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Groups with finitely many normalizers of subnormal subgroups

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Abstract

The structure of soluble groups in which normality is a transitive relation is known. Here, groups with finitely many normalizers of subnormal subgroups are investigated, and the behavior of the Wielandt subgroup of such groups is described; moreover, groups having only finitely many normalizers of infinite subnormal subgroups are considered.

Keywords: Subnormal subgroup; Normalizer subgroup; T-group

1. Introduction

A group G is called a T*-group* if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T-groups has been described by W. Gaschütz [8] in the finite case and by D.J.S. Robinson [15] for arbitrary groups. It turns out in particular that soluble groups with the property T are metabelian and hypercyclic, and that finitely generated soluble T-groups are either finite or Abelian. In recent years, many authors have investigated the structure of soluble groups in which normality

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is imposed only to certain relevant systems of subnormal subgroups; other classes of generalized T-groups can be introduced by imposing that the set of all subnormal nonnormal subgroups of the group is small in some sense.

The *Wielandt subgroup* $\omega(G)$ of a group G is defined to be the intersection of all the normalizers of subnormal subgroups of G. Clearly, $\omega(G)$ is a T-group, and G is a T-group if and only if it coincides with its Wielandt subgroup; thus the size of $G/\omega(G)$ can be considered as a measure of the distance of the group G from the property T. For instance, it is well known that if G is a group satisfying the minimal condition on subnormal subgroups, then $G/\omega(G)$ is finite (see [16, Part 1, Theorem 5.49]). Moreover, if G is a group such that $G/\omega(G)$ is finite, it is clear that each subnormal subgroup of G has only finitely many conjugates; conversely, C. Casolo proved that if a soluble group G has boundedly finite conjugacy classes of subnormal subgroups, then the Wielandt subgroup $\omega(G)$ has finite index in G (see [2, Theorem 4.8]). Of course, the finiteness of $G/\omega(G)$ also implies that the group G has finitely many normalizers of subnormal subgroups, and groups with this latter property generalize those in which normality is a transitive relation. The behavior of normalizers has often a strong influence on the structure of the group; for instance, Y.D. Polovickiĭ [14] has proved that if a group has finitely many normalizers of Abelian subgroups, then its center has finite index, and more recently groups with finitely many normalizers of subgroups with a given property χ have been studied for several different choices of χ (see [3–5]).

The aim of this paper is to investigate groups with finitely many normalizers of (infinite) subnormal subgroups, and in particular the behavior of the Wielandt subgroup of such groups. Among other results, we shall prove that in every finitely generated soluble-by-finite group with finitely many normalizers of infinite subnormal subgroups the Wielandt subgroup has finite index, and even the center has finite index if the groups is soluble. Moreover, it will be shown that also periodic soluble groups with finitely many normalizers of subnormal subgroups are finite over their Wielandt subgroup. Finally, in the last section we will study the inheritance properties of the above conditions to subgroups of finite index.

Most of our notation is standard and can for instance be found in [16].

2. General properties

It is well known that if G is an arbitrary T-group, then $G'' = G^{(3)}$; in this section it will be proved that also the derived series of a group with finitely many normalizers of subnormal subgroups stops after finitely many steps. The following result of finite soluble permutation groups is probably well known.

Lemma 2.1. *Let m be a positive integer, and let G be a soluble subgroup of the symmetric group* $\mathrm{Sym}(m)$. *Then G has derived length at most $m-1$.*

Proof. The statement is obvious if $m \leqslant 4$. Let $m > 4$, and suppose first that G is primitive. Then G is the split extension of an Abelian group by the stabilizer of a point (see [11, Satz II.3.2]), and hence by induction on m it turns out that G has derived length at most

$m-1$. Assume now that G is transitive but not primitive. In this case, G contains a proper normal subgroup N such that G/N is a permutation group of degree r, with $1<r<m$, and N can be embedded into the direct product of groups of degree s, where $rs=m$ (see [11, Satz II.1.2]); thus G has derived length at most $(r-1)+(s-1)<rs-1$. Finally, if G is intransitive, it is a subdirect product of groups of smaller degree, and once again the statement follows by induction on m. □

Lemma 2.2. *Let G be a group with $k>1$ normalizers of subnormal subgroups. If X is any subnormal subgroup of G, then G contains a normal subgroup $M(X)$ such that $|G:M(X)|\leqslant(k-1)!$ and $N_G(X)\cap M(X)$ is normal in $M(X)$. Moreover, if the factor group $G/M(X)$ is soluble, then it has derived length at most $k-2$.*

Proof. If the subgroup X is normal in G, it is obviously enough to put $M(X)=G$. Suppose that X is not normal in G. Then the normalizer $N_G(X)$ has at most $k-1$ conjugates in G and so $|G:N_G(N_G(X))|\leqslant k-1$; it follows that

$$M(X)=\bigcap_{g\in G}N_G\bigl(N_G(X)\bigr)^g$$

is a normal subgroup of G with $|G:M(X)|\leqslant(k-1)!$. Moreover, $M(X)$ is contained in $N_G(N_G(X))$, and hence $N_G(X)\cap M(X)$ is a normal subgroup of $M(X)$. Finally, as $G/M(X)$ is isomorphic to a permutation group of degree $k-1$, it follows from Lemma 2.1 that if $G/M(X)$ is soluble, then its derived length is at most $k-2$. □

