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Anti-affine algebraic groups

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ABSTRACT

We say that an algebraic group G over a field is anti-affine if every regular function on G is constant. We obtain a classification of these groups, with applications to the structure of algebraic groups in positive characteristics, and to the construction of many counterexamples to Hilbert's fourteenth problem.

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0. Introduction

We say that a group scheme G of finite type over a field k is anti-affine if $\mathcal{O}(G) = k$; then G is known to be smooth, connected and commutative. Examples include abelian varieties, their universal vector extensions (in characteristic 0 only) and certain semi-abelian varieties.

The classes of anti-affine groups and of affine (or, equivalently, linear) group schemes play complementary roles in the structure of group schemes over fields. Indeed, any connected group scheme G , of finite type over k , has a smallest normal subgroup scheme G_{ant} such that the quotient G/G_{ant} is affine. Moreover, G_{ant} is anti-affine and central in G (see [DG70]). Also, G has a smallest normal connected affine subgroup scheme G_{aff} such that G/G_{aff} is an abelian variety (as follows from Chevalley's structure theorem, see [BLR90]). This yields the *Rosenlicht decomposition*: $G = G_{\text{aff}}G_{\text{ant}}$ and $G_{\text{aff}} \cap G_{\text{ant}}$ contains $(G_{\text{ant}})_{\text{aff}}$ as an algebraic subgroup of finite index (see [Ro56]).

Affine group schemes have been extensively investigated, but little seems to be known about their anti-affine counterparts; they only appear implicitly in work of Rosenlicht and Serre (see [Ro58, Ro61, Se58a]). Here we obtain some fundamental properties of anti-affine groups, and reduce their structure to that of abelian varieties.

In Theorem 2.7, we classify anti-affine algebraic groups G over an arbitrary field k with separable closure k_s . In positive characteristics, G is a semi-abelian variety, parametrized by a pair (A, Λ) where A is an abelian variety over k , and Λ is a sublattice of $A(k_s)$, stable under the action of the Galois

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group. The classification is a bit more complicated in characteristic 0: the parameters are then triples (A, Λ, V) where A and Λ are as above, and V is a subspace of the Lie algebra of A . In both cases, A is the dual of the abelian variety G/G_{aff} .

We illustrate this classification by describing the universal morphisms from anti-affine varieties to commutative algebraic groups, as introduced by Serre (see [Se58a, Se58b]).

Together with the Rosenlicht decomposition, our classification yields structure results for several classes of group schemes. As a first consequence, any connected commutative group scheme over a perfect field k is the almost direct product of an anti-affine group, a torus, and a connected unipotent group scheme (see Theorem 3.4 for a precise statement).

In another direction, if the ground field k is finite, then any anti-affine group is an abelian variety. This gives back a remarkable result of Arima: any connected group scheme over a finite field has the decomposition $G = G_{\text{aff}}G_{\text{ab}}$ where G_{ab} is the largest abelian subvariety of G ; moreover, $G_{\text{aff}} \cap G_{\text{ab}}$ is finite (see [Ar60, Ro61]).

Arima's result does not extend to (say) uncountable and algebraically closed fields, as there exist semi-abelian varieties that are anti-affine but non-complete. Yet we obtain a structure result for connected algebraic groups over perfect fields of positive characteristics, namely, the decomposition $G = HS$ where $H \subset G_{\text{aff}}$ denotes the smallest normal connected subgroup such that G_{aff}/H is a torus, and $S \subset G$ is a semi-abelian subvariety; moreover, $H \cap S$ is finite (Theorem 3.7).

Our classification also has rather unexpected applications to Hilbert's fourteenth problem. In its algebro-geometric formulation, it asks if the coordinate ring of every quasi-affine variety is finitely generated (see [Za54], and [Win03] for the equivalence with the invariant-theoretic formulation). The answer is known to be negative, the first counterexample being due to Rees (see [Re58]). Here we obtain many counterexamples, namely, all \mathbb{G}_m -torsors associated to ample line bundles over anti-affine, non-complete algebraic groups (Theorem 3.9).

Some of the preceding statements bear a close analogy to known results on complex Lie groups. Specifically, any connected complex Lie group G has a smallest closed normal subgroup G_{tor} such that the quotient G/G_{tor} is Stein. Moreover, G_{tor} is connected and central in G , and every holomorphic function on G_{tor} is constant. The latter property defines the class of toroidal complex Lie groups, also known as Cousin groups, or quasi-tori, or (H.C) groups. Toroidal groups may be parametrized by pairs (T, Λ) where T is a complex torus, and Λ is a sublattice of the dual torus. Any connected commutative complex Lie group admits a unique decomposition $G = G_{\text{tor}} \times (\mathbb{C}^*)^m \times \mathbb{C}^n$ (see the survey [AK01] for these results). Yet this analogy is incomplete, as Chevalley's structure theorem admits no direct generalization to the setting of complex Lie groups. In fact, the maximal closed connected Stein subgroups of a connected Lie group need not be pairwise isomorphic (see [AK01] again), or normal, or co-compact.

Returning to the algebraic setting, our structure results have applications to homogeneous spaces, which will be developed elsewhere. A natural question asks for their generalizations to group schemes over (say) local artinian rings, or discrete valuation rings.

Notation and conventions. Throughout this article, we denote by k a field with separable closure k_s and algebraic closure \bar{k} . The Galois group of k_s over k is denoted by Γ_k . A Γ_k -lattice is a free abelian group of finite rank equipped with an action of Γ_k .

By a *scheme*, we mean a scheme of finite type over k , unless otherwise specified; a point of a scheme will always mean a closed point. Morphisms of schemes are understood to be k -morphisms, and products are taken over k . A *variety* is a separated, geometrically integral scheme.

We use [SGA3] as a general reference for *group schemes*. However, according to our conventions, any group scheme G is assumed to be of finite type over k . The group law of G is denoted multiplicatively, and e_G stands for the neutral element of $G(k)$, except for commutative groups where we use an additive notation. By an *algebraic group*, we mean a smooth group scheme G , possibly non-connected. An *abelian variety* is a connected and complete algebraic group. For these, we refer to [Mu70], and to [Bo91] for affine algebraic groups.

Given a connected group scheme G , we denote by G_{aff} the smallest normal connected affine subgroup scheme of G such that the quotient G/G_{aff} is an abelian variety, and by

$$\alpha_G : G \rightarrow G/G_{\text{aff}} =: A(G) \quad (0.1)$$

the quotient homomorphism. The existence of G_{aff} is due to Chevalley in the setting of algebraic groups over algebraically closed fields; then G_{aff} is an algebraic group as well, see [Ro56, Ch60]. Chevalley's result implies the existence of G_{aff} for any connected group scheme G , see [Ra70, Lem. IX.2.7] or [BLR90, Thm. 9.2.1].

Also, we denote by

$$\varphi_G : G \rightarrow \text{Spec } \mathcal{O}(G) \quad (0.2)$$

the canonical morphism, known as the affinization of G . Then φ_G is the quotient homomorphism by G_{ant} , the largest anti-affine subgroup scheme of G . Moreover, G_{ant} is a connected algebraic subgroup of the center of G (see [DG70, Sec. III.3.8] for these results).

1. Basic properties

1.1. Characterizations of anti-affine groups

Recall that a group scheme G over k is affine if and only if G admits a faithful linear representation in a finite-dimensional vector space; this is also equivalent to the affineness of the K -group scheme $G_K := G \otimes_k K$ for some field extension K/k . We now obtain analogous criteria for anti-affineness:

Lemma 1.1. *The following conditions are equivalent for a k -group scheme G :*

- (i) G is anti-affine.
- (ii) G_K is anti-affine for some field extension K/k .
- (iii) Every finite-dimensional linear representation of G is trivial.
- (iv) Every action of G on a variety X containing a fixed point is trivial.

Proof. (i) \Leftrightarrow (ii) follows from the isomorphism $\mathcal{O}(G_K) \simeq \mathcal{O}(G) \otimes_k K$.

(i) \Leftrightarrow (iii) follows from the fact that every linear representation of G factors through the affine quotient group scheme G/G_{ant} .

