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Ekedahl–Oort strata and Kottwitz–Rapoport strata

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ABSTRACT

We study Ekedahl–Oort strata on the moduli space \mathcal{A}_g of g -dimensional principally polarized abelian varieties in positive characteristic, and Kottwitz–Rapoport strata on its variants \mathcal{A}_J with parahoric level structure. First, we show that every Ekedahl–Oort stratum is isomorphic to a parahoric Kottwitz–Rapoport stratum. Second, both supersingular Ekedahl–Oort strata and supersingular Kottwitz–Rapoport strata are isomorphic to disjoint unions of Deligne–Lusztig varieties (see Hoeve (2010) [10] and Görtz and Yu (2010) [5], resp.), and here we compare these isomorphisms. Finally we give an explicit description of Kottwitz–Rapoport strata contained in the supersingular locus in the general parahoric case.

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1. Introduction

The moduli space \mathcal{A}_g of principally polarized abelian varieties is a central object in algebraic geometry, both classically and nowadays. To obtain a better understanding of its arithmetic properties, it is useful to consider this moduli space over fields of positive characteristic p . Then there are two interesting stratifications. The oldest is the *Newton stratification* which classifies the isogeny type of the underlying p -divisible group. The importance of the Newton stratification shows for instance (for a different kind of moduli space) in the work of Harris and Taylor about the local Langlands correspondence [8] and the subsequent refinements by Boyer [1]. Probably the most interesting Newton stratum is the *supersingular locus*, the closed subset of all supersingular abelian varieties inside the moduli space. The second stratification is the *Ekedahl–Oort stratification* which classifies the isomorphism class of the p -kernel.

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It is also interesting to consider variants of \mathcal{A}_g with deeper level structure at p . As a consequence, one has to consider singular spaces, too. So far, the only case where the geometric structure seems tractable is the case of parahoric level structure. This amounts to considering spaces of chains of p -power isogenies of abelian varieties (see Section 2.1 below for the precise definition). One can hope that certain facts which a priori involve even deeper level structure can be reduced to the parahoric case using base change methods; for instance, compare the method used by Taylor and Yoshida, [16]. For moduli spaces \mathcal{A}_J with a parahoric level structure of type J , it is natural to consider a third stratification, the *Kottwitz–Rapoport stratification*.

In the recent papers [5] and [6] C.-F. Yu and the first named author explicitly described supersingular Kottwitz–Rapoport strata (those completely contained in the supersingular locus) in the Iwahori case in terms of Deligne–Lusztig varieties. In [10] the second named author did the same for supersingular Ekedahl–Oort strata, in terms of fine Deligne–Lusztig varieties (see Section 2 for a summary of these results). This led to the question how these descriptions are related and, more generally, how the Ekedahl–Oort stratification and Kottwitz–Rapoport stratification are related. In this paper we try to answer these questions. Our main result is:

Theorem 1.1.

1. Each EO-stratum is isomorphic to a certain KR-stratum in the parahoric moduli space \mathcal{A}_J with J the type of the canonical filtration (Theorem 2.8).
2. The descriptions of supersingular KR-strata in the Iwahori case in [5] and supersingular EO-strata in [10] in terms of Deligne–Lusztig varieties are naturally compatible with the projection $\mathcal{A}_{I, w\tau} \rightarrow \text{EO}_w$ (Theorem 3.10).
3. A parahoric KR stratum is contained in the supersingular locus S_J of \mathcal{A}_J if and only if it is superspecial (Theorem 4.5) and in that case it is isomorphic to a disjoint union of Deligne–Lusztig varieties (Theorem 4.3). This generalizes results in [5,6] from the Iwahori case to the general parahoric case.

As an immediate corollary to (3) we obtain that all connected components of a KR stratum contained in the supersingular locus of \mathcal{A}_J are isomorphic. This alone is not at all obvious, and we do not know of another method to prove this fact directly.

2. Ekedahl–Oort and Kottwitz–Rapoport strata

In this section we review the definitions of the Kottwitz–Rapoport stratification and the Ekedahl–Oort stratification, and show that each Ekedahl–Oort stratum is isomorphic to a parahoric Kottwitz–Rapoport stratum. We work with schemes over an algebraic closure k of the finite field \mathbb{F}_p for some fixed prime p .