Corollary 2.3. *Let G be a group with finitely many normalizers of subnormal subgroups. Then G contains a normal subgroup M of finite index such that $N_M(X)$ is normal in M for each subnormal subgroup X of M.*

Proof. Let $\mathcal{H}$ be the set of all subnormal subgroups of G. By Lemma 2.2 for each element X of $\mathcal{H}$ there exists in G a normal subgroup $M(X)$ of finite index such that $N_G(X)\cap M(X)$ is normal in $M(X)$; moreover, if X and Y are elements of $\mathcal{H}$ such that $N_G(X)=N_G(Y)$, it follows from the definition that $M(X)=M(Y)$. Therefore also

$$M=\bigcap_{X\in\mathcal{H}}M(X)$$

is a normal subgroup of finite index of G. If X is any subnormal subgroup of M, then X is subnormal in G and

$$N_M(X)=N_G(X)\cap M=\bigl(N_G(X)\cap M(X)\bigr)\cap M$$

is normal in M. □

Recall that a group G is called *subsoluble* if it has an ascending series of subnormal subgroups whose factors are Abelian. Clearly, all soluble groups are subsoluble, and finite subsoluble groups are soluble. It turn out that, like T-groups, subsoluble groups with finitely many normalizers of subnormal subgroups are soluble.

Lemma 2.4. *Let G be a subsoluble group with k normalizers of subnormal subgroups. If the normalizer of each subnormal subgroup of G is normal, then G is soluble with derived length at most $2k$.*

Proof. If $k = 1$, the group G has the property T and hence it is metabelian. Suppose $k > 1$ and let X be any subnormal subgroup of G. Then $N_G(X)$ is a normal subgroup of G and $G/N_G(X)$ has at most $k - 1$ normalizers of subnormal subgroups, so that by induction it can be assumed that $G/N_G(X)$ is soluble with derived length at most $2(k - 1)$; it follows that $G^{(2(k-1))}$ is contained in the Wielandt subgroup $\omega(G)$. On the other hand, $\omega(G)$ is a soluble T-group, so that it is metabelian and hence G is soluble with derived length at most $2k$. □

Corollary 2.5. *Let G be a subsoluble group with finitely many normalizers of subnormal subgroups. Then G is soluble.*

Proof. The statement follows from Corollary 2.3 and Lemma 2.4. □

Theorem 2.6. *Let G be a group with $k > 1$ normalizers of subnormal subgroups. Then the derived series of G stops after at most $3k - 2$ steps, i.e. $G^{(3k-2)} = G^{(3k-1)}$. In particular, if G is soluble, its derived length is at most $3k - 2$.*

Proof. Replacing G by the factor group $G/G^{(3k-1)}$, it can be assumed without loss of generality that G is soluble. By Corollary 2.3, G contains a normal subgroup M of finite index such that $N_M(X)$ is normal in M for each subnormal subgroup X of M, and M has derived length at most $2k$ by Lemma 2.4. Moreover,

$$M = \bigcap_{X \in \mathcal{H}} M(X),$$

where $\mathcal{H}$ is the set of all subnormal subgroups of G. Moreover, each $G/M(X)$ has derived length at most $k-2$ by Lemma 2.2, and hence the derived length of G is at most $3k-2$. □

A group G is called an *IT-group* if all its infinite subnormal subgroups are normal; groups with the property *IT* have been investigated in [6] and [9]. It has been proved that infinite subsoluble *IT*-groups are soluble with derived length at most 3; moreover, any infinite periodic soluble *IT*-group either has the property T or is a finite extension of a group of type p^∞ (where p is a prime number), and in particular it is finite over its Wielandt subgroup.

For an arbitrary group G, we define $\bar{\omega}(G)$ to be the intersection of all the normalizers of infinite subnormal subgroups of G; clearly, $\bar{\omega}(G)$ is an *IT*-group and so it is soluble, provided that G is subsoluble. Moreover, the Wielandt subgroup $\omega(G)$ is contained in $\bar{\omega}(G)$, and the structure of the factor group $\bar{\omega}(G)/\omega(G)$ is quite restricted (see [1]). Replacing

$\omega(G)$ by $\bar{\omega}(G)$, the arguments used in this section also prove that every subsoluble group with finitely many normalizers of infinite subnormal subgroups is soluble.

3. Normalizers of subnormal subgroups

It was proved by D.J.S. Robinson [15] that a finitely generated soluble T-group is either finite or Abelian; this result cannot be extended to the case of soluble-by-finite groups, since the property T holds for any direct product of an infinite cyclic group and a finite simple non-Abelian group. On the other hand, we have the following result.