Since (iv) \Rightarrow (iii) is obvious, it remains to show (iii) \Rightarrow (iv). Let x be a G -fixed point in X with local ring \mathcal{O}_x and maximal ideal \mathfrak{m}_x . Then each quotient $\mathcal{O}_x/\mathfrak{m}_x^n$ is a finite-dimensional k -vector space on which G acts linearly, and hence trivially. Since $\bigcap_n \mathfrak{m}_x^n = \{0\}$, it follows that G fixes \mathcal{O}_x pointwise. Thus, G acts trivially on X . \square

Remark 1.2. The preceding argument yields another criterion for affineness of a group scheme; namely, the existence of a faithful action on a variety having a fixed point. This was first observed by Matsumura (see [Ma63]).

Next, we show that the class of anti-affine groups is stable under products, extensions and quotients:

Lemma 1.3. *Let G_1, G_2 be connected group schemes. Then:*

- (i) $G_1 \times G_2$ is anti-affine if and only if G_1 and G_2 are both anti-affine.
- (ii) Given an exact sequence of group schemes

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1,$$

if G is anti-affine, then so is G_2 . Conversely, if G_1 and G_2 are both anti-affine, then so is G .

Proof. (i) follows from the isomorphism $\mathcal{O}(G_1 \times G_2) \simeq \mathcal{O}(G_1) \otimes_k \mathcal{O}(G_2)$.

(ii) The isomorphism $\mathcal{O}(G_2) \simeq \mathcal{O}(G)^{G_1}$ (the algebra of invariants under the action of G_1 on $\mathcal{O}(G)$ via left multiplication) yields the first assertion.

If G_1 is anti-affine, then its action on $\mathcal{O}(G)$ is trivial (as $\mathcal{O}(G)$ is a union of finite-dimensional k - G_1 -submodules, and G_1 acts trivially on any such module by Lemma 1.1). Thus, $\mathcal{O}(G_2) \simeq \mathcal{O}(G)$ which implies the second assertion. \square

Anti-affineness is also stable under isogenies:

Lemma 1.4. *Let $f : G \rightarrow H$ be an isogeny of connected commutative algebraic groups. Then G is anti-affine if and only if so is H .*

Proof. If G is anti-affine, then so is H by Lemma 1.3(ii). For the converse, note that f induces an isogeny $G_{\text{ant}} \rightarrow I$, where I is a subgroup scheme of H , and in turn an isogeny $G/G_{\text{ant}} \rightarrow H/I$. As G/G_{ant} is affine, so is H/I . But H/I is also anti-affine, and hence is trivial. Thus, f restricts to an isogeny $G_{\text{ant}} \rightarrow H$. In particular, $\dim(G_{\text{ant}}) = \dim(H) = \dim(G)$, whence $G_{\text{ant}} = G$. \square

1.2. Rigidity

In this subsection, we generalize some classical properties of abelian varieties to the setting of anti-affine groups. Our results are implicit in [Ro56, Se58a]; we give full proofs for the sake of completeness.

Lemma 1.5. *Let G be an anti-affine algebraic group, and H a connected group scheme.*

- (i) *Any morphism (of schemes) $f : G \rightarrow H$ such that $f(e_G) = e_H$ is a homomorphism (of group schemes), and factors through H_{ant} ; in particular, through the center of H .*
- (ii) *The abelian group (for pointwise multiplication) of homomorphisms $f : G \rightarrow H$ is free of finite rank.*

Proof. (i) Consider the quotient homomorphism (0.1)

$$\alpha_H : H \rightarrow H/H_{\text{aff}} =: A(H).$$

By rigidity of abelian varieties (see e.g. [Co02, Lem. 2.2]), the composition $\alpha_H f : G \rightarrow A(H)$ is a homomorphism. Equivalently, the morphism

$$F : G \times G \longrightarrow H, \quad (x, y) \longmapsto f(xy)f(x)^{-1}f(y)^{-1}$$

factors through the affine subgroup scheme H_{aff} . As $G \times G$ is anti-affine, and $F(e_G, e_G) = e_H$, it follows that F factors through e_H ; thus, f is a homomorphism.

The composition of f with the homomorphism (0.2)

$$\varphi_H : H \rightarrow H/H_{\text{ant}}$$

is a homomorphism from G to an affine group scheme. Hence $\varphi_H f$ factors through $e_{H/H_{\text{ant}}}$, that is, f factors through H_{ant} .

(ii) We may assume that k is algebraically closed; then G_{aff} is a connected affine algebraic group. By [Co02, Lem. 2.3], it follows that any homomorphism $f : G \rightarrow H$ fits into a commutative square

$$\begin{array}{ccc}
 G & \xrightarrow{f} & H \\
 \alpha_G \downarrow & & \downarrow \alpha_H \\
 A(G) & \xrightarrow{\alpha(f)} & A(H)
 \end{array}$$

where $\alpha(f)$ is a homomorphism. This yields a homomorphism

$$\alpha : \operatorname{Hom}(G, H) \longrightarrow \operatorname{Hom}(A(G), A(H)), \quad f \longmapsto \alpha(f).$$

If $\alpha(f) = 0$, then f factors through H_{aff} , and hence is trivial. Thus, $\operatorname{Hom}(G, H)$ is identified to a subgroup of $\operatorname{Hom}(A(G), A(H))$; the latter is free of finite rank by [Mu70, p. 176]. \square

Next, we show that anti-affine groups are “divisible” (this property is the main ingredient of the classification of anti-affine groups in positive characteristics):

Lemma 1.6. *Let G be an anti-affine algebraic group, and n a non-zero integer. Then the multiplication map $n_G : G \rightarrow G$, $x \mapsto nx$ is an isogeny.*

Proof. Let H denote the cokernel of n_G ; then n_H is trivial. Hence the abelian variety H/H_{aff} is trivial, i.e., H is affine. But H is anti-affine as a quotient of G , so that H is trivial. \square

2. Structure

2.1. Semi-abelian varieties

Throughout this section, we consider connected group schemes G equipped with an isomorphism

$$\alpha : G/G_{\text{aff}} \xrightarrow{\sim} A$$

where A is a prescribed abelian variety. We then say that G is a *group scheme over A* .

Our aim is to classify anti-affine groups over A , up to isomorphism of group schemes over A (in an obvious sense). We begin with the case where G_{aff} is a torus, i.e., G is a semi-abelian variety. Then G is obtained as an extension of algebraic groups

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\alpha} A \longrightarrow 0 \quad (2.1)$$

where T is a torus. Moreover, as $T_{k_S} := T \otimes_k k_S$ is split, we have a decomposition of quasi-coherent sheaves on A_{k_S} :

$$\alpha_*(\mathcal{O}_{G_{k_S}}) = \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda \quad (2.2)$$

where Λ denotes the character group of T (so that Λ is a Γ_k -lattice), and \mathcal{L}_λ consists of all eigenvectors of T_{k_S} in $\alpha_*(\mathcal{O}_{G_{k_S}})$ with weight λ . Each \mathcal{L}_λ is an invertible sheaf on A_{k_S} , algebraically equivalent to 0. Thus, \mathcal{L}_λ yields a k_S -rational point $c(\lambda)$ of the dual abelian variety A^\vee . Moreover, the map

$$c : \Lambda \rightarrow A^\vee(k_S), \quad \lambda \mapsto c(\lambda) \quad (2.3)$$

is a Γ_k -equivariant homomorphism, which classifies the extension (2.1) up to isomorphism of extensions (as follows e.g. from [Se59, VII.3.16]). In other words, the extensions (2.1) are classified by the homomorphisms $T^\vee \rightarrow A^\vee$, where T^\vee denotes the Cartier dual of T .

Proposition 2.1.

- (i) *With the preceding notation, G is anti-affine if and only if c is injective.*
- (ii) *The isomorphism classes of anti-affine semi-abelian varieties over A correspond bijectively to the sub- Γ_k -lattices of $A^\vee(k_S)$.*

Proof. (i) By the decomposition (2.2), we have

$$\mathcal{O}(G_{k_S}) = H^0(A_{k_S}, \alpha_*(\mathcal{O}_{G_{k_S}})) = \bigoplus_{\lambda \in \Lambda} H^0(A_{k_S}, \mathcal{L}_\lambda)$$

and of course $H^0(A_{k_S}, \mathcal{L}_0) = \mathcal{O}(A_{k_S}) = k_S$. Thus, G is anti-affine if and only if $H^0(A_{k_S}, \mathcal{L}_\lambda) = 0$ for all $\lambda \neq 0$.

