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The set of fixed points of a unipotent group

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ABSTRACT

Let K be an algebraically closed field. Let G be a non-trivial connected unipotent group, which acts effectively on an affine variety X. Then every non-empty component R of the set of fixed points of G is a K-uniruled variety, i.e., there exist an affine cylinder $W \times K$ and a dominant, generically-finite polynomial mapping $\phi : W \times K \to R$. We show also that if an arbitrary infinite algebraic group G acts effectively on K^n and the set of fixed points contains a hypersurface H, then this hypersurface is K-uniruled.

1. Introduction

Let K be an algebraically closed field (of arbitrary characteristic). Let G be a connected unipotent algebraic group, which acts effectively on a variety X. The set of fixed points of this action was studied intensively (see, e.g., [1–4]). In particular Białynicki-Birula has proved that if X is an affine variety, then G has no isolated fixed points.

Here we consider the case when X is an affine variety. We generalize the result of Białynicki-Birula and we prove, that the set $Fix(G)$ of fixed points of G is in fact a K-uniruled variety. In particular for every point $x \in Fix(G)$ there is a polynomial curve $\phi : K \to Fix(G)$ such that $\phi(0) = x$.

We show also that if an arbitrary infinite algebraic group G acts effectively on K^n and the set of fixed points contains a hypersurface H, then this hypersurface is K-uniruled. This generalizes [6].

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2. Preliminaries

At the beginning we recall some basic facts about K-uniruled varieties (see [7]).

Proposition 2.1. *Let Γ be an affine curve. The following two statements are equivalent*:

(1) *there exists a regular bi-rational map $\phi : K \to \Gamma$*;
(2) *there exists a regular dominant map $\phi : K \to \Gamma$.*

Definition 2.2. Let Γ be an affine curve which has the property (1) (or (2)) from the above proposition. Then Γ will be called *an affine polynomial curve* and the mapping ϕ will be called *a parametrization of* Γ. A family $\mathcal{F}$ of affine polynomial curves on X is called *bounded* if there exist an embedding $i : X \subset K^N$ and a natural number D such that every curve $\Gamma \in \mathcal{F}$ has degree less then or equal to D in K^N.

Remark 2.3. It is easy to see that the definition of bounded family does not depend on an embedding $i : X \to K^N$.

Now we give the definition of a *K-uniruled* variety. We have introduced this notion for uncountable fields in [7]. However, here we work over any field and we need a refined version of the definition (it coincides with the older one for uncountable fields).

Proposition 2.4. *Let $X \subset K^N$ be an irreducible affine variety of dimension $\geqslant 1$. The following conditions are equivalent*:

(1) *there is a bounded family $\mathcal{F}$ of affine polynomial curves, such that for every point $x \in X$ there is a curve $l_x \in \mathcal{F}$ going through x*;
(2) *there is an open, non-empty subset $U \subset X$ and a bounded family $\mathcal{F}$ of affine polynomial curves, such that for every point $x \in U$ there is a curve $l_x \in \mathcal{F}$ going through x*;
(3) *there exists an affine variety W with* $\dim W = \dim X - 1$ *and a dominant polynomial mapping $\phi : W \times K \to X$.*

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) follows from [10]. (3) $\Rightarrow$ (2) is obvious. We prove (2) $\Rightarrow$ (1). Assume that $X \subset K^n$. Every curve $l_x \in \mathcal{F}$ is given by n polynomials of one variable:

$$l_x(t) = \left(x_1 + \sum_{i=1}^{D} a^{1,i} t^i, \ldots, x_n + \sum_{i=1}^{D} a^{n,i} t^i\right).$$

Let Δ denote an $nD-1$-dimensional weighted projective space with weights $1, 2, \ldots, D, \ldots, 1, 2, \ldots, D$. Hence we can associate l_x with one point

$$\left(x_1, \ldots, x_n; a^{1,1}, \ldots, a^{1,D}; a^{2,1}, \ldots, a^{2,D}; \ldots; a^{n,1}, \ldots, a^{n,D}\right) \in X \times \Delta.$$

