Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 314 (2007) 168–184
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

# On zero-divisor graphs of finite rings

S. Akbari [a,*], A. Mohammadian [b]

[a] *Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran*
[b] *Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran*

Received 30 July 2006
Available online 14 March 2007
Communicated by Luchezar L. Avramov

**Abstract**

The zero-divisor graph of a ring $R$ is defined as the directed graph $\Gamma(R)$ that its vertices are all non-zero zero-divisors of $R$ in which for any two distinct vertices $x$ and $y$, $x \to y$ is an edge if and only if $xy = 0$. Recently, it has been shown that for any finite ring $R$, $\Gamma(R)$ has an even number of edges. Here we give a simple proof for this result. In this paper we investigate some properties of zero-divisor graphs of matrix rings and group rings. Among other results, we prove that for any two finite commutative rings $R, S$ with identity and $n, m \geqslant 2$, if $\Gamma(M_n(R)) \simeq \Gamma(M_m(S))$, then $n = m$, $|R| = |S|$, and $\Gamma(R) \simeq \Gamma(S)$.


*Keywords:* Eulerian graph; Group ring; Matrix ring; Zero-divisor graph

## 1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see [1–4]. Throughout the paper, $R$ denotes a ring, not necessarily with identity; and $\mathcal{D}(R)$ denotes the set of all zero-divisors of $R$. If $X$ is either an element or a subset of $R$, then the *left annihilator* of $X$ is $\mathrm{Ann}_\ell(X) = \{a \in R \mid aX = 0\}$ and the *right annihilator* of $X$, denoted by $\mathrm{Ann}_r(X)$, is similarly defined. For any subset $Y$ of $R$, let $Y^* = Y \backslash \{0\}$. The *zero-divisor graph* of $R$, denoted by $\Gamma(R)$, is a directed graph with the vertex set $\mathcal{D}(R)^*$ in which for every two vertices $x$ and $y$, $x \to y$ is an edge if and only if $x \neq y$ and

* Corresponding author.
*E-mail addresses:* s_akbari@sharif.edu (S. Akbari), ali_m@mehr.sharif.edu (A. Mohammadian).

0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2007.02.051

$xy = 0$. It is well-known that if $R$ is not a domain, then $R$ is a finite ring if and only if $\Gamma(R)$ is a finite graph [2, Remark 1]. Also, for a ring $R$, we define a simple undirected graph $\overline{\Gamma}(R)$ with the vertex set $\mathcal{D}(R)^*$ in which two vertices $x$ and $y$ are adjacent if and only if $x \neq y$ and either $xy = 0$ or $yx = 0$.

In [15] it has been shown that for any ring $R$, every two vertices in $\overline{\Gamma}(R)$ are connected by a path of length at most 3. Moreover, in [15] it is shown that for any ring $R$, if $\overline{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\overline{\Gamma}(R)$, is at most 4. For other properties of zero-divisor graphs, see [2,3].

For every directed graph $\Gamma$ and any vertex $a$ of $\Gamma$, the number of the edges of $\Gamma$ of the form $x \to a$ is called the *in-degree* of $a$. The *out-degree* of $a$ is similarly defined. A directed graph $\Gamma$ is called *Eulerian* if for every vertex of $\Gamma$, the in-degree and the out-degree are the same. Two directed graphs $\Gamma_1$ and $\Gamma_2$ are said to be *isomorphic* if there is a bijective map $\varphi$ between the vertex set of $\Gamma_1$ and the vertex set of $\Gamma_2$ such that for any two vertices $x$ and $y$ of $\Gamma_1$, $x \to y$ is an edge in $\Gamma_1$ if and only if $\varphi(x) \to \varphi(y)$ is an edge in $\Gamma_2$. Similarly, two undirected graphs $G_1$ and $G_2$ are said to be *isomorphic* if there is a bijective map $\psi$ between the vertex set of $G_1$ and the vertex set of $G_2$ such that the adjacency relation is preserved. For an undirected graph $G$, the *degree* of a vertex $v$ of $G$ is the number of edges incident with $v$ and denoted by $\deg(v)$. Moreover, $\mathcal{N}(v)$ denotes the set of all vertices of $G$ adjacent to the vertex $v$.

For any ring $R$, we denote $M_n(R)$ and $M_{n\times m}(R)$ the ring of all $n \times n$ matrices over $R$ and the set of all $n \times m$ matrices over $R$, respectively. If $R$ has identity, then for any $i$ and $j$, $1 \leqslant i, j \leqslant n$, we denote by $E_{ij}$ that element of $M_n(R)$ whose $(i, j)$-entry is 1 and whose other entries are 0. The Jacobson radical of a ring $R$ denoted by $J(R)$. A ring $R$ is called *semi-simple* if $J(R) = \{0\}$. If $e$ is an idempotent element of $R$; that is $e^2 = e$, then by the Peirce Decomposition [12, (21.3)], we can write $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Notice that if $R$ has no identity, then $eR(1-e)$ denotes the subring $\{ex - exe \mid x \in R\}$; and similar definitions are used for $(1-e)Re$ and $(1-e)R(1-e)$. We recall that the idempotent $e$ is said to be *trivial* if $e \in \{0, 1\}$. Moreover, we say that a ring $R$ is *decomposable* if $R$ is isomorphic to a direct product of two non-zero rings. A ring $R$ is called *local* if $R$ has identity and $R/J(R)$ is a division ring. Also, $R$ is said to be *nilpotent* if $R^n = \{a_1 \cdots a_n \mid a_1, \ldots, a_n \in R\} = \{0\}$, for some $n \geqslant 1$.

## 2. The zero-divisor graphs of general rings

We begin this section with the investigation of finite rings whose zero-divisor graphs are Eulerian.

**Proposition 1.** *If $n \geqslant 2$ and $R_1, \ldots, R_n$ are finite rings and $R = R_1 \times \cdots \times R_n$, then $\Gamma(R)$ is an Eulerian graph if and only if for $i = 1, \ldots, n$, either $R_i$ is a field or $\Gamma(R_i)$ is an Eulerian graph.*

**Proof.** Clearly, if $S$ is a finite ring which is not a field, then $\Gamma(S)$ is an Eulerian graph if and only if $|\mathrm{Ann}_\ell(s)| = |\mathrm{Ann}_r(s)|$, for each element $s \in \mathcal{D}(S)$. First suppose that for $i = 1, \ldots, n$, either $R_i$ is a field or $\Gamma(R_i)$ is an Eulerian graph. Let $x = (x_1, \ldots, x_n)$ be a vertex in $\Gamma(R)$. If $R_i$ is not a field and $x_i$ is a vertex of $\Gamma(R_i)$, then we have $|\mathrm{Ann}_\ell(x_i)| = |\mathrm{Ann}_r(x_i)|$. Otherwise, $x_i$ is either zero or a unit. Thus we obtain that $|\mathrm{Ann}_\ell(x_i)| = |\mathrm{Ann}_r(x_i)|$, for $i = 1, \ldots, n$. Hence

$$\left|\mathrm{Ann}_\ell(x)\right| = \left|\mathrm{Ann}_\ell(x_1)\right| \cdots \left|\mathrm{Ann}_\ell(x_n)\right| = \left|\mathrm{Ann}_r(x_1)\right| \cdots \left|\mathrm{Ann}_r(x_n)\right| = \left|\mathrm{Ann}_r(x)\right|.$$

This implies that $\Gamma(R)$ is Eulerian. For the other direction, assume that for a fixed index $i$, $R_i$ is not a field and let $x_i$ be an arbitrary vertex of $\Gamma(R_i)$. Suppose that $\omega_i$ is that element of $R$ whose $i$th component is $x_i$ and other components are zero. Since $\Gamma(R)$ is Eulerian, $|\mathrm{Ann}_\ell(\omega_i)| = |\mathrm{Ann}_r(\omega_i)|$. This clearly yields that $|\mathrm{Ann}_\ell(x_i)| = |\mathrm{Ann}_r(x_i)|$ and so $\Gamma(R_i)$ is an Eulerian graph. Now, the result follows. □

**Lemma A.** *(See [2, Lemma 14].) Let $F$ be a finite field and $n \geqslant 2$. Suppose that $a \in \mathcal{D}(M_n(F))^*$ and* $\operatorname{rank} a = k$. *Then the in-degree and the out-degree of $a$ in $\Gamma(M_n(F))$ are $|F|^{n(n-k)} - \varepsilon$, and* $\deg(a) = 2|F|^{n(n-k)} - |F|^{(n-k)^2} - \varepsilon$ *in $\overline{\Gamma}(M_n(F))$, where $\varepsilon = 1$, unless $a^2 = 0$ and in this case $\varepsilon = 2$.*

**Theorem 2.** *Let $R$ be a finite semi-simple ring which is not a field. Then $\Gamma(R)$ is an Eulerian graph.*

**Proof.** By Proposition 1 and the Wedderburn–Artin Theorem [12, (3.5)], we may assume that $R = M_n(F)$, for a finite field $F$ and $n \geqslant 2$. By Lemma A, for any vertex $a$ of $\Gamma(R)$, $|\mathrm{Ann}_\ell(a)| = |\mathrm{Ann}_r(a)|$. This completes the proof. □

**Theorem 3.** *Let $K$ be a finite field and $G$ be a finite group. Then $\Gamma(KG)$ is an Eulerian graph.*

**Proof.** Assume that $\alpha = \sum_{g\in G} \alpha_g g$, $\beta = \sum_{g\in G} \beta_g g$, and $\gamma = \sum_{g\in G} \gamma_g g$ be three elements in $KG$ such that $\beta\alpha = \alpha\gamma = 0$. We have

$$\sum_{h\in G}\left(\sum_{g\in G} \beta_{g^{-1}}\alpha_{gh}\right)h = 0 \quad \text{and} \quad \sum_{h\in G}\left(\sum_{g\in G} \alpha_{hg}\gamma_{g^{-1}}\right)h = 0.$$

Therefore if we set $G = \{g_1, \ldots, g_n\}$, then the above relations shows that the vectors $(\beta_{g_1^{-1}}, \ldots, \beta_{g_n^{-1}})$ and $(\gamma_{g_1^{-1}}, \ldots, \gamma_{g_n^{-1}})$ are contained in the right kernels of the matrices $\mathscr{L} = [\alpha_{g_j g_i}]_{1\leqslant i,j\leqslant n}$ and $\mathscr{R} = [\alpha_{g_i g_j}]_{1\leqslant i,j\leqslant n}$, respectively. Conversely, for any two vectors $\mathcal{U}$ and $\mathcal{V}$ in the right kernels of $\mathscr{L}$ and $\mathscr{R}$, respectively, we can easily construct two elements $\beta_{\mathcal{U}} \in \mathrm{Ann}_\ell(\alpha)$ and $\gamma_{\mathcal{V}} \in \mathrm{Ann}_r(\alpha)$. Now, since $\mathscr{R}$ is the transpose of $\mathscr{L}$,

$$\dim_K \mathrm{Ann}_\ell(\alpha) = \operatorname{nullity} \mathscr{L} = \operatorname{nullity} \mathscr{R} = \dim_K \mathrm{Ann}_r(\alpha).$$

This yields that $|\mathrm{Ann}_\ell(\alpha)| = |\mathrm{Ann}_r(\alpha)|$ and so the theorem is proved. □

**Remark 4.** Suppose that $R$ is a left Artinian ring. A well-known theorem due to Brauer states that every left ideal of $R$ either is nilpotent or it contains a non-zero idempotent element [6, Theorem 13-1]. Hence $R$ has no non-trivial idempotent if and only if $R$ is either a nilpotent ring or a local ring. We will use of this fact frequently.

