associated mapping cones, which we denote $\tilde{C}$, is supported at p. Since the complex $\tilde{C}$ is built from the $C(\delta_i)$ by the same procedure as the complex in Section 6 was built from the $C(\mathfrak{p}_i; y_i)$, to deduce that $\tilde{C}$ defines the correct class in $K_0(Y)$, it suffices to show that

(1) $C(\delta_i)$ becomes quasi-isomorphic to $C(\mathfrak{p}_i; y_i)$ after setting $T = 1$, and
(2) $[C(\delta_i)] = [R[T]/(\mathfrak{p}_i, T)]$ in $K_0(Y)$.

To see the first statement, we note first that $C(\mathfrak{p}_i; T, y_i)$ becomes $C(\mathfrak{p}_i; y_i)$ when T is set equal to 1. Since $[C(\delta_i)] = [C(\mathfrak{p}_i; T, y_i)] - [Q_i]$ and Q_i is supported at $TR[T]$, this proves (1).

The second statement follows from the equalities

$$\big[C(\delta_i)\big] = \big[C(\mathfrak{p}_i; T, y_i)\big] - [Q_i] = \big[C(\mathfrak{p}_i; T, T)\big] - [Q_i] = \big[R[T]/(\mathfrak{p}_i, T)\big]$$

by Lemma 5.

8. Example: the case of $k[X, Y, Z, W]/(XW - YZ)$ worked out in detail

While the previous sections described a general method for constructing complexes, the results are unwieldy and not as efficient as they might be. In this section we describe a better procedure in the case where $R = k[X, Y, Z, W]/(XW - YZ)$ that, while following the same general idea, gives a simpler result.

In this case the element of the K-group that we begin with is $[R/(X, Y)] - [R/(X, Z)]$. The general method would be to take truncated resolutions of each of these terms and then to take the cones on natural maps from their tensor products with truncated Koszul complexes that can be split into sums. Finally, we would take another map on the complexes tensored with more truncated Koszul complexes to give the complex we want. Instead, we take another complex that is close to being a resolution but, like the truncated resolution, is not exact in two places; however, this complex will have support strictly smaller than that of the simply truncated resolution. A consequence is that we can use Koszul complexes on fewer elements, leading to a more manageable result.

To begin, we let E_Y denote the complex

$$0 \to R \xrightarrow{\begin{bmatrix} Z \\ X \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}} R^2 \xrightarrow{[X \quad Y]} R \to 0$$

and let E_Z denote the complex

$$0 \to R \xrightarrow{\begin{bmatrix} Y \\ X \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} -Z & W \\ X & -Y \end{bmatrix}} R^2 \xrightarrow{[X \quad Z]} R \to 0$$

We describe the computations on E_Y; those on E_Z are similar. The homology of E_Y is $R/(X, Y)$ in degree 0 and $R/(X, Z)$ in degree 2. Let K^+ denote the truncated Koszul complex on $W, Y+Z$; note that these elements generate the maximal ideal in both $R/(X,Y)$ and $R/(X,Z)$.

The next step is to split multiplication by W and by $Y+Z$ into a sum satisfying the properties of Lemma 2. We recall that the splitting in general is begun by decomposing multiplication by x_j^n on E_i into a sum for certain n and i; here $i \in \{W, Y+Z\}$. In this case we take $i = Y$ and $n = 1$. The splitting of multiplication by W is done using the decomposition

$$\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} = \begin{bmatrix} W & 0 \\ -Z & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Z & W \end{bmatrix}.$$

This decomposes multiplication by W on E_Y into the sum of the two maps

$$\begin{array}{ccccccccc}
0 \to & R & \xrightarrow{\begin{bmatrix} Z \\ X \end{bmatrix}} & R^2 & \xrightarrow{\begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}} & R^2 & \xrightarrow{\begin{bmatrix} X & Y \end{bmatrix}} & R & \to 0 \\
 & \Big\downarrow W & & \Big\downarrow \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} & & \Big\downarrow \begin{bmatrix} W & 0 \\ -Z & 0 \end{bmatrix} & & \Big\downarrow 0 & \\
0 \to & R & \xrightarrow[\begin{bmatrix} Z \\ X \end{bmatrix}]{} & R^2 & \xrightarrow[\begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}]{} & R^2 & \xrightarrow[\begin{bmatrix} X & Y \end{bmatrix}]{} & R & \to 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 \to & R & \xrightarrow{\begin{bmatrix} Z \\ X \end{bmatrix}} & R^2 & \xrightarrow{\begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}} & R^2 & \xrightarrow{\begin{bmatrix} X & Y \end{bmatrix}} & R & \to 0 \\
 & \Big\downarrow 0 & & \Big\downarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & & \Big\downarrow \begin{bmatrix} 0 & 0 \\ Z & W \end{bmatrix} & & \Big\downarrow W & \\
0 \to & R & \xrightarrow[\begin{bmatrix} Z \\ X \end{bmatrix}]{} & R^2 & \xrightarrow[\begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}]{} & R^2 & \xrightarrow[\begin{bmatrix} X & Y \end{bmatrix}]{} & R & \to 0
\end{array}$$