Let $\{f_i = 0,\ i = 1, \ldots, m\}$ ($f_i \in K[x_1, \ldots, x_n]$) be equations of the variety X. The condition $l_x \subset X$ is equivalent to conditions $f_i(l_x(t)) = 0$, $i = 1, \ldots, m$. The last equations are in fact equivalent to a finite number of polynomial equations

$$h_\alpha\left(x_1, \ldots, x_n; a^{1,1}, \ldots, a^{1,D}; a^{2,1}, \ldots, a^{2,D}; \ldots; a^{n,1}, \ldots, a^{n,D}\right) = 0,$$

which are weighted homogeneous with respect to $a^{1,1}, \ldots, a^{1,D}; a^{2,1}, \ldots, a^{2,D}; \ldots; a^{n,1}, \ldots, a^{n,D}$. Let $W \subset X \times \Delta$ be a variety described by polynomials h_α and let $\pi : X \times \Delta \to X$ be the projection. The mapping π is proper, in particular the set $\pi(W)$ is closed. Since $\pi(W)$ contains the dense subset U, we have $\pi(W) = X$. □

Now we can state:

Definition 2.5. An affine irreducible variety X is called *K-uniruled* if it is of dimension $\geqslant 1$, and satisfies one of equivalent conditions (1)–(3) listed in Proposition 2.4.

If the field K is uncountable we have stronger result (see [10]):

Proposition 2.6. *Let K be an uncountable field. Let X be an irreducible affine variety of dimension $\geqslant 1$. The following conditions are equivalent*:

(1) *X is K-uniruled*;
(2) *for every point $x \in X$ there is a polynomial affine curve in X going through x*;
(3) *there exists a Zariski-open, non-empty subset U of X, such that for every point $x \in U$ there is a polynomial affine curve in X going through x*;
(4) *there exists an affine variety W with* $\dim W = \dim X - 1$ *and a dominant polynomial mapping $\phi : W \times K \to X$.*

Let X be a smooth projective surface and let $D = \sum_{i=1}^{n} D_i$ be a simple normal crossing (s.n.c.) divisor on X (here we consider only reduced divisors). Let *graph*(D) be a graph of D, i.e., a graph with one vertex Q_i for each irreducible component D_i of D, and one edge between Q_i and Q_j for each point of intersection of D_i and D_j.

Definition 2.7. Let D be a simple normal crossing divisor on a smooth surface X. We say that D is a tree if *graph*(D) is connected and acyclic.

We have the following fact which is obvious from graph theory:

Proposition 2.8. *Let X be a smooth projective surface and let divisor $D \subset X$ be a tree. Assume that $D', D'' \subset D$ are connected divisors without common components. Then D' and D'' have at most one common point.*

Definition 2.9. Let X, Y be affine varieties and $f : X \to Y$ be a regular mapping. We say that f *is finite* at a point $y \in Y$ if there exists an open neighborhood U of y such that $\mathrm{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is a finite map.

It is well known that if f is generically finite, then the set of points at which f is not finite is either empty or it is a hypersurface in $\overline{f(X)}$ (for details see [7,8]). We denote this set by S_f. Now we can formulate the following useful:

Theorem 2.10. *Let Γ be an affine curve. Let $\phi : \Gamma \times K \to K^N$ be a generically-finite mapping. Then the set S_ϕ is a union of finitely many (possibly empty) of affine polynomial curves.*

Proof. Taking a normalization we can assume that the curve Γ is smooth (note that a normalization is a finite mapping). Let $\overline{\Gamma}$ be a smooth completion of Γ and denote $\overline{\Gamma} \setminus \Gamma = \{a_1, \dots, a_l\}$. Let $X = \Gamma \times K$ and $\overline{X} = \overline{\Gamma} \times \mathbb{P}^1$ be a projective closure of X. The divisor $D = \overline{X} \setminus X = \overline{\Gamma} \times \infty + \sum_{i=1}^{l} \{a_i\} \times \mathbb{P}^1$ is a tree. Now we can resolve points of indeterminacy of the mapping ϕ:

$$\begin{tikzcd}
\overline{X}_m \arrow[d, "\pi_m"] \arrow[dddrr, "\phi'"] \arrow[ddd, phantom, bend right=70, "\pi"'] \\
\overline{X}_{m-1} \arrow[d, "\pi_{m-1}"] \\
\vdots \arrow[d, "\pi_1"] \\
\overline{X} \arrow[rr, dashed, "\phi"] & & \mathbb{P}^N
\end{tikzcd}$$

