Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 319 (2008) 1555–1584
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Ad-nilpotent ideals of a parabolic subalgebra

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Received 24 January 2007

Available online 3 December 2007

Communicated by Peter Littelmann

Abstract

We extend the results of Cellini and Papi [P. Cellini, P. Papi, Ad-nilpotent ideals of a Borel subalgebra, J. Algebra 225 (2000) 130–140; P. Cellini, P. Papi, Ad-nilpotent ideals of a Borel subalgebra II, J. Algebra 258 (2002) 112–121] on the characterizations of ad-nilpotent and abelian ideals of a Borel subalgebra to parabolic subalgebras of a simple Lie algebra. These characterizations are given in terms of elements of the affine Weyl group and faces of alcoves. In the case of a parabolic subalgebra of a classical simple Lie algebra, we give formulas for the number of these ideals.

Keywords: Ad-nilpotent ideal; Abelian ideal; Parabolic subalgebra; Lie algebra

1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l. Let $\mathfrak{h}$ be a Cartan subalgebra and Δ the associated root system. We fix a system of positive roots Δ^+. Denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the corresponding set of simple roots. Let V be the Euclidian space $\sum_{k=1}^{l} \mathbb{R}\alpha_k$. For each $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α.

For $I \subset \Pi$, set $\Delta_I = \mathbb{Z}I \cap \Delta$. We fix the corresponding standard parabolic subalgebra:

$$\mathfrak{p}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha \right).$$

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.005

An ideal $\mathfrak{i}$ of $\mathfrak{p}_I$ is ad-nilpotent if and only if for all $x \in \mathfrak{i}$, $\mathrm{ad}_{\mathfrak{p}_I} x$ is nilpotent. Since any ideal of $\mathfrak{p}_I$ is $\mathfrak{h}$-stable, we can deduce easily that an ideal is ad-nilpotent if and only if it is nilpotent. Moreover, we have $\mathfrak{i} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, for some subset $\Phi \subset \Delta^+ \setminus \Delta_I$.

The purpose of this paper is to characterize and to enumerate ad-nilpotent and abelian ideals of a parabolic subalgebra.

When $I = \emptyset$, $\mathfrak{p}_\emptyset$ is a Borel subalgebra of $\mathfrak{g}$. Peterson proved that the number of abelian ideals of $\mathfrak{p}_\emptyset$ is 2^l. Motived by this result, Cellini and Papi, Kostant, Panyushev, Sommers and Suter among others studied ad-nilpotent and abelian ideals of a Borel subalgebra.

In their articles [CP1] and [CP2], Cellini and Papi established different characterizations of the set $\mathcal{I}$ of ad-nilpotent ideals of a Borel subalgebra. They constructed a bijection between $\mathcal{I}$ and certain elements of the affine Weyl group $\widehat{W}$ associated to Δ, which we shall call $\emptyset$-compatible. These $\emptyset$-compatible elements are in turn characterized by elements of the coroot lattice. They established also, when $\mathfrak{g}$ is of classical type, a correspondence between ad-nilpotent ideals of $\mathfrak{g}$ and some diagrams. We extend here their theory to the case of parabolic subalgebras.

Fix $I \subset \Pi$, we establish a bijection between ad-nilpotent ideals of $\mathfrak{p}_I$ and what we call I-compatible elements of the affine Weyl group $\widehat{W}$. We identify $\widehat{W}$ with the group of affine transformations W_{aff} defined in [Bo] and we give a characterization of the I-compatible elements via the dimension of the intersection of the image of the fundamental alcove associated to W_{aff} with some affine hyperplanes of V.

Using this result, we obtain an identity (Theorem 4.7) which generalizes the result of Peterson. This identity links the number of abelian ideals and the coefficients of the simple roots in the highest root of Δ. This allows us to conclude that if $\mathfrak{g}$ is of type A or C, the number of abelian ideals of $\mathfrak{p}_I$ is $2^{l-\sharp I}$. It also explains why this result does not hold in general.

On the other hand, the enumeration of ad-nilpotent and abelian ideals of $\mathfrak{p}_I$, when $\mathfrak{g}$ is of classical type, is obtained using the diagrams given in [CP1], modified, by deleting some rows and columns and grouping together some boxes, according to the type of $\mathfrak{g}$. The formulas obtained depend on the decomposition in connected components of I. Note that the formulas obtained when $\mathfrak{g}$ is of type A or C are nicer than the ones obtained when $\mathfrak{g}$ is of type B or D (Theorems 4.7, 5.12 and Propositions 5.18, 5.20).

This paper is organized as follows: in Section 2, we recall some results on the affine Weyl group. In Section 3, we give different characterizations of I-compatible elements of $\widehat{W}$. The study, in Section 4, of the volume of the intersection of some affine hyperplanes on V gives the results stated above on abelians ideals. Section 5 deals with the enumeration of both ad-nilpotent and abelians ideals when $\mathfrak{g}$ is of classical type, using diagrams. We give some remarks concerning the exceptional cases and the relations with antichains in Section 6.

2. Generalities on the affine Weyl group

We shall conserve the notations given in the introduction. In this section, we shall recall some basic facts on the affine Weyl group associated to Δ. In particular, we need to recall two different realizations of this group. See [Bo,CP1,K] for more details.

We fix a scalar product $(.,.)$ on V. For $\alpha \in \Delta$, let

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

denote the corresponding coroot. Denote by $Q^\vee$ the coroot lattice of Δ.

Let W denote the Weyl group associated to Δ. We shall realize the affine Weyl group as a group of automorphisms of the affine root system associated to Δ. Let $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$. We extend the above bilinear form on V to a non-degenerate symmetric bilinear form on $\widehat{V}$, also denoted $(.,.)$, by setting:

$$(\lambda, \lambda) = (\delta, \delta) = (\lambda, V) = (\delta, V) = 0 \quad \text{and} \quad (\delta, \lambda) = 1.$$

Let $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$ be the set of (real) affine roots. We fix the following positive root system $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^*\delta)$. We shall write $\alpha > 0$ (respectively $\alpha < 0$) if $\alpha \in \widehat{\Delta}^+$ (respectively if $\alpha \in \widehat{\Delta}^- = -\widehat{\Delta}^+$). Let θ be the highest root of Δ, then $\widehat{\Pi} = \{\alpha_0 = -\theta + \delta, \alpha_1, \ldots, \alpha_l\}$ is the set of simple roots for $\widehat{\Delta}^+$.

Note that for any element $\beta + k\delta \in \widehat{\Delta}^+$, we have $(\beta + k\delta, \beta + k\delta) = (\beta, \beta) \neq 0$. For all $\alpha \in \widehat{\Delta}^+$, we denote by s_α the reflection of $\widehat{V}$ defined by

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha$$

for $x \in \widehat{V}$. The affine Weyl group $\widehat{W}$ is the subgroup of $\mathrm{Aut}(\widehat{V})$ generated by $\{s_\alpha; \alpha \in \widehat{\Pi}\}$. Observe that $w(\delta) = \delta$ for all $w \in \widehat{W}$, $s_\alpha(\lambda) = \lambda$, for all $\alpha \in \Pi$ and $s_{\alpha_0}(\lambda) = \lambda - \frac{2}{\|\theta\|^2}\alpha_0$, where $\|\theta\| = \sqrt{(\theta, \theta)}$.

Let $\tau \in Q^\vee$, we define the endomorphism t_τ of $\widehat{V}$ by:

$$t_\tau(x + a\delta + b\lambda) = x + a\delta + b\lambda + b\tau + \left(\frac{b}{2}(\tau, \tau) - (x, \tau)\right)\delta \tag{1}$$

for $x \in V$ and $a, b \in \mathbb{R}$. Let $S = \{t_\tau; \tau \in Q^\vee\}$, then the group $\widehat{W}$ is the semi-direct product of S by W.

Consider the $\widehat{W}$-invariant affine subspace

$$E = \{x \in \widehat{V}; (x, \delta) = 1\} = V \oplus \mathbb{R}\delta + \lambda.$$

Let $\pi : E \to V$ be the projection $ax + b\delta + \lambda \mapsto ax$ and

$$\begin{aligned} i : V &\to E, \\ v &\mapsto v + \lambda. \end{aligned}$$

For $w \in \widehat{W}$, we set $\overline{w} = \pi \circ w|_E \circ i$. The map $w \mapsto \overline{w}$ defines an injective morphism of groups from $\widehat{W}$ to $\mathrm{Aut}(V)$. We shall identify $\widehat{W}$ with its image W_{aff} under this map.

For $\alpha \in \Delta$, $\overline{s_\alpha}$ is the reflection s_α on V associated to α, and for $\tau \in Q^\vee$, $\overline{t_\tau}$ is the translation T_τ by the vector τ on V. For $\alpha \in \Delta^+$, $k \geqslant 0$, $x \in V$, we obtain that

$$\begin{aligned} \overline{s_{-\alpha+k\delta}}(x) &= x - \big((x, \alpha) - k\big)\alpha^\vee = T_{k\alpha^\vee} \circ s_\alpha(x), \\ \overline{s_{\alpha+k\delta}}(x) &= x - \big((x, \alpha) + k\big)\alpha^\vee = T_{-k\alpha^\vee} \circ s_\alpha(x). \end{aligned}$$

Thus $\overline{s_{-\alpha+k\delta}}$ and $\overline{s_{\alpha+k\delta}}$ are the orthogonal reflections with respect to $H_{\alpha,k} = \{x \in V; (x, \alpha) = k\}$ and $H_{\alpha,-k}$ respectively. It follows that W_{aff} is the semi-direct product of W by the group of translations T_τ, $\tau \in Q^\vee$.

Observe that for $v \in W$, $\tau \in Q^\vee$, $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we have

$$\overline{vt_\tau}(H_{\alpha,k}) = H_{v(\alpha),k+(\tau,\alpha)}.$$

Recall that the connected components of the complement in V of $\bigcup_{\alpha\in\Delta,k\in\mathbb{Z}} H_{\alpha,k}$ are called alcoves. The group W_{aff} acts simply transitively on the set of alcoves. We denote

$$C = \{x \in V; (\alpha_i, x) > 0 \text{ for all } \alpha_i \in \Pi\}, \qquad A = \{x \in C; (\theta, x) < 1\}$$

respectively the fundamental chamber and the fundamental alcove with respect to Π and $\widehat{\Pi}$.

We shall end this section by recording the following results:

Proposition 2.1. *For $w \in \widehat{W}$, let $N(w) = \{\beta \in \widehat{\Delta}^+; w^{-1}(\beta) < 0\}$ and denote by $\ell(w)$ the length of any reduced expression of w.*

(a) *We fix a reduced expression of $w = s_{\beta_1} \circ \cdots \circ s_{\beta_k}$ with $\beta_i \in \widehat{\Pi}$, then $N(w) = \{s_{\beta_1} \circ \cdots \circ s_{\beta_{p-1}}(\beta_p); 1 \leqslant p \leqslant k\}$. In particular, $N(w)$ contains a simple root.*
(b) *Let $w_1, w_2 \in \widehat{W}$, then $N(w_1) \subseteq N(w_2)$ if and only if, there exists $u \in \widehat{W}$ such that $w_2 = w_1 u$, and $\ell(w_2) = \ell(w_1) + \ell(u)$. In particular, w is uniquely determined by $N(w)$.*
(c) *If $N(w) \cap \Delta^+ \neq \emptyset$, then $N(w) \cap \Pi \neq \emptyset$.*

Proof. For parts (a) and (b), see for example [CP1]. Let us prove (c). The case $\widetilde{A_1}$ is clear. In the others cases, this is a direct consequence of the fact that $N(w)$ is a "compatible" set, by Theorem 1.3 from [CP1]. □

3. *I*-compatible elements in $\widehat{W}$

Let $I \subset \Pi$ and $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_I$. We set

$$\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \setminus \Delta_I; \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}.$$

Then $\mathfrak{i} = \bigoplus_{\alpha\in\Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$ and if $\alpha \in \Phi_{\mathfrak{i}}$, $\beta \in \Delta^+ \cup \Delta_I$ are such that $\alpha + \beta \in \Delta^+$, then $\alpha + \beta \in \Phi_{\mathfrak{i}}$.