On the other hand, $H^0(A_{k_S}, \mathcal{L}) = 0$ for any invertible sheaf \mathcal{L} on A_{k_S} which is algebraically trivial but non-trivial. (Otherwise, $\mathcal{L}_{\bar{k}} = \mathcal{O}_{A_{\bar{k}}}(D)$ for some non-zero effective divisor D on $A_{\bar{k}}$. We may find an integral curve C in $A_{\bar{k}}$ that meets properly $\text{Supp}(D)$. Then the pull-back of \mathcal{L} to C has positive degree, contradicting the algebraic triviality of \mathcal{L} .)

Thus, G is anti-affine if and only if \mathcal{L}_λ is non-trivial for any $\lambda \neq 0$.

(ii) Given two injective and Γ_k -equivariant homomorphisms

$$c_1, c_2 : \Lambda \longrightarrow A^\vee(k_S),$$

the corresponding anti-affine groups are isomorphic over A if and only if the corresponding extensions differ by an automorphism of T , i.e., there exists a Γ_k -equivariant automorphism f of Λ such that $c_2 = c_1 f$. This amounts to the equality $c_1(\Lambda) = c_2(\Lambda)$. \square

In positive characteristics, the preceding construction yields all anti-affine groups:

Proposition 2.2. *Any anti-affine algebraic group over a field of characteristic $p > 0$ (resp. over a finite field) is a semi-abelian variety (resp. an abelian variety).*

Proof. The multiplication map p_G is an isogeny by Lemma 1.6. In particular, the group $G(\bar{k})$ contains only finitely many points of order p . Thus, every unipotent subgroup of $G_{\bar{k}}$ is trivial. By [SGA3, Exp. XVII, Thm. 7.2.1], it follows that $(G_{\bar{k}})_{\text{aff}}$ is a torus, i.e., $G_{\bar{k}}$ is a semi-abelian variety. Hence G is a semi-abelian variety as well, see [BLR90, p. 178].

If k is finite (so that $k_S = k$), then the group $A^\vee(k_S)$ is the union of its subgroups $A^\vee(K)$, where K ranges over all finite subfields of k_S that contain k . As a consequence, every point of $A^\vee(k_S)$ has finite order. Hence any sublattice of $A^\vee(k_S)$ is trivial. \square

2.2. Vector extensions of abelian varieties

In this subsection, we assume that k has characteristic 0. Recall that every abelian variety A has a universal vector extension $E(A)$ by the k -vector space $H^1(A, \mathcal{O}_A)^*$ regarded as an additive group. In other words, any extension G of A by a vector group U fits into a unique commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H^1(A, \mathcal{O}_A)^* & \longrightarrow & E(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow \gamma & & \downarrow & & \downarrow \text{id} & & \\
 0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0
 \end{array} \quad (2.4)$$

(see [Ro58,Se59,MM74]). Note that $E(A)_{\text{aff}} = H^1(A, \mathcal{O}_A)^*$.

Proposition 2.3.

- (i) $E(A)$ is anti-affine.
- (ii) With the notation of the diagram (2.4), G is anti-affine if and only if the classifying map $\gamma : H^1(A, \mathcal{O}_A)^* \rightarrow U$ is surjective.
- (iii) The anti-affine groups over A obtained as vector extensions are classified by the subspaces of the k -vector space $H^1(A, \mathcal{O}_A)$.

Proof. (i) The affinization epimorphism (0.2)

$$\varphi = \varphi_{E(A)} : E(A) \rightarrow V$$

induces epimorphisms

$$H^1(A, \mathcal{O}_A)^* \rightarrow W, \quad A = E(A)/H^1(A, \mathcal{O}_A)^* \rightarrow V/W$$

where W is a subspace of V . The latter epimorphism must be trivial, and hence φ restricts to an epimorphism

$$\delta : H^1(A, \mathcal{O}_A)^* \rightarrow V.$$

Moreover, V is a vector group, and δ is k -linear. The extension given by the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H^1(A, \mathcal{O}_A)^* & \longrightarrow & E(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow & & \downarrow \text{id} & & \\
 0 & \longrightarrow & V & \longrightarrow & H & \longrightarrow & A & \longrightarrow & 0
 \end{array}$$

is split, as the map $-\varphi + \text{id} : E(A) \times V \rightarrow V$ factors through a retraction of H onto V . Since $E(A)$ is the universal extension, it follows that $\delta = 0$, i.e., $V = 0$.

(ii) The group G is the quotient of $E(A) \times U$ by the diagonal image of $H^1(A, \mathcal{O}_A)^*$. Since $\mathcal{O}(E(A)) = k$, it follows that $\mathcal{O}(G)$ is the algebra of invariants of $\mathcal{O}(U)$ under $H^1(A, \mathcal{O}_A)^*$ acting by translations via γ . This implies the assertion.

(iii) follows from (ii) by assigning to γ the image of the transpose map $\gamma^t : U^* \rightarrow H^1(A, \mathcal{O}_A)$. \square

Remark 2.4. In the preceding statement, the assumption of characteristic 0 cannot be omitted in view of Proposition 2.2. This may also be seen directly as follows. If k has characteristic $p > 0$, any vector extension $0 \rightarrow U \rightarrow G \rightarrow A \rightarrow 0$ splits after pull-back under the multiplication map $p_A : A \rightarrow A$ (since p_A is an isogeny, and $p_U = 0$). This yields an isogeny $U \times A \rightarrow G$. Thus, G cannot be anti-affine in view of Lemma 1.4.

2.3. Classification of anti-affine groups

To complete this classification, we may assume that k has characteristic 0, in view of Proposition 2.2.

Let G be an anti-affine algebraic group. Then G_{aff} is a connected commutative algebraic group, and hence admits a unique decomposition

$$G_{\text{aff}} = T \times U \quad (2.5)$$

where T is a torus, and U is connected and unipotent; U has then a unique structure of k -vector space. Thus, G/U is a semi-abelian variety (extension of A by T) and G/T is a vector extension of A by U . Moreover, the quotient homomorphisms $p_U : G \rightarrow G/U$, $p_T : G \rightarrow G/T$ fit into a cartesian square

$$\begin{array}{ccc} G & \xrightarrow{p_U} & G/U \\ p_T \downarrow & \alpha_{G/U} \downarrow & \\ G/T & \xrightarrow{\alpha_{G/T}} & A \end{array} \quad (2.6)$$

where $\alpha_{G/U}$ (resp. $\alpha_{G/T}$) is the quotient by T (resp. U). Moreover, $\alpha = \alpha_{G/T} p_T = \alpha_{G/U} p_U$. This yields a canonical isomorphism of algebraic groups over A :

$$G \xrightarrow{\cong} G/U \times_A G/T. \quad (2.7)$$

Proposition 2.5. *With the preceding notation, G is anti-affine if and only if G/U and G/T are both anti-affine.*

Proof. If G is anti-affine, then so are its quotient groups G/U and G/T .

For the converse, we may assume that k is algebraically closed in view of Lemma 1.1. Note that the diagram (2.6) yields an isomorphism of quasi-coherent sheaves on A :

$$\alpha_*(\mathcal{O}_G) \simeq \alpha_{G/U,*}(\mathcal{O}_{G/U}) \otimes_{\mathcal{O}_A} \alpha_{G/T,*}(\mathcal{O}_{G/T}). \quad (2.8)$$

Moreover, we have a decomposition

$$\alpha_{G/U,*}(\mathcal{O}_{G/U}) = \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda$$

as in (2.2), where $\mathcal{L}_0 = \mathcal{O}_A$ while $H^0(A, \mathcal{L}_\lambda) = 0$ for any $\lambda \neq 0$. On the other hand, the quasi-coherent sheaf $\alpha_{G/T,*}(\mathcal{O}_{G/T})$ admits an increasing filtration with subquotients isomorphic to \mathcal{O}_A , by the next lemma applied to the U -torsor $\alpha_{G/T} : G/T \rightarrow A$. It follows that

$$H^0(A, \mathcal{L}_\lambda \otimes_{\mathcal{O}_A} \alpha_{G/T,*}(\mathcal{O}_{G/T})) = 0$$

for any $\lambda \neq 0$. Thus,

$$\begin{aligned} \mathcal{O}(G) &= H^0(A, \alpha_*(\mathcal{O}_G)) \simeq \bigoplus_{\lambda \in \Lambda} H^0(A, \mathcal{L}_\lambda \otimes_{\mathcal{O}_A} \alpha_{G/T,*}(\mathcal{O}_{G/T})) \\ &= H^0(A, \alpha_{G/T,*}(\mathcal{O}_{G/T})) = \mathcal{O}(G/T) = k. \quad \square \end{aligned}$$