Lemma 3.1. *Let G be a finitely generated soluble-by-finite IT-group. Then the factor group $G/Z(G)$ is finite. Moreover, if G is soluble, then it is either finite or Abelian.*

Proof. Suppose first that G is soluble. Clearly, the commutator subgroup G' of G cannot be a group of type p^∞, so that G is either finite or a T-group (see [6, Theorem 1.11]) and hence it is even either finite or Abelian. Assume now that G is an infinite soluble-by-finite IT-group, and let K be a soluble normal subgroup of G such that G/K is finite. Then K is a finitely generated infinite soluble IT-group, and hence it is Abelian by the first part of the proof; as K can obviously be chosen torsion-free, all its subgroups are normal in G and so $G/C_G(K)$ is Abelian. Thus $[K, G'] = \{1\}$ and $K \cap G'$ is contained in $Z(G')$, so that $G'/Z(G')$ is finite and G'' is finite by Schur's theorem. It follows that G/G'' is a finitely generated infinite soluble IT-group and so it is Abelian. Therefore $G' = G''$ is finite and $G/Z(G)$ is also finite. □

Recall that the *norm* $N(G)$ of a group G is defined as the intersection of all the normalizers of subgroups of G. It is well known that the norm always lies in the second center of the group, and obviously we have $N(G) = \omega(G)$ if all cyclic subgroups of G are subnormal. Recall also that an automorphism of a group G is a *power automorphism* if it maps every subgroup of G onto itself. Thus an element x of a group G belongs to $N(G)$ if and only if x induces on G a power automorphism by conjugation.

Theorem 3.2. *Let G be a finitely generated soluble-by-finite group with finitely many normalizers of infinite subnormal subgroups. Then the factor group $G/\omega(G)$ is finite. Moreover, if G is soluble, $G/Z(G)$ is finite.*

Proof. Assume by contradiction that G is not Abelian-by-finite and let K be a soluble normal subgroup of G such that G/K is finite. It follows from Lemma 3.1 that G is not an IT-group and so we can choose a counterexample with a minimal number k of proper normalizers of infinite subnormal subgroups. If G contains an infinite subnormal nonnormal subgroup X such that the index $|G : N_G(X)|$ is finite, the core L of $N_G(X)$ has less than k proper normalizers of infinite subnormal subgroups, and hence it is Abelian-by-finite by the minimal choice of k; thus G itself is Abelian-by-finite, and this contradiction shows in particular that all subnormal nonnormal subgroups have infinite index. Clearly, this property is inherited by K, so that all finite homomorphic images of K are T-groups, and then K

is a T-group (see [17]). This new contradiction proves that G contains an Abelian normal subgroup A of finite index. In particular, G is polycyclic-by-finite and so it contains only finitely many finite subnormal subgroups. Let X be any subnormal subgroup of G; then $A \cap X$ is a normal subgroup of $H = AX$ and $X/A \cap X$ is a finite subnormal subgroup of $H/A \cap X$. As G is polycyclic-by-finite, it follows that $X^H/A \cap X$ is also finite and hence $N_H(X)$ has finite index in H and so also in G. Therefore all the normalizers of subnormal subgroups of G have finite index, and so $G/\omega(G)$ is finite.

Suppose now that G is soluble. In order to prove that $G/Z(G)$ is finite, by induction on the derived length of G we may suppose that its commutator subgroup G' is central-by-finite, so that G'' is finite. Since it is enough to prove that G is finite-by-Abelian, replacing G by G/G'' it can be assumed without loss of generality that G is metabelian. Moreover, as the largest periodic normal subgroup of G is finite, we may also suppose that G contains no periodic nontrivial normal subgroups. The Wielandt subgroup $\omega(G)$ is an infinite soluble finitely generated T-group, so that it is Abelian and $U = G'\omega(G)$ is a torsion-free nilpotent group. Let p be any prime number, and let P/U be the p-component of the finite Abelian group G/U. For each positive integer n, P/U^{p^n} is a finite p-group and so $\omega(G)$ induces on P/U^{p^n} a group of power automorphisms. Thus $\omega(G)U^{p^n}/U^{p^n}$ is contained in the norm of P/U^{p^n} and hence also in $Z_2(P/U^{p^n})$. Therefore $[\omega(G), P, P] \leqslant U^{p^n}$ and so

$$\big[\omega(G), P, P\big] \leqslant \bigcap_{n\geqslant 1} U^{p^n} = \{1\}.$$

It follows that $\omega(G)$ is contained in $Z_2(P)$, so that $P/Z_2(P)$ is finite and hence $\gamma_3(P) = \{1\}$. This shows that P is nilpotent, so that G is nilpotent. As G is torsion-free and Abelian-by-finite, it is even Abelian. The theorem is proved. □

It follows in particular from Theorem 3.2 that if G is a finitely generated soluble group whose Wielandt subgroup has finite index, then the factor group $G/Z(G)$ is finite.