Note that the divisor $D' = \pi^*(D)$ is a tree. Let $\overline{\Gamma} \times \infty'$ denote a proper transform of $\overline{\Gamma} \times \infty$. It is an easy observation that $\phi'(\overline{\Gamma} \times \infty') \subset H_\infty$, where H_∞ denotes the hyperplane at infinity of $\mathbb{P}^N$. Now $S_\phi = \phi'(D' \setminus \phi'^{-1}(H_\infty))$. The curve $L = \phi'^{-1}(H_\infty)$ is a complement of a semi-affine variety $\phi'^{-1}(K^N)$ hence it is connected (for details see [7, Lemma 4.5]). Now by Proposition 2.8 we have that every irreducible curve $Z \subset D'$ which does not belong to L has at most one common point with L. Let $S \subset S_\phi$ be an irreducible component. Hence S is a curve. There is a curve $Z \subset D'$, which has exactly one common point with L such that $S = \phi'(Z \setminus L) = \phi'(K)$. This completes the proof. $\square$

3. Main result

The aim of this section is to prove the following:

Theorem 3.1. *Let G be a non-trivial connected unipotent group which acts effectively on an affine variety X. Then every non-empty component R of the set of fixed points of G is a K-uniruled variety.*

Proof. First of all let us recall that a connected unipotent group has a normal series

$$0 = G_0 \subset G_1 \subset \cdots \subset G_r = G,$$

where $G_i/G_{i-1} \cong G_a = (K, +, 0)$. By induction on $\dim G$ we can easily reduce the general case to that of $G = G_a$.

First assume that the field K is uncountable. Take a point $a \in R$. By Proposition 2.6 it is enough to prove, that there is an affine polynomial curve $S \subset \mathit{Fix}(G)$ through a. Let L be an irreducible curve in X going through a, which is not contained in any orbit of G and it is not contained in $\mathit{Fix}(G)$. Consider a surface $Y = L \times G$. There is natural G action on Y: for $h \in G$ and $y = (l, g) \in Y$ we put $h(y) = (l, hg) \in Y$. Take a mapping

$$\Phi : L \times G \ni (x, g) \to g(x) \in X.$$

It is a generically-finite polynomial mapping. Observe that it is G-invariant, i.e., $\Phi(gy) = g\Phi(y)$. This implies that the set S_Φ of points at which the mapping Φ is not finite is G-invariant. Indeed, it is enough to show that the complement of this set is G-invariant. Let Φ be finite at $x \in X$. This means that there is an open neighborhood U of x such that the mapping $\Phi : \Phi^{-1}(U) \to U$ is finite. Now we have the following diagram:

$$\begin{tikzcd}
\Phi^{-1}(U) \arrow[r, "g"] \arrow[d, "\Phi"] & \Phi^{-1}(gU) = g\Phi^{-1}(U) \arrow[d, "\Phi"] \\
U \arrow[r, "g"] & gU
\end{tikzcd}$$

This diagram shows that the mapping Φ is finite over gU if it is finite over U. In particular this implies that the set S_Φ is G-invariant. Let $S_\Phi = S_1 \cup S_2 \cup \cdots \cup S_k$ be a decomposition of S_Φ in (irreducible) affine polynomial curves (see Theorem 2.10). Since the set S_Φ is G-invariant, we have that each curve S_i is also G-invariant. Note that the point a belongs to S_Φ, because the fiber over a has infinite number of points. We can assume that $a \in S_1$. Let $x \in S_1$, we want to show that $x \in Fix(G)$. Indeed, otherwise $G.x = S_1$ and a would be in the orbit of x—a contradiction. Hence $S_1 \subset Fix(G)$ and we conclude our result by Theorem 2.10.