Lemma 2.6. *Let $\pi : X \rightarrow Y$ be a torsor under a non-trivial connected unipotent algebraic group U . Then the quasi-coherent sheaf $\pi_*(\mathcal{O}_X)$ admits an infinite increasing filtration with subquotients isomorphic to \mathcal{O}_Y .*

Proof. We claim that there is an isomorphism of quasi-coherent sheaves over Y :

$$u : \pi_*(\mathcal{O}_X) \xrightarrow{\simeq} \pi_*(\mathcal{O}_X \otimes_k \mathcal{O}(U))^U.$$

Here the right-hand side denotes the subsheaf of U -invariants in the quasi-coherent sheaf $\pi_*(\mathcal{O}_X \otimes_k \mathcal{O}(U))$, where U acts via its natural action on \mathcal{O}_X and its action on $\mathcal{O}(U)$ by left multiplication.

The assertion of the lemma follows from that claim, as the U -module $\mathcal{O}(U)$ admits an infinite increasing filtration with trivial subquotients.

To prove the claim, we first construct a natural isomorphism

$$u_M : M \xrightarrow{\simeq} (M \otimes_k \mathcal{O}(U))^U$$

for any U -module M . Indeed, the right-hand side may be regarded as the space of U -equivariant morphisms $f : U \rightarrow M$. Any such morphism is of the form $f_m : u \rightarrow u \cdot m$ for a unique $m \in M$, namely, $m = f(e_U)$. We then set $u_M(m) := f_m$.

Next, if the U -module M is also a k -algebra where U acts by algebra automorphisms, then u_M is an isomorphism of M^U -algebras, where the algebra of invariants M^U acts on $(M \otimes_k \mathcal{O}(U))^U$ via multiplication on M . Moreover, u_M commutes with localization by elements of M^U . Thus, the isomorphisms $u_{\mathcal{O}(\pi^{-1}(Y_i))}$, where $(Y_i)_{i \in I}$ is an affine open covering of Y , may be glued to yield the desired isomorphism. \square

Combining the results of Propositions 2.2, 2.3 and 2.5, we obtain the following classification:

Theorem 2.7.

- (i) *In positive characteristics, the isomorphism classes of anti-affine groups over an abelian variety A correspond bijectively to the sub- Γ_k -lattices $\Lambda \subset A^\vee(k_S)$.*
- (ii) *In characteristic 0, the isomorphism classes of anti-affine groups over A correspond bijectively to the pairs (Λ, V) , where Λ is as in (i), and V is a subspace of $H^1(A, \mathcal{O}_A)$.*

Remark 2.8.

- (i) The preceding classification may be formulated in terms of the dual variety A^\vee only, as $H^1(A, \mathcal{O}_A)$ is naturally isomorphic to the tangent space $T_0(A^\vee)$ (the Lie algebra of A^\vee); see e.g. [Mu70, p. 130].
- (ii) To classify the anti-affine groups G without prescribing an isomorphism $G/G_{\text{aff}} \simeq A$, it suffices to replace the sublattices Λ (resp. the pairs (Λ, V)) with their isomorphism classes under the natural action of the automorphism group $\text{Aut}(A)$ of the abelian variety A (resp. of the natural action of $\text{Aut}(A) \times \text{Aut}(A)$ on pairs).

2.4. Universal morphisms

Throughout this subsection, we assume that the ground field k is perfect. We investigate morphisms from a prescribed variety to anti-affine algebraic groups, by adapting results and methods of Serre (see [Se58a, Se58b]).

Consider a variety X equipped with a k -rational point x . Then there exists a universal morphism to a semi-abelian variety

$$\sigma_{X,x} : X \longrightarrow S, \quad x \longmapsto e_S.$$

Indeed, this is a special case of [Se58a, Thm. 7] when k is algebraically closed, and the case of a perfect field follows by Galois descent as in [Se59, Sec. V.4] (see [Wit06, Thm. A.1] for a generalization to an arbitrary field).

We say that $\sigma_{X,x}$ is the *generalized Albanese morphism* of the pointed variety (X, x) , and $S = S_X$ is the *generalized Albanese variety*, which indeed depends only on X . The formation of $\sigma_{X,x}$ commutes with base change to perfect field extensions.

Recalling the extension of algebraic groups (2.1)

$$1 \longrightarrow T \longrightarrow S \xrightarrow{\alpha_S} A \longrightarrow 0,$$

the composite morphism

$$\alpha_{X,x} := \alpha_S \sigma_{X,x} : X \longrightarrow A = A_X$$

is the *Albanese morphism* of X , i.e., the universal morphism to an abelian variety that maps x to the origin.

We note that the pull-back map

$$\alpha_{X,x}^* : A^\vee(k) \subset \text{Pic}(A) \longrightarrow \text{Pic}(X)$$

is independent of the choice of $x \in X(k)$ (indeed, any two Albanese morphisms differ by a translation by a k -rational point of A , and the translation action of A on A^\vee is trivial); we denote that map by α_X^* . Likewise, the analogous map $H^1(A, \mathcal{O}_A) \rightarrow H^1(X, \mathcal{O}_X)$ is independent of x .

We also record the following observation:

Lemma 2.9. *Let X be a complete variety equipped with a k -rational point. Then the pull-back maps $A^\vee(k) \rightarrow \text{Pic}(X)$ and $H^1(A, \mathcal{O}_A) \rightarrow H^1(X, \mathcal{O}_X)$ are both injective.*

Proof. This follows from general results on the Picard functor (see [BLR90, Chap. 8]); we provide a direct argument. We may assume that k is algebraically closed. Let $L \in A^\vee(k)$ such that $\alpha_X^*(L) = 0$. Consider the corresponding extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \xrightarrow{\alpha} A \longrightarrow 0 \tag{2.9}$$

as a \mathbb{G}_m -torsor over A . Then the pull-back of this torsor under $\alpha_{X,x}$ is trivial, that is, the projection $X \times_A G \rightarrow X$ has a section. Thus, $\alpha_{X,x}$ lifts to a morphism $\gamma : X \rightarrow G$ and hence to a morphism $X \rightarrow H$ where $H \subset G$ denotes the algebraic subgroup generated by the image of γ . Since X is complete, H is an abelian variety (as follows e.g. from [SGA3, Exp. VIB, Prop. 7.1]). Thus, α restricts to an isogeny $\beta : H \rightarrow A$. By the universal property of the Albanese morphism, it follows that β is an isomorphism. Thus, the extension (2.9) is split; in other words, L is trivial.

Likewise, given $u \in H^1(A, \mathcal{O}_A)$ such that $\alpha_X^*(u) = 0$, one checks that $u = 0$ by considering the associated extension

$$0 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow A \longrightarrow 0. \quad \square$$

We now obtain a criterion for anti-affineness of the generalized Albanese variety:

Proposition 2.10. *Given a pointed variety (X, x) , the associated semi-abelian variety $S = S_X$ is anti-affine if and only if $\mathcal{O}(X_{\bar{k}})^* = \bar{k}^*$.*

Under that assumption, S is classified by the pair (A, Λ) , where A is the Albanese variety of X , and Λ is the kernel of the pull-back map

$$\alpha_X^* : A^\vee(\bar{k}) \longrightarrow \text{Pic}(X_{\bar{k}}).$$

In particular, this kernel is a Γ_k -lattice.

Proof. Denote by $\varphi : S \rightarrow S/S_{\text{ant}}$ the affinization morphism (0.2). Then S/S_{ant} is affine and semi-abelian, hence a torus. Clearly, the composite $\varphi\sigma_{X,x}$ is the universal morphism from X to a torus, that maps x to the neutral element.