2.1. The Kottwitz–Rapoport stratification

In this paper we study moduli spaces of abelian varieties with parahoric level structure. From the perspective of Shimura varieties the algebraic group underlying Siegel moduli spaces is the group GSp_{2g} of symplectic similitudes. We can identify the set of simple reflections in its affine Weyl group with $I = \{0, \dots, g\}$. The type of a parahoric subgroup is given by a subset $J = \{j_0 < j_1 < \dots < j_r\} \subset I$. The moduli space \mathcal{A}_J over k with parahoric level structure of type J at p parametrizes chains of g -dimensional polarized abelian varieties

$$(\underline{A}_{j_0} \xrightarrow{\alpha} \underline{A}_{j_1} \xrightarrow{\alpha} \dots \xrightarrow{\alpha} \underline{A}_{j_r}, \eta)$$

where

1. $\underline{A}_{j_v} = (A_{j_v}, \lambda_{j_v})$ with A_{j_v} a g -dimensional abelian variety and λ_{j_v} a polarization of degree $p^{2(g-j_v)}$ whose kernel is contained in $A_{j_v}[p]$,
2. α is an isogeny that pulls back $\lambda_{j_{v+1}}$ to λ_{j_v} ,

3. η is a symplectic level N -structure for $N \geq 3$ coprime to p , with respect to a fixed primitive N -th root of unity.

When $j_0 = 0$, we have $\ker(\lambda_0) = A_0[p]$. So $p^{-1}\lambda_0$ is a principal polarization. The only reason why we restrict to the special type of level structure as in (3) is that this makes the whole moduli space connected. Otherwise we would have to reformulate statements about certain strata being connected in a more technical way.

We can (and often do) extend the given chain to a periodic chain with index set $\{j + kg; j \in J, k \in \mathbb{Z}\}$ by inserting the dual abelian schemes and such that for every j , the map $A_j \rightarrow A_{2g+j}$ is multiplication by p .

The relative position of the chain of Hodge filtrations to the first de Rham cohomology is given by an element in the extended affine Weyl group, see [5, Section 2.4]. Let us describe this group for GSp_{2g} . The finite Weyl group W_g of GSp_{2g} can be realized as the group of permutations w of $\{1, \dots, 2g\}$ such that $w(2g+1-i) = 2g+1-w(i)$ for all i . It is generated by the simple reflections

$$s_g = (g, g+1) \quad \text{and} \quad s_i = (i, i+1)(2g-i, 2g+1-i) \quad \text{for } i = 1, \dots, g-1.$$

The cocharacter group of GSp_{2g} is

$$X_* = \{(u_1, \dots, u_{2g}) \in \mathbb{Z}^{2g} \mid u_1 + u_{2g} = u_2 + u_{2g-1} = \dots = u_g + u_{g+1}\}.$$

The Weyl group W_g acts on it by permuting the coordinates and we can realize the extended affine Weyl group \tilde{W} as the semi-direct product $X_* \rtimes W_g$. The affine Weyl group W_g^a is the subgroup of \tilde{W} generated by the simple affine reflections s_0, \dots, s_g with

$$s_0 = ((-1, 0, \dots, 0, 1), (1, 2g)) \in X_* \rtimes W_g$$

and s_1, \dots, s_g as above. Note that $i \mapsto s_i$ does indeed give a bijection from I to the set of simple affine reflections as mentioned above. For $J \subset I$ we let W_J be the subgroup of W_g^a generated by all s_i with i not in J .

Warning 2.1. This differs from convention; usually one defines W_J as the subgroup generated by s_j for $j \in J$.

Let μ be the dominant minuscule coweight $(1^{(g)}, 0^{(g)})$ in $X_* \subset \tilde{W}$. The admissible set $\mathrm{Adm}(\mu)$ consists of all elements in \tilde{W} less than some element in the W_g -orbit of t^μ with respect to the Bruhat order. Let $\mathrm{Adm}_J(\mu)$ be the image of $\mathrm{Adm}(\mu)$ under $\tilde{W} \rightarrow W_J \backslash \tilde{W} / W_J$. For $x \in \mathrm{Adm}(\mu)$, we denote by \bar{x} its image in $\mathrm{Adm}_J(\mu)$. The relative position of the Hodge flag to the first de Rham cohomology of a point of \mathcal{A}_J is given by an element in $\mathrm{Adm}_J(\mu)$. Grouping points by this relative position we get the *Kottwitz–Rapoport stratification* of \mathcal{A}_J .

$$\mathcal{A}_J = \coprod_{\bar{x} \in \mathrm{Adm}_J(\mu)} \mathcal{A}_{J, \bar{x}}.$$

This stratification was first considered by Genestier and Ngô [4], see also [5].

Let us make the definition explicit, for easy reference in the sequel. See also [6]. We start with a chain $A_{j_0} \rightarrow \dots \rightarrow A_{j_r}$ of abelian varieties over k that belongs to a k -valued point of \mathcal{A}_J . Since the above description of \tilde{W} implicitly builds on the standard embedding $\mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$, it is easiest to first extend the given chain by duality and using the given polarizations to a chain

$$A_{j_0} \rightarrow \dots \rightarrow A_{j_r} \rightarrow A_{2g-j_r} := A_{j_r}^\vee \rightarrow \dots \rightarrow A_{2g-j_0} := A_{j_0}^\vee,$$

i.e., we now have a chain with index set $\tilde{J} := J \cup \{2g-j; j \in J\}$.