In the next theorem, we characterize all zero-divisor graphs whose out-degrees of all vertices are the same.

**Theorem 5.** *Let $R$ be a finite ring. If all vertices of $\Gamma(R)$ have the same out-degrees, then either $\mathcal{D}(R)^2 = \{0\}$ or there exists a finite field $F$ such that $R \simeq F \times F$.*

**Proof.** If there is a vertex $v$ in $\Gamma(R)$ such that for any vertex $x \neq v$, $v \to x$ is an edge in $\Gamma(R)$, then $\overline{\Gamma}(R)$ is a complete graph and so by [2, Theorem 5], we are done. So we may assume that $R$ is neither a nilpotent ring nor a local ring. Thus by Remark 4, $R$ has a non-trivial idempotent, say $e$, and therefore $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$.

First we show that $eRe$ is a field. It is enough to show that $eRe$ is a domain. Assume that there exist two elements $x, y \in eRe^*$ such that $xy = 0$. We have $\mathrm{Ann}_r(e) \subseteq \mathrm{Ann}_r(x)$, so noting the out-degrees of $e$ and $x$ are the same and $ey \neq 0$, we find $y = x$ and $\mathrm{Ann}_r(x) = \mathrm{Ann}_r(e) \cup \{x\}$. Since $\mathrm{Ann}_r(e)$ and $\mathrm{Ann}_r(x)$ are two additive subgroups of $R$ and $|\mathrm{Ann}_r(x) \setminus \mathrm{Ann}_r(e)| = 1$, we conclude that $\mathrm{Ann}_r(e) = \{0\}$. This implies that the out-degree of any vertex of $\Gamma(R)$ is zero. Since $\overline{\Gamma}(R)$ is a connected graph [15, Theorem 3.2] and $\Gamma(R)$ has at least two vertices $e$ and $x$, we get a contradiction. Thus $\mathcal{D}(eRe) = \{0\}$ and so $eRe$ is a field. Next, we show that $(1-e)Re = \{0\}$. Consider an element $a \in (1-e)Re^*$. Since the out-degrees of $e$ and $a$ are the same and as well as $\{a\} \subseteq \mathrm{Ann}_r(e) \subseteq \mathrm{Ann}_r(a)$, both sets $\mathrm{Ann}_r(e)$ and $\mathrm{Ann}_r(a)$ are two non-trivial additive subgroups of $R$ with $|\mathrm{Ann}_r(a) \setminus \mathrm{Ann}_r(e)| = 1$, which is impossible.

Now, for simplification, let $S = (1-e)R(1-e)$ and $T = eRe \oplus eR(1-e)$. Suppose that there exist two elements $s_1, s_2 \in S^*$ such that $s_1^2 = 0$ and $s_2^2 \neq 0$. It is easily checked that the out-degree of $s_1$ is $|T||\mathrm{Ann}_r(s_1) \cap S| - 2$ and the out-degree of $s_2$ is $|T||\mathrm{Ann}_r(s_2) \cap S| - 1$. The equality of these numbers shows that $|T|(|\mathrm{Ann}_r(s_1) \cap S| - |\mathrm{Ann}_r(s_2) \cap S|) = 1$. This is a contradiction and so either $S$ is a nilpotent ring or a reduced ring. In the first case, since $S$ is finite, there exists at least one element $c \in S^* \cap \mathrm{Ann}_r(S)$. The equality $R = T \oplus S$ follows that for each vertex $v \neq c$ in $\Gamma(R)$, there exists an edge from $c$ to $v$, a contradiction. Hence $S$ is a finite reduced ring and so has identity, say $g$. Suppose that $(e+g)z = 0$, for some $z \in R$. Then multiplying on the left by $e$ yields $ez = 0$. This shows that $\mathrm{Ann}_r(e+g) \subseteq \mathrm{Ann}_r(e)$. Since $g \in \mathrm{Ann}_r(e) \setminus \mathrm{Ann}_r(e+g)$, $e+g$ cannot be a vertex of $\overline{\Gamma}(R)$. Thus $e+g$ is an idempotent in $R \setminus \mathcal{D}(R)$ and so $1 = e+g$ is the identity of $R$. Hence $1-e \in R$. So, if we apply the method used in the previous paragraph for $g$ instead of $e$, then we conclude that $S$ is a field and $eR(1-e) = \{0\}$. Hence $T = eRe$ is a field and therefore the equality $R = T \oplus S$ completes the proof. □

Note that if we replace the out-degrees by the in-degrees in Theorem 5, by considering $R^{\mathrm{op}}$ instead of $R$, then the assertion is still valid. The following theorem has been established in [3] for commutative rings with identity. We want to generalize this result for any arbitrary ring.

**Theorem 6.** *Let $R$ be a finite ring. If all vertices of $\overline{\Gamma}(R)$ have the same degrees, then either $\mathcal{D}(R)^2 = \{0\}$ or there exists a finite field $F$ such that $R \simeq F \times F$.*

**Proof.** If there exists a vertex $v$ in $\overline{\Gamma}(R)$ adjacent to all other vertices of the graph, then $\overline{\Gamma}(R)$ is a complete graph and so by [2, Theorem 5], we are done. So assume that there is no such a vertex. Thus $R$ is neither a nilpotent ring nor a local ring. Hence by Remark 4, $R$ has a non-trivial idempotent $e$ and therefore we can write $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$, where $R_1 = eRe$, $R_2 = eR(1-e)$, $R_3 = (1-e)Re$, and $R_4 = (1-e)R(1-e)$. We show that $R_1$ is a field. Suppose that $xy = 0$, for some elements $x, y \in R_1^*$. Since $\deg(e) = \deg(x)$ and $\mathcal{N}(e) \subseteq \mathcal{N}(x)$, $\mathcal{N}(e) = \mathcal{N}(x)$. Also, since $ye = ey = y$, $y = x$ and $\mathrm{Ann}_\ell(x) \setminus \mathrm{Ann}_\ell(e) = \mathrm{Ann}_r(x) \setminus \mathrm{Ann}_r(e) = \{x\}$. Now, $\mathrm{Ann}_\ell(x), \mathrm{Ann}_\ell(e), \mathrm{Ann}_r(x)$ and $\mathrm{Ann}_r(e)$ are additive subgroups of $R$, thus we conclude that $\mathrm{Ann}_\ell(e) = \mathrm{Ann}_r(e) = \{0\}$. So $R = R_1$, which contradicts $e$ is a non-trivial idempotent. Thus $R_1$ is a field.

First assume that $R_4$ has no non-zero idempotent. Then $R_4$ is a nilpotent ring. For any two elements $a \in R_2$ and $b \in R_3$, we have $ba \in R_4$ and thus $ab$ is a nilpotent element of $R_1$. Since $R_1$

is a field, $ab = 0$. Suppose that $R_4 = \{0\}$. Since $e \neq 1$, at least one of the rings $R_2$ and $R_3$ is non-zero. Also, for any two elements $a \in R_2$ and $b \in R_3$, we have $R \subseteq \mathrm{Ann}_r(a)$ and $R \subseteq \mathrm{Ann}_\ell(b)$. So we find a vertex adjacent to all other vertices of $\overline{\Gamma}(R)$, a contradiction. Hence $R_4 \neq \{0\}$ and since $R_4$ is nilpotent, there is an element $c \in R_4^* \cap \mathrm{Ann}_\ell(R_4) \cap \mathrm{Ann}_r(R_4)$. Noting $c^2 = 0$, we find that

$$\begin{aligned}\deg(c) = {} & |R_1|\big|\mathrm{Ann}_\ell(c) \cap R_2\big||R_3||R_4| + |R_1||R_2|\big|\mathrm{Ann}_r(c) \cap R_3\big||R_4| \\ & - |R_1|\big|\mathrm{Ann}_\ell(c) \cap R_2\big|\big|\mathrm{Ann}_r(c) \cap R_3\big||R_4| - 2.\end{aligned}$$

Note that if $|R_2| \geqslant |R_3|$, then obviously we have

$$\big|\mathrm{Ann}_\ell(c) \cap R_2\big|\big(|R_3| - \big|\mathrm{Ann}_r(c) \cap R_3\big|\big) + |R_2|\big(\big|\mathrm{Ann}_r(c) \cap R_3\big| - 1\big) + 1 \geqslant |R_3|.$$

If $|R_3| \geqslant |R_2|$, then we obtain a similar inequality. Thus

$$\big|\mathrm{Ann}_\ell(c) \cap R_2\big||R_3| + |R_2|\big|\mathrm{Ann}_r(c) \cap R_3\big| - \big|\mathrm{Ann}_\ell(c) \cap R_2\big|\big|\mathrm{Ann}_r(c) \cap R_3\big| \geqslant |R_2| + |R_3| - 1.$$

Since $\deg(e) = |R_2||R_4| + |R_3||R_4| - |R_4| - 1$, the above inequality shows that $\deg(c) + 2 \geqslant |R_1|(\deg(e) + 1)$. This is a contradiction, since $\deg(e) = \deg(c) \geqslant 1$ and $|R_1| \geqslant 2$.

Next assume that $R_4$ has a non-zero idempotent, say $g$. Clearly, we have $\mathrm{Ann}_\ell(e+g) \subseteq \mathrm{Ann}_\ell(e)$ and $\mathrm{Ann}_r(e+g) \subseteq \mathrm{Ann}_r(e)$. Therefore, if $e+g$ is a vertex of $\overline{\Gamma}(R)$, then $\mathcal{N}(e+g) \subseteq \mathcal{N}(e)$. Since $\deg(e) = \deg(e+g)$, $\mathcal{N}(e) = \mathcal{N}(e+g)$. But $g \in \mathcal{N}(e) \setminus \mathcal{N}(e+g)$, a contradiction. Thus $e+g$ is an idempotent in $R \setminus \mathcal{D}(R)$ and so $1 = e+g$ is the identity of $R$. Hence $1 - e \in R$, so with a similar argument given in the first paragraph of the proof, we find that $R_4$ is a field. Since $R_1$ and $R_4$ are fields, it is straightforward to verify that $eR$ and $gR$ are two simple right $R$-modules. Since $R = eR \oplus gR$,

$$R \simeq \mathrm{End}(R_R) \simeq \begin{bmatrix} \mathrm{Hom}_R(eR, eR) & \mathrm{Hom}_R(eR, gR) \\ \mathrm{Hom}_R(gR, eR) & \mathrm{Hom}_R(gR, gR) \end{bmatrix}$$

and hence $R$ is isomorphic to either the $2 \times 2$ full matrix ring over a finite field or the direct product of two finite fields; depends on $eR \simeq gR$ as right $R$-modules or not. By Lemma A, $\deg(E_{11}) = 2|K|^2 - |K| - 1$ and $\deg(E_{12}) = 2|K|^2 - |K| - 2$ in $\overline{\Gamma}(M_2(K))$, for any finite field $K$. Therefore there exist two finite fields $F_1$ and $F_2$ such that $R \simeq F_1 \times F_2$. Now, $\deg(1, 0) = |F_2| - 1$ and $\deg(0, 1) = |F_1| - 1$ are equal, thus we find that $|F_1| = |F_2|$. Hence $F_1 \simeq F_2$ and the proof is complete. □

The following beautiful theorem has been proved by Redmond in [16]. His proof is rather long. Here we give a simple proof for Redmond's result.