Conversely, set

$$\mathcal{F}_I = \{\Phi \subset \Delta^+ \setminus \Delta_I;\ \text{if } \alpha \in \Phi,\ \beta \in \Delta^+ \cup \Delta_I,\ \alpha + \beta \in \Delta^+,\ \text{then } \alpha + \beta \in \Phi\}.$$

Then for $\Phi \in \mathcal{F}_I$, $\mathfrak{i}_\Phi = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{p}_I$.

We obtain therefore a bijection

$$\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \to \mathcal{F}_I,$$
$$\mathfrak{i} \mapsto \Phi_{\mathfrak{i}}.$$

For $\Phi \in \mathcal{F}_I$, we define $\Phi^1 = \Phi$, $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$, for $k \geqslant 2$ and

$$L_\Phi = \bigcup_{k\in\mathbb{N}^*} (-\Phi^k + k\delta).$$

Since any ad-nilpotent ideal of $\mathfrak{p}_I$ is an ad-nilpotent ideal of the Borel subalgebra $\mathfrak{p}_\emptyset = \mathfrak{b}$, we have by [CP1] the following proposition:

Proposition 3.1. *Let $\Phi \in \mathcal{F}_I$, then there exists a unique $w_\Phi \in \widehat{W}$ such that $L_\Phi = N(w_\Phi)$.*

Thus we have the following injective map:

$$\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \to \widehat{W},$$
$$\mathfrak{i} \mapsto w_{\Phi_\mathfrak{i}}.$$

Recall from [CP1] the following characterization of the image of the above map when $I = \emptyset$.

Proposition 3.2. *Let $w \in \widehat{W}$, then there exists an ideal $\mathfrak{i}$ of $\mathfrak{b}$ such that $N(w) = L_{\Phi_\mathfrak{i}}$ if and only if*

(a) $w^{-1}(\alpha) > 0$, *for all* $\alpha \in \Pi$.
(b) *If* $w(\alpha) < 0$ *for some* $\alpha \in \widehat{\Pi}$, *then* $w(\alpha) = \beta - \delta$ *for some* $\beta \in \Delta^+$.

If these conditions are verified, we say that w is Borel-compatible *or* $\emptyset$-compatible.

For $w \in \widehat{W}$, let $\Phi_w = \{\alpha \in \Delta; -\alpha + \delta \in N(w)\}$. It follows that if w is $\emptyset$-compatible, then $\Phi_w \subset \Delta^+$.

Theorem 3.3. *Let $w \in \widehat{W}$ be Borel-compatible and $I \subset \Pi$. The following conditions are equivalent*:

(a) $\mathfrak{i}_{\Phi_w}$ *is an ad-nilpotent ideal of* $\mathfrak{p}_I$.
(b) $s_\alpha(\Phi_w) = \Phi_w$, *for all* $\alpha \in I$.
(c) $s_\alpha(L_{\Phi_w}) = L_{\Phi_w}$, *for all* $\alpha \in I$.
(d) $N(s_\alpha w) = N(w) \cup \{\alpha\}$, *for all* $\alpha \in I$.
(e) $w^{-1}(\alpha) \in \widehat{\Pi}$, *for all* $\alpha \in I$.

If the hypothesis and these conditions are verified, we say that w is I-compatible.

Proof. (a) $\Rightarrow$ (b). By assumption, we have $\Phi_w \in \mathcal{F}_I$. Let $\beta \in \Phi_w$, then $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, hence $s_\alpha(\Phi_w) \subset \Phi_w$, for all $\alpha \in I$. Moreover, since s_α is an involution, we obtain that $s_\alpha(\Phi_w) = \Phi_w$.

(b) $\Rightarrow$ (c). Since $w(\delta) = \delta$, for all $w \in \widehat{W}$, this is clear (by induction on k or just remark that $\Phi_w^k \in \mathcal{F}_I$).

(c) $\Rightarrow$ (d). Let $\alpha \in I$, by assumption, we have $s_\alpha(N(w)) = N(w)$, hence for $\beta \in N(w)$, we have $s_\alpha(\beta) \in N(w)$. So $w^{-1}s_\alpha(\beta) < 0$ and $\beta \in N(s_\alpha w)$. We have proved that $N(w) \subset N(s_\alpha w)$. Since $\sharp N(w) = \ell(w)$ and $\ell(s_\alpha w) = \ell(w) \pm 1$, by Proposition 2.1, we obtain that $\sharp N(s_\alpha w) = \sharp N(w) + 1$. Moreover we have $(s_\alpha w)^{-1}(\alpha) = w^{-1}(-\alpha) < 0$, hence $N(s_\alpha w) = N(w) \cup \{\alpha\}$.

(d) $\Rightarrow$ (e). Let $\alpha \in I$. By assumption, we have $N(w) \subset N(s_\alpha w)$, hence by Proposition 2.1, there exists $\beta \in \widehat{\Pi}$ such that,

$$N(s_\alpha w) = N(ws_\beta) = N(w) \cup \{w(\beta)\} = N(w) \cup \{\alpha\}.$$

Consequently, we have $w^{-1}(\alpha) = \beta \in \widehat{\Pi}$.

(e) ⇒ (a). Let $\alpha \in I$ and assume that $w^{-1}(\alpha) \in \widehat{\Pi}$. Let $\beta \in \Phi_w$ be such that $\beta - \alpha \in \Delta^+$. We have

$$w^{-1}\big(-(\beta-\alpha)+\delta\big) = w^{-1}(-\beta+\delta) + w^{-1}(\alpha) \in (\widehat{\Delta}^- + \widehat{\Pi}) \cap \widehat{\Delta}.$$

It follows that $w^{-1}(-(\beta-\alpha)+\delta) < 0$. Moreover, $w^{-1}(-\alpha+\delta) = w^{-1}(-\alpha)+\delta > 0$ hence $\alpha \notin \Phi_{\mathfrak{i}_w}$. We obtain that $\Phi_w \in \mathcal{F}_{\alpha_i}$, for all $\alpha_i \in I$, hence Φ_w belongs to $\mathcal{F}_I$ and $\mathfrak{i}_{\Phi_w}$ is an ideal of $\mathfrak{p}_I$. □

Another characterization of ad-nilpotent ideals in $\mathfrak{b}$ is given in [CP2] via the set $D = \{\tau \in Q^\vee;\ (\tau, \alpha_j) \leqslant 1,\ j = 1, \ldots, l \text{ and } (\tau, \theta) \geqslant -2\}$. Let $\widetilde{D} = \{(\tau, v) \in D \times W;\ vt_\tau(A) \subset C\}$. We can state this characterization in the following way:

Proposition 3.4. *The following map is bijective*:

$$\begin{aligned} \widetilde{D} &\to \{w \in \widehat{W}, \emptyset\text{-compatible}\},\\ (\tau, v) &\mapsto vt_\tau. \end{aligned}$$

Remark 3.5. In [CP2], the above correspondence is not viewed in the same way since the elements of $\widehat{W}$ are written $t_\tau v = vt_{v^{-1}(\tau)}$ instead of vt_τ, for $w \in W$ and $\tau \in Q^\vee$.

Let $w \in \widehat{W}$ be Borel-compatible, then $I_w = \{\alpha \in \Pi;\ w^{-1}(\alpha) \in \widehat{\Pi}\}$ is the unique maximal element of $\{I \subset \Pi;\ w \text{ is } I\text{-compatible}\}$. For $\tau \in Q^\vee$, set

$$D_\tau = \begin{cases} \{\alpha \in \Pi;\ (\alpha, \tau) = 0\} \cup \{-\theta\} & \text{if } (\theta, \tau) = -1,\\ \{\alpha \in \Pi;\ (\alpha, \tau) = 0\} & \text{if } (\theta, \tau) \neq -1. \end{cases}$$

Proposition 3.6. *Let $(\tau, v) \in \widetilde{D}$, and $w = vt_\tau \in \widehat{W}$. Then $v(D_\tau) = I_w$. In particular, w is I-compatible if and only if $I \subset v(D_\tau)$.*

Proof. Let $\alpha \in I_w$, then

$$w^{-1}(\alpha) = t_{-\tau} v^{-1}(\alpha) = v^{-1}(\alpha) + \big(v^{-1}(\alpha), \tau\big)\delta \in \widehat{\Pi}.$$

If $w^{-1}(\alpha) \in \Pi$, then we have $v^{-1}(\alpha) \in \Pi$ and $(v^{-1}(\alpha), \tau) = 0$, hence $v^{-1}(\alpha) \in D_\tau$. If $w^{-1}(\alpha) = \alpha_0$, then we have $v^{-1}(\alpha) = -\theta$ and $(\theta, \tau) = -1$, hence $-\theta = v^{-1}(\alpha) \in D_\tau$.

Conversely, let $\alpha \in D_\tau \cap \Pi$, then $vt_\tau(\alpha) = v(\alpha) \in \Delta^+$, because w is Borel-compatible. Then we have $N(ws_\alpha) = N(w) \cup \{w(\alpha)\}$, and by part (3) of Proposition 2.1, there exists a simple root $\beta \in \Pi$ such that $\beta \in N(ws_\alpha)$. Since $N(w) \cap \Delta^+ = \emptyset$, we obtain that $w(\alpha) = \beta$ and $v(\alpha) \in I_w$.

Assume now that $-\theta \in D_\tau$. Since w is Borel-compatible, $vt_\tau(\alpha_0) = -v(\theta) \in \Delta^+$. As above we have $N(ws_{\alpha_0}) = N(w) \cup \{w(\alpha_0)\}$, and by part (3) of Proposition 2.1, there exists a simple root $\beta \in \Pi$ such that $\beta \in N(ws_{\alpha_0})$. Since $N(w) \cap \Delta^+ = \emptyset$, we obtain that $w(\alpha_0) = \beta$ and $v(-\theta) \in I_w$.

We have therefore proved that $v(D_\tau) = I_w$, which concludes the proof. □

Let us denote $H_\alpha = H_{\alpha,0}$ for $\alpha \in \Pi$, and $H_{\alpha_0} = H_{\theta,1}$. Let $\{\omega_1, \ldots, \omega_l\}$ be elements of V such that $(\omega_i, \alpha_j) = \delta_{ij}$. Set $n_0 = 1$ and let n_i, $i = 1, \ldots, l$, be the strictly positive integers such that $\theta = \sum_{i=1}^{l} n_i\alpha_i$. Let $\varpi_i = \omega_i/n_i$, $i = 1, \dots, l$, and $\varpi_0 = 0$. Then the closure $\overline{A}$ of A is the convex hull $\mathrm{Conv}(\varpi_0, \varpi_1, \dots, \varpi_l)$ of $\varpi_0, \dots, \varpi_l$. For $k \in \mathbb{N}^*$, the convex hull (respectively the image by $\overline{w} \in W_{\mathrm{aff}}$ of the convex hull) of $(k+1)$ points in $\{\varpi_0, \varpi_1, \dots, \varpi_l\}$ is called a k-face of $\overline{A}$ (respectively of $\overline{w}(\overline{A})$). For example, $H_{\alpha_i} \cap \overline{A} = \mathrm{Conv}(\varpi_0, \dots, \varpi_{i-1}, \varpi_{i+1}, \dots, \varpi_l)$ is an $(l-1)$-face of $\overline{A}$.

We shall give yet another characterization of ad-nilpotent ideals of $\mathfrak{p}_I$ which shall be useful in enumerating abelian ideals when $\mathfrak{g}$ is of type A or C.

Proposition 3.7. *Let $w \in \widehat{W}$ be Borel-compatible and $I \subset \Pi$. Then, $\mathfrak{i}_{\Phi_w}$ is an ideal of $\mathfrak{p}_I$ if and only if for all $\alpha \in I$, $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.*

Proof. Assume that $w \in \widehat{W}$ is I-compatible. Let $(\tau, v) \in \widetilde{D}$ be such that $w = vt_\tau$. By Proposition 3.6, $I \subset v(D_\tau)$, and $v^{-1}(\alpha) \in D_\tau$, for all $\alpha \in I$. Let $\alpha \in I$, we distinguish two cases:

If $v^{-1}(\alpha) = \beta \in \Pi$, then $(\beta, \tau) = 0$. We obtain that

$$\overline{vt_\tau}(H_\beta) = H_{v(\beta),(\tau,\beta)} = H_\alpha.$$

Hence $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.