This chain gives rise to a chain $(H_{\text{DR}}^1(A_j))_{j \in \tilde{J}}$ (of $2g$ -dimensional k -vector spaces). Inside each $H_{\text{DR}}^1(A_j)$ we have the Hodge filtration ω_j , a g -dimensional subspace. We can choose bases e_1^j, \dots, e_{2g}^j of the $H_{\text{DR}}^1(A_j)$ such that for $j' \leq j \in \tilde{J}$ the map $H_{\text{DR}}^1(A_j) \rightarrow H_{\text{DR}}^1(A_{j'})$ in the chain above with respect to these bases is described by the diagonal matrix $\text{diag}(1^{(2g-j)}, 0^{(j-j')}, 1^{(j')})$. Then the KR stratum of the above chain of abelian varieties is given by the unique element $\bar{x} \in \text{Adm}_J(\mu)$ such that for all $j \in \tilde{J}$,

$$\omega_j = \langle e_i^j; ((-1^j, 0^{2g-j}) - x \cdot (-1^j, 0^{2g-j}))(i) = 0 \rangle.$$

Here $x \in \tilde{W}$ is a lift of \bar{x} , $x \cdot$ denotes the obvious action of \tilde{W} on \mathbb{Z}^{2g} , and $v(i)$ denotes the i -th component of a vector $v \in \mathbb{Z}^{2g}$.

Equivalently, we can also obtain \bar{x} as the relative position of the chain of Dieudonné modules $M_j := M(A_j[p^\infty])$, $j \in \tilde{J}$, of the p -divisible groups $A_j[p^\infty]$ and the chain of VM_j , the images of Verschiebung. The reason is that by work of Oda, we can identify $H_{\text{DR}}^1(A) \cong M(A[p])^{(p)} = (M(A[p^\infty])/p)^{(p)}$, such that the Hodge filtration corresponds to the image of Verschiebung. (We use contravariant Dieudonné theory throughout this paper.) With this formulation, $x \in W_J \setminus \tilde{W}/W_J$ is the “usual relative position” $\text{inv}(M_\bullet, VM_\bullet)$ of lattice chains (which can be expressed in an analogous way as above, choosing a standard basis of the chain of Dieudonné modules).

Let τ be the unique element of length 0 in the admissible set $\text{Adm}(\mu)$, see [5, Section 3]. In terms of the above identification of \tilde{W} with $X_* \rtimes W_g$, we have

$$\tau = ((0^{(g)}, 1^{(g)}), (1, g+1)(2, g+2) \cdots (g, 2g)).$$

We can also consider τ as an element of $\text{GSp}_{2g}(\mathbb{Z}_p)$, namely as the matrix

$$\tau = \begin{pmatrix} 0 & I_g \\ pI_g & 0 \end{pmatrix},$$

where I_g denotes the $g \times g$ unit matrix. Then τ acts on the set of lattice chains, and its effect on the standard lattice chain is simply a shift by g steps. In particular, for any two lattice chains $\mathcal{L}_\bullet, \mathcal{L}'_\bullet$,

$$\text{inv}(\mathcal{L}_\bullet, (\mathcal{L}'_{i+g})_i) = \text{inv}(\mathcal{L}_\bullet, \mathcal{L}'_\bullet) \tau^{-1}. \quad (2.1.1)$$

Using the above notation, we have

Lemma 2.2. *Let $x \in \text{Adm}(\mu)$, and assume that $x\tau^{-1} \in W_g$. Assume that $0 \in J$. Then a point $(A_\bullet)_\bullet \in \mathcal{A}_J(k)$ lies in $A_{J, \bar{x}}$ if and only if $VM_g = M_{2g}$ and the relative position of the flags $M_j/M_{2g} \subset M_0/M_{2g}$ and $VM_{j+g}/M_{2g} \subset M_0/M_{2g}$ is equal to $x\tau^{-1}$.*

Note that $M_{2g} = pM_0$, so we have a σ -linear bijection between $H_{\text{DR}}^1(A_0)$ and M_0/M_{2g} . We omit the easy proof of the lemma.

We have the natural projection $\pi_{J,I} : \mathcal{A}_I \rightarrow \mathcal{A}_J$, and for each $x \in \text{Adm}(\mu)$,

$$\pi_{J,I}^{-1}(\mathcal{A}_{J, \bar{x}}) = \coprod_{v \in W_J x W_J \cap \text{Adm}(\mu)} \mathcal{A}_{I, v}. \quad (2.1.2)$$

We identify $\mathcal{A}_g = \mathcal{A}_{\{0\}}$, and write π for the projection $\mathcal{A}_I \rightarrow \mathcal{A}_g$.

We insert another lemma which completes the result of [5, Corollary 6.5] and whose statement was not given explicitly in the discussion in [5].