**Theorem 7.** *(See [16, Theorem 5.1].) Let $R$ be a finite ring which is not a field. Then $\Gamma(R)$ has an even number of edges.*

**Proof.** We prove that the total sum of the out-degrees in $\Gamma(R)$ is even. For every vertex $x$, we have $\mathrm{Ann}_r(x) = \mathrm{Ann}_r(-x)$ and so the out-degrees of $x$ and $-x$ are the same. Hence it is enough to verify that the total sum of the out-degrees of the vertices $x$ with this property that $x = -x$, is even. Moreover, if $2x = 0$ and $x \to y$ is an edge, then $x \to -y$ is also an edge of $\Gamma(R)$. Therefore it is sufficient to show that the induced subgraph on all vertices $x$ in which $2x = 0$, has an even number of edges. So we may assume that $\operatorname{char} R = 2$. Clearly, it is enough to show that $\sum_{x\in R} |\mathrm{Ann}_r(x)\backslash \mathrm{Ann}_\ell(x)|$ is even. For any $x \in R$, we have

$$\bigl|\mathrm{Ann}_r(x)\backslash \mathrm{Ann}_\ell(x)\bigr| = \bigl(\bigl[\mathrm{Ann}_r(x) : \mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)\bigr] - 1\bigr)\bigl|\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)\bigr|.$$

Since $|R|$ is a 2-power, $|\mathrm{Ann}_r(x)\backslash \mathrm{Ann}_\ell(x)|$ is even if and only if $\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x) \neq \{0\}$. Thus, to complete the proof, we show that the set

$$\mathcal{T} = \bigl\{x \in R \mid \mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x) = \{0\} \text{ and } \mathrm{Ann}_r(x) \neq \{0\}\bigr\}$$

has an even number of elements. Assume that $a$ is an arbitrary element of $\mathcal{T}$. Since $R$ is finite, there are two natural numbers $m > n$ such that $a^m = a^n$. Thus $a^{n-1}(a^{m-n+1} - a) = 0$. On the other hand, $\mathrm{Ann}_\ell(a) \cap \mathrm{Ann}_r(a) = \{0\}$, so we conclude that $a^{m-n+1} = a$. Thus $e = a^{m-n}$ is an idempotent element and it is easily checked that $\mathrm{Ann}_\ell(e) = \mathrm{Ann}_\ell(a)$ and $\mathrm{Ann}_r(e) = \mathrm{Ann}_r(a)$. Therefore $e \in \mathcal{T}$. Since $\mathrm{Ann}_\ell(e) \cap \mathrm{Ann}_r(e) = \{0\}$, $(1 - e)R(1 - e) = \{0\}$. On the other hand, $R = eRe \oplus eR(1-e) \oplus (1-e)Re$ implies that $\mathrm{Ann}_r(e) = (1-e)Re$. For any $h \in \mathrm{Ann}_r(e)$, since $((1-e)Re)^2 = \{0\}$ and $ae = a$, we have $\mathrm{Ann}_r(e) \subseteq \mathrm{Ann}_r(a+h)$. Now, if $(a+h)y = 0$, for some $y \in R$, then multiplying this equation by $e$, we conclude that $ay = 0$ and so $ey = a^{m-n-1}ay = 0$. This shows that $\mathrm{Ann}_r(e) = \mathrm{Ann}_r(a + h)$. Also, we have $\mathrm{Ann}_\ell(a + h) \cap \mathrm{Ann}_r(a + h) = \{0\}$. To see this, suppose that $z(a + h) = (a + h)z = 0$, for some $z \in R$. Thus $z \in (1 - e)Re$ and so $zh = hz = 0$. Hence $za = az = 0$ and this implies that $z = 0$. So we obtain that $a + \mathrm{Ann}_r(e) \subseteq \mathcal{T}$.

Let $U$ be the family of all sets of the form $a + \mathrm{Ann}_r(e)$, where $a, e \in \mathcal{T}$, $e^2 = e$, and $e = a^k$, for some $k \geqslant 1$. Thus $\mathcal{T}$ is the union of all elements of $U$. We show that every two distinct elements $a + \mathrm{Ann}_r(e)$ and $a' + \mathrm{Ann}_r(e')$ of $U$ are disjoint. Indeed, if $a + w = a' + w'$ is an element in their intersection, then $\mathrm{Ann}_r(e) = \mathrm{Ann}_r(a + w) = \mathrm{Ann}_r(a' + w') = \mathrm{Ann}_r(e')$ and so $a + \mathrm{Ann}_r(e) = a' + \mathrm{Ann}_r(e')$. Moreover, for any $e \in \mathcal{T}$, $\mathrm{Ann}_r(e)$ is a non-trivial additive subgroup of $R$. Hence every element of $U$ has even size and therefore $|\mathcal{T}|$ is even. The proof is complete. □

## 3. The zero-divisor graphs of matrix rings

A subset $\Omega$ of the vertex set of a directed graph $\Gamma$ is called *clique* if $x \to y$ and $y \to x$ are two edges of $\Gamma$, for each pair of distinct vertices $x, y \in \Omega$. The maximum size of a clique in a directed graph $\Gamma$ is called the *clique number* of $\Gamma$ and denoted by $\omega(\Gamma)$. In the next theorem, we calculate $\omega(\Gamma(M_n(F)))$, for any finite field $F$ and $n \geqslant 2$.

**Theorem 8.** *If $F$ is a finite field and $n \geqslant 2$, then $\omega(\Gamma(M_n(F))) = |F|^{\lfloor \frac{n^2}{4} \rfloor} - 1$, unless $n = 2$ and $|F| = 2$ and in this case $\omega(\Gamma(M_n(F))) = 2$.*

**Proof.** Clearly, the set

$$\Omega_0 = \left\{ \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \;\middle|\; X \in M_{\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil}(F)^* \right\}$$

is a clique of size $|F|^{\lfloor \frac{n^2}{4} \rfloor} - 1$ in $\Gamma(M_n(F))$. Let $\Omega$ be a clique of maximum size in $\Gamma(M_n(F))$. For any $x, y \in \Omega$, we have $xy = yx = 0$. By a theorem due to Schur [13], every commuting subset of $M_n(F)$ contains at most $m = \lfloor \frac{n^2}{4} \rfloor + 1$ linearly independent matrices over $F$. Assume that $|\Omega| > |\Omega_0|$. So $\Omega$ has exactly $m$ linearly independent matrices, say $a_1, \ldots, a_m$. Then $I$ is contained in the vector space generated by $\Omega$. This means that

$$I = \lambda_1 a_1 + \cdots + \lambda_m a_m, \tag{1}$$

where $\lambda_i \in F$, for any $i$. We have

$$\ker a_1 \supseteq \ker a_1 \cap \ker a_2 \supseteq \cdots \supseteq \bigcap_{i=1}^{m} \ker a_i. \tag{2}$$

If $n \geqslant 4$, then $m > n$. Now, the relation (2) implies that there exists some index $t$ such that $\bigcap_{i=1}^{t} \ker a_i = \bigcap_{i=1}^{t+1} \ker a_i$. Since all columns of $a_{t+1}$ are contained in $\bigcap_{i=1}^{t} \ker a_i$, $a_{t+1}^2 = 0$. Now, multiplying Eq. (1) by $a_{t+1}$ yields $a_{t+1} = 0$, a contradiction. Therefore $n \leqslant 3$. If there exists a matrix $a \in \Omega$ whose minimal polynomial has degree 3, then $n = 3$ and by [11, Corollary 4.4.18], every matrix in $\Omega$ is a polynomial in $a$. Let $b = \lambda a^2 + \mu a + \nu I$ be an arbitrary element in $\Omega$. Clearly, in this case $|\Omega_0| \geqslant 3$ and so there exists a matrix $c \in \Omega \backslash \{a, b\}$. We have $ca = cb = 0$. But $cb = \nu c$ and so $\nu = 0$. Hence we have $|\Omega| \leqslant |F|^2 - 1$, a contradiction. Thus the minimal polynomial of every matrix in $\Omega$ has degree 2. If there exists a matrix $z \in \Omega$ such that $z^2 = 0$, then we may suppose that $a_1 = z$. By multiplying Eq. (1) by $a_1$, we obtain again a contradiction. So $\omega^2 \neq 0$, for every $\omega \in \Omega$. Since each element of $\Omega$ has zero as an eigenvalue, the minimal polynomial of any element of $\Omega$ has simple roots. Thus $\Omega$ is simultaneously diagonalizable. Furthermore, since the product of every two distinct elements of $\Omega$ is zero, $|\Omega| \leqslant n$. On the other hand, $|\Omega| > |F|^{\lfloor \frac{n^2}{4} \rfloor} - 1$, so we conclude that $n = 2$ and $|F| = 2$. Now, since $\{E_{11}, E_{22}\}$ is a clique in $\Gamma(M_n(F))$, the proof is complete. □

Now we want to verify for which full matrix rings over a finite commutative ring with identity, the zero-divisor graph determines the corresponding ring.

**Lemma 9.** *Let $F$ and $E$ be two finite fields with* $\operatorname{char} F = \operatorname{char} E$ *and $n, m \geqslant 2$. If*

$$\left|\mathcal{D}(M_n(F))\right| = \left|\mathcal{D}(M_m(E))\right|,$$

*then $n = m$ and $F \simeq E$.*

**Proof.** Let $p = \operatorname{char} F$. Since $|\mathcal{D}(M_n(F))| = |\mathcal{D}(M_m(E))|$, using [12, (9.20)] we have

$$|F|^{n^2} - \prod_{i=0}^{n-1} \left(|F|^n - |F|^i\right) = |E|^{m^2} - \prod_{i=0}^{m-1} \left(|E|^m - |E|^i\right).$$

By considering the $p$-powers in the left side and the right side of the above equality we find

$$|F|^{n(n-1)/2} = |E|^{m(m-1)/2}, \tag{3}$$

and also

$$|F|^{n(n+1)/2} - |E|^{m(m+1)/2} = \prod_{i=1}^{n}\big(|F|^i - 1\big) - \prod_{i=1}^{m}\big(|E|^i - 1\big). \tag{4}$$

By contradiction, assume that $|F| > |E|$. So by (3) we have $n < m$. If $n$ and $m$ have the same parities, then $|E|^{m(m+1)/2}$ and $|E|$ are the biggest $p$-powers that divide the left side and the right side of (4), respectively, which contradicts $m \geqslant 2$. Thus the parities of $n$ and $m$ are not the same. Since the left side and the right side of (4) are congruent to 0 and $(-1)^n - (-1)^m$ modulo of $|E|$, respectively, we conclude that $|E| = 2$ and therefore there exists an integer $t$ such that $|F| = 2^t$. If $t \geqslant 3$, then it is not hard to verify that the left side and the right side of (4) are congruent to 0 and 4 modulo of 8, respectively, which is impossible. Hence $t = 2$. By a result of [7], for any integer $\alpha \geqslant 2$, $\alpha \neq 6$, the number $2^\alpha - 1$ has a prime factor, say $q$, such that for every integer $\beta$, $1 \leqslant \beta \leqslant \alpha - 1$, $2^\beta - 1$ is not divisible by $q$. Equation (3) follows that $2n(n-1) = m(m-1)$ and hence $n \neq 6$ and $2n > m$. So (4) yields that

$$2^{m(m+1)/2}\big(2^{2n-m} - 1\big) = \prod_{i=1}^{n}\big(2^i + 1\big)\prod_{i=1}^{n}\big(2^i - 1\big) - \prod_{i=1}^{m}\big(2^i - 1\big).$$