If $v^{-1}(\alpha) = -\theta$, then $(\theta, \tau) = -1$. We obtain that

$$\overline{vt_\tau}(H_{\alpha_0}) = H_{v(\theta),(\tau,\theta)+1} = H_\alpha.$$

Hence, $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.

Conversely, let $v \in W$, $\tau \in Q^\vee$ be such that $w = vt_\tau \in \widehat{W}$ is Borel-compatible. By assumption, for all $\alpha \in I$, there exists $\beta \in \widehat{\Pi}$ such that $\overline{w}(H_\beta) = H_\alpha$.

If $\beta \in \Pi$, then

$$\overline{vt_\tau}(H_\beta) = H_{v(\beta),(\tau,\beta)} = H_\alpha$$

hence $(\tau, \beta) = 0$, and $w^{-1}(\alpha) = \pm\beta$. Since w is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$.

If $\beta = \alpha_0$, then

$$\overline{vt_\tau}(H_{\alpha_0}) = H_{v(\theta),(\tau,\theta)+1} = H_\alpha$$

hence $(\tau, \theta) = -1$, and $w^{-1}(\alpha) = \pm(\theta - \delta)$. Since w is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$. We have proved that $I \subset v(D_\tau)$, and by Proposition 3.6, w is I-compatible. □

Let $H_\emptyset = V$. For $J \subset \widehat{\Pi}$ non-empty, denote $H_J = \bigcap_{\alpha \in J} H_\alpha$. By the proposition above, if w is I-compatible, then we have $\overline{w}(\overline{A}) \cap H_I = \overline{w}(\overline{A} \cap H_{w^{-1}(I)})$.

4. Volume of the faces of the fundamental alcove

Recall from [CP1] and [Ko], that $w \in \widehat{W}$ is Borel-compatible and the ideal $\mathfrak{i}_{\Phi_w}$ of $\mathfrak{b}$ is abelian if and only if $\overline{w}(A) \subset 2A$. As a consequence, we have the following remarkable result of Peterson: the number of abelian ideals of $\mathfrak{b}$ is 2^l. Observe that the above result says that the number of abelian ideals in $\mathfrak{b}$ depends only on the rank of $\mathfrak{g}$. In the case of parabolic algebras, we shall see in this section to what extent this result can be extended.

For $J \subset \widehat{\Pi}$, let $F_J = \overline{A} \cap H_J = \mathrm{Conv}(\overline{\omega}_j; \alpha_j \notin J)$. Observe that the F_J are the faces of $\overline{A}$. Let $w \in \widehat{W}$, if $\overline{w}(\overline{A}) \cap H_J$ is an $(l - \sharp J)$-face of $\overline{w}(\overline{A})$, then we shall call $\overline{w}(\overline{A}) \cap H_J$ an $(l - \sharp J)$-*alcove* of H_J.

Proposition 4.1.

(a) *Let $w \in \widehat{W}$ and $I \subset \Pi$, if $\overline{w}(A) \subset 2A$ and $\overline{w}(\overline{A}) \cap H_I$ is an $(l - \sharp I)$-alcove of H_I, then w is I-compatible.*
(b) *Let $I \subset \Pi$ and $w, w' \in \widehat{W}$ be I-compatible. If $\overline{w}(A) \subset 2A$, $\overline{w'}(A) \subset 2A$ and $\overline{w}(\overline{A}) \cap H_I = \overline{w'}(\overline{A}) \cap H_I$, then $w = w'$.*

Proof. (a) Let $w \in \widehat{W}$ and $I \subset \Pi$ be of cardinality r. If $\overline{w}(A) \subset 2A$, then w is Borel-compatible and the ideal $\mathfrak{i}_{\Phi_w}$ is abelian.

Set $N = l - r + 1$. Since $\overline{w}(\overline{A}) \cap H_I$ is an $(l - r)$-alcove of H_I, there exist N vertices $\overline{\omega}_{i_1}, \dots, \overline{\omega}_{i_N}$ of $\overline{A}$ such that $\overline{w}(\overline{\omega}_{i_1}), \dots, \overline{w}(\overline{\omega}_{i_N})$ belong to $\overline{w}(\overline{A}) \cap H_I$.

There exist r distinct reflecting affine hyperplanes $H'_1, \dots, H'_r$ of the form H_α, for $\alpha \in \widehat{\Pi}$, such that $\bigcap_{j=1}^r H'_j$ contains $\overline{\omega}_{i_1}, \dots, \overline{\omega}_{i_N}$. For $j = 1, \dots, r$, $H_I \cap \overline{w}(H'_j)$ contains $\overline{w}(\overline{\omega}_{i_1}), \dots, \overline{w}(\overline{\omega}_{i_N})$. Since the dimension of H_I is $N - 1$, it follows that $H_I \subset \overline{w}(H'_j)$.

The hyperplane H_I is defined by the equations $(x, \alpha) = 0$ for all $\alpha \in I$, it follows that $\overline{w}(H'_j)$ is an hyperplane of the form $H_{\beta,0}$, where β is a linear combination of elements of I.

Assume that $\beta \notin I$. Then, the intersection of $H_{\beta,0}$ with the closure of the fundamental chamber C is of dimension at most $l - 2$. Since by construction $H_{\beta,0}$ contains an $(l - 1)$-face of $\overline{w}(\overline{A})$, and $\overline{w}(\overline{A}) \subset \overline{C}$, we obtain a contradiction. It follows that $\beta \in I$.

Set $w = vt_\tau$. We then have that for each $\beta \in I$:

$$w^{-1}(H_{\beta,0}) = H_{v^{-1}(\beta),(\tau, v^{-1}(\beta))} = H_\alpha$$

for some $\alpha \in \widehat{\Pi}$. If $\alpha \in \Pi$, then $v^{-1}(\beta) = \pm\alpha$ and $(\tau, v^{-1}(\beta)) = 0$. Since w is Borel-compatible, we obtain that $w^{-1}(\beta) = \alpha$.

If $\alpha = \alpha_0$, we obtain that $v^{-1}(\beta) = \pm\theta$ and $(\tau, v^{-1}(\beta)) = \pm 1$. Since w is Borel-compatible, we finally obtain that $w^{-1}(\beta) = \alpha_0$. Thus, $w^{-1}(I) \subset \widehat{\Pi}$, and w is I-compatible as required.

(b) Let $I \subset \Pi$ and $w, w' \in \widehat{W}$ be I-compatible. Let $\alpha \in I$, then w is $I \setminus \{\alpha\}$-compatible. It follows by Proposition 3.7 that $\overline{w}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ is an $(l - \sharp I + 1)$-alcove of $H_{I\setminus\{\alpha\}}$ and it is the convex hull of $\overline{w}(\overline{A}) \cap H_I$ and a vertex of $H_{I\setminus\{\alpha\}} \cap \overline{w}(\overline{A})$, which is not in $H_I \cap \overline{w}(\overline{A})$. In the same way $\overline{w'}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ is an $(l - \sharp I + 1)$-alcove of $H_{I\setminus\{\alpha\}}$ and it is the convex hull of $\overline{w'}(\overline{A}) \cap H_I$ and a vertex of $H_{I\setminus\{\alpha\}} \cap \overline{w'}(\overline{A})$, which is not in $H_I \cap \overline{w}(\overline{A})$. Since $\overline{w}(\overline{A}) \subset 2\overline{A}$, there is a unique vertex in $H_{I\setminus\{\alpha\}}$ satisfying these conditions. So, $\overline{w}(\overline{A}) \cap H_{I\setminus\{\alpha\}} = \overline{w'}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ and by induction, we have $\overline{w}(\overline{A}) = \overline{w'}(\overline{A})$. Hence $w = w'$. $\square$

Let $F'_J = \overline{2A} \cap H_J = \mathrm{Conv}(2\overline{\omega}_j; \alpha_j \notin J)$. It is clear that F'_J is a union of $(l - \sharp J)$-alcoves of H_J. Let

$$\mathcal{A}b_I = \{w \in \widehat{W}; \mathfrak{i}_{\Phi_w} \text{ is an abelian ideal of } \mathfrak{p}_I\}.$$

By the above proposition and by Proposition 3.7, we obtain the following result:

Theorem 4.2. *Let $I \subset \Pi$, then the map $w \mapsto \overline{w}(\overline{A}) \cap H_I$ is a bijection between $\mathcal{A}b_I$ and the set of all the $(l - \sharp I)$-alcoves of F'_I.*

Remark 4.3. The above theorem can be viewed as a generalization of Peterson's result.

In order to determine $\sharp \mathcal{A}b_I$, we are reduced to computing the volume of the $(l - \sharp I)$-alcoves of F'_I. Furthermore, to compute the volume of the $(l - \sharp I)$-alcoves of F'_I, it suffices to compute the volume of the $(l - \sharp I)$-faces of $\overline{A}$.

Let $d(x, H_\alpha)$ denote the distance from $x \in V$ to the affine hyperplane H_α, for $\alpha \in \widehat{\Pi}$. For B a k-alcove, let $\mathrm{Vol}_k(B)$ be the k-volume of B. By [Be], the volume of the fundamental alcove is

$$\mathrm{Vol}_l(A) = \frac{1}{l} \times d(0, H_{\alpha_0}) \times \mathrm{Vol}_{l-1}(F_{\alpha_0}).$$

Since the projection of 0 on H_{α_0} is $\frac{\theta}{\|\theta\|^2}$, we have $d(0, H_{\alpha_0}) = \frac{1}{\|\theta\|}$. We obtain that $\mathrm{Vol}_l(A) = \frac{1}{l\|\theta\|}\mathrm{Vol}_{l-1}(F_{\alpha_0})$. Moreover, by [CLO],

$$\mathrm{Vol}_l(A) = \frac{1}{l!}|\overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_l|.$$

Let $D = |\overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_l|$, then

$$\mathrm{Vol}_{l-1}(F_{\alpha_0}) = \frac{D}{(l-1)!} n_0 \|\theta\|. \tag{2}$$

To compute the $(l-1)$-volume of the faces F_{α_i}, $i = 1, \ldots, l$, we compute the l-volume of the convex hull of $(\{\overline{\omega}_1, \ldots, \overline{\omega}_l\} \setminus \{\overline{\omega}_i\}) \cup \{\frac{\alpha_i}{\|\alpha_i\|}\}$. Thus, we have:

$$\mathrm{Vol}_{l-1}(F_{\alpha_i}) = \frac{1}{(l-1)!}\left|\overline{\omega}_1 \wedge \cdots \wedge \frac{\alpha_i}{\|\alpha_i\|} \wedge \cdots \wedge \overline{\omega}_l\right|.$$

Since $\alpha_i = \sum_{k=1}^{l} (\alpha_i, \alpha_k)\omega_k$,

$$\mathrm{Vol}_{l-1}(F_{\alpha_i}) = \frac{D}{(l-1)!} n_i \|\alpha_i\|. \tag{3}$$

We have therefore computed the $(l-1)$-volume of the $(l-1)$-faces of $\overline{A}$. In particular, we have:

Lemma 4.4. *Let $\alpha_i, \alpha_j \in \widehat{\Pi}$, be such that $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, then:*

$$n_i \mathrm{Vol}_{l-1}(F_j) = n_j \mathrm{Vol}_{l-1}(F_i).$$

This lemma also appears as Proposition 26 in [Sut]. We shall generalize this result. For $I \subset \widehat{\Pi}$, let $n_I = 1$ if $I = \emptyset$, and $n_I = \prod_{\alpha_i \in I} n_i$ otherwise. We shall prove the following result:

Proposition 4.5. *Let $I \subset \Pi$ and $w \in \widehat{W}$ be such that $w^{-1}(I) = J \subset \widehat{\Pi}$. Then, we have*:

$$n_I \mathrm{Vol}_{l-\sharp J}(F_J) = n_J \mathrm{Vol}_{l-\sharp I}(F_I).$$

To prove this proposition, we need the following technical lemma:

Lemma 4.6. *Let $I \subset \Pi$ be such that $\sharp I \leqslant l-1$. Let $w \in \widehat{W}$ be such that $w^{-1}(I) = J \subset \widehat{\Pi}$. Let α_j be any element of J if $\alpha_0 \notin J$, and $\alpha_j = \alpha_0$ if $\alpha_0 \in J$. Set $\alpha_i = w(\alpha_j)$. Then we have*:

$$n_i d(\overline{\omega}_i, H_I) = n_j d(\overline{\omega}_j, H_J).$$

Proof. The result is clear if $J = \emptyset$. We may therefore assume that $1 \leqslant \sharp J \leqslant l-1$.