Given a Γ_k -lattice M , the morphisms from X to the dual torus M^\vee correspond bijectively to the Γ_k -equivariant homomorphisms $M \rightarrow \mathcal{O}(X_{\bar{k}})^*$. Moreover, the exact sequence of Γ_k -modules

$$1 \longrightarrow \bar{k}^* \longrightarrow \mathcal{O}(X_{\bar{k}})^* \longrightarrow \mathcal{O}(X_{\bar{k}})^*/\bar{k}^* \longrightarrow 1$$

is split by the evaluation map at $x \in X(k)$, and $\mathcal{O}(X_{\bar{k}})^*/\bar{k}^*$ is a Γ_k -lattice. Thus, the morphisms of pointed varieties

$$(X, x) \longrightarrow (M^\vee, e_{M^\vee})$$

correspond bijectively to the Γ_k -equivariant homomorphisms

$$M \longrightarrow \mathcal{O}(X_{\bar{k}})^*/\bar{k}^*.$$

In particular, there is a universal such morphism, to the dual torus of the lattice $\mathcal{O}(X_{\bar{k}})^*/\bar{k}^*$. This yields the first assertion.

Assuming that $\mathcal{O}(X_{\bar{k}})^* = \bar{k}^*$, consider a sub- Γ_k -lattice $M \subset A^\vee(\bar{k})$ and the corresponding extension

$$1 \longrightarrow M^\vee \longrightarrow G \longrightarrow A \longrightarrow 0.$$

We regard G as an M^\vee -torsor over A . Then the morphisms $\gamma : X \rightarrow G$ that lift the Albanese morphism $\alpha_{X,x} : X \rightarrow A$ are identified to the sections of the pull-back M^\vee -torsor $X \times_A S \rightarrow X$, as in the proof of Lemma 2.9. Such sections exist if and only if the pull-back map $M \rightarrow \text{Pic}(X_{\bar{k}})$ is trivial; moreover, any two sections differ by a morphism $X \rightarrow T$, i.e. by a k -rational point of T . Thus, there exists a unique section such that the associated morphism γ maps x to e_S .

As a consequence, the liftings of $\alpha_{X,x}$ to semi-abelian varieties over A are classified by the sub- Γ_k -lattices of $\Lambda := \ker(\alpha_X^*)$. We now show that Λ is a Γ_k -lattice, thereby completing the proof. For this, we may assume that k is algebraically closed.

If X is complete, then Λ is trivial by Lemma 2.9. In the general case, let $i : X \rightarrow \bar{X}$ be an open immersion into a complete variety. We may assume that $\alpha_{X,x}$ extends to a morphism $\alpha_{\bar{X},x} : \bar{X} \rightarrow A$; then $\alpha_{\bar{X},x}$ is the Albanese morphism of $(\bar{X}, i(x))$. Since α_X^* is injective, Λ is identified to a subgroup of the kernel of

$$i^* : \text{Pic}(\bar{X}) \rightarrow \text{Pic}(X).$$

But $\ker(i^*)$ is the group of Cartier divisors with support in $\bar{X} \setminus X$, as $\mathcal{O}(X)^* = k^*$. In particular, the abelian group $\ker(i^*)$ is free of finite rank, and hence so is Λ . \square

By another result of Serre (see [Se58a, Thm. 8]), a pointed variety (X, x) admits a universal morphism to a commutative algebraic group,

$$\gamma_{X,x} : X \longrightarrow G, \quad x \longmapsto e_G$$

if and only if $\mathcal{O}(X) = k$, that is, X is anti-affine. Then G is also anti-affine, as this group is generated over \bar{k} by the image of X . In positive characteristics, the universal group G is just the generalized Albanese variety, by Proposition 2.2. In characteristic 0, this group may be described as follows:

Proposition 2.11. *Let (X, x) be a pointed anti-affine variety over a field k of characteristic 0 and let G be the associated anti-affine group. Then G is classified by the triple (A, Λ, V) where A and Λ are as in the preceding proposition, and V is the kernel of the pull-back map $\alpha_X^* : H^1(A, \mathcal{O}_A) \longrightarrow H^1(X, \mathcal{O}_X)$.*

The proof is analogous to that of Proposition 2.10, taking into account the isomorphism (2.7) and the structure of anti-affine vector extensions of A .

Remarks 2.12. (i) The associated data A, Λ, V may be described explicitly in terms of completions, for smooth varieties in characteristic 0. Namely, given such a variety X , there exists an open immersion $i : X \rightarrow \bar{X}$ where \bar{X} is a smooth complete variety. Then $\text{Pic}^0(\bar{X})$ is an abelian variety with dual the Albanese variety $A_X = A_{\bar{X}}$.

If $\mathcal{O}(X_k)^* = \bar{k}^*$, then the Γ_k -lattice Λ of Proposition 2.10 is the group of divisors supported in $\bar{X} \setminus X$ and algebraically equivalent to 0, by the arguments in [Se58b, Sec. 1].

Under the (stronger) assumption that $\mathcal{O}(X) = k$, the subspace $V \subset H^1(A, \mathcal{O}_A)$ of Proposition 2.11 equals

$$H^1_{\bar{X} \setminus X}(\bar{X}, \mathcal{O}_{\bar{X}}) = H^0(\bar{X}, i_*(\mathcal{O}_X)/\mathcal{O}_{\bar{X}}),$$

as follows from similar arguments.

(ii) Dually, one may also consider morphisms from varieties, or schemes, to a prescribed anti-affine group G . In fact, such a group admits a modular interpretation, which generalizes the duality of abelian varieties.

To state it, recall that any abelian variety A classifies the invertible sheaves on A^\vee , algebraically equivalent to 0 and equipped with a rigidification along the zero section.

The universal extension $E(A)$ has also a modular interpretation: it classifies the algebraically trivial invertible sheaves on A^\vee , equipped with a rigidification along the first infinitesimal neighborhood $T_0(A^\vee)$ (see [MM74, Prop. 2.6.7]).

It follows that the algebraically trivial invertible sheaves on A^\vee , equipped with rigidifications along a basis of the lattice Λ and along the subspace $V \subset T_0(A^\vee)$, are classified by an anti-affine algebraic group over A with data (Λ, V) .

3. Some consequences

3.1. The Rosenlicht decomposition

We first obtain a variant of a structure theorem for algebraic groups due to Rosenlicht (see [Ro56, Cor. 5, p. 440]), in the setting of group schemes.

Proposition 3.1. *Let G be a connected group scheme over a field k . Then:*

(i) *The group law of G yields an exact sequence of group schemes*

$$1 \longrightarrow G_{\text{aff}} \cap G_{\text{ant}} \longrightarrow G_{\text{aff}} \times G_{\text{ant}} \longrightarrow G \longrightarrow 1. \quad (3.1)$$

In other words, we have the decomposition $G = G_{\text{aff}}G_{\text{ant}}$.

- (ii) The connected subgroup scheme $(G_{\text{ant}})_{\text{aff}} \subset G_{\text{ant}}$ is an algebraic group, contained in G_{aff} ; moreover, the quotient $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is finite.
- (iii) The quotient group scheme $G' := G/(G_{\text{ant}})_{\text{aff}}$ has the decomposition $G' = G'_{\text{ab}}G'_{\text{aff}}$ where $G'_{\text{ab}} = G_{\text{ant}}/(G_{\text{ant}})_{\text{aff}}$ is the largest abelian subvariety of G' , and $G'_{\text{aff}} = G_{\text{aff}}/(G_{\text{ant}})_{\text{aff}}$.
- (iv) Any subgroup scheme $H \subset G$ such that $G = G_{\text{aff}}H$ contains G_{ant} .

Proof. (i) Since G_{aff} is a normal subgroup scheme of G , and G_{ant} is contained in the center of G , we see that the multiplication map $G_{\text{aff}} \times G_{\text{ant}} \rightarrow G$ is a homomorphism with kernel isomorphic to $G_{\text{aff}} \cap G_{\text{ant}}$; the image $G_{\text{aff}}G_{\text{ant}}$ is a normal subgroup scheme of G by [SGA3, Exp. VIA 5.3, 5.4]. The quotient $G/(G_{\text{aff}}G_{\text{ant}})$ is affine, as a quotient of G/G_{ant} ; but it is also an abelian variety, as a quotient of G/G_{aff} . Thus, this quotient is trivial.

(ii) The smoothness of $(G_{\text{ant}})_{\text{aff}}$ follows from Proposition 2.2. By rigidity (see e.g. [Co02, Lem. 2.2]), every homomorphism from $(G_{\text{ant}})_{\text{aff}}$ to an abelian variety is trivial. As a consequence, $(G_{\text{ant}})_{\text{aff}} \subset G_{\text{aff}}$.