Corollary 3.3. *Let G be a finitely generated torsion-free soluble group with finitely many normalizers of subnormal subgroups. Then G is Abelian.*

Finally, it is easy to show that a finitely generated soluble-by-finite group with finitely many normalizers of subnormal subgroups need not be central-by-finite. In fact, let G be the standard wreath product of an infinite cyclic group and a finite simple non-Abelian group; then every subnormal nonnormal subgroup of G is contained in the base group W, so that $\omega(G) = W$ and G has finitely many normalizers of subnormal subgroups.

The following result plays a central role in the study of groups with finitely many normalizers of subgroups with a given property. It was proved by B.H. Neumann [13] in the more general case of groups covered by finitely many cosets.

Lemma 3.4. *Let the group $G = X_1 \cup \cdots \cup X_t$ be the union of finitely many subgroups $X_1, \ldots, X_t$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_t$ has finite index in G.*

Recall that the *FC-center* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates, and a group G is called an *FC-group* if G coincides with its FC-center. Thus a group G has the property FC if and only if the centralizer $C_G(x)$ has finite index in G for each element x. Recall also that the *Baer radical* of a group G is the subgroup generated by all Abelian subnormal subgroups of G, and G is called a *Baer group* if it coincides with its Baer radical. In particular, every Baer group is locally nilpotent and all cyclic subgroups of a Baer group are subnormal.

Lemma 3.5. *Let G be a group with finitely many normalizers of infinite subnormal subgroups, and let B be the Baer radical of G. If x is an element of B such that $\langle x\rangle^G$ is finitely generated, then x belongs to the FC-center of G.*

Proof. It can obviously be assumed that x has infinite order, so that $\langle x\rangle^G$ is a finitely generated infinite nilpotent group. Then

$$\langle x\rangle = \bigcap_{X\in\mathcal{F}} X,$$

where $\mathcal{F}$ is the set of all subgroups of finite index of $\langle x\rangle^G$ containing x, and hence

$$\bigcap_{X\in\mathcal{F}} N_G(X) \leqslant N_G\bigl(\langle x\rangle\bigr).$$

Let X be any element of $\mathcal{F}$; then $\langle x\rangle^G/X_G$ is finite, so that also X^G/X_G is finite and X has finitely many conjugates in G. On the other hand, each element of $\mathcal{F}$ is an infinite subnormal subgroup of G, so that the set

$$\bigl\{N_G(X) \mid X\in\mathcal{F}\bigr\}$$

is finite; it follows that the index $|G : N_G(\langle x\rangle)|$ is finite. Therefore x has finitely many conjugates in G. $\square$

Our next main theorem shows in particular that a periodic soluble group G has finitely many normalizers of subnormal subgroups if and only if $G/\omega(G)$ is finite.

Theorem 3.6. *Let G be a soluble group with finitely many normalizers of subnormal subgroups. If G locally satisfies the maximal condition on subgroups, then the Wielandt subgroup $\omega(G)$ has finite index in G.*

Proof. Assume for a contradiction that the index $|G:\omega(G)|$ is infinite, and let $N_G(X_1)$, $\ldots, N_G(X_k)$ be the normalizers of infinite index of subnormal subgroups of G. Then $N_G(X_1)\cup\cdots\cup N_G(X_k)$ is properly contained in G by Lemma 3.4. Let g be an element of the set

$$G\setminus\bigl(N_G(X_1)\cup\cdots\cup N_G(X_k)\bigr),$$

and let x be any element of the Baer radical B of G. Then $\langle x\rangle^{\langle g\rangle}$ is a finitely generated subnormal subgroup of G which is normalized by g and so $N_G(\langle x\rangle^{\langle g\rangle})$ has finite index in G; it follows that $\langle x\rangle^G = (\langle x\rangle^{\langle g\rangle})^G$ is finitely generated and hence B is contained in the FC-center of G by Lemma 3.5. As the hypotheses are inherited from homomorphic images, we obtain that G has an ascending normal series with polycyclic factors. Let X be any subnormal subgroup of G with infinitely many conjugates, and let

$$\{1\} = C_0 < C_1 < \cdots < C_\alpha < C_{\alpha+1} < \cdots < C_\tau = X$$

be an ascending series with cyclic factors consisting of subnormal subgroups. Consider the least ordinal $\mu \leqslant \tau$ such that the index $|G : N_G(C_\mu)|$ is infinite. Suppose first that μ is not a limit ordinal, so that $H = N_G(C_{\mu-1})$ has finite index in G; since $C_\mu/C_{\mu-1}$ is a cyclic subnormal subgroup of $C_\mu H_G/C_{\mu-1}$, the first part of the proof yields that C_μ has finitely many conjugates in $C_\mu H_G$ and so also in G. This contradiction shows that μ is a limit ordinal, so that

$$C_\mu = \bigcup_{\alpha<\mu} C_\alpha$$

and hence

$$\bigcap_{\alpha<\mu} N_G(C_\alpha) \leqslant N_G(C_\mu).$$

As $N_G(C_\alpha)$ has finite index in G for each $\alpha < \mu$ and G has finitely many normalizers of subnormal subgroups, it follows that also the index $|G : N_G(C_\mu)|$ is finite, and this last contradiction completes the proof of the theorem. $\square$

In the special case of Baer groups, Theorem 3.6 has the following consequence. Observe that the existence of insoluble locally nilpotent T-groups (see, for instance, [12, Theorem 4.1]) shows that it cannot be extended to the case of locally nilpotent groups.