Step 1: Assume that $\alpha_0 \in J$. We shall determine the distance $d(\overline{\omega}_0, H_J)$.

Let J_0 be the connected component of J containing α_0. Set $r = \sharp J_0$.

If $J_0 = \{\alpha_0\}$, then the projection of 0 on H_J is $\frac{\theta}{\|\theta\|^2}$. Therefore, the distance $d(\overline{\omega}_0, H_J)$ is $\frac{1}{\|\theta\|}$.

Now assume that $J_0 \neq \{\alpha_0\}$. Then, $J_0 \setminus \{\alpha_0\}$ contains one or two roots β such that $(\beta, \theta) \neq 0$. Set $J_0 = \{\beta_1, \dots, \beta_r\}$, $\alpha_0 = \beta_k$ and $V_{J_0} = \bigoplus_{\beta_i \in J_0 \setminus \{\alpha_0\}} \mathbb{R}\beta_i$.

First of all, assume that $J_0 \setminus \{\alpha_0\}$ contains only one root β_t such that $(\beta_t, \theta) \neq 0$. Let $\gamma_t \in V_{J_0}$ be such that $(\gamma_t, \beta_t) = 1$ and $(\gamma_t, \beta_i) = 0$ for all $\beta_i \in J_0 \setminus \{\beta_t, \beta_k\}$. Let $\mu_t = (\|\theta\|^2(1 - \frac{(\gamma_t, \theta)}{2}))^{-1}$ and $\beta = \mu_t(\theta - (\theta, \beta_t)\gamma_t)$. Then, we have $(\beta, \alpha) = 0$ for all $\alpha \in J_0 \setminus \{\alpha_0\}$ and

$$(\beta, \theta) = \mu_t\big[\|\theta\|^2 - (\theta, \beta_t)(\gamma_t, \theta)\big] = \mu_t \|\theta\|^2 \left[1 - \frac{(\gamma_t, \theta)}{2}\right] = 1.$$

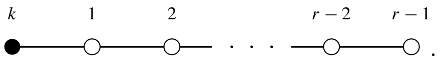
For all $x \in H_J$, we have $(\gamma_t, x) = 0$, and so

$$\begin{aligned}(\beta - x, \beta) &= \mu_t\big(\theta - (\theta, \beta_t)\gamma_t, \beta - x\big) \\ &= \mu_t\big[(\theta, \beta) - (\theta, x)\big] \\ &= 0.\end{aligned}$$

We have proved that β is the projection of $\overline{\omega}_0$ in H_J. It follows that by taking any $x \in H_J$, we have $d(\overline{\omega}_0, H_J)^2 = \|\beta\|^2 = (x, \beta) = \mu_t(x, \theta) = \mu_t$.

Since $I \subset \Pi$ and $J = w^{-1}(I)$ and I have the same Dynkin diagram, we have by a case by case consideration that J_0 is of type A_r, C_r, or D_r.

If J_0 is of type A_r, then by renumbering the roots β_i, the Dynkin diagram of J_0 is of the form:



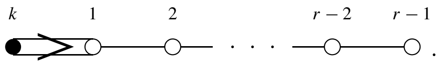
Then $t = 1$, and take

$$\gamma_t = \frac{2}{r\|\beta_1\|^2}\big((r-1)\beta_1 + (r-2)\beta_2 + \cdots + \beta_{r-1}\big).$$

So $(\gamma_t, \theta) = \frac{r-1}{r}$, and we have

$$\mu_t = \frac{2r}{(r+1)\|\theta\|^2}. \tag{4}$$

If J_0 is of type C_r, then the Dynkin diagram of J_0 is of the form:



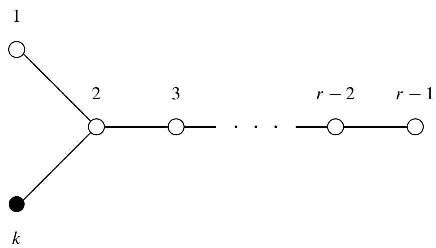
Again $t = 1$, and take

$$\gamma_t = \frac{2}{r\|\beta_1\|^2}\big((r-1)\beta_1 + (r-2)\beta_2 + \cdots + \beta_{r-1}\big).$$

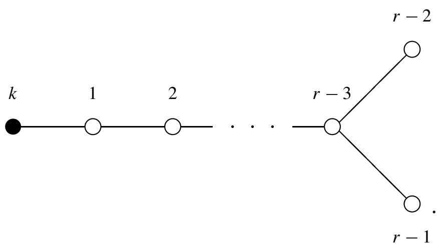
So $(\gamma_t, \theta) = \frac{2(r-1)}{r}$, and we have

$$\mu_t = \frac{r}{(r+1)\|\theta\|^2}. \tag{5}$$

If J_0 is of type D_r, then the Dynkin diagram of J_0 is of the form:



or of the form:



In the first case we have $t = 2$, and we take

$$\gamma_t = \frac{2}{r\|\beta_2\|^2}\big((r-2)\beta_1 + 2(r-2)\beta_2 + 2(r-3)\beta_3 + \cdots + 2\beta_{r-1}\big).$$

Thus $(\gamma_t, \theta) = \frac{2(r-2)}{r}$, and we have

$$\mu_t = \frac{r}{2\|\theta\|^2}. \tag{6}$$

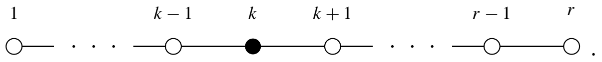
In the second case, we have $t = 1$ and we take

$$\gamma_t = \frac{1}{\|\beta_1\|^2}(2\beta_1 + 2\beta_2 + \cdots + 2\beta_{r-3} + \beta_{r-2} + \beta_{r-1}).$$

Thus we have

$$\mu_t = \frac{2}{\|\theta\|^2}. \tag{7}$$

Assume now that J_0 contains two roots α such that $(\alpha, \theta) \neq 0$. Then the Dynkin diagram of J_0 is of type A_r and these two roots are β_{k-1}, β_{k+1}:



Let $\eta, \eta' \in V_{J_0}$ be such that $(\eta, \beta_{k-1}) = 1 = (\eta', \beta_{k+1})$ and $(\eta, \beta_i) = 0$ (respectively $(\eta', \beta_i) = 0$) for all $\beta_i \in J_0 \setminus \{\beta_{k-1}, \beta_k\}$ (respectively $\beta_i \in J_0 \setminus \{\beta_k, \beta_{k+1}\}$). Let $\mu = (\|\theta\|^2(1 - \frac{(\eta+\eta',\theta)}{2}))^{-1}$ and $\beta = \mu(\theta - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta'))$. Then we have $(\beta, \alpha) = 0$ for all $\alpha \in J_0 \setminus \{\alpha_0\}$ and

$$(\beta, \theta) = \mu\big[\|\theta\|^2 - \big((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta', \theta\big)\big] = 1,$$
$$(\beta - x, \beta) = \mu\big(\theta - \big((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta'\big), \beta - x\big) = 0$$

for all $x \in H_J$. We obtain that β is the projection of 0 on H_J. Take

$$\eta = \frac{2}{k\|\beta_{k-1}\|^2}\big((k-1)\beta_{k-1} + (k-2)\beta_{k-2} + \cdots + \beta_1\big),$$
$$\eta' = \frac{2}{(r-k+1)\|\beta_{k+1}\|^2}\big(\beta_r + 2\beta_{r-1} + \cdots + (r-k)\beta_{k+1}\big)$$

then $(\eta + \eta', \theta) = \frac{k-1}{k} + \frac{r-k}{r-k+1}$. We obtain that:

$$d^2(\bar{\omega}_0, H_J) = \|\beta\|^2 = \mu = \frac{2k(r-k+1)}{n_0^2(r+1)\|\theta\|^2}. \tag{8}$$

Observe that the formulas (4) and (8) generalize the formula obtained when $J_0 = \{\alpha_0\}$. Let k be the position of $\alpha_0 \in J_0$, then we can sum up the above results in the following table, when $1 \leqslant \sharp J \leqslant l - 1$:

Table 1

J_0	A_r	C_r	D_r $t = 2$	D_r $t = 1$
$d(\bar{\omega}_0, H_J)^2$	$\frac{2k(r-k+1)}{n_0^2(r+1)\|\theta\|^2}$	$\frac{r}{n_0^2(r+1)\|\theta\|^2}$	$\frac{r}{2n_0^2\|\theta\|^2}$	$\frac{2}{n_0^2\|\theta\|^2}$

Step 2: Assume that $J \subset \Pi$. Let $\alpha_j \in J$. We shall determine the distance $d(\bar{\omega}_j, H_J)$.

We have $H_J = \mathrm{Vect}(\bar{\omega}_t; t$ such that $\alpha_t \notin J) \subset H_{J\setminus\{\alpha_j\}} \subset V$. Let $H_J^{\perp} = \{x \in V; (x, \bar{\omega}_t) = 0$ for all t such that $\alpha_t \notin J\}$, then $H_J^{\perp} = \mathrm{Vect}(\alpha_t; \alpha_t \in J)$, and $\dim(H_J^{\perp} \cap H_{J\setminus\{\alpha_j\}}) = 1$. Since

$$H_{J\setminus\{\alpha_j\}} \cap H_J^{\perp} = \Big\{x = \sum_{\alpha_t \in J} \tau_t\alpha_t;\ (x, \beta) = 0 \text{ for all } \beta \in J \setminus \{\alpha_j\}\Big\},$$

there exists $\gamma \in V$ such that $H_{J\setminus\{\alpha_j\}} \cap H_J^{\perp} = \mathrm{Vect}(\gamma)$, and $(\gamma, \alpha_j) \neq 0$. Thus, we have $H_{J\setminus\{\alpha_j\}} = H_J \oplus \mathbb{C}\gamma$. It follows that there exists $\mu \in \mathbb{C}^*$ such that $\overline{\omega}_j + \mu\gamma \in H_J$.

Let J_j be the connected component of J which contains α_j. Set $J_j = \{\beta_1, \ldots, \beta_r\}$, with $\alpha_j = \beta_k$. Set $V_{J_j} = \bigoplus_{\beta_j \in J_j} \mathbb{R}\beta_j$. We may choose γ such that $(\gamma, \alpha_j) = 1$ and $(\gamma, \alpha) = 0$ for all $\alpha \in J_j \setminus \{\alpha_j\}$. Since $(\overline{\omega}_j + \mu\gamma, \alpha_j) = 0$, we obtain that $\mu = -\frac{1}{n_j}$, and hence

$$d(\overline{\omega}_j, H_J) = \frac{\|\gamma\|}{n_j}, \tag{9}$$

where γ depends only on the position of α_j in the Dynkin diagram of J_j.

Finally, we need to compute explicitly $d(\overline{\omega}_j, H_J)$ in some particular cases. We use the numbering of [TY, Chapter 18].

If $J_j = A_r$, take

$$\gamma = \frac{2}{(r+1)\|\beta_k\|^2}\bigl[(r-k+1)\beta_1 + 2(r-k+1)\beta_2 + \cdots + (k-1)(r-k+1)\beta_{k-1} + k(r-k+1)\beta_k + k(r-k)\beta_{k+1} + \cdots + k\beta_r\bigr].$$

If $J_j = C_r$, and $k = r$, take

$$\gamma = \frac{2}{\|\beta_r\|^2}\left(\beta_1 + 2\beta_2 + \cdots + (r-1)\beta_{r-1} + \frac{r}{2}\beta_r\right).$$

If $J_j = D_r$, take

$$\gamma = \frac{1}{\|\beta_r\|^2}\left[\beta_1 + 2\beta_2 + \cdots + (r-2)\beta_{r-2} + \frac{1}{2}\bigl[(r-2)\beta_{r-1} + r\beta_r\bigr]\right] \quad \text{if } k = r,$$
$$\gamma = \frac{1}{\|\beta_r\|^2}\left[\beta_1 + 2\beta_2 + \cdots + (r-2)\beta_{r-2} + \frac{1}{2}\bigl[r\beta_{r-1} + (r-2)\beta_r\bigr]\right] \quad \text{if } k = r-1,$$
$$\gamma = \frac{1}{\|\beta_1\|^2}[2\beta_1 + 2\beta_2 + \cdots + 2\beta_{r-2} + \beta_{r-1} + \beta_r] \quad \text{if } k = 1.$$

In these particular cases, we obtain the following result:

Table 2

J_j	A_r	C_r $k = r$	D_r $k = r-1, r$	D_r $k = 1$
$d(\overline{\omega}_j, H_J)^2$	$\frac{2k(r-k+1)}{n_j^2(r+1)\|\alpha_j\|^2}$	$\frac{r}{n_j^2(r+1)\|\alpha_j\|^2}$	$\frac{r}{2n_j^2\|\alpha_j\|^2}$	$\frac{2}{n_j^2\|\alpha_j\|^2}$

Final step: We are now in a position to prove the lemma. Let I_i be the connected component of I containing α_i. If $J \subset \Pi$, then we have the result by (9), since α_j and α_i have the same position in the Dynkin diagram of J_j and I_i respectively.