Now, $2n - m < n$, so there exists a prime number $q$ such that divides $2^n - 1$ and does not divide $2^{2n-m} - 1$. Since $n < m$, we get a contradiction. Thus $|F| = |E|$. Now, Eq. (3) implies that $n = m$, as desired. □

**Theorem 10.** *Let $R$ and $S$ be two finite commutative local rings and $n, m \geqslant 2$. If $\Gamma(M_n(R)) \simeq \Gamma(M_m(S))$, then $n = m$, $|R| = |S|$, and $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$.*

**Proof.** If one of the rings $R$ and $S$ is a field, then by [2, Theorem 20], there is nothing to prove. So suppose that $J(R)$ and $J(S)$ are non-zero. Assume that $\mathcal{A}$ is the set of all vertices $v$ of $\Gamma(M_n(R))$ such that $v \to x$ is an edge in $\Gamma(M_n(R))$ if and only if $x \to v$ is an edge. Let $F = R/J(R)$ and $Z = \mathrm{Ann}_\ell(J(R))$. Also, for any element $x \in M_n(R)$, suppose that $\bar{x}$ is the element of $M_n(F)$ corresponding to $x$. It is straightforward to see that for any matrix $\bar{x} \in \mathcal{D}(M_n(F))^*$, $\mathrm{Ann}_r(\bar{x}) \setminus \mathrm{Ann}_\ell(\bar{x}) \neq \emptyset$. Hence if there exists a vertex $v \in \mathcal{A}$ with $\bar{v} \neq 0$, then there is a matrix $\overline{w} \in M_n(F)$ such that $\bar{v}\,\overline{w} = 0$ and $\overline{w}\,\bar{v} \neq 0$. Now, for an element $z \in Z^*$, $v \to wz$ is an edge in $\Gamma(M_n(R))$ and there is no edge from $wz$ to $v$. This contradicts the definition of $\mathcal{A}$ and shows that $\mathcal{A} \subseteq M_n(J(R))$. Let $\mathcal{B} = \mathrm{Ann}_\ell(\mathcal{A})^*$. Since $\mathcal{A} \subseteq M_n(J(R))$, $M_n(Z)^* \subseteq \mathcal{B}$. Furthermore, $\{xI \mid x \in J(R)^*\} \subseteq \mathcal{A}$. Therefore $\mathcal{B} \subseteq M_n(Z)$ and so $\mathcal{B} = M_n(Z)^*$. We have $M_n(J(R)) \subseteq \mathrm{Ann}_\ell(\mathcal{B})$. On the other hand, $\{xI \mid x \in Z^*\} \subseteq \mathcal{B}$ and thus $\mathrm{Ann}_\ell(\mathcal{B}) \subseteq M_n(J(R))$. So $\mathrm{Ann}_\ell(\mathcal{B}) = M_n(J(R))$. By $\Gamma(M_n(R)) \simeq \Gamma(M_m(S))$, we conclude that

$$\big|J(R)\big|^{n^2} = \big|J(S)\big|^{m^2}. \tag{5}$$

Since the order of any finite local ring is a prime power, Eq. (5) yields that $\operatorname{char} F = \operatorname{char} E$, where $E = S/J(S)$. Moreover, if $GL_n(F)$ denotes the unit group of $M_n(F)$, then by [8, Theorem 1], the number of units of $M_n(R)$ is $|J(R)|^{n^2}|GL_n(F)|$. Therefore the number of vertices of $\Gamma(M_n(R))$ is $|R|^{n^2} - |J(R)|^{n^2}|GL_n(F)| - 1 = |J(R)|^{n^2}|\mathcal{D}(M_n(F))| - 1$. Since $\Gamma(M_n(R)) \simeq \Gamma(M_m(S))$, the equality (5) concludes that $|\mathcal{D}(M_n(F))| = |\mathcal{D}(M_m(E))|$. By Lemma 9, $n = m$ and $|F| = |E|$. Again applying (5), we obtain that $|J(R)| = |J(S)|$ and therefore $|R| = |S|$.

To complete the proof, we show that $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$. Let $a$ be the vertex of $\Gamma(M_n(S))$ corresponding to the vertex $E_{11}$ in $\Gamma(M_n(R))$. Then by Lemma A, the out-degree of $a$ is equal to $|R|^{n(n-1)} - 1$. Since $\mathrm{Ann}_\ell(\mathcal{B}) = M_n(J(R))$ and $\Gamma(M_n(R)) \simeq \Gamma(M_n(S))$, the induced subgraph of $\Gamma(M_n(R))$ on $M_n(J(R))^*$ is isomorphic to the induced subgraph of $\Gamma(M_n(S))$ on $M_n(J(S))^*$. Hence, since $E_{11} \notin M_n(J(R))$, at least one of the entries of $a$ is unit. Up to matrix similarity, we may assume that at least one of the entries of the first row of $a$ is unit. There is a permutation matrix $p$ such that

$$ap = \begin{bmatrix} a_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where $a_1$ is a unit in $S$. Suppose that

$$\begin{bmatrix} x_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathrm{Ann}_r(ap),$$

where $x_1 \in S$. Thus $[x_1 \quad X_2] = -a_1^{-1}A_2[X_3 \quad X_4]$ and $(A_4 - a_1^{-1}A_3A_2)[X_3 \quad X_4] = 0$. This implies that

$$\left|\mathrm{Ann}_r(ap)\right| = \left|\left\{M \in M_{(n-1)\times n}(S) \mid \left(A_4 - a_1^{-1}A_3A_2\right)M = 0\right\}\right| \leqslant |S|^{n(n-1)}.$$

On the other hand, $|\mathrm{Ann}_r(ap)| = |\mathrm{Ann}_r(a)| = |S|^{n(n-1)}$. Therefore $(A_4 - a_1^{-1}A_3A_2)M = 0$, for each $M \in M_{(n-1)\times n}(S)$. This yields that $A_4 - a_1^{-1}A_3A_2 = 0$. If we put

$$q = \begin{bmatrix} 1 & 0 \\ -A_3 & a_1 I_{n-1} \end{bmatrix},$$

then for $i = 2, \ldots, n$, the $i$th row of $qap$ is zero. Obviously, this property holds for the matrix $qaq^{-1}$. Up to matrix similarity, we may assume that $a = a_{11}E_{11} + \cdots + a_{1n}E_{1n}$. We show that $a_{11}$ is unit. Clearly, $\mathrm{Ann}_\ell(E_{11})$ is the set of all matrices in $M_n(R)$ whose first columns are zero. Since at least one of the entries of the first row of $a$ is unit, a similar property holds for $\mathrm{Ann}_\ell(a)$. Hence $\mathrm{Ann}_r(\mathrm{Ann}_\ell(E_{11}))$ and $\mathrm{Ann}_r(\mathrm{Ann}_\ell(a))$ are the sets of all matrices in $M_n(R)$ and $M_n(S)$, respectively, whose $i$th rows are zero, for $i = 2, \ldots, n$. Indeed, $\mathcal{C} = \{xE_{11} \mid x \in \mathrm{Ann}_r(a_{11})\}$ is a subset of $\mathrm{Ann}_r(\{a\} \cup \mathrm{Ann}_\ell(a))$. Since $\mathrm{Ann}_r(\{E_{11}\} \cup \mathrm{Ann}_\ell(E_{11})) = \{0\}$, $\mathcal{C} = \{0\}$. This implies that $a_{11}$ is unit, as desired. Obviously, two matrices $a$ and $a_{11}E_{11}$ are similar and hence we may assume that $a = a_{11}E_{11}$. Suppose that $\Gamma_R$ is the induced subgraph of $\Gamma(M_n(R))$ on the subset

$$\left\{x \in M_n(R)^* \mid \mathrm{Ann}_\ell(E_{11}) \subseteq \mathrm{Ann}_\ell(x) \text{ and } \mathrm{Ann}_r(E_{11}) \subseteq \mathrm{Ann}_r(x)\right\},$$

and similarly define the subgraph $\Gamma_S$ of $\Gamma(M_n(S))$. Clearly, the vertex sets of $\Gamma_R$ and $\Gamma_S$ are $\{rE_{11} \mid r \in R^*\}$ and $\{sE_{11} \mid s \in S^*\}$, respectively. By $\Gamma(M_n(R)) \simeq \Gamma(M_n(S))$, we have $\Gamma_R \simeq \Gamma_S$. Suppose that $\psi$ is a graph isomorphism from $\Gamma_R$ to $\Gamma_S$. Note that for any $r \in R$, $r \in \mathcal{D}(R)$ if and only if $\mathrm{Ann}_\ell(rE_{11}) \neq \mathrm{Ann}_\ell(E_{11})$; and a similar fact holds for $S$. So we can define a map $f : \mathcal{D}(R)^* \to \mathcal{D}(S)^*$ in which $f(z)$ is the $(1,1)$-entry of the matrix $\psi(zE_{11})$, for any $z \in \mathcal{D}(R)^*$. Now, it is easily seen that $f$ is a graph isomorphism from $\overline{\Gamma}(R)$ to $\overline{\Gamma}(S)$. $\square$

**Theorem 11.** *Let $n \geqslant 2$ and $S_1, \ldots, S_n$ be finite indecomposable rings with identity. Suppose that $S = S_1 \times \cdots \times S_n$ and $R$ is a ring with identity such that $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$. Then there exist indecomposable rings $R_1, \ldots, R_n$ such that $R \simeq R_1 \times \cdots \times R_n$ and for $i = 1, \ldots, n$, $|R_i| = |S_i|$ and $\overline{\Gamma}(R_i) \simeq \overline{\Gamma}(S_i)$, unless $S \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $S \simeq \mathbb{Z}_6$.*

**Proof.** Let $\varphi$ be a graph isomorphism from $\overline{\Gamma}(R)$ to $\overline{\Gamma}(S)$. If $S_1, \ldots, S_n$ are fields, then using [2, Theorem 17], the assertion is proved. So, after a suitable reindexing, we may assume that $S_1, \ldots, S_t$ are not fields and $S_{t+1}, \ldots, S_n$ are fields, for some $t$, $1 \leqslant t \leqslant n$. Consider the vertices $\tilde{x} = (1, 0, \ldots, 0)$ and $\tilde{y} = (0, 1, \ldots, 1)$ in $\overline{\Gamma}(S)$ and let $x = \varphi^{-1}(\tilde{x})$ and $y = \varphi^{-1}(\tilde{y})$. Since $\tilde{x}$–$\tilde{y}$ is an edge of $\overline{\Gamma}(S)$ and $\mathcal{N}(\tilde{x}) \cap \mathcal{N}(\tilde{y}) = \emptyset$, we have $\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$ and without loss of generality, we may assume that $xy = 0$. Let $S_1' = S_2 \times \cdots \times S_n$.