Lemma 2.3. *Let $c \leq g/2$, and let $J = \{c, g - c\}$. Then*

$$\pi_{J,I}(\mathcal{A}_{I,\tau}) = \mathcal{A}_{J,\bar{\tau}}.$$

Proof. Clearly the left-hand side is contained in the right-hand side. On the other hand, let $(A_c \rightarrow A_{g-c}) \in \mathcal{A}_{J,\bar{\tau}}(k)$. We have to show that we can extend this chain to a complete chain of abelian varieties which lies in $\mathcal{A}_{I,\tau}$. As before, we extend the chain to a chain with index set $J \cup \{g+c, 2g-c\}$. First, we will consider the chain of Dieudonné modules

$$M(A_{2g-c}) \rightarrow M(A_{g+c}) \rightarrow M(A_{g-c}) \rightarrow M(A_c)$$

and the images under Verschiebung.

The fact that the chosen point is in the $\bar{\tau}$ -stratum is equivalent to the property that

$$VM(A_{g-c}) = M(A_{2g-c}), \quad VM(A_c) = M(A_{g+c}).$$

In particular, this implies that $V^2M(A_c) = pM(A_c)$, and similarly for the other abelian varieties in the chain, so they are all superspecial. It is then easy to see that the chain can be completed to a complete chain of Dieudonné modules, and hence of abelian varieties $(A_i)_i$ with the property that for all i , $VM(A_i) = M(A_{g+i})$. This is exactly the property characterizing chains in $\mathcal{A}_{I,\tau}$. \square

2.2. The Ekedahl–Oort stratification

The space $\mathcal{A}_g = \mathcal{A}_{[0]}$ is just the moduli space of principally polarized abelian varieties. On this space the Kottwitz–Rapoport stratification consists of only one stratum. We get more information from another stratification, the Ekedahl–Oort stratification.

The Ekedahl–Oort stratification looks at the p -kernel of an abelian variety A over k , see [15] and [3]. We define the EO strata by requiring that two k -valued points A, B lie in the same stratum if and only if the p -torsion groups $A[p], B[p]$ together with the isomorphisms with their respective Cartier duals given by restricting the polarizations are isomorphic. There are several other ways to describe this stratification.

First one constructs the canonical filtration $\mathcal{F}_\bullet \subseteq M(A[p])$, the smallest chain of sub-Dieudonné modules that is stable under taking images by the Frobenius F and inverse images by the Verschiebung V . The relative position of the canonical filtration to the kernel of F determines an element in the set of final elements $W_{g,\text{final}} \subset W_g$, the minimal length representatives for the cosets in W_g/S_g , where S_g is the subgroup S_g of W_g generated by s_1, \dots, s_{g-1} (it is isomorphic to the symmetric group on g letters). This construction gives a bijection from the set of isomorphism classes of p -kernels to $W_{g,\text{final}}$, see [14, Theorem 4.7 and Theorem 5.5].

Instead of the relative position of the canonical filtration to the kernel of F , we can equivalently use the relative position of the canonical filtration to its conjugate filtration \mathcal{F}_\bullet^c with $\mathcal{F}_{g+i}^c = V^{-1}(\mathcal{F}_i^{(p)})$, $\mathcal{F}_{g-i}^c = (\mathcal{F}_{g+i}^c)^\perp$, $i \in J$ (see [3, 3.1]).

Yet another way to describe the isomorphism class of $A[p]$ is by its *canonical type* as defined by Oort [15]. Slightly adapting his terminology to ours, the canonical type is given by the type $J \subseteq \{0, \dots, g\}$ of the canonical filtration, together with the function v such that $V^{-1}(\mathcal{F}_i^{(p)}) = \mathcal{F}_{v(i)}$ for all $i \in J$.

To simplify giving references to these results, let us collect the different descriptions in the following lemma:

Lemma 2.4. *The isomorphism class of $(A[p], \lambda_{|A[p]})$ (as finite group schemes over k together with an isomorphism with the Cartier dual), for principally polarized abelian varieties $(A, \lambda) \in \mathcal{A}_g(k)$, is characterized by any of the following data:*

1. *The isomorphism class of $A[p]$ (as finite group schemes over k).*

2. The corresponding element in $W_g/S_g \cong W_{g,\text{final}}$.
3. The corresponding canonical type (v, J) , $J \subseteq \{0, \dots, g\}$ with $0 \in J$, $v: J \rightarrow J \cup \{2g - i; i \in J\}$.
4. The relative position of the canonical filtration and its conjugate filtration (an element in $W_J \setminus W_g/W_J$, where J is the type of the canonical flag).

Proof. For (1), which is only listed for reference and not used in this article, and for (2) see [14] Theorem 5.5 and, for $p = 2$, Remark 5.6. Note however that Moonen uses a different normalization; cf. [6, 2.4].