If $\alpha_0 \in J$, then the connected component J_0 of J containing α_0 is of the type A_r, C_r, or D_r. Again since $w^{-1}(\alpha_0)$ and α_0 have the same position in the respective Dynkin diagram, we obtain the result by inspecting the correspondence between Tables 1 and 2. $\square$

Proof of Proposition 4.5. The case $\sharp I = 0$ is trivial since in this case, $F_I = F_J = \overline{A}$.

Let $I \subset \Pi$. Let us proceed by induction on $\sharp I$. If $\sharp I = 1$, the result is proved in Lemma 4.4. Assume that $l > \sharp I > 1$ and that the claim is true for $\sharp I - 1$. Let α_j be any element of J if $\alpha_0 \notin J$, and $\alpha_j = \alpha_0$ if $\alpha_0 \in J$. Set $\alpha_i = w(\alpha_j)$. Then, we have by Lemma 4.6,

$$\begin{aligned} n_J \mathrm{Vol}_{l-\sharp I}(F_I) &= n_J(l-\sharp I+1)\mathrm{Vol}_{l-\sharp I+1}(F_{I\setminus\{\alpha_i\}}) \times \frac{1}{d(\overline{\omega}_i, H_I)} \\ &= n_j(l-\sharp I+1)\frac{n_I}{n_i}\mathrm{Vol}_{l-\sharp I+1}(F_{J\setminus\{\alpha_j\}}) \times \frac{n_i}{n_j d(\overline{\omega}_j, H_J)} \\ &= n_I(l-\sharp I+1)\mathrm{Vol}_{l-\sharp I+1}(F_{J\setminus\{\alpha_j\}}) \times \frac{1}{d(\overline{\omega}_j, H_J)} \\ &= n_I \mathrm{Vol}_{l-\sharp I}(F_J). \end{aligned}$$

Finally, the result is clear if $\sharp I = l$ since in this case F_I (respectively F_J) is a single point. □

Observe that for $I \subset \Pi$, $F'_I = 2F_I$, so

$$\mathrm{Vol}_{l-\sharp I}\big(F'_I\big) = 2^{l-\sharp I}\mathrm{Vol}_{l-\sharp I}(F_I). \tag{10}$$

We obtain a generalization of Peterson's result:

Theorem 4.7. *Let $I \subset \Pi$, then*

$$\frac{1}{n_I}\sum_{w\in\mathcal{A}b_I} n_{w^{-1}(I)} = 2^{l-\sharp I}.$$

Proof. Let $I \subset \Pi$ and $w \in \widehat{W}$. By Propositions 3.7 and 4.1, then

$$\sum_{w\in\mathcal{A}b_I} \mathrm{Vol}_{l-\sharp I}\big(\overline{w}(\overline{A}) \cap H_I\big) = \mathrm{Vol}_{l-\sharp I}\big(F'_I\big).$$

Observe that for $w \in \mathcal{A}b_I$ we have $\overline{w}(\overline{A}) \cap H_I = \overline{w}(F_{w^{-1}(I)})$, and $\mathrm{Vol}_{l-\sharp I}(\overline{w}(F_{w^{-1}(I)})) = \mathrm{Vol}_{l-\sharp I}(F_{w^{-1}(I)})$. So by Proposition 4.5 and by (10), we obtain that

$$\sum_{w\in\mathcal{A}b_I} \frac{n_{w^{-1}(I)}}{n_I}\mathrm{Vol}_{l-\sharp I}(F_I) = 2^{l-\sharp I}\mathrm{Vol}_{l-\sharp I}(F_I).$$

Thus, we have the result. □

Theorem 4.8. *Let $I \subset \Pi$, if $\mathfrak{g}$ is of type A_l or C_l, then the parabolic subalgebras $\mathfrak{p}_I$ have exactly $2^{l-\sharp I}$ abelian ideals.*

Proof. If $\mathfrak{g}$ is of type A_l or C_l, the numbers n_i, for $i = 0, \dots, l$, depends only on the length of α_i. It follows that for any $w \in \mathcal{A}b_I$, $n_I = n_{w^{-1}(I)}$. So by Theorem 4.7, we obtain the result. □

Remark 4.9. The fact that the integers n_i, for $i = 0, \dots, l$, depends only on the length of α_i is false when $\mathfrak{g}$ is not of type A or C. Indeed, Theorem 4.8 is false in general. For example, in B_3, the parabolic subalgebra $\mathfrak{p}_{\{\alpha_1\}}$ has only 3 abelian ideals. We shall see in the next section another way to count the number of abelian ideals in cases B and D.

5. Enumeration of ideals via diagrams

In this section, we shall determine, via diagram enumeration, the number of ad-nilpotent (respectively abelian) ideals of $\mathfrak{p}_I$, for $I \subset \Pi$, when $\mathfrak{g}$ is simple and of classical type. We shall use the numbering of simple roots of [TY, Chapter 18].

Recall the following partial order on Δ^+: $\alpha \leqslant \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}_\emptyset$ if and only if for all $\alpha \in \Phi$, $\beta \in \Delta^+$, such that $\alpha \leqslant \beta$, then $\beta \in \Phi$. When $\mathfrak{g}$ is of type A, B, C or D, we can display the positive roots into a diagram of suitable shape, as in [CP1]. Then, they established a bijection between elements of $\mathcal{F}_\emptyset$ and certain subdiagrams.

Let $I \subset \Pi$. In order to adapt this construction in the parabolic case $\mathfrak{p}_I$, we shall use a similar construction, but our diagram will depend not only on the type of $\mathfrak{g}$, but also on I.

Let $I \subset \Pi$ and $\gamma, \beta \in \Delta^+$. We say that $\beta \xrightarrow{I} \gamma$ if there exists $\eta \in I$ such that $\beta + \eta = \gamma$. Define an equivalence relation on $\Delta^+ \setminus \Delta_I$: for $I \subset \Pi$, $\gamma \sim_I \beta$ if there exist $\beta_1, \dots, \beta_s \in \Delta^+ \setminus \Delta_I$ such that

(i) $\beta = \beta_1$, $\gamma = \beta_s$,
(ii) either $\beta_i \xrightarrow{I} \beta_{i+1}$ or $\beta_{i+1} \xrightarrow{I} \beta_i$, for $i = 1, \dots, s-1$.

As the standard Levi factor of $\mathfrak{p}_I$ acts in a reductive way on the nilpotent radical, the fact that two roots β, γ are $\sim_I$ equivalent means that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are in the same simple submodule.

Let X be the type of $\mathfrak{g}$. The idea is to start by displaying the positive roots Δ^+ in a diagram T_X of a suitable shape as in [CP1]: that is, we assign to each box labeled (i, j) in T_X, a positive root $t_{i,j}$. The shape and the filling of T_X are chosen such that we obtain a bijection between elements of $\mathcal{F}_\emptyset$ and the northwest flushed subdiagrams, henceforth nw-diagrams, of T_X (in type D, we need to include also nw-diagrams modulo a permutation of certain columns). Then, for $I \subset \Pi$, we delete the boxes containing elements of Δ_I. Observe that the set of boxes of the same equivalent class is connected. Therefore, we can regroup into a big box all the roots of the same equivalent class. We obtain a new diagram denoted by T_X^I. Then, we count the nw-diagrams of T_X^I (again in type D, we need to count also nw-diagrams modulo a permutation of certain columns), which are clearly in bijection with the elements of $\mathcal{F}_I$.

5.1. Type A_l

If $\mathfrak{g}$ is of type A_l, then T_{A_l} is a diagram of shape $[l, l-1, \dots, 1]$. The label (i, j) means a box in the ith row and the jth column. The boxes (i, j) of T_{A_l} are filled by the positive roots $t_{i,j} = \alpha_i + \dots + \alpha_{l-j+1}$, $1 \leqslant i, j \leqslant l$. For example, for $l = 5$, we have:

$$\overbrace{\begin{array}{|c|c|c|c|c|}\hline t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} \\ \hline t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & \multicolumn{1}{c}{} \\ \cline{1-4} t_{3,1} & t_{3,2} & t_{3,3} & \multicolumn{2}{c}{} \\ \cline{1-3} t_{4,1} & t_{4,2} & \multicolumn{3}{c}{} \\ \cline{1-2} t_{5,1} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}}^{l}$$

Let $I \subset \Pi$. We first delete the boxes containing elements of Δ_I. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I$, we regroup the $(l-i+1)$th and the $(l-i+2)$th columns if $i \neq 1$, and the rows $i, i+1$ if $i \neq l$, on T_{A_l}. At the end, we obtain that $T_{A_l}^I$ is a diagram of shape $[l - \sharp I, l - \sharp I - 1, \ldots, 1]$. For example, for A_5 and $I = \{\alpha_2, \alpha_3\}$, we have:

$$\overbrace{\begin{array}{|c|c|c|}\hline t_{1,1} & t_{1,2} & t_{1,3}\ t_{1,4}\ t_{1,5} \\ \hline t_{2,1} & t_{2,2} & \multicolumn{1}{c}{} \\ t_{3,1} & t_{3,2} & \multicolumn{1}{c}{} \\ t_{4,1} & t_{4,2} & \multicolumn{1}{c}{} \\ \cline{1-2} t_{5,1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}}^{l}$$

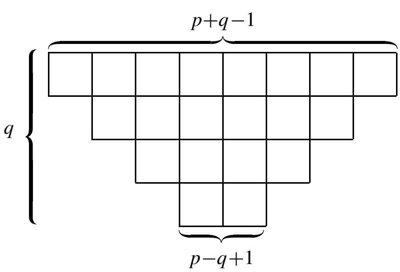
Let $\mathcal{C}_l = \frac{1}{l+1}\binom{2l}{l}$ denote the lth Catalan number.

Proposition 5.1. *Let $I \subset \Pi$. Let $\mathcal{S}_{A_l}^I$ be the set of all nw-diagrams of $T_{A_l}^I$. Then, the cardinality of $\mathcal{S}_{A_l}^I$ is $\mathcal{C}_{l-\sharp I+1}$.*

Proof. Let $I \subset \Pi$, then $T_{A_l}^I$ if of shape $[l - \sharp I, l - \sharp I - 1, \ldots, 1]$, so by [S, 8, 6.19 vv.], we obtain that the cardinality of the set of nw-diagrams of $T_{A_l}^I$ is $\mathcal{C}_{l-\sharp I+1}$. $\square$

5.2. Type C_l

Definition 5.2. Let p, q be two integers such that $q \leqslant p$. Let $T_{p,q}$ be the (shifted) diagram of shape $[p+q-1, p+q-3, \ldots, p-q+1]$ arranged in the following way:



If $\mathfrak{g}$ is of type C_l, then T_{C_l} is the diagram $T_{l,l}$, and the boxes (i,j) of T_{C_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l, & 1 \leqslant j \leqslant l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

Let $I \subset \Pi$, we first delete the boxes containing elements of Δ_I. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup column $2l-i$ and column $2l-i+1$ if $i \neq 1$, then we regroup the ith and $(i+1)$th columns and also the ith and $(i+1)$th rows on T_{C_l}. If $\alpha_l \in I$, we regroup also the columns l and $l+1$. We obtain at the end that $T^I_{C_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}$ if $\alpha_l \notin I$ and of shape $T_{l-\sharp I+1, l-\sharp I}$, if $\alpha_l \in I$.