We show that $x$ is not a nilpotent element. To get a contradiction, assume that $x^k = 0$, where $k$ is the smallest natural number with this property. First we claim that $k = 2$. Toward a contradiction, suppose $k \neq 2$. Then $x^{k-1} \neq x$. Since $x^{k-1} \in \mathrm{Ann}_\ell(\{x, y\})$ and $\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$, $x^{k-1} = y$. This yields that $\mathcal{N}(x) = \{y\}$ and so $\mathcal{N}(\tilde{x}) = \{\tilde{y}\}$. Hence $S_1' \simeq \mathbb{Z}_2$. We show that $\mathrm{Ann}_\ell(y) = \mathrm{Ann}_r(y)$. Assume that $z \in \mathrm{Ann}_\ell(y)^*$. Since $zx^{k-1} = 0$, there exists an integer $i$, $1 \leqslant i \leqslant k-1$, such that $zx^i = 0$ and $zx^{i-1} \neq 0$. We note that $zx^{i-1} \in \mathrm{Ann}_\ell(\{x, y\})$, $x^2 \neq 0$, and $\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$, so $zx^{i-1} = y$ and this implies that $(xz)x^{i-1} = 0$. By repeating this procedure (at most $k-1$ times), we obtain an integer $j$, $1 \leqslant j \leqslant k-1$, such that $x^j z = 0$. Therefore $yz = 0$ and so $\mathrm{Ann}_\ell(y) \subseteq \mathrm{Ann}_r(y)$. Noting $xy = yx = 0$, the converse is similarly proved and thus $\mathrm{Ann}_\ell(y) = \mathrm{Ann}_r(y)$, as desired. If $R$ is a local ring, then $\mathcal{D}(R)$ is a nilpotent ideal. Thus there exists a non-zero element $w \in \mathrm{Ann}_\ell(\mathcal{D}(R)) \cap \mathrm{Ann}_r(\mathcal{D}(R))$. Indeed, $w$ is adjacent to all other vertices of $\overline{\Gamma}(R)$. Since $S_1$ is not field, it is easily seen that $\overline{\Gamma}(S)$ contains no such a vertex. Thus $R$ is not a local ring. By Remark 4, $R$ has a non-trivial idempotent $e$ and therefore we can write $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Let $u_e = (u_1, u_2)$ and $v_e = (v_1, v_2)$ be those vertices of $\overline{\Gamma}(S)$ such that $\varphi(e) = u_e$ and $\varphi(1-e) = v_e$. Since $e$ and $1-e$ are adjacent, $u_2 v_2 = 0$. If $u_2 = v_2 = 0$, then $\tilde{y} \in \mathcal{N}(u_e) \cap \mathcal{N}(v_e)$ and so $y \in \mathcal{N}(e) \cap \mathcal{N}(1-e)$. Since $\mathrm{Ann}_\ell(y) = \mathrm{Ann}_r(y)$, $ye = y(1-e) = 0$, which is impossible. With no loss of generality, assume that $u_2 = 0$ and $v_2 = 1$. Note that $y^2 = 0$, so $y \neq 1-e$ and therefore $v_1 \neq 0$. Moreover, $u_e$ is adjacent to $\tilde{y}$ in $\overline{\Gamma}(S)$. This implies that $e$–$y$ is an edge of $\overline{\Gamma}(R)$ and since $\mathrm{Ann}_\ell(y) = \mathrm{Ann}_r(y)$, $ey = ye = 0$ and hence $y \in (1-e)R(1-e)$.

Let $\widetilde{z_e} = (v_1, 0)$ and $z_e = \varphi^{-1}(\widetilde{z_e})$. It is easy to verify that $\mathcal{N}(v_e) \setminus \{\widetilde{z_e}\} = \mathcal{N}(\tilde{y}) \cap \mathcal{N}(\widetilde{z_e}) \setminus \{v_e\}$ and so

$$\mathcal{N}(1-e) \setminus \{z_e\} = \big(\mathcal{N}(y) \cap \mathcal{N}(z_e)\big) \setminus \{1-e\}. \tag{6}$$

If $e \neq z_e$, then the equality (6) shows that $e$ and $z_e$ are adjacent. Moreover, since $\tilde{y}$ is adjacent to $\widetilde{z_e}$, $yz_e = z_e y = 0$. Now, $e + y \in (\mathcal{N}(y) \cap \mathcal{N}(z_e)) \setminus \mathcal{N}(1-e)$, a contradiction. Therefore $e = z_e$. For every three elements $a \in eRe^*$, $b \in eR(1-e)$, and $c \in (1-e)Re$, we have $\{a, e+b, e+c\} \subseteq \mathcal{N}(1-e) \setminus \mathcal{N}(e)$. Hence the equality (6) implies that $\{a, e+b, e+c\} = \{e\}$. This shows that $eRe = \{0, e\}$ and $eR(1-e) = (1-e)Re = \{0\}$. If $\overline{\Gamma}((1-e)R(1-e))$ has more than one vertex, then by connectivity [15, Theorem 3.2], $\overline{\Gamma}((1-e)R(1-e))$ has at least one vertex $d$ adjacent to $y$. Now, $d \in (\mathcal{N}(y) \cap \mathcal{N}(e)) \setminus \mathcal{N}(1-e)$, which contradicts (6). Hence $\overline{\Gamma}((1-e)R(1-e))$ has exactly one vertex and using [2, Corollary 4], we find that $(1-e)R(1-e) \simeq \mathbb{Z}_4$ or $\mathbb{Z}_2[\mathrm{X}]/(\mathrm{X}^2)$. Therefore $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathrm{X}]/(\mathrm{X}^2)$. But these rings have just one non-zero nilpotent element, while $x$ and $y$ are two non-zero nilpotent elements of $R$. This contradiction establishes the claim, i.e., $x^2 = 0$.

Since $x^2 = xy = 0$, $Rx\backslash\{0, x, y\} \subseteq \mathcal{N}(x) \cap \mathcal{N}(y)$. But $\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$, so we have $Rx \subseteq \{0, x, y\}$. If $Rx = \{0, x, y\}$, then $x + y = 0$. Moreover, if $\mathcal{D}(R)^* \neq \{x, y\}$, then the connectivity of $\overline{\Gamma}(R)$ [15, Theorem 3.2] yields that there exists a vertex $z \in (\mathcal{N}(x) \cup \mathcal{N}(y)) \setminus \{x, y\}$. By $x + y = 0$, we deduce that $z \in \mathcal{N}(x) \cap \mathcal{N}(y)$, which is impossible. Thus $\Gamma(R)$ has exactly two vertices. Using [2, Theorem 17], we find that $S \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., $S_1$ is a field, a contradiction. Hence $Rx = \{0, x\}$ and so $[R : \mathrm{Ann}_\ell(x)] = 2$. Indeed, we have $\deg(x) \geqslant |\mathrm{Ann}_\ell(x)| - 2 = \frac{|R|}{2} - 2$ and $\deg(\tilde{x}) = |S'_1| - 1$. On the other hand, $\overline{\Gamma}(R)$ contains at most $|R| - 1$ vertices and the number of vertices of $\overline{\Gamma}(S)$ is $|S_1||\mathcal{D}(S'_1)| + |\mathcal{D}(S_1)||S'_1| - |\mathcal{D}(S_1)||\mathcal{D}(S'_1)| - 1$. Since $\deg(x) = \deg(\tilde{x})$ and the vertex sets of $\overline{\Gamma}(R)$ and $\overline{\Gamma}(S)$ have the same sizes, we find

$$|S_1|\big|\mathcal{D}(S'_1)\big| + \big|\mathcal{D}(S_1)\big||S'_1| - \big|\mathcal{D}(S_1)\big|\big|\mathcal{D}(S'_1)\big| - 1 \leqslant |R| - 1 \leqslant 2|S'_1| + 1.$$

This implies that

$$\big|\mathcal{D}(S'_1)\big|\big(|S_1| - \big|\mathcal{D}(S_1)\big|\big) + |S'_1|\big(\big|\mathcal{D}(S_1)\big| - 2\big) \leqslant 2. \tag{7}$$

By the Wedderburn–Artin Theorem [12, (3.5)], it is not hard to see that a finite ring which contains exactly one non-zero-divisor is isomorphic to the direct product of finitely many $\mathbb{Z}_2$. Since $S_1$ is not a field, $|S_1| - |\mathcal{D}(S_1)| \geqslant 2$. Now, inequality (7) yields that $S'_1$ is a field and $|\mathcal{D}(S_1)| = 2$. By [2, Corollary 4], $S_1$ is isomorphic to one of the rings $\mathbb{Z}_4$ or $\mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2)$. Therefore $\overline{\Gamma}(S)$ is a bipartite graph which is not a complete bipartite graph. Hence by [2, Theorem 11], $R$ is isomorphic to one of the rings $\mathbb{Z}_4 \times S'_1$ or $\mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2) \times S'_1$. Since each of these rings has just one non-zero nilpotent element, we find that $\deg(x) = 2|S'_1| - 2$ in $\overline{\Gamma}(R)$. But $\deg(\tilde{x}) = |S'_1| - 1$, a contradiction. This shows that $x$ is not nilpotent.

Since $\overline{\Gamma}(S)$ is a finite graph and $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$, [2, Remark 1] implies that $R$ is a finite ring and so there are natural numbers $r > s$ such that $x^r = x^s$. Assume that $s$ is the smallest possible natural number with this property. We claim that $s = 1$. If not, $x^{r-1} - x^{s-1} \in \mathrm{Ann}_\ell(\{x, y\})$. We note that $x^2 \neq 0$ and $\mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset$, so $x^{r-1} - x^{s-1} = y$. This implies that $\mathcal{N}(x) = \{y\}$ and so $\mathcal{N}(\tilde{x}) = \{\tilde{y}\}$. Thus $S'_1 \simeq \mathbb{Z}_2$. By [17, p. 55], there is a natural number $m$ such that $e' = x^m$ is a non-trivial idempotent. Since $e'y = ye' = 0$, $y \in (1 - e')R(1 - e')$. Thus $\varphi(e') = (u'_1, 0)$, for some $u'_1 \in S_1^*$. Since $y^2 = 0$, $y \neq 1 - e'$. Moreover, there is no edge between $y$ and $1 - e'$ in $\overline{\Gamma}(R)$, so $\varphi(1 - e') = (v'_1, 1)$, for some $v'_1 \in S_1^*$. Now, if we define $u_{e'} = \varphi(e')$, $v_{e'} = \varphi(1 - e')$, $\widetilde{z_{e'}} = (v'_1, 0)$, and $z_{e'} = \varphi^{-1}(\widetilde{z_{e'}})$, then by a similar argument given in the third paragraph of the proof, we find that $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2)$. Thus from [2, Theorem 13], we conclude that $S_1 \simeq \mathbb{Z}_4$ or $S_1 \simeq \mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2)$ and therefore $\deg(\tilde{y}) = 3$. Furthermore, since each of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2)$ has just one non-zero nilpotent element, we have $\deg(y) = 2$ in $\overline{\Gamma}(R)$, which contradicts $\deg(\tilde{y}) = 3$.