Characterization (3) is discussed in [15, (2.3), (5.7), (9.4)]. Finally, for (4) see [3, Proposition 4.5]. \square

Now let EO_w be the locally closed subset of \mathcal{A}_g consisting of all abelian varieties over k whose p -kernels is in the isomorphism class attached to $w \in W_{g,\text{final}}$. In this way we get the *Ekedahl–Oort stratification*

$$\mathcal{A}_g = \coprod_{w \in W_{g,\text{final}}} \text{EO}_w.$$

Example 2.5. On the EO-stratum of abelian varieties with a -number 1 and p -rank f , the type of the canonical filtration is $\{0, f, f+1, \dots, 2g-f-1, 2g-f, 2g\}$, see [3, Example 3.4]. On the stratum of abelian varieties with a -number a and p -rank $g-a$ the canonical type is $\{0, g-a, g, g+a, 2g\}$.

Example 2.6. In the case $g = 2$, we have four final elements, corresponding to the superspecial locus (id), the supersingular locus without the superspecial points (s_2), the p -rank 1 locus (s_1s_2) and the p -rank 2 locus ($s_2s_1s_2$). The corresponding canonical filtrations are given by

$$J = \{0, 2\}, \quad \{0, 1, 2\}, \quad \{0, 1, 2\}, \quad \{0, 2\},$$

respectively.

Remark 2.7. The definition of $W_{g,\text{final}}$ as the set of minimal length representatives of W_g/S_g shows that every finite element is less or equal than $w_\emptyset := (1, g+1)(2, g+2) \cdots (g, 2g)$, the finite part of τ .

2.3. EO strata as parahoric KR strata

Fix a final element $w \in W_{g,\text{final}}$. Let $J \subseteq I$ be the type of the corresponding canonical filtration. We have $0 \in J$, and, since the canonical filtration always contains a maximal totally isotropic subspace, we also have $g \in J$.

Theorem 2.8. Let $w \in W_{g,\text{final}}$. The natural map $\pi_J: \mathcal{A}_J \rightarrow \mathcal{A}_g$ restricts to an isomorphism $\mathcal{A}_{J, \overline{w\tau}} \rightarrow \text{EO}_w$. Its inverse maps a point $A \in \text{EO}_w$ to $(A, \mathcal{F}_\bullet^{\text{can}} \subset A[p])$, where $\mathcal{F}_\bullet^{\text{can}}$ denotes the canonical filtration of $A[p]$.

Proof. First note that the assumption $w \in W_{g,\text{final}}$ implies that $w\tau \in \text{Adm}(\mu)$; see Remark 2.7 and [6, Lemma 9.1].

Because over EO_w the canonical filtration can be constructed globally (see [15, Proposition (3.2)]) we have a section $s: \text{EO}_w \rightarrow \mathcal{A}_J$ of π_J .

We claim that its image is in $\mathcal{A}_{J, \overline{w\tau}}$. In fact, by one of the characterizations of the EO stratification given above, we know that for all k -valued points in the image of s the relative position of the flag $M_J/M_{2g} \subset M_0/M_{2g}$ to its conjugate flag VM_{J+g}/M_{2g} is w . Here we use the notation set up in Sections 2.1 and 2.2. But by Lemma 2.2 and Lemma 2.4 this condition ensures that the image of s is contained in $\mathcal{A}_{J, \overline{w\tau}}$.

To finish the proof, we show that for each point $(A_\bullet) \in \mathcal{A}_{J, \overline{w\tau}}(k)$, M_J/M_{2g} is the canonical filtration of $M_0/M_{2g} = M(A_0[p])$. Let (v, J) be the canonical type of w as in Section 2.2, in particular

Lemma 2.4. We know that for all points in the image of s we have $V(M_j/M_{2g}) = M_{v(j)}/M_{2g}$. But then this must hold for all points in $\mathcal{A}_{J, \overline{w\tau}}(k)$, since the relative position of $V(M_\bullet)$ and M_\bullet is constant on $\mathcal{A}_{J, \overline{w\tau}}(k)$. This implies that M_j/M_{2g} is stable under taking images of V and under duality and that it is the coarsest filtration with this property, hence that M_j/M_{2g} is the canonical filtration. \square

There is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{I, w\tau} & \xrightarrow{\pi_{J, I}} & \mathcal{A}_{J, \overline{w\tau}} \\ & \searrow \pi & \swarrow \cong \\ & \text{EO}_w & \end{array} \quad (2.3.3)$$

Since the image of each KR stratum is a union of EO strata [6, Proposition 3.3], the map π in the diagram is surjective. So for w final and J as above the horizontal map is surjective. Furthermore, because w is final, π in the diagram is finite (and étale), see [6, Corollary 9.7]. Since all KR strata in $\pi_{J, I}^{-1}(\mathcal{A}_{J, \overline{w\tau}})$ map (necessarily surjectively) to EO_w , they all have dimension at least $\dim \text{EO}_w = \ell(w)$, therefore $w\tau$ is the unique element of minimal length in $W_J w\tau W_J$. So we are in a very special situation.