By [Pr], we obtain directly that the number of nw-diagram of $T_{p,q}$ is $\binom{p+q}{p}$. Consequently, we have the following proposition:

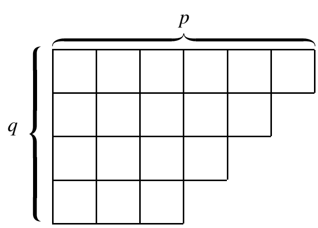
Proposition 5.3. *Let $I \subset \Pi$. Let $\mathcal{S}^I_{C_l}$ be the set of all nw-diagrams of $T^I_{C_l}$. Then, the cardinality of $\mathcal{S}^I_{C_l}$ is*

$$(l - \sharp I + 1)\mathcal{C}_{l-\sharp I} \quad \textit{if } \alpha_l \notin I, \quad \textit{and} \quad \frac{l - \sharp I + 2}{2}\mathcal{C}_{l-\sharp I+1} \quad \textit{if } \alpha_l \in I.$$

5.3. Type B_l and D_l

Let $I \subset \Pi$. Assume that $\mathfrak{g}$ is of type $X = B_l$ or D_l. Then the shape of T^I_X is more complicated than in the case A or C, so we need more combinatorial results on diagrams.

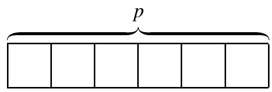
Definition 5.4. Let p, q be two integers such that $q \leqslant p$. Let $T'_{p,q}$ be the diagram of q rows of the shape $[p, p-1, \ldots, p-q+1]$ arranged in the following way:



Proposition 5.5. *Let p, q be two integers such that $q \leqslant p$. Then, the number of nw-diagrams of $T'_{p,q}$ is*

$$\mathcal{T}'_{p,q} = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$

Proof. Let $D_{p,q}$ be the set of nw-diagrams of $T'_{p,q}$. We shall proceed by induction on q. If $q = 1$, then $T'_{p,1}$ is



so we have

$$\sharp D_{p,q} = \mathcal{T}'_{p,q} = p + 1 = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$

Assume that $q > 1$ and the claim is true for $q - 1$. For $1 \leqslant k \leqslant p - q + 1$, let

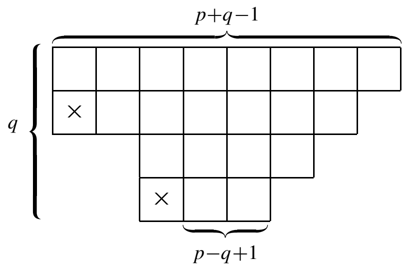
$$S_k = \bigl\{S \in D_{p,q};\ (q,k) \in S \text{ and } (q,k+1) \notin S\bigr\}.$$

Then, $\sharp S_k = \mathcal{T}'_{p-k,q-1}$ and $L = \bigcup_{k=1}^{p-q+1} S_k$ is the set of nw-diagrams containing at least a box in the last row of $T'_{p,q}$. Since $D_{p,q}$ is the disjoint union of $D_{p,q-1}$ and L, we obtain that:

$$\begin{aligned}
\mathcal{T}'_{p,q} &= \mathcal{T}'_{p,q-1} + \sharp L = \sum_{i=0}^{p-q+1} \mathcal{T}'_{p-i,q-1} \\
&= \sum_{k=q-1}^{p} \mathcal{T}'_{k,q-1} = \sum_{k=q-1}^{p} \frac{(k+q)!(k-q+3)}{(q-1)!(k+2)!} \\
&= \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}
\end{aligned}$$

where the last equality is a simple induction on $p \geqslant q$. $\square$

Definition 5.6. Let $p \geqslant q$ be two positive integers and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant q + 1$ be some other integers. Denote by $T_{p,q}(l_1, l_2, \ldots, l_s)$ the new diagram obtained by adding to $T_{p,q}$ the boxes $(l_i, l_i - 1)$, for $1 \leqslant i \leqslant s$. For example, $T_{5,4}(2,4)$ is:



where the added boxes are marked with a $\times$.

Proposition 5.7. *Let $p \geqslant q$ be two positive integers and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant q + 1$ be some other integers, then the number of nw-diagrams of $T_{p,q}(l_1, l_2, \dots, l_s)$ is*

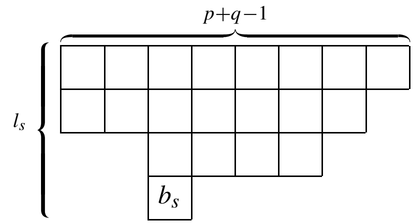
$$\binom{p+q}{p} + \sum_{j=1}^{s} \mathcal{T}'_{p+q-l_j, l_j-1}.$$

Proof. Let $D_{p,q}(l_1, \dots, l_s)$ be the set of nw-diagrams of $T_{p,q}(l_1, \dots, l_s)$ and $\mathcal{D}_{p,q}(l_1, \dots, l_s)$ be its cardinality. Let $b_s = (l_s, l_s - 1)$. Set

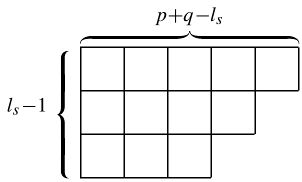
$$\begin{aligned}
E &= \big\{S \in D_{p,q}(l_1, \dots, l_s);\ b_s \notin S\big\},\\
F &= \big\{S \in D_{p,q}(l_1, \dots, l_s);\ b_s \in S \text{ and } S \setminus \{b_s\} \in E\big\},\\
G &= \big\{S \in D_{p,q}(l_1, \dots, l_s);\ b_s \in S \text{ and } S \setminus \{b_s\} \notin E\big\}.
\end{aligned}$$

Then, we have clearly $\mathcal{D}_{p,q}(l_1, \dots, l_s) = \sharp E + \sharp F + \sharp G$.

If $S \in F$, then S contains all the boxes north-west of b_s and the other boxes of S are strictly north-east of b_s, so there exists a bijection between F and the set of nw-diagrams of T'_{p+q-l_s, l_s-1}. For the example in Definition 5.6, if $S \in F$, S is a nw-diagram of:



containing b_s. Hence it suffices to count the nw-diagrams of the subdiagram strictly north-east of b_s:



So by Proposition 5.5, the cardinality of F is $\mathcal{T}'_{p+q-l_s, l_s-1}$.

If $S \in G$, then $S \setminus \{b_s\}$ is a nw-diagram of T where $T = T_{p,q}$ if $s = 1$ and $T = T_{p,q}(l_1, \dots, l_{s-1})$ if $s > 1$. So the cardinality of G is the cardinality of the set of nw-diagrams in T minus the cardinality of the set H of nw-diagrams having at most $l_s - 1$ rows. Observe that the elements of H correspond to those of E. Hence, (by [Pr])

$$\sharp G = \begin{cases} \mathcal{D}_{p,q}(l_1, \dots, l_{s-1}) - \sharp E & \text{if } s > 1,\\ \binom{p+q}{p} - \sharp E & \text{if } s = 1. \end{cases}$$

The result now follows easily by induction on s. $\square$

Notations 5.8. Fix $I \subset \Pi$. Let $I_1, \dots, I_s$ be the connected components of I of cardinality $r_1, \dots, r_s$ respectively. For each connected component I_j, set $m_j = \min\{i; \alpha_i \in I_j\}$. Without loss of generality, we shall assume that $m_1 < m_2 < \cdots < m_s$.

If $\mathfrak{g}$ is of type B_l, then T_{B_l} is $T_{l,l}$ and the boxes (i, j) of T_{B_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_l), & 1 \leqslant j \leqslant l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

As before, for $I \subset \Pi$, we delete the boxes containing elements of Δ_I. For $j = 1, \dots, s$, set

$$l_j = m_j - \sum_{k=1}^{j-1} r_k. \tag{11}$$

Regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup rows i and $i+1$ and if $i \neq 1$, we regroup column $2l-i$ and column $2l-i+1$, then the columns $i-1$ and i. If $\alpha_l \in I$, we also regroup the columns $l-1$, l and $l+1$. We obtain that $T^I_{B_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}(l_1, \dots, l_n)$, where the l_i are defined as above and, $n = s-1$ if $\alpha_l \in I$ and $n = s$ if $\alpha_l \notin I$. For example, for B_5 and $I = \{\alpha_2, \alpha_3, \alpha_5\}$, we have:

$$\begin{array}{|ccc|ccc|ccc|} \hline t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} & t_{1,7} & t_{1,8} & t_{1,9} \\ \hline & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} & & & \\ & & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} & & & \\ & & & t_{4,4} & t_{4,5} & t_{4,6} & & & \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|} \hline & & \\ \hline b_1 & & \\ \cline{1-2} \end{array}$$

It follows from Proposition 5.7 that:

Proposition 5.9. *Let $I \subset \Pi$ be of cardinality r. Let $\mathcal{S}^I_{B_l}$ be the set of all nw-diagrams of $T^I_{B_l}$. Then, the cardinality of $\mathcal{S}^I_{B_l}$ is*

$$(l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{n} \mathcal{T}'_{2(l-r)-l_j, l_j-1}$$

where $n = s-1$ if $\alpha_l \in I$, and $n = s$ otherwise.

If $\mathfrak{g}$ is of type D_l, then T_{D_l} is $T_{l,l-1}$, and the boxes (i, j) of T_{D_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l, & 1 \leqslant j \leqslant l-2, \\ \alpha_i + \cdots + \alpha_{l-2} + \alpha_l, & j = l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

For $I \subset \Pi$, we first delete the boxes containing elements of Δ_I. For $j = 1, \dots, s$, set

$$l_j = \begin{cases} m_j - \sum_{k=1}^{j-1} r_k & \text{if } j \neq s \text{ or } I_s \neq \{\alpha_l\}, \\ m_j - \sum_{k=1}^{j-1} r_k - 1 & \text{if } j = s \text{ and } I_s = \{\alpha_l\}. \end{cases} \tag{12}$$

Regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I \setminus \{\alpha_{l-1}, \alpha_l\}$, we first regroup the rows i and $i+1$ and if $i \neq 1$, we regroup column $2l - i - 1$ and column $2l - i$, and then the columns $i - 1$ and i.

If $\alpha_{l-1} \in I$, but $\alpha_l \notin I$, then we regroup the columns $l-2, l-1$ and columns $l, l+1$.

If $\alpha_l \in I$, but $\alpha_{l-1} \notin I$, then we first reverse the columns $l-1$ and l, and then we regroup the (new) columns $l-2, l-1$ and columns $l, l+1$.

If $\{\alpha_{l-1}, \alpha_l\} \subset I$, then we regroup the four columns $l-2, l-1, l$ and $l+1$.

We obtain that if $\{\alpha_{l-1}, \alpha_l\} \not\subset I$, then $T^I_{D_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I-1}(l_1, \dots, l_s)$, where the l_i are defined as above. If $\{\alpha_{l-1}, \alpha_l\} \subset I$, then $T^I_{D_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}(l_1, \dots, l_{s-1})$.

In the following examples, we denote by i the simple root α_i and by i^2 the element $2\alpha_i$. We consider, $X = D_5$ and I is respectively $\{\alpha_1, \alpha_2, \alpha_5\}$ and $\{\alpha_2, \alpha_4, \alpha_5\}$:

12^23^245	123^245	12345	1234	1235	123
	23^245	2345	234	235	23
		345	34	35	3
			4		

$\longleftrightarrow$

b_1		
	b_2	

12^23^245	123^245	12345	1235	1234	123	12	1
	23^245	2345	235	234	23		
		345	35	34	3		

$\longleftrightarrow$

b_1		

Definition 5.10. For a subdiagram L of $T^I_{D_l}$, we shall denote by $L^\bullet$ the set of boxes of L obtained from L by exchanging columns $l - r - 1$ and $l - r$ (respectively $l - r$ and $l - r + 1$) if $\alpha_1 \notin I$ (respectively if $\alpha_1 \in I$).

If $L^\bullet$ is a nw-diagram of $T^I_{D_l}$, then we say that L is a $\bullet$-nw-diagram of $T^I_{D_l}$.