Corollary 3.7. *Let G be a Baer group with k normalizers of subnormal subgroups. Then G is nilpotent and its nilpotency class is at most $2k$.*

Proof. Clearly, the Wielandt subgroup and the norm of G coincide, so that by Theorem 3.6 the norm $N(G)$ has finite index in G; it follows that $G/Z_2(G)$ is finite and G is nilpotent. If $k = 1$, we have that G is a Dedekind group and so it has class at most 2. Suppose $k > 1$, so that $N(G)$ is properly contained in G and hence also $N(G) < N_2(G)$, where $N_2(G)/N(G) = N(G/N(G))$. It follows that $G/N(G)$ has less than k normalizers, and so by induction it has nilpotency class at most $2(k-1)$. Therefore the nilpotency class of G is at most $2k$. $\square$

It is well known that torsion-free soluble T-groups are Abelian, and Corollary 3.3 proves that this property also holds in the case of finitely generated soluble groups with finitely many normalizers of subnormal subgroups. This is no longer true for soluble groups with finitely many normalizers of subnormal subgroups, even in the case of metabelian residually finite minimax groups, as the following example shows. Let

$$U = \langle u_n \mid n \in \mathbb{N}_0,\ u_{n+1}^2 = u_n \rangle$$

and

$$V = \langle v_n \mid n \in \mathbb{N}_0,\ v_{n+1}^2 = v_n \rangle$$

be two (multiplicative) copies of the additive group $\mathbb{Q}_2$ of rational numbers whose denominators are powers of 2, and let $W = U \times V$. Consider the semidirect product $G = \langle x \rangle \ltimes W$, where x is an element of infinite order such that $u_n^x = v_n$ and $v_n^x = u_n$ for all n. Then $A = \langle W, x^2 \rangle$ is Abelian and G is a torsion-free residually finite minimax metabelian group; moreover,

$$G' = \langle u_n^{-1} v_n \mid n \in \mathbb{N}_0 \rangle \simeq \mathbb{Q}_2$$

and x acts as the inversion on G'. It follows that each subnormal nonnormal subgroup of G is contained in A, and hence A is the unique proper normalizer of subnormal subgroups of G.

4. Normalizers of infinite subnormal subgroups

The first lemma of this section shows in particular that a Baer group with finitely many normalizers of infinite subnormal subgroups is nilpotent and finite-by-Abelian.

Lemma 4.1. *Let G be a Baer group with finitely many normalizers of infinite subnormal subgroups. Then the factor group $G/Z(G)$ is finite.*

Proof. Assume for a contradiction that G is not nilpotent, and choose a counterexample with a minimal number k of proper normalizers of infinite subnormal subgroups. Since Baer groups with the property *IT* are nilpotent (see [6, Proposition 1.3]), G is not an *IT*-group and so $k \geqslant 1$. Let $N_G(X_1), \dots, N_G(X_k)$ be the proper normalizers of infinite subnormal subgroups of G. Clearly, each $N_G(X_i)$ has finitely many conjugates in G and so it is subnormal; thus $N_G(X_i)$ has less than k proper normalizers of infinite subnormal subgroups, and hence it is nilpotent. It follows that the index $|G : N_G(X_i)|$ is infinite for all i, and so $N_G(X_1) \cup \dots \cup N_G(X_k)$ is properly contained in G by Lemma 3.4. Consider an element g in the set

$$G \setminus \big(N_G(X_1) \cup \dots \cup N_G(X_k)\big).$$

Suppose first that G is not periodic. If x is any element of infinite order of G, $\langle x \rangle^{\langle g \rangle}$ is an infinite subnormal subgroup of G which is normalized by g, so that $\langle x \rangle^G = \langle x \rangle^{\langle g \rangle}$ is finitely generated and the index $|G : N_G(\langle x \rangle)|$ is finite by Lemma 3.5; then the subgroup