Thus $x^r = x$ and so $g = x^{r-1}$ is a non-trivial idempotent of $R$. It is straightforward to see that $\mathrm{Ann}_\ell(g) = \mathrm{Ann}_\ell(x)$ and $\mathrm{Ann}_r(g) = \mathrm{Ann}_r(x)$. Therefore $\mathcal{N}(g) \cap \mathcal{N}(y) = \emptyset$. Since $xy = 0$, $gy = 0$. Moreover, $(1 - g)Rg \subseteq \mathrm{Ann}_\ell(y) \cap \mathrm{Ann}_r(g)$, so $(1 - g)Rg \subseteq \{0, y\}$. If $y \in (1 - g)Rg$, then $(1 - g)R(1 - g) \subseteq \mathrm{Ann}_r(g) \cap \mathrm{Ann}_r(y)$ and thus $(1 - g)R(1 - g) = \{0\}$. This implies that $gR(1 - g) \subseteq \mathrm{Ann}_\ell(g) \cap \mathrm{Ann}_r(y)$ and therefore $gR(1 - g) = \{0\}$. So $R = gRg \oplus (1 - g)Rg$ and hence $Ry = \{0\}$, which contradicts that $\tilde{y}$ is not adjacent to all other vertices of $\overline{\Gamma}(S)$. This shows that $(1 - g)Rg = \{0\}$ and hence $y \in (1 - g)R(1 - g)$. We have $gR(1 - g) \subseteq \mathrm{Ann}_\ell(g) \cap \mathrm{Ann}_r(y)$ and thus $gR(1 - g) = \{0\}$. Hence $R = R_1 \oplus R'_1$, where $R_1 = gRg$ and $R'_1 = (1 - g)R(1 - g)$. If $\{0, y\} \subsetneqq \mathcal{D}(R'_1)$, then by connectivity [15, Theorem 3.2], $\overline{\Gamma}(R'_1)$ has at least one vertex $d'$

adjacent to $y$, which clearly contradicts $\mathcal{N}(g) \cap \mathcal{N}(y) = \emptyset$. If $\mathcal{D}(R_1') = \{0, y\}$, then using [2, Corollary 4], we find that $R_1' \simeq \mathbb{Z}_4$ or $R_1' \simeq \mathbb{Z}_2[\mathrm{x}]/(\mathrm{x}^2)$. Now, it is easily checked that if $w$ is a vertex of $\overline{\Gamma}(R)$ with $\mathcal{N}(y) \subseteq \mathcal{N}(w)$, then $w = y$. A similar property is also valid for $\overline{\Gamma}(S)$, so we conclude that $|S_1'| = 2$. On the other hand, $\deg(x) = \deg(g) = 3$ and $\deg(\tilde{x}) = |S_1'| - 1$ are equal, i.e., $|S_1'| = 4$, a contradiction. Hence $y$ is a unit in $R_1'$. This implies that $\deg(y) = |R_1| - 1$. Since $\deg(\tilde{y}) = |S_1| - 1$, we find that $|R_1| = |S_1|$. Furthermore, $\deg(x) = |R_1'| - 1$ and $\deg(\tilde{x}) = |S_1'| - 1$ are the same, so we have $|R_1'| = |S_1'|$.

Let $\tilde{g} = \varphi(g)$. Since $\mathcal{N}(\tilde{x}) = \mathcal{N}(\tilde{g})$, the first component of $\tilde{g}$ is a unit in $S_1$ and whose other components are zero. Let $\widetilde{\mathcal{A}}$ be the set of all vertices in $\overline{\Gamma}(S)$, adjacent to all neighbors of $\tilde{g}$ such that their degrees are more than $\deg(\tilde{g})$. Clearly, $\widetilde{\mathcal{A}} = \{(z_1, 0, \ldots, 0) \mid z_1 \in \mathcal{D}(S_1)^*\}$. Also, let $\widetilde{\mathcal{B}}$ be the set of all vertices in $\overline{\Gamma}(S)$ adjacent to $\tilde{g}$ and adjacent to at least one of the neighbors of $\tilde{g}$. Hence the first component of any vertex of $\widetilde{\mathcal{B}}$ is 0 and at least one of its other components is a zero-divisor. Now, if the induced subgraphs of $\overline{\Gamma}(S)$ on $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are denoted by $\Gamma_{\widetilde{\mathcal{A}}}$ and $\Gamma_{\widetilde{\mathcal{B}}}$, respectively, then $\Gamma_{\widetilde{\mathcal{A}}} \simeq \overline{\Gamma}(S_1)$ and $\Gamma_{\widetilde{\mathcal{B}}} \simeq \overline{\Gamma}(S_1')$.

Let $\mathcal{A} = \varphi^{-1}(\widetilde{\mathcal{A}})$ and $\mathcal{B} = \varphi^{-1}(\widetilde{\mathcal{A}})$. Clearly, $R_1'^*$ is the set of all neighbors of $g$ in $\overline{\Gamma}(R)$. Assume that $a = a_1 + a_1'$ is an arbitrary element of $\mathcal{A}$, where $a_1 \in R_1$ and $a_1' \in R_1'$. So, by the definition of $\mathcal{A}$, $a_1 + a_1'$ is adjacent to all vertices of $R_1'^*$. This implies that $R = \mathrm{Ann}_\ell(a_1') \cup \mathrm{Ann}_r(a_1')$. Since none of the vertices of $\overline{\Gamma}(R)$ is adjacent to all other vertices of the graph, $a_1' = 0$. Moreover, $\deg(a) > \deg(g)$ yields that $a \in \mathcal{D}(R_1)^*$. So we have $\Gamma_{\mathcal{A}} \simeq \overline{\Gamma}(R_1)$, where $\Gamma_{\mathcal{A}}$ is the induced subgraph of $\overline{\Gamma}(R)$ on $\mathcal{A}$. Suppose that $b = b_1 + b_1'$ is an arbitrary element of $\mathcal{B}$, where $b_1 \in R_1$ and $b_1' \in R_1'$. Since $b$ is adjacent to $g$, $b_1 = 0$. Also, $b$ is adjacent to at least one vertex contained in $R_1'^*$, so $b \in \mathcal{D}(R_1')^*$. Thus it is evident that $\Gamma_{\mathcal{B}} \simeq \overline{\Gamma}(R_1')$, where $\Gamma_{\mathcal{B}}$ is the induced subgraph of $\overline{\Gamma}(R)$ on $\mathcal{B}$.

By $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$, we conclude that $\overline{\Gamma}(R_1) \simeq \overline{\Gamma}(S_1)$ and $\overline{\Gamma}(R_1') \simeq \overline{\Gamma}(S_1')$. Since $\overline{\Gamma}(R_1) \simeq \overline{\Gamma}(S_1)$ and $S_1$ is indecomposable, by repeating the above proof for the rings $R_1$ and $S_1$ instead of the rings $S$ and $R$, respectively, we find that $R_1$ should be an indecomposable ring. Moreover, since $\overline{\Gamma}(R_1') \simeq \overline{\Gamma}(S_1')$, by continuing the above method $t$ times, we find indecomposable rings $R_1, \ldots, R_t$ and a ring $R_t'$ such that $R \simeq R_1 \times \cdots \times R_t \times R_t'$ and for $i = 1, \ldots, t$, $|R_i| = |S_i|$ and $\overline{\Gamma}(R_i) \simeq \overline{\Gamma}(S_i)$. If $t < n$, then $|R_t'| = |S_t'|$ and $\overline{\Gamma}(R_t') \simeq \overline{\Gamma}(S_t')$, where $S_t' = S_{t+1} \times \cdots \times S_n$. Since $S_t'$ is a reduced ring and the excluding cases in [2, Theorem 17] have different orders, the aforementioned theorem implies that $R_t' \simeq S_t'$. This completes the proof. □

**Remark 12.** Let $S$ be a finite decomposable ring with identity. Assume that $S = S_1 \times \cdots \times S_n$ is the decomposition of $S$ in which $S_1, \ldots, S_n$ are indecomposables rings. Suppose that $R$ is a ring with identity and $\varphi : \Gamma(R) \to \Gamma(S)$ is a graph isomorphism. Then $\varphi$ is also a graph isomorphism from $\overline{\Gamma}(R)$ to $\overline{\Gamma}(S)$. If $R_1, \ldots, R_n$ are those rings are obtained in the previous theorem, then for simplification, we may suppose that $R = R_1 \times \cdots \times R_n$. Note that in the above proof, the isomorphism $\Gamma_{\mathcal{A}} \simeq \Gamma_{\widetilde{\mathcal{A}}}$ is induced by $\varphi$. Thus we find a bijective map

$$\mathcal{D}(R_1)^* \times \{0\} \times \cdots \times \{0\} \longrightarrow \mathcal{D}(S_1)^* \times \{0\} \times \cdots \times \{0\}$$

which preserves zero products. This shows that $\Gamma(R_1) \simeq \Gamma(S_1)$. Therefore by the proof of the previous theorem, we conclude that $\Gamma(R_i) \simeq \Gamma(S_i)$, for each $i$.

**Theorem 13.** *Let $R$ and $S$ be two finite commutative rings with identity and $n, m \geqslant 2$. If $\Gamma(M_n(R)) \simeq \Gamma(M_m(S))$, then $n = m$, $|R| = |S|$, and $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$.*

**Proof.** It is well known that every commutative Artinian ring is isomorphic to the direct product of finitely many local rings [5, p. 90]. Suppose that $R = R_1 \times \cdots \times R_r$ and $S = S_1 \times \cdots \times S_s$, where $R_i$ and $S_j$ are finite local rings, for all $i$ and $j$. We have $\Gamma(M_n(R_1) \times \cdots \times M_n(R_r)) \simeq \Gamma(M_m(S))$. For $i = 1, \ldots, r$, since $R_i$ is a local ring, $R_i$ contains no non-trivial idempotent and thus $M_n(R_i)$ has no non-trivial central idempotent. So $M_n(R_1), \ldots, M_n(R_r)$ are indecomposable rings. Hence by Theorem 11 and Remark 12, there exist indecomposable rings $S'_1, \ldots, S'_r$ such that $M_m(S) \simeq S'_1 \times \cdots \times S'_r$, $|M_n(R_i)| = |S'_i|$, and $\Gamma(M_n(R_i)) \simeq \Gamma(S'_i)$, for all $i$. On the other hand, $M_m(S) \simeq M_m(S_1) \times \cdots \times M_m(S_s)$. Using the Krull–Schmidt Theorem [12, (19.22)], we have $r = s$ and $S'_i \simeq M_m(S_i)$, for $i = 1, \ldots, r$. Thus, after a suitable reindexing, we may assume that $|M_n(R_i)| = |M_m(S_i)|$ and $\Gamma(M_n(R_i)) \simeq \Gamma(M_m(S_i))$, for each $i$. Now, using Theorem 10, we obtain that $n = m$, $|R_i| = |S_i|$, and $\overline{\Gamma}(R_i) \simeq \overline{\Gamma}(S_i)$, for all $i$. Clearly, $|R| = |S|$ and by [3, Theorem 4], we conclude that $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$. The proof is complete. □

## 4. The zero-divisor graphs of group rings

**Theorem 14.** *Let $K$ and $K_1$ be two finite fields and $G, G_1$ be two finite groups. If $\overline{\Gamma}(KG) \simeq \overline{\Gamma}(K_1G_1)$, then $K \simeq K_1$ and $|G| = |G_1|$.*