The scheme $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is affine (as a quotient of a subgroup scheme of G_{aff}) and proper (as a subgroup scheme of the abelian variety $G_{\text{ant}}/(G_{\text{ant}})_{\text{aff}}$). Hence this scheme is finite.

(iii) follows readily from (i) and (ii).

(iv) Note that $G = G_{\text{aff}}H^0$, as G is connected. Thus,

$$G = G_{\text{aff}}H_{\text{aff}}^0H_{\text{ant}}^0$$

and $H_{\text{ant}}^0 \subset G_{\text{ant}}$; in particular, H_{ant}^0 is contained in the center of G . On the other hand, $G_{\text{aff}}H_{\text{aff}}^0$ is affine, so that

$$G/H_{\text{ant}}^0 \simeq (G_{\text{aff}}H_{\text{aff}}^0)/(H_{\text{ant}}^0 \cap (G_{\text{aff}}H_{\text{aff}}^0))$$

is affine as well. Since the quotient homomorphism $G \rightarrow G/G_{\text{ant}}$ is the affinization, it follows that H_{ant}^0 contains G_{ant} . \square

Next, we consider the functorial properties of the Rosenlicht decomposition. By the results of [DG70, III.3.8], the formation of G_{ant} commutes with base change to arbitrary field extensions, and with homomorphisms of group schemes. Also, note that the homomorphism (0.2) $\varphi_G : G \rightarrow \text{Spec } \mathcal{O}(G) = G/G_{\text{ant}}$ depends only on G regarded as a scheme. In particular, G_{ant} depends only on the pointed scheme (G, e_G) .

These properties are also satisfied by G_{aff} under additional assumptions. Specifically, if G is a connected algebraic group over a perfect field k , then G_{aff} is the largest connected affine algebraic subgroup of G ; the formation of G_{aff} commutes with base change to any perfect field extension of k and with homomorphisms of algebraic groups (see [Co02] for these results). The quotient homomorphism $\alpha_G : G \rightarrow G/G_{\text{aff}}$ is the Albanese morphism of the pair (G, e_G) . In particular, G_{aff} depends only on the pointed variety (G, e_G) .

The assumption that k is perfect cannot be omitted in view of the following example, obtained by a construction of Raynaud (see [SGA3, Exp. XVII, App. III, Prop. 5.1]):

Example 3.2. Let k be a non-perfect field of characteristic $p > 0$ and choose a finite, non-trivial field extension K/k such that $K^p \subset k$. Given a non-trivial abelian variety A over k , let $A_K := A \otimes_k K$ (a non-trivial abelian variety over K) and

$$G := \Pi_{K/k}(A_K)$$

where $\Pi_{K/k}$ denotes the Weil restriction; in other words, G is the unique k -scheme such that

$$G(R) = A_K(R \otimes_k K) \tag{3.2}$$

for any k -algebra R . Then G is a commutative connected algebraic k -group, as follows e.g. from the results of [Oe84, A.2] that we shall use freely.

We claim that

$$G_K = U \times A_K \quad (3.3)$$

where U is a connected unipotent algebraic K -group; in particular, $(G_K)_{\text{aff}} = U$ and $(G_K)_{\text{ant}} = A_K$. Moreover, $G_{\text{ant}} = A$ but G_{aff} is not smooth, and $(G_{\text{aff}})_K \neq (G_K)_{\text{aff}}$.

Indeed, for any K -algebra R , we have

$$G_K(R) = G(R) = A_K(R \otimes_K (K \otimes_K K))$$

and $K \otimes_K K$ is a finite-dimensional K -algebra. The multiplication map $\mu : K \otimes_K K \rightarrow K$ yields an exact sequence

$$0 \rightarrow \mathfrak{m} \rightarrow K \otimes_K K \rightarrow K \rightarrow 0$$

and the ideal \mathfrak{m} is nilpotent, as $(x \otimes 1 - 1 \otimes x)^p = 0$ for any $x \in K$. This yields a functorial morphism $G_K(R) \rightarrow A_K(R)$ and, in turn, an extension of K -group schemes

$$1 \longrightarrow U \longrightarrow G_K \xrightarrow{\alpha} A_K \longrightarrow 0 \quad (3.4)$$

where U has a filtration with subquotients isomorphic to the Lie algebra of A_K . In particular, U is smooth, connected and unipotent. Moreover, α is the Albanese morphism of (G_K, e_{G_K}) .

For any k -scheme S , the map $G(S) \rightarrow A_K(S_K)$ that sends any $f : S \rightarrow G$ to $\alpha f_K : S_K \rightarrow A_K$ is bijective. This yields a morphism $\beta : A \rightarrow G$ such that $\alpha \beta_K$ is the identity map of A_K . It follows that β is a closed immersion of group schemes; we shall identify A with $\beta(A)$, and likewise A_K with $\beta_K(A_K)$. As β_K splits the extension (3.4), this yields the decomposition (3.3).

As a consequence, $G_{\text{ant}} = A$ and hence $G = G_{\text{aff}}A$. Also, G_{aff} is not smooth; indeed, any morphism from a connected affine algebraic group to G is constant, as follows from the equality (3.2) together with [Co02, Lem. 2.3]. Thus, the finite group scheme $G_{\text{aff}} \cap A$ is non-trivial: otherwise, $G \simeq G_{\text{aff}} \times A$, so that G_{aff} would be smooth. In particular, $(G_{\text{aff}})_K \neq U = (G_K)_{\text{aff}}$.

In particular, G/G_{aff} is the quotient of A by a non-trivial subgroup scheme. On the other hand, the quotient map $\alpha_G : G \rightarrow G/G_{\text{aff}}$ is easily seen to be the Albanese morphism of (G, e_G) considered in [Wit06]. Thus, the formation of the Albanese morphism does not commute with arbitrary field extensions.

3.2. Structure of connected commutative algebraic groups

We first obtain a simple characterization of non-affine group schemes that are minimal for this property:

Proposition 3.3. *The following conditions are equivalent for a non-trivial group scheme G :*

- (i) G is non-affine and every subgroup scheme $H \subset G$, $H \neq G$ is affine.
- (ii) G is anti-affine and has no non-trivial anti-affine subgroup.
- (iii) G is anti-affine and the abelian variety $A(G) = G/G_{\text{aff}}$ is simple.

If one of these conditions holds, then either G is an abelian variety or G contains no complete subvariety of positive dimension.

Proof. (i) \Leftrightarrow (ii) follows easily from the fact that a group scheme H is affine if and only if H_{ant}^0 is trivial.

(ii) \Rightarrow (iii) Assume that $A(G)$ contains a non-trivial abelian variety B , and denote by H the pull-back of B in G . Then H_{ant} is a non-trivial subgroup of G , a contradiction.

(iii) \Rightarrow (ii) Let H be an anti-affine subgroup of G . Then $H_{\text{aff}} \subset G_{\text{aff}}$, as G_{aff} is the largest connected affine subgroup of G ; hence $A(H)$ is identified with a subgroup of $A(G)$. Thus, either $A(H)$ is trivial so that H is affine, or $A(H) = A(G)$ so that $G_{\text{aff}}H = G$. In the latter case, $H = G$ by Proposition 3.1.

Under one of these conditions, consider the algebraic subgroup $H \subset G$ generated by a complete subvariety of G . Then H is complete as well (see e.g. [SGA3, Exp. VIB, Prop. 7.1]); thus, either $H = G$ or H is trivial. \square

Next, we obtain a decomposition of connected commutative group schemes over perfect fields:

Theorem 3.4. *Let G be a connected commutative group scheme over a perfect field k . Then there exist a subtorus $T \subset G$ and a connected unipotent subgroup scheme $U \subset G$ such that the group law of G induces an isogeny*

$$f : G_{\text{ant}} \times T \times U \longrightarrow G. \quad (3.5)$$

Moreover, T is unique up to isogeny, and U is unique up to isomorphism; if G is an algebraic group, then so is U .