Proposition 5.11. *Let $I \subset \Pi$ be of cardinality r. Let $\mathcal{S}^I_{D_l}$ be the set of nw-diagrams of $T^I_{D_l}$ if $\{\alpha_{l-1}, \alpha_l\} \cap I \neq \emptyset$, and be the union of the set of nw-diagrams and the set of $\bullet$-nw-diagrams of $T^I_{D_l}$ if $\{\alpha_{l-1}, \alpha_l\} \cap I = \emptyset$. Then, the cardinality of $\mathcal{S}^I_{D_l}$ is*

$$\text{(i)} \qquad \big(3(l-r)-2\big)\mathcal{C}_{l-r-1} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1, l_j-2} + \mathcal{T}'_{2(l-r)-l_j-1, l_j-1}, \quad \textit{if } t = 0,$$

$$\text{(ii)} \qquad \frac{l-r+1}{2}\mathcal{C}_{l-r} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1, l_j-1}, \quad \textit{if } t = 1,$$

(iii) $$(l-r+1)\mathcal{C}_{l-r}+\sum_{j=1}^{s-1}\mathcal{T}'_{2(l-r)-l_j,l_j-1}, \quad \textit{if } t=2,$$

where $t=\sharp(\{\alpha_{l-1},\alpha_l\}\cap I)$.

Proof. Assume first that $\{\alpha_{l-1},\alpha_l\}\cap I=\emptyset$. Note that, the elements of $\mathcal{F}_I$ are in bijection with the subdiagrams S of $T^I_{D_l}=T_{l-r,l-r-1}(l_1,\dots,l_s)$ such that either S or $S^\bullet$ is a nw-diagram. Let

$$E_1= \text{ the set of nw-diagrams of } T^I_{D_l},$$
$$E_2=\big(\text{the set of }\bullet\text{-nw-diagrams of } T^I_{D_l}\big)\setminus E_1.$$

So $S^I_{D_l}=E_1\cup E_2$ (disjoint union). By Proposition 5.7, we have:

$$\sharp E_1=\frac{l-r+1}{2}\mathcal{C}_{l-r}+\sum_{j=1}^{s}\mathcal{T}'_{2(l-r)-l_j-1,l_j-1}.$$

On the other hand, the number of elements of E_2 is $\sharp E_1-\sharp F$, where F is the set of elements of E_1 having columns $l-r-1$ and $l-r$ (respectively $l-r$ and $l-r+1$) of the same length if $\alpha_1\notin I$ (respectively if $\alpha_1\in I$).

Clearly, the number of elements of F is exactly the number of nw-diagrams of the diagram obtained from $T^I_{D_l}$ by removing the $(l-r)$th (respectively $(l-r+1)$th) column if $\alpha_1\notin I$ (respectively if $\alpha_1\in I$). So, by Proposition 5.7,

$$\sharp F=(l-r)\mathcal{C}_{l-r-1}+\sum_{j=1}^{s}\mathcal{T}'_{2(l-r)-l_j-2,l_j-1}.$$

We obtain therefore the result since we have the equality:

$$\mathcal{T}'_{2(l-r)-l_j-1,l_j-1}-\mathcal{T}'_{2(l-r)-l_j-2,l_j-1}=\mathcal{T}'_{2(l-r)-l_j-1,l_j-2}.$$

If α_{l-1} or $\alpha_l\in I$, then there is no column reversing. Then the result follows from Proposition 5.7 according to the shape of $T^I_{D_l}$. $\square$

As in [CP1], we have clearly a bijection between $\mathcal{F}_I$ and $\mathcal{S}^I_X$. It follows from Propositions 5.1, 5.3, 5.9, 5.11, that we have:

Theorem 5.12. *Let $I\subset\Pi$ of cardinality r, X be the type of $\mathfrak{g}$ and s,l_j as defined in* 5.8, (11) *and* (12).

If $X=A_l$, then

$$\sharp\mathcal{F}_I=\mathcal{C}_{l-r+1}.$$

If $X = B_l$, then

$$\sharp\mathcal{F}_I = (l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{n} \mathcal{T}'_{2(l-r)-l_j,l_j-1},$$

where $n = s-1$ if $\alpha_l \in I$, and $n = s$ otherwise.

If $X = C_l$, then

$$\sharp\mathcal{F}_I = \begin{cases} (l-r+1)\mathcal{C}_{l-r} & \text{if } \alpha_l \notin I, \\ \frac{l-r+2}{2}\mathcal{C}_{l-r+1} & \text{if } \alpha_l \in I. \end{cases}$$

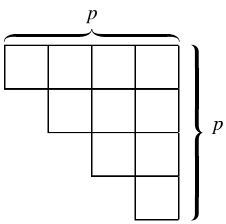
If $X = D_l$, then

$$\sharp\mathcal{F}_I = \begin{cases} \frac{l-r+1}{2}\mathcal{C}_{l-r} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & \text{if } \sharp\{\alpha_{l-1},\alpha_l\} \cap I = 1, \\ (l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{s-1} \mathcal{T}'_{2(l-r)-l_j,l_j-1} & \text{if } \{\alpha_{l-1},\alpha_l\} \subset I, \\ (3(l-r)-2)\mathcal{C}_{l-r-1} & \\ \quad + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-2} + \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & \text{otherwise.} \end{cases}$$

5.4. *Abelian ideals*

We have already determined in Theorem 4.8 the number of abelian ideals for type A and C. We shall now enumerate the abelian ideals of $\mathfrak{p}_I$ using diagrams when $\mathfrak{g}$ is of type B or D. Observe that a similar argument could be used to enumerate abelian ideals in type A and C.

Definition 5.13. Let p be a positive integer and R_p be the diagram of shape $[p, p-1, \ldots, 1]$ arranged in the following way:



Proposition 5.14. *The number of nw-diagrams of R_p is 2^p.*

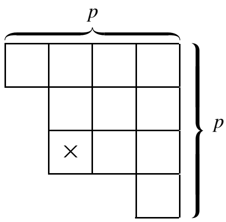
Proof. We shall proceed by induction on p. If $p = 1$, the result is clear. Assume that $p > 1$ and the claim is true for $p - 1$. Let b be the box $(1, p)$ and

$$E = \text{the set of nw-diagrams of } R_p \text{ which do not contain } b,$$

$$F = \text{the set of nw-diagrams of } R_p \text{ which contain } b.$$

Then, the number of nw-diagrams of R_p is $\sharp E + \sharp F$. Furthermore, by the induction hypothesis, we have $\sharp E = 2^{p-1} = \sharp F$, and we obtain the result. $\square$

Definition 5.15. Let p be a positive integer and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant p+1$ be some other integers. Denote by $R_p(l_1, l_2, \ldots, l_s)$ the new diagram obtained by adding to R_p the boxes $(l_i, l_i - 1)$, for $1 \leqslant i \leqslant s$. For example, $R_4(3)$:



Proposition 5.16. *Let p be a positive integer and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant p+1$ be some other integers, then the number of nw-diagrams of $R_p(l_1, l_2, \ldots, l_s)$ is*

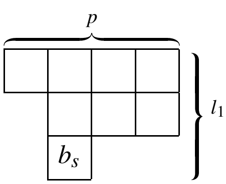
$$2^p + \sum_{j=1}^{s} \binom{p}{l_j - 1}.$$

Proof. Let $D_p(l_1, \ldots, l_s)$ be the set of nw-diagrams of $R_p(l_1, \ldots, l_s)$ and $\mathcal{D}_p(l_1, \ldots, l_s)$ be its cardinality. Let $b_s = (l_s, l_s - 1)$. Set

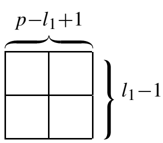
$$\begin{aligned} E &= \{S \in D_p(l_1, \ldots, l_s);\, b_s \notin S\}, \\ F &= \{S \in D_p(l_1, \ldots, l_s);\, b_s \in S \text{ and } S \setminus \{b_s\} \in E\}, \\ G &= \{S \in D_p(l_1, \ldots, l_s);\, b_s \in S \text{ and } S \setminus \{b_s\} \notin E\}. \end{aligned}$$

Then we have clearly $\mathcal{D}_p(l_1, \ldots, l_s) = \sharp E + \sharp F + \sharp G$.

If $S \in F$, then S contains all the boxes north-west of b_s and the other boxes of S are strictly north-east of b_s, so there exists a bijection between F and the set of nw-diagrams of T where T is a diagram whose shape is a rectangle containing $p - l_j + 1$ columns and $l_j - 1$ rows. For the example in Definition 5.15, if $S \in F$, S is a nw-diagram of:



containing b_s. Hence it suffices to count the nw-diagrams of the rectangular subdiagram strictly north-east of b_s:



So by [Pr] the cardinality of F is $\binom{p}{l_j-1}$.

If $S \in G$, then $S \setminus \{b_s\}$ is a nw-diagram of L where $L = R_p$ if $s = 1$ and $L = R_p(l_1, \ldots, l_{s-1})$ if $s > 1$. So the cardinality of G is the cardinality of the set of nw-diagrams in L minus the cardinality of the set H of nw-diagrams having at most $l_s - 1$ rows. Observe that the elements of H correspond to those of E. Hence, by Proposition 5.14

$$\sharp G = \begin{cases} \mathcal{D}_{p,q}(l_1, \ldots, l_{s-1}) - \sharp E & \text{if } s > 1, \\ 2^p - \sharp E & \text{if } s = 1. \end{cases}$$

The result now follows easily by induction on s. $\square$

Let $\mathcal{F}_I^{ab} = \{\Phi \in \mathcal{F}_I;\ \mathfrak{i}_\Phi \text{ is abelian}\}$. If S is a subdiagram of a diagram, let

$$\tau_h^S = \max\{k;\ (h,k) \in S\},$$

so (h, τ_h^S) is the right most box in the hth row of S.

Proposition 5.17. *Assume that $\mathfrak{g}$ is of type B_l. Let $I \subset \Pi$ be of cardinality r. Consider $\Phi \in \mathcal{F}_I$ and S its corresponding nw-diagram in $T_{B_l}^I$. Then $\Phi \in \mathcal{F}_I^{ab}$ if and only if*

(a) $\tau_1^S \leqslant l - r$ *if* $\alpha_1 \in I$,
(b) $\tau_1^S + \tau_2^S \leqslant 2(l-r) - 1$ *if* $\alpha_1 \notin I$.

Proof. Let S_0 be the corresponding nw-diagram of Φ in $T_{B_l}^{\emptyset}$, then by [CP1], we have $\Phi \in \mathcal{F}_{\emptyset}^{ab}$ if and only if $\tau_1^{S_0} + \tau_2^{S_0} \leqslant 2l - 1$.

If $\alpha_1 \in I$, then $\tau_1^{S_0} = \tau_2^{S_0}$. The regrouping process reduces the number of columns on the left of column l of $T_{B_l}^{\emptyset}$ by one for each simple root in $I \setminus \{\alpha_1\}$. It follows that $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_1^S \leqslant l - r$.

The argument is similar for the case $\alpha_1 \notin I$. $\square$

Proposition 5.18. *Assume that $\mathfrak{g}$ is of type B_l. Let $I \subset \Pi$ be of cardinality r, and $l_1, \ldots, l_s$ be as defined in* (11). *Then we have*:

$$\sharp \mathcal{F}_I^{ab} = \begin{cases} 2^{l-r} + \sum_{j=1}^{n} 2\binom{l-r-1}{l_j-1} & \text{if } \alpha_1 \notin I, \\ 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j-1} & \text{if } \alpha_1 \in I, \end{cases}$$

where $n = s$ if $\alpha_l \notin I$ and $n = s - 1$ if $\alpha_l \in I$.

Proof. Recall that $T_{B_l}^I$ is of shape $T_{l-r,l-r}(l_1, \ldots, l_n)$, where $n = s$ if $\alpha_l \notin I$ and $n = s - 1$ if $\alpha_l \in I$.

Let $\Phi \in \mathcal{F}_I$ and S be the nw-diagram of $T_{B_l}^I$ corresponding to Φ.

Assume that $\alpha_1 \in I$, then $l_1 = 1$. By Proposition 5.17, S is in the left-hand half of $T_{B_l}^I$, so it is a nw-diagram of $R_{l-r-1}(1, \ldots, l_n)$. We then obtain the result by Proposition 5.16.