Similarly, the decomposition

$$\begin{bmatrix} Y+Z & 0 \\ 0 & Y+Z \end{bmatrix} = \begin{bmatrix} Y & -W \\ -X & Z \end{bmatrix} + \begin{bmatrix} Z & W \\ X & Y \end{bmatrix}$$

gives a decomposition of multiplication by $Y+Z$ into a sum of two maps. We denote the two maps on E_Y decomposing multiplication by W by ϕ'_W and ϕ''_W, and those for $Y+Z$ by ϕ'_{Y+Z} and ϕ''_{Y+Z}.

Since the Koszul complex we are using is on only two elements, there is only one more step in constructing the maps from $K^+ \otimes E$ to E, and that is to lift the differences

$$(Y+Z)\phi'_W - W\phi'_{Y+Z} \quad \text{and} \quad (Y+Z)\phi''_W - W\phi''_{Y+Z}.$$

We have

$$(Y+Z)\phi'_W - W\phi'_{Y+Z} = (Y+Z)\begin{bmatrix} W & 0 \\ -Z & 0 \end{bmatrix} - W\begin{bmatrix} Y & -W \\ -X & Z \end{bmatrix},$$
$$\begin{bmatrix} ZW & W^2 \\ -Z^2 & -WZ \end{bmatrix} = \begin{bmatrix} -Y & W \\ X & -Z \end{bmatrix}\begin{bmatrix} 0 & 0 \\ Z & W \end{bmatrix}.$$

Thus the map from E to $E[1]$ inserted to make the maps from $K^+ \otimes E$ to E compatible is given by the matrix $\left[\begin{smallmatrix} 0 & 0 \\ Z & W \end{smallmatrix}\right]$ from E_1 to E_2 and zero, elsewhere. Note that the condition that $d_E\phi'_{12}$ that arose in the proof of Proposition 1 holds in this case. The corresponding map for ϕ'' will, as in the general construction, be given by the negative of this matrix, or $\left[\begin{smallmatrix} 0 & 0 \\ -Z & -W \end{smallmatrix}\right]$.

We now, denoting the two maps from $K^+ \otimes E_Y$ to E_Y by Φ'_Y and Φ''_Y, take the mapping cone of the sum $\Phi'_Y + Z\Phi''_Y$. This is the representation of $R/((X,Y)+(Z))$ by a perfect complex. To display the resulting mapping cone, it is convenient to let $V = Y+Z$, and we only show the nonzero portions, let

$$\Gamma = \begin{bmatrix} Z & 0 & -W & 0 \\ X & 0 & 0 & -W \\ 0 & Z & V & 0 \\ 0 & X & 0 & V \\ 0 & 0 & Y & -W \\ 0 & 0 & -X & Z \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -Y & W & 0 & 0 & -W & 0 \\ X & -Z & 0 & 0 & 0 & -W \\ 0 & 0 & -Y & W & V & 0 \\ 0 & 0 & X & -Z & 0 & V \\ 0 & 0 & 0 & 0 & -X & -Y \end{bmatrix},$$
$$\Sigma = \begin{bmatrix} V & 0 & W & 0 & 0 & 0 \\ 0 & V & 0 & W & Z-Z^2 & W-ZW \end{bmatrix},$$
$$\Pi = \begin{bmatrix} Y+Z^2 & ZW-W & W & 0 & 0 \\ XZ-X & Z+YZ & Z^2-Z & ZW & 0 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} X & Y & 0 & 0 & -W \\ 0 & 0 & X & Y & V \end{bmatrix}.$$

$$\begin{array}{ccccccccc}
R & \xrightarrow{\left[\begin{smallmatrix} -W \\ V \\ -Z \\ -X \end{smallmatrix}\right]} & R^2 \oplus R^2 & \xrightarrow{\Gamma} & R^4 \oplus R^2 & \xrightarrow{\Lambda} & R^4 \oplus R & \xrightarrow{\Theta} & R^2 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle [V\ W\ 0\ 0]} & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle [YZ+Z^2\ \ ZW]} \\
0 & \longrightarrow & R & \xrightarrow[\left[\begin{smallmatrix} Z \\ X \end{smallmatrix}\right]]{} & R^2 & \xrightarrow[\left[\begin{smallmatrix} -Y & W \\ X & -Z \end{smallmatrix}\right]]{} & R^2 & \xrightarrow[{[X\ \ Y]}]{} & R
\end{array}$$

The mapping cone of the above map of complexes is then the complex $C(\mathfrak{p}_1; y_1) = C((X,Y);Z)$ of the general construction; while the complex E_Y that we used here is not exactly the same as the truncated complexes in the general case, the rest of the construction is identical. The complex $C(\mathfrak{p}_2; y_2) = C((X,Z);Y)$ is defined similarly as the mapping cone of $\Phi'_Z + Y\Phi''_Z$; in fact, by the symmetry of the situation it suffices to interchange Y and Z in $C((X,Y);Z)$.