Proof. The Rosenlicht decomposition yields an exact sequence of group schemes

$$1 \longrightarrow G_{\text{aff}} \cap G_{\text{ant}} \longrightarrow G_{\text{aff}} \xrightarrow{\psi} G/G_{\text{ant}} \longrightarrow 1.$$

Moreover, we have unique decompositions $G_{\text{aff}} = T' \times U'$ and $G/G_{\text{ant}} = T'' \times U''$, where T', T'' are tori and U', U'' are connected unipotent group schemes. This yields epimorphisms $\psi_s : T' \rightarrow T'', \psi_u : U' \rightarrow U''$. Thus, we may find a subtorus $T \subset T'$ such that ψ_s restricts to an isogeny $T \rightarrow T''$.

If k has characteristic 0, we may also find a (connected) unipotent subgroup $U \subset U'$ such that ψ_u restricts to an isomorphism $U \rightarrow U''$, as U' and U'' are vector groups. Then the homomorphism f induces an isogeny $T \times U \rightarrow G/G_{\text{ant}}$. Thus, f is an isogeny, and T, U are unique up to isogeny; hence the vector group U is uniquely determined.

In positive characteristics, $G_{\text{aff}} \cap G_{\text{ant}}$ contains the torus $(G_{\text{ant}})_{\text{aff}}$ and the quotient is finite; hence ψ_u is an isogeny. Thus, our statement holds with $U = U'$, but for no other choice of U . \square

The assumption that k is perfect cannot be omitted in the preceding result, as shown by Example 3.2.

3.3. Further decompositions in positive characteristics

In this subsection, we combine the Rosenlicht decomposition with the particularly simple structure of anti-affine algebraic groups in positive characteristics, to obtain information on general algebraic groups.

We begin with the case where the field k is finite. Then Propositions 2.2 and 3.1 immediately imply the following result, due to Arima in the setting of algebraic groups (see [Ar60, Thm. 1] and also [Ro61, Thm. 4]):

Proposition 3.5. *Let G be a connected group scheme over a finite field k . Then $G = G_{\text{aff}}G_{\text{ab}}$ where G_{ab} denotes the largest abelian subvariety of G . Moreover, $G_{\text{aff}} \cap G_{\text{ab}}$ is finite.*

In particular, the Albanese morphism of (G, e_G) is trivialized by the finite cover $G_{\text{aff}} \times G_{\text{ab}} \rightarrow G$ (possibly non-étale).

Returning to a possibly infinite field k , we record the following preliminary result:

Lemma 3.6. *Let G be a connected algebraic group over a perfect field k . Then:*

- (i) *There exists a smallest normal connected algebraic group $H \subset G$ such that G/H is a semi-abelian variety. The quotient homomorphism $G \rightarrow G/H$ is the generalized Albanese morphism of the pointed variety (G, e_G) .*
- (ii) *We have*

$$H = R_u(G_{\text{aff}})[G, G] = R_u(G_{\text{aff}})[G_{\text{aff}}, G_{\text{aff}}] \quad (3.6)$$

where $R_u(G_{\text{aff}})$ denotes the unipotent radical of G_{aff} , and $[G, G]$ the derived group.

- (iii) *The formation of H commutes with perfect field extensions.*
- (iv) *The group $H_{\bar{k}}$ is generated by all connected unipotent subgroups of $G_{\bar{k}}$.*

Proof. By the Rosenlicht decomposition, we have $[G, G] = [G_{\text{aff}}, G_{\text{aff}}]$. Define H by the equality (3.6); then H is a connected normal subgroup of G . Moreover, the quotient G_{aff}/H is a connected commutative reductive group, i.e., a torus. Thus, G/H is a semi-abelian variety.

Consider a morphism $f: G \rightarrow S$, where S is a semi-abelian variety, and $f(e_S) = e_G$. Then f is a homomorphism by [Ro61, Thm. 3]. Hence f factors through $G/R_u(G_{\text{aff}})$ (as every unipotent subgroup of S is trivial) and also through $G/[G, G]$ (as S is commutative). Thus, f factors through G/H . This proves (i) and (ii), while (iii) and (iv) are obtained by similar arguments. \square

Under the assumptions of the preceding lemma, we say that H is *geometrically unipotently generated*, and write $H := G_{\text{gug}}$.

Also, given a group scheme G , a normal subgroup scheme $H \subset G$ and a subgroup scheme $S \subset G$, we say that S is a *quasi-complement* to H in G if $G = HS$ and $H \cap S$ is finite; equivalently, the natural map $S \rightarrow G/H$ is an isogeny. We may now state our structure result:

Theorem 3.7. *Let G be a connected algebraic group over a perfect field k of positive characteristic and let T be a maximal torus of the radical $R(G_{\text{aff}})$. Then:*

- (i) *T is a quasi-complement to G_{gug} in G_{aff} .*
- (ii) *$S := TG_{\text{ant}}$ is a quasi-complement to G_{gug} in G , and is a semi-abelian subvariety of G with maximal torus T .*
- (iii) *The generalized Albanese morphism of (G, e_G) is trivialized by the finite cover $G_{\text{gug}} \times S \rightarrow G$.*

Proof. (i) By the structure of affine algebraic groups (see [Bo91]) and the equality (3.6), we have

$$\begin{aligned} G_{\text{aff}} &= R(G_{\text{aff}})[G_{\text{aff}}, G_{\text{aff}}] = R_u(G_{\text{aff}})T[G_{\text{aff}}, G_{\text{aff}}] \\ &= R_u(G_{\text{aff}})[G_{\text{aff}}, G_{\text{aff}}]T = G_{\text{gug}}T. \end{aligned}$$

We now show the finiteness of $G_{\text{gug}} \cap T$. For this, we may assume that G is affine. Since the homomorphism

$$G_{\text{gug}} \cap T \rightarrow (G_{\text{gug}} \cap R(G))/R_u(G) \subset G/R_u(G)$$

is finite, we may also assume G to be reductive. Then $G_{\text{gug}} = [G, G]$ is semi-simple and T is the largest central torus, so that their intersection is indeed finite.

(ii) By the Rosenlicht decomposition and (i), $G = G_{\text{gug}}S$. Also, the quotient $(G_{\text{gug}} \cap S)/(G_{\text{gug}} \cap T)$ is finite, since $G_{\text{gug}} \cap S$ is affine and $T = S_{\text{aff}}$. Thus, $G_{\text{gug}} \cap S$ is finite, i.e., S is a quasi-complement to G_{gug} in G .

We know that G_{ant} is a semi-abelian variety contained in the center of G . Thus, S is a semi-abelian variety as well. Moreover, the maximal torus $(G_{\text{ant}})_{\text{aff}}$ of G_{ant} is a central subtorus of G_{aff} , and hence is contained in T . Thus, T is the maximal torus of S .

(iii) follows readily from (ii). \square

Remarks 3.8. (i) The quasi-complements constructed in the preceding theorem are all conjugate under $R_U(G_{\text{aff}})$. But G_{gug} may admit other quasi-complements in G ; namely, all subgroups $T'G_{\text{ant}}$ where T' is a quasi-complement of G_{gug} in G_{aff} . Such a subtorus T' need not be contained in $R(G_{\text{aff}})$, e.g., when G_{aff} is reductive and non-commutative.

(ii) With the notation and assumptions of the preceding theorem, G_{ant} also admits quasi-complements in G , namely, the subgroups $T'G_{\text{gug}}$ where T' is a quasi-complement to $(G_{\text{ant}})_{\text{aff}}$ in T .

In contrast, G_{aff} may admit no quasi-complement in G . Indeed, such a quasi-complement S comes with a finite surjective morphism to G/G_{aff} , and hence is an abelian variety. Thus, S exists if and only if G_{ant} is an abelian variety, and then $S = G_{\text{ant}}$. Equivalently, $G = G_{\text{aff}}G_{\text{ab}}$ as in Proposition 3.5.

(iii) In characteristic 0, the group G_{gug} still admits quasi-complements in G_{aff} , but may admit no quasi-complement in G .

For example, let C be an elliptic curve, $E(C)$ its universal extension, H the Heisenberg group of upper triangular 3×3 matrices with diagonal entries 1, and $G = (H \times E(C))/\mathbb{G}_a$ where the additive group \mathbb{G}_a is embedded in H as the center, and in $E(C)$ as $E(C)_{\text{aff}}$. Then G is a connected algebraic group; moreover, $G_{\text{gug}} = G_{\text{aff}} \cong H$ and $G_{\text{ant}} \cong E(C)$. Since G_{ant} is non-complete, there exist no quasi-complement to G_{aff} in G .