$\langle x\rangle$ must be normal in G and so $x \in Z(G)$. Since G is generated by its elements of infinite order, it follows that G is Abelian, and this contradiction shows that G is periodic. Put $L = \gamma_{n+1}(G)$, where $n = 2k + 2$. Let H be any infinite normal subgroup of G; then the factor group G/H has at most $k + 1$ normalizers of subnormal subgroups, so that it is nilpotent with class at most n by Corollary 3.7 and hence L is contained in H. It follows that L has no infinite proper G-invariant subgroups. In particular, L' is finite and replacing G by G/L' it can be assumed that L is Abelian; moreover, either L is a group of type p^∞ or it has exponent p, where p is a prime number. In the first case, L is contained in the center of G and G is nilpotent, a contradiction. Suppose finally that L has exponent p, and let V be a proper subgroup of finite index of L. Since V has finite index in $K = \langle L, g\rangle$, the core V_K is an infinite subnormal subgroup of G which is normalized by g. It follows that V_K is a normal subgroup of G, and this last contradiction proves that G is nilpotent. Therefore G has finitely many normalizers of infinite subgroups and hence $G/Z(G)$ is finite (see [4, Theorem B]). □

Lemma 4.2. *Let G be a group in which every subnormal subgroup with finitely many conjugates is normal, and let A be a periodic Abelian normal subgroup of G whose socle S is infinite. If G contains an element g of finite order normalizing no infinite subnormal nonnormal subgroups, then all subgroups of A are normal in G.*

Proof. Let H be any subgroup of finite index of S. Then the index $|\langle g, S\rangle : H|$ is finite and so the core $K = H_{\langle g,S\rangle}$ of H in $\langle g, S\rangle$ has finite index in H; moreover, K is a subnormal subgroup of G normalized by g, so that K is normal in G and H/H_G is finite. It follows that H^G/H_G is finite, so that H has finitely many conjugates and hence it is normal in G. On the other hand, each subgroup of S can be obtained as intersection of subgroups of finite index, and so all subgroups of S are normal in G. If a is any element of A, there exist infinite subgroups S_1 and S_2 of S such that

$$\langle a\rangle = \langle a, S_1\rangle \cap \langle a, S_2\rangle.$$

Clearly,

$$S_i \leqslant \langle a, S_i\rangle_G \leqslant \langle a, S_i\rangle \leqslant \langle a, S_i\rangle^G = \langle a, S_i\rangle^{\langle g\rangle} \leqslant \langle g, a, S_i\rangle,$$

so that $\langle a, S_i\rangle^G/\langle a, S_i\rangle_G$ is finite and the conjugacy class of $\langle a, S_i\rangle$ in G is finite (for $i = 1, 2$). Thus $\langle a, S_1\rangle$ and $\langle a, S_2\rangle$ are normal in G, and hence also $\langle a\rangle$ is a normal subgroup of G. Therefore all subgroups of A are normal in G. □

Lemma 4.3. *Let G be a soluble group with finitely many normalizers of infinite subnormal subgroups. If G contains a subgroup U of finite index which is a T-group, then $G/\omega(G)$ is finite.*

Proof. Since the subgroup U is hypercyclic, G locally satisfies the maximal condition on subgroups. Moreover, the core U_G of U is clearly a normal subgroup of finite index of G

with the property T, so that without loss of generality it can be assumed that U is normal in G. Let X be any infinite subnormal subgroup of G. Then $X \cap U$ is an infinite normal subgroup of $K = XU$; as K is subnormal in G, the group $K/X \cap U$ has finitely many normalizers of subnormal subgroups and hence its Wielandt subgroup has finite index by Theorem 3.6. It follows that X has finitely many conjugates in K and so also in G. Therefore the factor group $G/\bar{\omega}(G)$ is finite, and hence also $\omega(G)$ has finite index in G (see [1, Corollary 5]). □

We can now prove the main result of this section.

Theorem 4.4. *Let G be a periodic soluble group with finitely many normalizers of infinite subnormal subgroups. Then the factor group $G/\omega(G)$ is finite.*

Proof. Assume for a contradiction that the statement is false and choose a counterexample for which the set $\{N_G(X_1), \dots, N_G(X_k)\}$ of all proper normalizers of infinite subnormal subgroups has minimal order k. Clearly, G is not an *IT*-group, and hence $k \geqslant 1$. Suppose first that the index $|G : N_G(X_i)|$ is finite for some $i \leqslant k$, and let L be the core of $N_G(X_i)$ in G. Then L has less than k proper normalizers of infinite subnormal subgroups, so that $L/\omega(L)$ is finite and $\omega(G)$ has finite index in G by Lemma 4.3. This contradiction shows that all subgroups $N_G(X_1), \dots, N_G(X_k)$ have infinite index in G, i.e. all infinite subnormal subgroups of G with finitely many conjugates are normal. The set $N_G(X_1) \cup \dots \cup N_G(X_k)$ is properly contained in G by Lemma 3.4, and so we may consider an element

$$g \in G \setminus \big(N_G(X_k) \cup \dots \cup N_G(X_k)\big);$$

then g normalizes no infinite subnormal nonnormal subgroups of G. Clearly, G is not a Černikov group, so that also its Baer radical B is not a Černikov group; moreover, $B/Z(B)$ is finite by Lemma 4.1, so that both groups $Z(B)$ and B/B' have infinite socle. If H is any subgroup of B, it follows from Lemma 4.2 that $H \cap Z(B)$ and HB' are normal subgroups of G, so that H^G/H_G is finite and H has finitely many conjugates in G. In particular, all infinite subgroups of B are normal in G. Let X be an infinite subnormal nonnormal subgroup of G; then $X \cap B$ is infinite and so it is normal in G. Moreover, the factor group $G/X \cap B$ has finitely many normalizers of subnormal subgroups and hence its Wielandt subgroup has finite index by Theorem 3.6. Therefore X has finitely many conjugates in G, so that X is normal in G and this last contradiction completes the proof of the theorem. □

The above theorem admits the following slight extension.