Now assume that the field K is countable. Let $X \subset K^n$. Let T be uncountable algebraically closed extension of K. By the base change the group G acts on $\overline{X} \subset T^n$. Moreover, the variety $\overline{R} \subset T^n$ is a component of the set of fixed points of G (because the set R is dense in $\overline{R}$). By the first part of our proof the variety $\overline{R}$ is T-uniruled. In particular there exists a number D such that for every point $x \in \overline{R}$ there is a polynomial affine curve $l_x \subset \overline{R} \subset T^n$, of degree at most D, going through x. Note that it is true for every point $x \in R$.

Every such curve l_x is given by n polynomials of one variable:

$$l_x(t) = \left(x_1 + \sum_{i=1}^{d} a^{1,i} t^i, \dots, x_n + \sum_{i=1}^{d} a^{n,i} t^i\right),$$

where $d \leqslant D$. Hence we can associate l with one point

$$\left(a^{1,0}, a^{1,1}, \dots, a^{1,d}; a^{2,0}, \dots, a^{2,d}; \dots; a^{n,0}, \dots, a^{n,d}\right) \in T^n.$$

We can assume without loss of generality that $a^{1,d} = 1$. Let $\{f_i = 0,\ i = 1, \dots, m\}$ ($f_i \in K[x_1, \dots, x_n]$) be equations of the variety S. The condition $l_x \subset \overline{R}$ is equivalent to conditions $f_i(l(t)) = 0$, $i = 1, \dots, m$. The last equations are in fact equivalent to a finite number of polynomial equations

$$h_\alpha\left(a^{1,0}, a^{1,1}, \dots, a^{1,d}; a^{2,0}, \dots, a^{2,d}; \dots; a^{n,0}, \dots, a^{n,d}\right) = 0,$$

where $h_\alpha \in K[y_1, \dots, y_N]$. Equations $h_\alpha = 0$ plus extra conditions $a^{i,0} = x_i$, $i = 1, \dots, n$, and $a^{1,d} = 1$ have solutions in the field T, hence they have also solutions in the field K.

This means that we can find an affine polynomial curve l_x over the field K of degree at most D, which is contained in R and goes through x. Consequently the variety R is K-uniruled. The proof of Theorem 3.1 is complete. □

Corollary 3.2. *(See Białynicki-Birula, [1].) Let G be a non-trivial connected unipotent group which acts effectively on an affine variety X. Then G has no isolated fixed points.*

Theorem 3.1 (or rather its proof) suggests the following generalization of [6]:

Theorem 3.3. *Let G be an infinite algebraic group which acts effectively on K^n, $n \geqslant 2$. Assume that an irreducible hypersurface W is contained in the set of fixed points of G. Then W is K-uniruled.*

Proof. Since G acts effectively on affine space K^n we can assume by the Chevalley Theorem (see [9, Theorem C, p. 190]) that the group G is affine. In particular it contains either the subgroup $G_m = (K^*, \cdot, 1)$ or the subgroup $G_a = (K, +, 0)$ (see, e.g., [5]). Thus we can assume that G is either G_m or it is G_a.

As before we can assume that the field K is uncountable. Take a point $a \in W$. By Proposition 2.6 it is enough to prove, that there is an affine parametric curve $S \subset W$ through a. Let L be a line in K^n going through a such that the set $L \cap Fix(G)$ is finite. Set $L \cap W = \{a, a_1, \ldots, a_m\}$. Now consider a mapping

$$\phi : L \times G \ni (x, g) \to g(x) \in K^n.$$

Observe that $\phi(L \times G)$ is a union of disjoint orbits of G. This implies $\phi(L \times G) \cap W = \{a, a_1, \ldots, a_m\}$. Take $X = \overline{\phi(L \times G)}$. Note that $X \cap W$ is a union of curves. This means that there is a curve $S \subset X \cap W$, which contains the point a. However $S \subset \overline{X \setminus \phi(L \times G)}$. This implies that $S \subset S_\phi$ and we conclude by Theorem 2.10. □

To finish this note we state:

Conjecture. *Let K be an algebraically closed field. Let G be an algebraic group, which acts effectively on K^n. If S is an irreducible component of the set of fixed points of G, then S is either a point or it is a K-uniruled variety.*

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