3. Supersingular strata

We say that a Kottwitz–Rapoport or Ekedahl–Oort stratum is *supersingular* if it is completely contained in the supersingular locus. In [5] it was shown that supersingular Kottwitz–Rapoport can be described using Deligne–Lusztig varieties, and in [10] it was shown that supersingular Ekedahl–Oort strata can be described using fine Deligne–Lusztig varieties. We now compare these two descriptions.

3.1. Deligne–Lusztig varieties

Let G be a connected reductive group over k , defined over a finite field \mathbb{F}_q of characteristic p . Let $\sigma : G \rightarrow G$ be its q -power Frobenius. Let W be the Weyl group.

We denote with $\mathcal{B}(G)$ the variety of Borel subgroups in G . For any w in W we have a *Deligne–Lusztig variety* (see [2])

$$X_G(w, \sigma)(k) = \{B \in \mathcal{B}(G)(k) \mid \text{inv}(B, \sigma B) = w\}.$$

Here $\text{inv} : \mathcal{B}(G) \times \mathcal{B}(G) \rightarrow W$ is the relative position map. If we fix a Borel subgroup B , then $\mathcal{B}(G) \cong G/B$ and inv sends $(g_1 B, g_2 B) \in G/B \times G/B$ to the unique $w \in W$ such that $g_1^{-1} g_2 \in BwB$.

For parabolic subgroups we can define Deligne–Lusztig varieties in the same way. But in our situation it is more interesting to use refinement of parabolic subgroups, which leads to the definition of fine Deligne–Lusztig varieties (these were first studied by Lusztig [13], see also [10, Section 2.2]). Let S be the set of simple reflection in W . For $I \subset S$ let \mathcal{P}_I be the variety of parabolic subgroups of type I . Given a P in $\mathcal{P}_I(k)$, define a sequence of parabolic subgroups (P_n) by $P_0 := P$ and $P_n = (P_{n-1} \cap \sigma(P_{n-1}))R_u(P_{n-1})$, with R_u the unipotent radical. This sequence stabilizes to a parabolic subgroup P_∞ . The *fine Deligne–Lusztig variety* $X_G^I(w, \sigma)$ for $w \in {}^I W$ (the set of minimal length representatives of the cosets $\langle s_i; i \in I \rangle \backslash W$) consists of all P with $\text{inv}(P_\infty, \sigma(P_\infty)) = \overline{w}$. (The notation here is a little sloppy, because the type of P_∞ , and hence the double quotient of W in which \overline{w} lies, depends on w .)

3.2. Supersingular Kottwitz–Rapoport strata

We summarize the description of supersingular Kottwitz–Rapoport strata in [5] and [6].

Lemma 3.1. Suppose that $c \leq g/2$. The stratum $\mathcal{A}_{\{c, g-c\}, \bar{\tau}}$ consists of all chains $(\underline{A}_c \rightarrow \underline{A}_{g-c}, \eta)$ such that A_c and A_{g-c} are superspecial and the isogeny $A_c \rightarrow A_{g-c}^\vee$ is isomorphic to the Frobenius of A_c .

Proof. The projection $\pi_{\{c, g-c\}} : \mathcal{A}_I \rightarrow \mathcal{A}_{\{c, g-c\}}$ gives a surjective morphism $\mathcal{A}_{I, \tau} \rightarrow \mathcal{A}_{\{c, g-c\}, \bar{\tau}}$ by Lemma 2.3. Now the lemma follows from the description of $\mathcal{A}_{I, \tau}$ in Remark 3.6 in [5]. \square

Corollary 3.2. For a Kottwitz–Rapoport stratum $\mathcal{A}_{I, x}$ the following are equivalent:

1. $\pi_{\{c, g-c\}}(\mathcal{A}_{I, x}) \subset \mathcal{A}_{\{c, g-c\}, \bar{\tau}}$.
2. For each chain (\underline{A}_I, η) in $\mathcal{A}_{I, x}$ the abelian varieties A_c and A_{g-c} are superspecial and the isogeny $A_c \rightarrow A_{g-c}^\vee$ is isomorphic to the Frobenius of A_c .
3. x is in $W_{\{c, g-c\}} \tau$.

Proof. The equivalence of (1) and (2) follows from the lemma and the definition of $\pi_{\{c, g-c\}}$. Points (1) and (3) are equivalent by Eq. (2.1.2), because $W_{\{c, g-c\}} \tau W_{\{c, g-c\}} = W_{\{c, g-c\}} \tau$ as $\tau s_i = s_{g-i} \tau$ for all simple roots s_i . \square

Definition 3.3. A Kottwitz–Rapoport stratum is c -superspecial if it satisfies the conditions in the corollary. It is superspecial if it is c -superspecial for some $c \in I$.

Theorem 3.4. A Kottwitz–Rapoport stratum is supersingular if and only if it is superspecial.