The final step is to take the mapping cone of a map from $K^+ \otimes E_Z$ to E_Y for a suitable truncated Koszul complex K^+. In this case, since both $C((X,Y);Z)$ and $C((X,Z);Y)$

have support in $\mathrm{Spec}(R/(X,Y,Z))$, it suffices to take a Koszul complex on one element, a power of W. The homology of both complexes has finite length everywhere except in degree zero, where the homology modules are $R/(X,Y,Z^2,ZW)$ and $R/(X,Z,Y^2,YW)$, respectively. It follows that we can find a map of complexes that lifts multiplication by W^m for some m, and the mapping cone of the resulting map from $C((X,Z);Y)$ to $C((X,Y);Z)$ will have homology of finite length. Let C denote this mapping cone. We now show directly that C will have the required intersection multiplicity.

We will show that $\chi(R/(Z,W),C)=-1$. Since C is defined as the mapping cone of a map from $C((X,Z);Y)$ to $C((X,Y);Z)$, it suffices to show that

$$\chi\bigl(R/(Z,W),C\bigl((X,Y);Z\bigr)\bigr)-\chi\bigl(R/(Z,W),C\bigl((X,Z);Y\bigr)\bigr)=-1.$$

To accomplish this, we tensor the diagrams for the maps $\Phi'_Y+Z\Phi''_Y$ and $\Phi'_Z+Y\Phi''_Z$ with $R/(Z,W)$ and compute the homology. Since after tensoring with $R/(Z,W)$ we are left with fairly simple complexes with entries in a polynomial ring in two variables, it is not terribly difficult. We let $S=R/(Z,W)$ and let k be the residue field of S. Tensoring the diagram defining the map $\Phi'_Y+Z\Phi''_Y$ with $R/(Z,W)$ and taking homology in the rows we obtain

$$\begin{array}{ccccc} 0 & 0 & S/XS\oplus k & k & k\oplus k\\ 0 & 0 & S/XS & 0 & k \end{array}$$

To complete the computation we need to compute the vertical map in the third position. The element of homology that generates the component S/XS is $(0,1,0,0,0,0)$, and computing its image we obtain Y in the lower copy of S/XS. Thus we are left with k in each row in the third column, and the Euler characteristic of $C((X,Y);Z)\otimes R/(Z,W)$ is $1-2+2-1=0$.

If we carry out the same computation tensoring $C((X,Z);Y)$ with $R/(Z,W)$ (which we do by tensoring $C((X,Y);Z)$ with $R/(Y,W)$, so now $S=R/(Y,W)$), we obtain

$$\begin{array}{ccccc} 0 & 0 & k & k\oplus k & S/XS\oplus k\\ 0 & 0 & 0 & k & S/XS \end{array}$$

In this case the vertical map from S/XS to S/XS in the fifth column is multiplication by Y^2. Hence the Euler characteristic is $2-2+2-1=1$. Thus the Euler characteristic of $C\otimes R/(Z,W)$ is $0-1=-1$.

References

[1] S.P. Dutta, M. Hochster, J.E. McLaughlin, Modules of finite projective dimension with negative intersection multiplicities, Invent. Math. 79 (1985) 253–291.

[2] H.-B. Foxby, K-theory for complexes with finite length homology, Kobenhauns Universitet Preprint Ser. (1982).

[3] H. Gillet, C. Soulé, K-théorie et nullité des multiplicités d'intersection, C.R. Acad. Sci. Paris Sér. I Math. 300 (3) (1985) 71–74.

[4] H. Gillet, C. Soulé, Intersection theory using Adams operations, Invent. Math. 90 (1987) 243–277.

[5] R. Hartshorne, Algebraic Geometry, Springer-Verlag, Berlin, 1977.
[6] K. Kurano, A remark on the Riemann–Roch formula on affine schemes associated with noetherian local rings, Tohoku Math. J. 48 (1996) 121–138.
[7] C. Miller, A. Singh, Intersection multiplicities over Gorenstein rings, Math. Ann. 317 (2000) 155–171.
[8] P. Roberts, The vanishing of intersection multiplicities of perfect complexes, Bull. Amer. Math. Soc. 13 (1985) 127–130.
[9] P. Roberts, Intersection theorems, in: Commutative Algebra, Proceedings of an MSRI Microprogram, Springer-Verlag, Berlin, 1989, pp. 417–436.
[10] P. Roberts, V. Srinivas, Modules of finite length and finite projective dimension, Invent. Math. 151 (2003) 1–27.
[11] J.-P. Serre, Algèbre Locale—Multiplicités, Lecture Notes in Math., vol. 11, Springer-Verlag, New York, 1961.
[12] R.W. Thomason, T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in: The Grothendieck Festschrift, vol. III, in: Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.