The same example shows that G_{ant} may admit no quasi-complement in G . Yet such a quasi-complement does exist when G is commutative, by Theorem 3.4.

3.4. Counterexamples to Hilbert's fourteenth problem

In this subsection, we construct a class of smooth quasi-affine varieties having a non-noetherian coordinate ring.

Recall that every connected algebraic group G is quasi-projective, i.e., G admits an ample invertible sheaf \mathcal{L} (see e.g. [Ra70, Cor. V 3.14]). Clearly, the associated \mathbb{G}_m -torsor over G (that is, the complement of the zero section in the total space of the associated line bundle $\mathbb{V}(\mathcal{L})$) is a smooth quasi-affine variety. This simple construction yields our examples:

Theorem 3.9. *Let $\pi : X \rightarrow G$ denote the \mathbb{G}_m -torsor associated to an ample invertible sheaf \mathcal{L} on a non-complete anti-affine algebraic group. Then the ring $\mathcal{O}(X)$ is not noetherian.*

Proof. As $X = \text{Spec}_{\mathcal{O}_G}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n)$, we have $\mathcal{O}(X) = \bigoplus_{n \in \mathbb{Z}} H^0(G, \mathcal{L}^n)$. Moreover, $H^0(G, \mathcal{O}_G) = k$ by assumption, and the k -vector space $H^0(G, \mathcal{L}^n)$ is infinite-dimensional for any $n > 0$ by the next lemma. Since $\mathcal{O}(X)$ is a domain, it follows that $H^0(G, \mathcal{L}^n) = 0$ for any $n < 0$, i.e., the algebra $\mathcal{O}(X)$ is positively graded. Clearly, this algebra is not finitely generated, and hence non-noetherian by the graded version of Nakayama's lemma. \square

Lemma 3.10. *Let \mathcal{L} be an ample invertible sheaf on an anti-affine algebraic group G . If G is non-complete, then the k -vector space $H^0(G, \mathcal{L})$ is infinite-dimensional.*

Proof. We may assume that k is algebraically closed. The quotient homomorphism $\alpha = \alpha_G : G \rightarrow A(G) =: A$ is a torsor under the connected commutative affine algebraic group G_{aff} . Since the Picard group of G_{aff} is trivial, it follows that $\mathcal{L} = \alpha^*(\mathcal{M})$ for some invertible sheaf \mathcal{M} on A . Moreover, \mathcal{M} is ample by the ampleness of \mathcal{L} together with [Ra70, Lem. XI 1.11.1]. We have

$$H^0(G, \mathcal{L}) \simeq H^0(A, \mathcal{M} \otimes \alpha_*(\mathcal{O}_G)). \quad (3.7)$$

In the case where G is a semi-abelian variety, Eqs. (2.2) and (3.7) yield the decomposition

$$H^0(G, \mathcal{L}) \simeq \bigoplus_{\lambda \in \Lambda} H^0(A, \mathcal{M} \otimes \mathcal{L}_\lambda).$$

As each \mathcal{L}_λ is algebraically trivial, $\mathcal{M} \otimes \mathcal{L}_\lambda$ is ample, and hence admits non-zero global sections (see [Mu70, p. 163]); this yields our statement in this case.

In the general case, we may assume in view of Proposition 2.2 and the isomorphism (2.8) that k has characteristic 0, and G_{aff} is a non-zero vector space U . Then $\mathcal{M} \otimes \alpha_*(\mathcal{O}_G)$ admits an infinite increasing filtration with subquotients isomorphic to \mathcal{M} , by Lemma 2.6. Since $H^1(A, \mathcal{M}) = 0$ (see [Mu70, p. 150]), it follows that $H^0(G, \mathcal{L})$ admits an infinite increasing filtration with subquotients isomorphic to $H^0(A, \mathcal{M})$, a non-zero vector space. \square

Example 3.11. The smallest examples arising from the preceding construction are threefolds; they may be described as follows.

Consider an invertible sheaf \mathcal{L} of positive degree on an elliptic curve C . If k has characteristic 0, let $\pi : G \rightarrow C$ denote the \mathbb{G}_a -torsor associated to the canonical generator of $H^1(C, \mathcal{O}_C) \simeq H^0(C, \mathcal{O}_C)^*$. Then G is the universal extension $E(C)$, and the \mathbb{G}_m -torsor on G associated to the ample invertible sheaf $\pi^*(\mathcal{L})$ yields the desired example X .

When $k = \mathbb{C}$, the analytic manifolds associated to G and X are both Stein; see [Ne88] which also contains an analytic proof of the fact that $\mathcal{O}(X)$ is not finitely generated. More generally, the universal extension $E(A)$ of a complex abelian variety of dimension g is analytically isomorphic to $(\mathbb{C}^*)^{2g}$, see e.g. [Ne88, Rem. 7.7]. In particular, the complex manifold associated to $E(A)$ is Stein.

Returning to a field k of arbitrary characteristics, assume that the elliptic curve C has a k -rational point x of infinite order (such curves exist if k contains either \mathbb{Q} or $\mathbb{F}_p(t)$, see [ST67]). Denote by \mathcal{M} the invertible sheaf on C associated to the divisor $(x) - (0)$. Then \mathcal{M} is algebraically trivial and has infinite order. Thus, $G := \text{Spec}_{\mathcal{O}_C}(\bigoplus_{n \in \mathbb{Z}} \mathcal{M}^n)$ is an anti-affine semi-abelian variety, and

$$X := \text{Spec}_{\mathcal{O}_C} \left(\bigoplus_{(m,n) \in \mathbb{Z}^2} \mathcal{L}^m \otimes_{\mathcal{O}_A} \mathcal{M}^n \right)$$

is the desired example.

It should be noted that $\mathcal{O}(X)$ is finitely generated for any smooth surface X , as shown by Zariski (see [Za54]). Also, Kuroda has constructed counterexamples to Hilbert's original problem, in dimension 3 and characteristic 0 (see [Ku05]).

Another consequence of Lemma 3.10 is the following:

Proposition 3.12. *For any completion \bar{G} of a connected algebraic group G , the boundary $\bar{G} \setminus G$ is either empty or of codimension 1.*

Proof. We argue by contradiction, and assume that $\bar{G} \setminus G$ is non-empty of codimension ≥ 2 . We may further assume that \bar{G} is normal; then the map $i^* : \mathcal{O}(\bar{G}) \rightarrow \mathcal{O}(G)$ is an isomorphism, where $i : G \rightarrow \bar{G}$ denotes the inclusion. It follows that G is anti-affine and non-complete.

Choose an ample invertible sheaf \mathcal{L} on G . Then $i_*(\mathcal{L})$ is the sheaf of sections of some Weil divisor on \bar{G} ; in particular, this sheaf is coherent. Thus, the k -vector space $H^0(\bar{G}, i_*(\mathcal{L})) = H^0(G, \mathcal{L})$ is finite-dimensional, contradicting Lemma 3.10. \square

Remark 3.13. With the assumptions of the preceding proposition, one may show (by completely different methods) that the boundary has pure codimension 1. For a G -equivariant completion \bar{G} (that is,

the action of G on itself by left multiplication extends to \overline{G} , this follows easily from [Br07, Thm. 3]. Namely, we may assume that k is algebraically closed and \overline{G} is normal; then $\overline{G} \simeq G \times^{G_{\text{aff}}} \overline{G}_{\text{aff}}$, and $\overline{G}_{\text{aff}} \setminus G_{\text{aff}}$ has pure codimension 1 in $\overline{G}_{\text{aff}}$, as G_{aff} is affine.

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After a first version of this text was posted on arXiv, I was informed by Carlos Sancho de Salas of his much earlier book [Sa01], where the classification of anti-affine groups over algebraically closed fields is obtained. Subsequently, he extended this classification to arbitrary fields in [SS08], jointly with Fernando Sancho de Salas. The approach of [Sa01, SS08] differs from the present one, their key ingredient being the classification of certain torsors over anti-affine varieties. The terminology is also different: the variedades cuasi-abelianas of [Sa01], or quasi-abelian varieties of [SS08], are called anti-affine groups here. I warmly thank Carlos Sancho de Salas for making me aware of his work, and for many interesting exchanges.

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