**Proof.** For simplification, let $n = |G|$ and define $\varepsilon = 2$, if $\operatorname{char} K$ divides $n$; and otherwise $\varepsilon = 1$. Consider the element $\sigma = \sum_{g\in G} g$. We have $(h - 1)\sigma = 0$, for all $h \in G$. Thus $\sigma^2 = n\sigma$ and so $\deg(\sigma) = |K|^{n-1} - \varepsilon$. We claim that the maximum degree in $\overline{\Gamma}(KG)$ is $|K|^{n-1} - \varepsilon$. Let $a = \sum_{g\in G} a_g g$ be a vertex with the maximum degree in $\overline{\Gamma}(KG)$. By Theorem 3, there exists an integer $s$, $1 \leqslant s \leqslant n - 1$, such that $|\mathrm{Ann}_\ell(a)| = |\mathrm{Ann}_r(a)| = |K|^s$. We know that if $\mathcal{A}$ is a finite-dimensional algebra over a field $F$ and $z \in \mathcal{D}(\mathcal{A})$, then by considering the minimal polynomial of $z$ over $F$, we find an element $z' \in \mathcal{A}^*$ such that $zz' = z'z = 0$. This yields that there exists an integer $t$, $1 \leqslant t \leqslant s$, such that $|\mathrm{Ann}_\ell(a) \cap \mathrm{Ann}_r(a)| = |K|^t$. Now, the maximality of $\deg(a)$ implies that $2|K|^s - |K|^t - 1 \geqslant |K|^{n-1} - 2$. Hence $|K|^t(|K|^{n-t-1} - 2|K|^{s-t} + 1) \leqslant 1$. This follows that $s = n - 1$. Using the fact $[KG : \mathrm{Ann}_\ell(a)] = |KG \cdot a|$, we conclude that $\dim_K KG \cdot a = 1$. Thus for any element $h \in G$, there is a unique scalar $\lambda(h) \in K^*$ such that $ha = \lambda(h)a$. It is easy to check that $\lambda : G \to K^*$ is a group homomorphism. By considering the coefficients of $h$ in two sides of the latter equality we have $a_1 = \lambda(h)a_h$, where $1 \in G$ is identity. Thus $a = a_1 \sum_{g\in G} \lambda(g^{-1})g$. Since $\lambda$ is fixed on each of conjugacy classes of $G$, using [14, Lemma 4.1.1], $a$ is contained in the center of $KG$ and hence $\mathrm{Ann}_\ell(a) = \mathrm{Ann}_r(a)$. Now, since $a^2 = a_1 \sum_{g\in G} \lambda(g^{-1})ga = a_1 \sum_{g\in G} \lambda(g^{-1})\lambda(g)a = na_1a$, $\deg(a) = |K|^{n-1} - \varepsilon$. This proves the claim.

We proved that the set of all vertices in $\overline{\Gamma}(KG)$ with the maximum degree is contained in

$$\mathcal{M} = \left\{ \mu \sum_{g\in G} \varphi\big(g^{-1}\big)g \;\middle|\; \mu \in K^* \text{ and } \varphi : G \to K^* \text{ is a group homomorphism} \right\}.$$

Next we show that each element of $\mathcal{M}$ has maximum degree. Obviously, it is sufficient to prove that for any homomorphism $\varphi : G \to K^*$, the vertex $x_\varphi = \sum_{g\in G} \varphi(g^{-1})g$ has the maximum degree. Since $[KG : \mathrm{Ann}_\ell(x_\varphi)] = |KG \cdot x_\varphi|$, it is enough to show that $\dim_K KG \cdot x_\varphi = 1$. For any element $h \in G$, we have

$$hx_\varphi = \varphi(h) \sum_{g\in G} \varphi\big((hg)^{-1}\big)(hg) = \varphi(h)x_\varphi.$$

This follows that $KG \cdot x_\varphi = K \cdot x_\varphi$, as desired. Define an equivalence relation $\sim$ on $\mathcal{M}$ as $x \sim y$ if $\mathcal{N}(x) \setminus \{y\} = \mathcal{N}(y) \setminus \{x\}$. Suppose that $\varphi_1, \varphi_2 : G \to K^*$ are two distinct homomorphisms and $c$ is an element of $G$ such that $\varphi_1(c) \neq \varphi_2(c)$. This implies that $n \geqslant 3$. We have $1 - \varphi_1(c^{-1})c \in \mathcal{N}(x_{\varphi_1}) \setminus \mathcal{N}(x_{\varphi_2})$. Thus for any two vertices $x, y \in \mathcal{M}$, we have $x \sim y$ if and only if there exists $\nu \in K^*$ such that $x = \nu y$. Hence every equivalence class of $\sim$ has $|K^*|$ elements. Since $\overline{\Gamma}(KG) \simeq \overline{\Gamma}(K_1G_1)$, $|K^*| = |K_1^*|$ and therefore $K \simeq K_1$. Now, by considering the maximum degrees of graphs $\overline{\Gamma}(KG)$ and $\overline{\Gamma}(K_1G_1)$, we find that $|G| = |G_1|$. The proof is complete. □

Next, we require to state the beautiful following theorem. We recall that for each group ring $KG$, the *augmentation ideal* of $KG$ is defined to be

$$\omega(KG) = \left\{ \sum_{g \in G} \lambda_g g \in KG \;\Big|\; \sum_{g \in G} \lambda_g = 0 \right\}.$$

**Theorem B.** *(See [9, Theorem 5.2].) Let $K$ be a finite field and $G$ be a finite group which is not Abelian. Then for every element $x \in KG$, $\mathrm{Ann}_\ell(x) = \mathrm{Ann}_r(x)$, if and only if $|K| = 2^r$ and $G \simeq Q \times H$, where $r$ is an odd integer, $Q = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ is the group of the quaternions, and $H$ is an Abelian group whose order divides $2^m - 1$, for some odd integer $m$.*

To prove the next theorem, we also need the following lemma that appeared in [10, Corollary 14] for $K \simeq \mathbb{Z}_p$.

**Lemma 15.** *Let $K$ be a finite field and $G$ be a cyclic $p$-group with $p = \mathrm{char}\, K$. Then for any $d$, $0 \leqslant d \leqslant |G|$, $KG$ has exactly one ideal of dimension $d$ over $K$.*

**Proof.** If $|G| = n$, then $KG \simeq K[\mathrm{X}]/(\mathrm{X}^n - 1)$. Since $K[\mathrm{X}]$ is a principle ideal ring and $n$ is a $p$-power, $\mathrm{X}^n - 1 = (\mathrm{X} - 1)^n$ and so every ideal of $K[\mathrm{X}]/(\mathrm{X}^n - 1)$ has the form $((\mathrm{X} - 1)^t)/(\mathrm{X}^n - 1)$ for some $t$, $0 \leqslant t \leqslant n$. This proves the lemma. □

**Theorem 16.** *Let $K_1, K_2$ be two finite fields and $G_1, G_2$ be two finite groups such that $\overline{\Gamma}(K_1G_1) \simeq \overline{\Gamma}(K_2G_2)$. Then if $G_1$ is an Abelian group so does $G_2$.*

**Proof.** By Theorem 14, $K_1 \simeq K_2$. Thus, in the sequel, we assume that these fields are the same and we denote them by $K$. Note that for every vertex $v$ in $\overline{\Gamma}(KG_1)$, $\deg(v) = |\mathrm{Ann}_\ell(v)| - \varepsilon$, where $\varepsilon = 1$ or $2$. Therefore, for any vertex $x$ of $\overline{\Gamma}(KG_2)$, at least one of the numbers $\deg(x) + 1$ or $\deg(x) + 2$ is a power of $p = \mathrm{char}\, K$. Moreover, by the same argument given in the proof of Theorem 14, for any element $x \in \mathcal{D}(KG_2)^*$, $\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x) \neq \{0\}$. Also, we know that for any element $x \in \mathcal{D}(KG_2)^*$, $\deg(x) = |\mathrm{Ann}_\ell(x)| + |\mathrm{Ann}_r(x)| - |\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)| - \varepsilon$, where $\varepsilon = 1$ or $2$. Hence $|\mathrm{Ann}_\ell(x)| + |\mathrm{Ann}_r(x)| - |\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)|$ is a $p$-power. This obviously yields that $|\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)| = |\mathrm{Ann}_\ell(x)|$ or $|\mathrm{Ann}_\ell(x) \cap \mathrm{Ann}_r(x)| = |\mathrm{Ann}_r(x)|$. On the other hand, Theorem 3 implies that $|\mathrm{Ann}_\ell(x)| = |\mathrm{Ann}_r(x)|$. Thus, for any element $x \in KG_2$, we have $\mathrm{Ann}_\ell(x) = \mathrm{Ann}_r(x)$.

Toward a contradiction, suppose that $G_2$ is not Abelian. Using Theorem B, $p = 2$ and there exists an Abelian group $B$ with odd order such that $G_2 \simeq Q \times B$. Clearly, $KG_2$ can be considered as the group rings of the group $Q$ over the ring $KB$. Also, by Maschke's Theorem [12, (6.1)] and the Wedderburn–Artin Theorem [12, (3.5)], there are finite fields $E_1, \dots, E_m$ such that $KB \simeq E_1 \times \cdots \times E_m$. This yields that $KG_2 \simeq (KB)Q \simeq (E_1 \times \cdots \times E_m)Q \simeq E_1Q \times \cdots \times E_mQ$. By Theorem 14, $|G_1| = |G_2|$ and since $G_1$ is Abelian, then there exist two Abelian groups $H$ and $C$ such that $G_1 \simeq H \times C$, $|C| = |B|$ and $|H| = 8$. With a similar argument, there are finite fields $F_1, \dots, F_n$ such that $KG_1 \simeq F_1H \times \cdots \times F_nH$. Hence

$$\overline{\Gamma}(F_1H \times \cdots \times F_nH) \simeq \overline{\Gamma}(E_1Q \times \cdots \times E_mQ).$$

We know that for every finite field $F$ and any $p$-group $P$ with $p = \operatorname{char} F$, the group ring $FP$ is a local ring [12, (19.10)]. Thus Theorem 11 and the Krull–Schmidt Theorem [12, (19.22)] imply that $\overline{\Gamma}(F_1H) \simeq \overline{\Gamma}(E_1Q)$. By Theorem 14, $F_1 \simeq E_1$ and we assume that these fields are the same and denote them by $T$ and note that $\operatorname{char} T = 2$. Moreover, since $H$ is an Abelian group of order 8, $H$ is isomorphic to one of the groups $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

First suppose that $H \simeq \mathbb{Z}_8$. By Lemma 15, it is not hard to see that for every two distinct vertices $u$ and $v$ in $\overline{\Gamma}(TH)$ with the same degrees, $\mathcal{N}(u) \setminus \{v\} = \mathcal{N}(v) \setminus \{u\}$. On the other hand, there exists a ring automorphism of $TQ$ which maps $\mathsf{i}$ to $\mathsf{j}$, so $\deg(1 + \mathsf{i}) = \deg(1 + \mathsf{j})$ in $\overline{\Gamma}(TQ)$. But we have $(1 + \mathsf{i})^3 \in \mathcal{N}(1 + \mathsf{i}) \setminus \mathcal{N}(1 + \mathsf{j})$, a contradiction. Next assume that $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since each element of $H$ has order 2 and $\mathcal{D}(TH) = \omega(TH)$ [12, (8.8)], for any $x \in \mathcal{D}(TH)$ we have $x^2 = 0$. This yields that the degree of any vertex of $\overline{\Gamma}(TH)$ is even. But $(1 + \mathsf{i})^2 \neq 0$, so $\deg(1 + \mathsf{i})$ in $\overline{\Gamma}(TQ)$ is odd, a contradiction. Finally, suppose that $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$. To obtain a contradiction, we show that the number of vertices of even degrees in $\overline{\Gamma}(TH)$ and $\overline{\Gamma}(TQ)$ is $|T|^6 - 1$ and $|T|^4 - 1$, respectively. For this, we should find the number of all elements with squares 0 in the rings $TH$ and $TQ$. If

$$\left(\sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_4} \lambda_g g\right)^2 = 0,$$

then

$$\left(\lambda^2_{(0,0)} + \lambda^2_{(0,2)} + \lambda^2_{(1,0)} + \lambda^2_{(1,2)}\right)(0,0) + \left(\lambda^2_{(0,1)} + \lambda^2_{(0,3)} + \lambda^2_{(1,1)} + \lambda^2_{(1,3)}\right)(0,2) = 0.$$