Proof. This is Theorem 1.3 in [6]. \square

Now suppose that w is in $W_{\{c, g-c\}}$, so that $\mathcal{A}_{I, w\tau}$ is c -superspecial. Then Corollary 3.2 gives a morphism $\mathcal{A}_{I, w\tau} \rightarrow \mathcal{A}_{\{c, g-c\}, \bar{\tau}}$. We can describe the fibres of this map as Deligne–Lusztig varieties as follows.

Write G' for the inner form (over \mathbb{Q}_p) of the derived group Sp_{2g} of GSp_{2g} that arises as the automorphism group of a superspecial abelian variety (together with a principal polarization), see [5, Section 6.1]. For $c \in \{0, \dots, [g/2]\}$ the subset $\{c, g-c\} \subset I$ gives a parahoric subgroup $P'_{\{c, g-c\}}$ of G' . We denote the maximal reductive quotient of $P'_{\{c, g-c\}}$ by \bar{G}'_c (see [5], where this group is denoted by $\bar{G}'_{\{c, g-c\}}$). This quotient is an algebraic group over \mathbb{F}_p which splits over \mathbb{F}_{p^2} . Its Dynkin diagram is obtained from the extended Dynkin diagram of Sp_{2g} by removing the vertices $c, g-c$. Frobenius acts on the Dynkin diagram by $i \mapsto g-i$. Denote by σ' the Frobenius on $\bar{G}'_{c, k}$ of the non-split form \bar{G}'_c .

Up to central isogeny, we can identify \bar{G}'_c with the maximal reductive quotient of the automorphism group $\mathrm{Aut}(A_c \rightarrow A_{g-c}) \otimes_{\mathbb{Z}} \mathbb{F}_p$. In particular, we obtain an isomorphism between the “flag varieties”, i.e., the varieties of Borel subgroups of these two groups, and we fix, for each point $(A_c \rightarrow A_{g-c}) \in \mathcal{A}_{\{c, g-c\}, \bar{\tau}}$ an isomorphism

$$\mathcal{B}(\mathrm{Aut}(A_c \rightarrow A_{g-c}) \otimes_{\mathbb{Z}} \mathbb{F}_p)^{\mathrm{red}} \cong \mathcal{B}(\bar{G}'_c). \quad (3.2.4)$$

See the discussion of the case $c = 0$ in [5, 3.1, 3.2].

Note that $\mathcal{A}_{\{c, g-c\}, \bar{\tau}}$ is 0-dimensional. Fix a k -valued point $y_0 = (\underline{A}'_c \rightarrow \underline{A}'_{g-c}, \eta)$ in this stratum. Suppose that S is a scheme over k and $y = (\underline{A}_I, \eta)$ is an S -valued point of $\mathcal{A}_{I, w\tau}$ such that $\pi_{c, g-c}(y) = y_0$. Then we get trivializations $A_c = S \times_k A'_c$ and $A_{g-c} = S \times_k A'_{g-c}$. Let $\omega_i \subset H_{\mathrm{DR}}^1(A_i)$ be the Hodge filtration of A_i and ω'_i that of A'_i . We get flags

$$0 \subsetneq \alpha(\omega_{j_1-1}) \subsetneq \alpha(\omega_{j_1-2}) \subsetneq \dots \subsetneq \alpha(\omega_{j_0+1}) \subsetneq \omega_{j_0}/\omega_{j_1} = \mathcal{O}_S \otimes (\omega'_{j_0}/\omega'_{j_1})$$

where (j_0, j_1) is equal to $(-c, c)$, $(c, g - c)$ or $(g - c, g + c)$ and we abusively write α for all the maps induced by $A_i \rightarrow A_{j_0}$. Using the fixed isomorphism (3.2.4), the stabilizer of these flags is an S -valued point $\phi(y)$ of $\mathcal{B}(\overline{G}'_c)$, and we choose this isomorphism such that under the identification

$$\overline{G}'_c \otimes k \cong \mathrm{Sp}_{2c,k} \times \mathrm{SL}_{g-2c,k} \times \mathrm{Sp}_{2c,k}. \quad (3.2.5)$$

according to the decomposition of the Dynkin diagram of \overline{G}'_c into connected components, the first factor corresponds to $\omega'_{g-c}/\omega'_{g+c}$, the second to ω'_c/ω'_{g-c} , and the third one to ω'_{-c}/ω'_c . This is the same normalization as in [5] (where $\omega'_{j_0}/\omega'_{j_1}$ was viewed as $H^1_{\mathrm{DR}}(A_{j_0+g})/H^1_{\mathrm{DR}}(A_{j_1+g})$).