Corollary 4.5. *Let G be a soluble group with finitely many normalizers of infinite subnormal subgroups. If the factor group $G/Z(G)$ is periodic, then $G/\omega(G)$ is finite. In particular, G has finitely many normalizers of subnormal subgroups.*

Proof. Clearly, G locally satisfies the maximal condition on subgroups. If the center of G is periodic, then G itself is periodic and so $G/\omega(G)$ is finite by Theorem 4.4. Suppose that $Z(G)$ is not periodic, and let X be any infinite subnormal subgroup of G. If $X \cap Z(G)$

is infinite, the factor group $G/X \cap Z(G)$ has finitely normalizers of subnormal subgroups and hence its Wielandt subgroup has finite index by Theorem 3.6; in particular, X has finitely many conjugates in G in this case. Assume now that $X \cap Z(G)$ is finite, so that X is periodic, and let a be an element of infinite order of $Z(G)$. The subnormal subgroup $\langle X, a\rangle$ has finitely many conjugates in G by the above argument; as X is characteristic in $\langle X, a\rangle$, also its conjugacy class is finite. Therefore all infinite subnormal subgroups of G have finitely many conjugates, so that $G/\bar{\omega}(G)$ is finite and hence $G/\omega(G)$ is finite. □

5. Subgroups of finite index

It is well known that every subgroup of a finite soluble T-group is likewise a T-group, but this property is no longer true for infinite soluble T-groups. However, H. Heineken and J.C. Lennox [10] proved that the property T is inherited by subgroups of finite index of arbitrary soluble T-groups. An example given in that paper can be used to show that subgroups of finite index of a soluble group with finitely many normalizers of subnormal subgroups may do not have the same property. In fact, let G be the group generated by elements a, b, c_n, d_n $(n \in \mathbb{N})$ with the relations

$$a^2 = b^3 = (ab)^2 = c_n^2 = d_n^2 = [c_m, c_n] = [d_m, d_n] = [c_m, d_n] = 1,$$
$$c_n^a = c_n^b = d_n, \qquad d_n^b = c_n d_n \quad \text{for all } m, n \in \mathbb{N}.$$

Then $G = H \ltimes K$, where $K = \langle c_n, d_n \mid n \in \mathbb{N}\rangle$ has exponent 2 and $H = \langle a, b\rangle$ is isomorphic to the symmetric group of degree 3; moreover, any subnormal subgroup of G either is contained in K or contains $\langle b, K\rangle$, so that $\omega(G) = K$ and G has finitely many normalizers of subnormal subgroups. On the other hand, $L = \langle a, K\rangle$ is a nilpotent subgroup of finite index of G and $\langle a\rangle$ has infinitely many conjugates in L, so that L has infinitely many normalizers of (subnormal) subgroups by Theorem 3.6.

In this section we will prove that a result corresponding to the theorem of Heineken and Lennox for groups with finitely many normalizers of subnormal subgroups holds at least within the universe of groups with finite conjugacy classes. Note that if G is a soluble FC-group with finitely many normalizers of infinite subnormal subgroups, it follows from Corollary 4.5 that also the set of all normalizers of subnormal subgroups of G is finite, since any FC-group is periodic over its center.

The first lemma proves in particular that a soluble FC-group with the property IT is either periodic or Abelian.

Lemma 5.1. *Let G be a soluble nonperiodic IT-group with a covering consisting of finitely generated normal subgroups. Then G is Abelian.*

Proof. Let a be an element of infinite order of G. If x and y are arbitrary elements of G, there exists a finitely generated normal subgroup E of G containing $\langle a, x, y\rangle$. As E is an IT-group, it is Abelian by Lemma 3.1. Therefore G is Abelian. □

It follows from the definition that every Abelian-by-finite FC-group is central-by-finite. We shall prove that a similar result holds replacing commutativity by the property IT and the center by the Wielandt subgroup; this is of independent interest, but will be used in the proof of the main theorem of this section. In our situation it turns out that all subnormal subgroups are normal-by-finite (recall here that a subgroup X of a group G is said to be *normal-by-finite* if X/X_G is finite).