This implies that $\lambda_{(0,0)} + \lambda_{(0,2)} + \lambda_{(1,0)} + \lambda_{(1,2)} = 0$ and $\lambda_{(0,1)} + \lambda_{(0,3)} + \lambda_{(1,1)} + \lambda_{(1,3)} = 0$. So $\dim_T\{x \in TH \mid x^2 = 0\} = 6$, as desired. Now, assume that the square of the element $a = \sum_{g \in Q} \lambda_g g \in TQ$ is 0. The coefficient of $-1$ on the left side of the equality $a^2 = 0$ is

$$\lambda_1\lambda_{-1} + \lambda_{-1}\lambda_1 + \lambda_{\mathsf{i}}^2 + \lambda_{-\mathsf{i}}^2 + \lambda_{\mathsf{j}}^2 + \lambda_{-\mathsf{j}}^2 + \lambda_{\mathsf{k}}^2 + \lambda_{-\mathsf{k}}^2 = 0.$$

Since $\mathcal{D}(TQ) = \omega(TQ)$ [12, (8.8)], $\sum_{g \in Q} \lambda_g = 0$. Hence the above equality implies that $\lambda_1 = \lambda_{-1}$. Moreover, the coefficient of $\mathsf{i}$ on the left side of the equality $a^2 = 0$ is

$$\lambda_1\lambda_{\mathsf{i}} + \lambda_{-1}\lambda_{-\mathsf{i}} + \lambda_{\mathsf{i}}\lambda_1 + \lambda_{-\mathsf{i}}\lambda_{-1} + \lambda_{\mathsf{j}}\lambda_{\mathsf{k}} + \lambda_{-\mathsf{j}}\lambda_{-\mathsf{k}} + \lambda_{\mathsf{k}}\lambda_{-\mathsf{j}} + \lambda_{-\mathsf{k}}\lambda_{\mathsf{j}} = 0.$$

This follows that $(\lambda_{\mathsf{j}} + \lambda_{-\mathsf{j}})(\lambda_{\mathsf{k}} + \lambda_{-\mathsf{k}}) = 0$. Similarly, by considering the coefficients $\mathsf{j}$ and $\mathsf{k}$ on two sides of the equality $a^2 = 0$, we find that $(\lambda_{\mathsf{i}} + \lambda_{-\mathsf{i}})(\lambda_{\mathsf{j}} + \lambda_{-\mathsf{j}}) = 0$ and $(\lambda_{\mathsf{i}} + \lambda_{-\mathsf{i}})(\lambda_{\mathsf{k}} + \lambda_{-\mathsf{k}}) = 0$. On the other hand, $(\lambda_{\mathsf{i}} + \lambda_{-\mathsf{i}}) + (\lambda_{\mathsf{j}} + \lambda_{-\mathsf{j}}) + (\lambda_{\mathsf{k}} + \lambda_{-\mathsf{k}}) = 0$, so we conclude that $\lambda_{\mathsf{i}} = \lambda_{-\mathsf{i}}$, $\lambda_{\mathsf{j}} = \lambda_{-\mathsf{j}}$, and $\lambda_{\mathsf{k}} = \lambda_{-\mathsf{k}}$. Therefore $|\{x \in TQ \mid x^2 = 0\}| = |T|^4$ and the proof is complete. $\square$

**Theorem 17.** *Let $K_1$ and $K_2$ be two finite fields and $G_1, G_2$ be two finite groups such that $\overline{\Gamma}(K_1G_1) \simeq \overline{\Gamma}(K_2G_2)$. If $G_1$ is a cyclic group, then $G_1 \simeq G_2$.*

**Proof.** By Theorem 14, $K_1 \simeq K_2$ and thus we may suppose that these fields are the same and we denote them by $K$. Also, by Theorem 16, $G_2$ is Abelian. Hence if $p = \operatorname{char} K$ and $P_1, P_2$ are Sylow $p$-subgroups of $G_1, G_2$, respectively, then there are two subgroups $\widetilde{P_1} \subseteq G_1$ and $\widetilde{P_2} \subseteq G_2$ of orders coprime to $p$ such that $G_1 \simeq P_1 \times \widetilde{P_1}$ and $G_2 \simeq P_2 \times \widetilde{P_2}$. By Maschke's Theorem [12, (6.1)] and the Wedderburn–Artin Theorem [12, (3.5)], we may write $K\widetilde{P_1} \simeq F_1 \times \cdots \times F_n$ and $K\widetilde{P_2} \simeq E_1 \times \cdots \times E_m$, where all $F_i$ and $E_j$ are fields. This yields that $KG_1 \simeq F_1P_1 \times \cdots \times F_nP_1$ and $KG_2 \simeq E_1P_2 \times \cdots \times E_mP_2$. Hence $\overline{\Gamma}(F_1P_1 \times \cdots \times F_nP_1) \simeq \overline{\Gamma}(E_1P_2 \times \cdots \times E_mP_2)$. Since all $F_iP_1$ and $E_jP_2$ are local rings [12, (19.10)], Theorem 11 and the Krull–Schmidt Theorem [12, (19.22)] imply that $n = m$ and, after a suitable reindexing, $\overline{\Gamma}(F_iP_1) \simeq \overline{\Gamma}(E_iP_2)$, for $i = 1, \dots, n$. Using Theorem 14, for any $i$, $F_i \simeq E_i$ and so $K\widetilde{P_1} \simeq K\widetilde{P_2}$. We know that a finite group $G$ is cyclic if and only if $G$ contains no two distinct subgroups of the same order. Combining this fact and a result by Perlis and Walker [9, Theorem 2.2], it is not hard to see that $\widetilde{P_2}$ is a cyclic group. But $|\widetilde{P_1}| = |\widetilde{P_2}|$, so we deduce that $\widetilde{P_1} \simeq \widetilde{P_2}$.

By Theorem 14, we have $|G_1| = |G_2|$. Therefore to continue the proof, we assume that $P_1$ and $P_2$ are non-trivial subgroups. By $F_1 \simeq E_1$, we may suppose that these fields are the same and we denote them by $T$. We have $\overline{\Gamma}(TP_1) \simeq \overline{\Gamma}(TP_2)$. Using Lemma 15, it is easily checked that for every two distinct vertices $u$ and $v$ in $\overline{\Gamma}(TP_1)$, if $\deg(u) = \deg(v)$, then $\mathcal{N}(u) \setminus \{v\} = \mathcal{N}(v) \setminus \{u\}$. Assume that $H$ is a subgroup of $P_2$ and let $\sigma_H = \sum_{h \in H} h \in TP_2$. If $\Lambda$ is a right transversal for $H$ in $P_2$, then by [14, Lemma 3.1.2], we have

$$\operatorname{Ann}_\ell(\sigma_H) = \left\{ \sum_{g \in P_2} \lambda_g g \;\middle|\; \sum_{h \in H} \lambda_{hx} = 0, \text{ for all } x \in \Lambda \right\}.$$

We note that $\operatorname{char} T$ divides $|H|$, so $\sigma_H^2 = |H|\sigma_H = 0$ and therefore $\deg(\sigma_H) = |T|^{|P_2| - [P_2 : H]} - 2$ in $\overline{\Gamma}(TP_2)$. To complete the proof, since $|P_1| = |P_2|$, it is enough to show that $P_2$ is cyclic. Suppose that $H_1$ and $H_2$ are two proper subgroups of $P_2$ with $|H_1| = |H_2|$. The relations $\overline{\Gamma}(TP_1) \simeq \overline{\Gamma}(TP_2)$ and $\deg(\sigma_{H_1}) = \deg(\sigma_{H_2})$ imply that $\mathcal{N}(\sigma_{H_1}) \setminus \{\sigma_{H_2}\} = \mathcal{N}(\sigma_{H_2}) \setminus \{\sigma_{H_1}\}$. Assume that there exists an element $h_1 \in H_1 \setminus H_2$. Since $P_2$ is Abelian, for any element $g \in P_2 \setminus H_1$, it is straightforward to check that $g - gh_1 \in \mathcal{N}(\sigma_{H_1}) \setminus \mathcal{N}(\sigma_{H_2})$, a contradiction. Thus $H_1 \subseteq H_2$ and so $H_1 = H_2$. Hence $P_2$ contains no two distinct subgroups of the same order and so $P_2$ is cyclic, as desired. Since $G_1$ and $G_2$ are two cyclic groups of the same order, $G_1 \simeq G_2$ and hence the proof is complete. □

## Acknowledgments

The authors are indebted to the Institute for Studies in Theoretical Physics and Mathematics (IPM) for support. The research of the first author was in part supported by a grant (No. 85160211) from IPM.

## References

[1] S. Akbari, M. Ghandehari, M. Hadian, A. Mohammadian, On commuting graphs of semisimple rings, Linear Algebra Appl. 390 (2004) 345–355.

[2] S. Akbari, A. Mohammadian, Zero-divisor graphs of non-commutative rings, J. Algebra 296 (2006) 462–479.

[3] S. Akbari, A. Mohammadian, On the zero-divisor graph of a commutative ring, J. Algebra 274 (2004) 847–855.
[4] S. Akbari, A. Mohammadian, H. Radjavi, P. Raja, On the diameters of commuting graphs, Linear Algebra Appl. 418 (2006) 161–176.
[5] M.F. Atiyah, I.G. MacDonald, Introduction to Commutative Algebra, Addison–Wesley, Reading, MA, 1969.
[6] D.M. Burton, A First Course in Rings and Ideals, Addison–Wesley, Reading, MA, 1970.
[7] L.E. Dickson, On the cyclotomic function, Amer. Math. Monthly 12 (4) (1905) 86–89.
[8] H. Feng, Orders of classical groups over finite rings, J. Math. Res. Exposition 18 (4) (1998) 507–512.
[9] M. Gutan, A. Kisielewicz, Reversible group rings, J. Algebra 279 (2004) 280–291.
[10] E.T. Hill, The annihilator of radical powers in the modular group ring of a $p$-group, Proc. Amer. Math. Soc. 25 (1970) 811–815.
[11] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[12] T.Y. Lam, A First Course in Noncommutative Rings, Springer, New York, 2001.
[13] M. Mirzakhani, A simple proof of a theorem of Schur, Amer. Math. Monthly 105 (3) (1998) 260–262.
[14] D.S. Passman, The Algebraic Structure of Group Rings, Wiley-Interscience, New York, 1977.
[15] S.P. Redmond, The zero-divisor graph of a non-commutative ring, Int. J. Commut. Rings 1 (4) (2002) 203–211.
[16] S.P. Redmond, Structure in the zero-divisor graph of a noncommutative ring, Houston J. Math. 30 (2) (2004) 345–355.
[17] S. Singh, Q. Zameeruddin, Modern Algebra, 3rd reprint, Vikas Publishing House Pvt. Ltd., Delhi, 1995.