Assume that $\alpha_1 \notin I$. Let E be the set of nw-diagrams of $T_{B_l}^I$ associated to elements of $\mathcal{F}_I^{ab}$. Set

$$P = \{S \in E;\, \tau_1^S \leqslant l-r-1\},$$
$$Q = \{S \in E;\, \tau_1^S > l-r-1\}.$$

Then, we have $\sharp E = \sharp P + \sharp Q$.

If $S \in P$, then S is included in the left-hand half of $T_{B_l}^I$, so

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j-1}$$

by Proposition 5.16.

For $i = l-r, \dots, 2(l-r)-1$, let $Q_i = \{S \in Q;\, \tau_1^S = i\}$ and $P_i = \{S \in P;\, \tau_1^S = 2(l-r)-1-i\}$. We then have:

$$Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i.$$

For $i = l-r, \dots, 2(l-r)-1$, we have an obvious bijection between P_i and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \dots, (1, i)$. Therefore $\sharp P = \sharp Q$ and the result follows. $\square$

Proposition 5.19. *Assume that $\mathfrak{g}$ is of type D_l. Let $I \subset \Pi$ be of cardinality r. Consider $\Phi \subset \mathcal{F}_I$ and S_Φ its corresponding subdiagram in $T_{D_l}^I$. Set $S = S_\Phi$ if S_Φ is a nw-diagram and $S = S_\Phi^\bullet$ if $S_\Phi^\bullet$ is a nw-diagram. Then $\Phi \in \mathcal{F}_I^{ab}$ if and only if*

(a) $\tau_1^S \leqslant l-r$ *if* $\alpha_1 \in I$,
(b) $\tau_1^S + \tau_2^S \leqslant 2(l-r)-2$ *if* $\alpha_1 \notin I$.

Proof. If $I = \emptyset$, set $S = S_0$, then by [CP1], we have $\Phi \in \mathcal{F}_\emptyset^{ab}$ if and only if $\tau_1^{S_0} + \tau_2^{S_0} \leqslant 2l-2$.

Assume that $\alpha_1 \in I$, then $\tau_1^{S_0} = \tau_2^{S_0}$. The regrouping process reduces the number of columns of the left of column l of $T_{D_l}^\emptyset$ by one for each simple root in $I \setminus \{\alpha_1\}$. It follows that $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_1^S \leqslant l-r$.

The argument is similar for the case $\alpha_1 \notin I$. $\square$

Proposition 5.20. *Assume that $\mathfrak{g}$ is of type D_l. Let $I \subset \Pi$ be of cardinality r and $l_1, \dots, l_s$ be as defined in* (12). *Set $t = \sharp(\{\alpha_{l-1}, \alpha_l\} \cap I)$. If $\alpha_1 \in I$, then the cardinality of $\mathcal{F}_I^{ab}$ is:*

(i) $$2^{l-r} - 2^{l-r-2} + \sum_{j=1}^{s} \left[2\binom{l-r-1}{l_j-1} - \binom{l-r-2}{l_j-1} \right], \quad \text{if } t = 0,$$

(ii) $$2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1}, \quad \text{if } t = 1,$$

(iii) $$2^{l-r-1} + \sum_{j=1}^{s-1} \binom{l-r-1}{l_j-1}, \quad \text{if } t = 2.$$

If $\alpha_1 \notin I$, then the cardinality of $\mathcal{F}_I^{ab}$ is:

$$2^{l-r} + \sum_{j=1}^{s} 2\binom{l-r-1}{l_j-1}, \quad \textit{if } t=0, \tag{iv}$$

$$2^{l-r-1} + 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1} + \sum_{j=1}^{s-1} \binom{l-r-2}{l_j-1}, \quad \textit{if } t=1, \tag{v}$$

$$2^{l-r} + 2\sum_{j=1}^{s-1} \binom{l-r-1}{l_j-1}, \quad \textit{if } t=2. \tag{vi}$$

Proof. We proceed as in the case of type B_l but here, we need to take into account column reversing.

Recall that if $t=0$ or 1, $T_{D_l}^I$ is of shape $T_{l-r,l-r-1}(l_1,\dots,l_s)$ and if $t=2$, $T_{D_l}^I$ is of shape $T_{l-r,l-r}(l_1,\dots,l_{s-1})$.

Let $\mathcal{S}_I^{ab}$ be the set of subdiagrams of $T_{D_l}^I$ corresponding to elements of $\mathcal{F}_I^{ab}$. The shape of elements of $\mathcal{S}_I^{ab}$ is conditioned by Proposition 5.19. Let

$$E_1 = \text{the set of nw-diagrams in } \mathcal{S}_I^{ab},$$
$$E_2 = \big(\text{the set of } \bullet\text{-nw-diagrams in } \mathcal{S}_I^{ab}\big) \setminus E_1.$$

Consider $\Phi \in \mathcal{F}_I^{ab}$ and S its corresponding subdiagram in $\mathcal{S}_I^{ab}$.

First assume that $\alpha_1 \in I$, then $l_1 = 1$. If $S \in E_1$, by Proposition 5.19, S is in the left-hand half of $T_{D_l}^I$, so it is a nw-diagram of $R_{l-r-1}(1,\dots,l_n)$, where $n=s$ if $t=0,1$ and $n=s-1$ if $t=2$. Hence, by Proposition 5.16, we have

$$\sharp E_1 = 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j-1}.$$

If $t \neq 0$, there is no column reversing, so $E_2 = \emptyset$. If $t=0$, the number of elements of E_2 is $\sharp E_1 - \sharp(F \cap E_1)$, where F is the set of nw-diagrams of $T_{D_l}^I$ having columns $l-r$ and $l-r+1$ of the same length.

Clearly, the number of elements of F is exactly the number of nw-diagrams of the diagram obtained from $T_{D_l}^I$ by removing the $(l-r+1)$th column. So, by Proposition 5.19, the set of elements which are in $F \cap E_1$ is in bijection with the set of nw-diagrams of $R_{l-r-2}(1,\dots,l_s)$. So by Proposition 5.16, we obtain:

$$\sharp F = 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-2}{l_j-1}.$$

We obtain therefore the result.

Now assume that $\alpha_1 \notin I$. Set

$$\begin{aligned} P &= \{S \in E_1; \tau_1^S \leqslant l-r-1\}, \\ \widetilde{P} &= \{S \in E_1; \tau_1^S \leqslant l-r-2\}, \\ Q &= \{S \in E_1; \tau_1^S > l-r-1\}. \end{aligned}$$

Then, we have $\sharp E_1 = \sharp P + \sharp Q$.

First assume that $t = 0$ or 1. If $S \in P$, then S is included in the left-hand half of $T_{D_l}^I$, so by Proposition 5.16, we have

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j - 1}.$$

For $i = l-r, \dots, 2(l-r)-2$, let $Q_i = \{S \in Q; \tau_1^S = i\}$ and $\widetilde{P}_i = \{S \in P; \tau_1^S = 2(l-r)-2-i\}$. We then have:

$$Q = \bigcup_{i=l-r}^{2(l-r)-2} Q_i \quad \text{and} \quad \widetilde{P} = \bigcup_{i=l-r}^{2(l-r)-2} \widetilde{P}_i.$$

For $i = l-r, \dots, 2(l-r)-2$, we have an obvious bijection between $\widetilde{P}_i$ and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \dots, (1, i)$. Therefore $\sharp\widetilde{P} = \sharp Q$ and by Proposition 5.16, we have

$$\sharp \widetilde{P} = 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-2}{l_j - 1}.$$

If $t = 1$, there is no column reversing, so we have the result. If $t = 0$, then the number of elements of E_2 is $\sharp E_1 - \sharp F$, where $F = E_1 \cap \{\bullet\text{-nw-diagrams of } \mathcal{S}_I^{ab}\}$. By Proposition 5.19, we have $F = Q \cup \widetilde{P}$, so by the consideration above, we have $\sharp F = 2\sharp Q$. It follows that $\sharp E_1 + \sharp E_2 = 2\sharp P$.

For the last case $t = 2$, the shape of $T_{D_l}^I$ is $T_{l-r,l-r}(l_1, \dots, l_{s-1})$. If $S \in P$, then S is included in the left-hand half of $T_{D_l}^I$, so by Proposition 5.16, we have

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{s-1} \binom{l-r-1}{l_j - 1}.$$

We have $E_2 = \emptyset$, and Q_i is defined for $i = l-r, \dots, 2(l-r)-1$. Set $P_i = \{S \in P; \tau_1^S = 2(l-r)-1-i\}$. We then have:

$$Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i.$$

As above, for $i = l-r, \dots, 2(l-r)-1$, we have an obvious bijection between P_i and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \dots, (1, i)$. Therefore $\sharp P = \sharp Q$ and the result follows. $\square$

Remark 5.21. All the results above depend on the numbering of simple roots.

6. Remarks

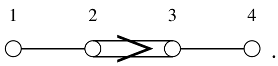
6.1. *Exceptional types*

In the exceptional types E, F and G, the number of ad-nilpotent and abelian ideals has been determined by using GAP 4.

The following tables give the cardinality of $\mathcal{F}_I$ and $\mathcal{A}b_I$ for the types F_4 and G_2. The subset I of Π is described by the symbol $\bullet$ in the Dynkin diagram without arrow.

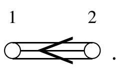
I	$\sharp\mathcal{F}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{F}_I$	$\sharp\mathcal{A}b_I$
○ ○ ○ ○	105	16	● ○ ○ ○	24	6
○ ● ○ ○	35	12	○ ○ ● ○	32	10
○ ○ ○ ●	49	9	● ● ○ ○	10	5
● ○ ● ○	8	4	● ○ ○ ●	12	4
○ ● ● ○	14	7	○ ● ○ ●	14	6
○ ○ ● ●	10	4	● ● ● ○	4	3
● ● ○ ●	5	3	● ○ ● ●	3	2
○ ● ● ●	3	2	● ● ● ●	1	1

where we use the following orientation for the Dynkin diagram of F_4:



I	$\sharp\mathcal{F}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{F}_I$	$\sharp\mathcal{A}b_I$
○ ○	8	4	● ○	3	2
○ ●	4	3	● ●	1	1

where we use the following orientation for the Dynkin diagram of G_2:



6.2. *Relation with antichains*

For $\Phi \in \mathcal{F}_\emptyset$, let

$$\Phi_{\min} = \left\{\beta \in \Phi;\, \beta - \alpha \notin \Phi,\ \text{for all } \alpha \in \Delta^+\right\}$$

be the set of minimal roots of Φ, also called an antichain of $(\Delta^+, \leqslant)$, see [P]. It is clear that each antichain corresponds to an element of $\mathcal{F}_\emptyset$ and vice versa.

By a similar proof as in [P], we obtain the following proposition:

Proposition 6.1. *Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$, then we have $\sharp\Phi_{\min} \leqslant l - r$.*

Proof. Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$. Set $\Gamma = \Phi_{\min} \cup I = \{\gamma_1, \dots, \gamma_t\}$. Let $\gamma_i, \gamma_j \in \Gamma$, then $\gamma_i - \gamma_j \notin \Delta$ by the definition of $\Phi_{\min}$ and the fact that $\Phi \in \mathcal{F}_I$. Thus the angle between any pair of distinct elements of Γ is non-acute and since all the γ_i's lie in an open half space of V, they are linearly independent. Consequently, we have $\sharp\Gamma \leqslant r$, and hence $\sharp\Phi_{\min} \leqslant l - r$. $\square$

Remarks 6.2.

(i) Recall from [CP1], that an antichain $\Gamma \subset \Delta^+$ is of cardinality l if and only if $\Gamma = \Pi$. This result has no equivalence in the general parabolic case. For example, in B_2, the set $\Phi = \{\alpha_1 + 2\alpha_2\}$ is an ad-nilpotent ideal of $\mathfrak{p}_{\alpha_1}$ such that $\Phi_{\min} = \Phi$ and $\sharp\Phi_{\min} = 1$.
(ii) Let $\mathfrak{g}$ be of type A_l. Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$, then it is possible to show that $\sharp\Phi_{\min} = l - r$ if and only if $\Phi_{\min} = \Pi \setminus I$.

Acknowledgment

The author would like to thank the referee for her/his comments and remarks, and in particular for indicating the correct approach to the proof of Proposition 4.1.

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