Lemma 5.2. *Let G be an FC-group containing a subgroup of finite index which is a T-group (an IT-group). Then X/X_G is finite for each (infinite) subnormal subgroup X of G.*

Proof. Let K be a subgroup of finite index of G with the property T. As G is an FC-group, there exists a finitely generated normal subgroup E of G such that $G = EK$ and the centralizer $C = C_K(E)$ is a subgroup of G with finite index. If X is any subnormal subgroup of G, the subgroup $X \cap C$ is subnormal, and so even normal in K; it follows that $X \cap C$ is normal in G and hence X/X_G is finite. A similar argument proves the statement when G contains a subgroup of finite index with the property IT. □

Corollary 5.3. *Let G be a locally nilpotent FC-group containing a subgroup of finite index which is a T-group. Then the factor group $G/Z(G)$ is finite.*

Proof. Clearly, G is a Baer group. Moreover, all subnormal subgroups of G are normal-by-finite by Lemma 5.2 and hence G contains an Abelian subgroup of finite index (see [7, Corollary 3.3]). Therefore the factor group $G/Z(G)$ is finite. □

Let G be a periodic group, and for every prime number p let $O_{p'}(G)$ be the largest normal subgroup of G with no elements of order p. We shall denote by $\omega_p(G)$ the normal subgroup of G defined by the position

$$\omega\big(G/O_{p'}(G)\big) = \omega_p(G)/O_{p'}(G).$$

It has been proved by C. Casolo that if G is any periodic soluble group, then

$$\omega(G) = \bigcap_p \omega_p(G)$$

(see [2, Lemma 4.1]). Note also that if G does not contain nontrivial normal p'-subgroups, $\omega_p(G)$ obviously coincides with the Wielandt subgroup of G.

Lemma 5.4. *Let G be a periodic soluble FC-group containing a subgroup of finite index which is a T-group. Then the index $|G : \omega_p(G)|$ is finite for each prime number p.*

Proof. Clearly, G contains a normal subgroup K of finite index with the property T. Replacing G by the factor group $G/O_{p'}(G)$, we may suppose without loss of generality that $O_{p'}(G) = \{1\}$, so that in particular the Fitting subgroup F of K is a p-group. Then F is a Dedekind group and $K/C_K(F)$ is isomorphic to a periodic group of power automorphisms of F, so that $K/C_K(F)$ is finite. Therefore K/F is finite, so that G is Abelian-by-finite and hence even central-by-finite. It follows that $\omega_p(G) = \omega(G)$ has finite index in G. □

Theorem 5.5. *Let G be a subsoluble FC-group containing a subgroup of finite index which is an IT-group. Then the Wielandt subgroup $\omega(G)$ has finite index in G.*

Proof. Clearly, the group G is soluble and contains a normal subgroup K of finite index with the property *IT*. Since periodic soluble *IT*-group either have the property T or are Abelian-by-finite (see [6, Theorem 1.10]), by Lemma 5.1 it can be assumed that K is a periodic T-group. In particular, the commutator subgroup K' of K is Abelian; moreover, G/K' is an Abelian-by-finite *FC*-group, and hence $Z/K' = Z(G/K')$ has finite index in G/K'. For each prime number p which does not divide the order of the finite group G/K, put $L_p = O_{p'}(G)$ and let N_p/L_p be the largest normal p-subgroup of G/L_p. Then $K' \leqslant N_p \leqslant KL_p$. Let X be any subnormal subgroup of G containing L_p, and put $Y = (X \cap K)N_p$ and $V = (X \cap K)L_p$. Since K is a T-group, $X \cap K$ is normal in K and hence V is a normal subgroup of $KL_p = KN_p$; in particular, V is normal in XN_p. Moreover, Y/V is a normal p-subgroup of XN_p/V and

$$X/V \simeq XK/L_pK$$

is a p'-group. It follows that X/V is a characteristic subgroup of $XN_p/V = (Y/V)(X/V)$. As K' is contained in N_p, we have also that $[Z, G]$ lies in N_p, so that in particular Z normalizes XN_p; on the other hand, $K \cap Z$ is contained in $N_G(V)$, so that $K \cap Z \leqslant N_G(X)$. Therefore $K \cap Z$ is a subgroup of $\omega_p(G)$, and it follows that

$$\bigcap_{p \notin \pi(G/K)} \omega_p(G)$$

has finite index in G. Finally, by Lemma 5.4 also

$$\bigcap_{p \in \pi(G/K)} \omega_p(G)$$

is a subgroup of finite index, so that

$$\omega(G) = \bigcap_p \omega_p(G)$$

has finite index in G and the theorem is proved. □

We can finally prove our result on subgroups of finite index of soluble groups with finitely many normalizers of subnormal subgroups.

Theorem 5.6. *Let G be a soluble FC-group with finitely many normalizers of subnormal subgroups, and let X be a subgroup of finite index of G. Then X has finitely many normalizers of subnormal subgroups.*

Proof. As G is an FC-group, it locally satisfies the maximal condition on subgroups, and hence the index $|G:\omega(G)|$ is finite by Theorem 3.6. Then also $X \cap \omega(G)$ has finite index in G, and hence it is a T-group (see [10, Theorem A]). Therefore $X/\omega(X)$ is finite by Theorem 5.5, and so X has finitely many normalizers of subnormal subgroups. □

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