Theorem 3.5. *For w in $W_{\{c,g-c\}}$ the morphism $\pi_{\{c,g-c\},I} \times \phi$ is an isomorphism*

$$\mathcal{A}_{I,w\tau} \xrightarrow{\sim} \mathcal{A}_{\{c,g-c\},\bar{\tau}} \times X_{\overline{G}'_c}(w^{-1}, \sigma').$$

Proof. This follows from Proposition 6.1 and Corollary 6.5 in [5]. \square

Since $\mathcal{A}_{\{c,g-c\},\bar{\tau}}$ is a finite set of points, $\mathcal{A}_{I,w\tau}$ is a finite disjoint union of Deligne–Lusztig varieties.

3.3. Supersingular Ekedahl–Oort strata

We now summarize the description of supersingular Ekedahl–Oort strata in [10]. In the case $g = 1$, which is more or less trivial from this point of view, one has to make some obvious modifications, so we exclude it from the discussion.

For $c \leq g$ we see W_c as a subgroup of W_g via the natural map induced by the inclusion of Dynkin diagrams, explicitly given by $s_{c+1-i} \mapsto s_{g+1-i}$ for $i = 1, \dots, c$. The inclusion $W_c \subset W_g$ maps $W_{c,\mathrm{final}}$ to $W_{g,\mathrm{final}}$. An Ekedahl–Oort stratum EO_w is supersingular if and only if w is in $W_{c,\mathrm{final}}$ for some $c \leq g/2$ (see [7, Remark 2.5.7]).

Lemma 3.6. *Suppose that $c \leq g/2$, $w \in W_{c,\mathrm{final}}$, so that EO_w is supersingular and let $x = (A, \lambda)$ be an S -valued point of EO_w for some connected k -scheme S . There exists a unique point $(\cdots \rightarrow A_c \rightarrow A_{g-c} \rightarrow \cdots) \in \mathcal{A}_{\{c,g-c\},\bar{\tau}}$ such that there exists an isogeny $\rho: A_{-c} \times S \rightarrow A$ with the following properties:*

1. *The pullback $\rho^*(p\lambda)$ is the polarization of A_{-c} .*
2. *The pullback of the level structure under ρ is the level structure on A_{-c} .*

Proof. Note that if we want to view the principally polarized abelian scheme A as an element of $\mathcal{A}_{\{0\}}$, then we have to use $p\lambda$ as the polarization, according to our definitions; this explains why we use $p\lambda$ in the second condition.

The lemma follows from [10, Theorem 1.2], together with the fact that Lemma 3.1 allows us to identify $\mathcal{A}_{\{c,g-c\},\bar{\tau}}$ with the set of isomorphism classes of superspecial abelian varieties (over k) with a polarization with kernel α_p^{2c} and a level N structure, by sending $(\underline{A}_c \rightarrow \underline{A}_{g-c})$ to \underline{A}_{g-c} . (The latter set was denoted by $\Lambda_{g,c}$ in [10]; see also the discussion in [10, Section 3.2].) \square

It follows that all geometric fibers of $\ker \rho$ are isomorphic to α_p^c , and in particular $\deg \rho = p^c$. Because of the presence of level structures, ρ is unique.

Let $\omega(-)$ denote the Hodge filtration of an abelian variety. Given $A \in \mathrm{EO}_w(S)$ and $(A_c \rightarrow A_{g-c})$ as in the lemma, we get a subbundle

$$\rho^*(\omega(A)) \subsetneq \mathcal{O}_S \otimes (\omega(A_{-c})/\omega(A_c)).$$

In fact this is a bundle of c -dimensional isotropic subspaces in a $2c$ -dimensional symplectic vector space. We identify the corresponding symplectic group and its flag variety with Sp_{2c} and $\mathcal{B}(\mathrm{Sp}_{2c})$

using the identifications (3.2.4), (3.2.5). Let $\psi(x)$ be the stabilizer of this subbundle in $\mathrm{Sp}_{2c} \times S$. This is an S -valued point in the variety $X_{\mathrm{Sp}_{2c}}^J$ of parabolic subgroups of type $J = \{1, 2, \dots, c-1\}$ in Sp_{2c} . We write $X_{\mathrm{Sp}_{2c}}^J(w, \sigma)$ for the fine Deligne–Lusztig variety attached to $w \in W_{c, \mathrm{final}}$ in $X_{\mathrm{Sp}_{2c}}^J$ (in [10] this variety was denoted by $X_c\{w\}$).

Theorem 3.7. *Suppose that w is in $W_{c, \mathrm{final}}$ for some $c \leq g/2$ and $w \notin W_{c-1}$. Then the morphism $x \mapsto ((E^g, \mu), \psi(x))$ is an isomorphism*

$$\mathrm{EO}_w \xrightarrow{\sim} \mathcal{A}_{\{c, g-c\}, \bar{\tau}} \times X_{\mathrm{Sp}_{2c}}^J(w^{-1}, \sigma^2).$$

Proof. This is Theorem 1.2 in [10], except for a few superficial differences.

First of all in [10, Lemma 3.1] the isomorphism is constructed with Dieudonné theory. If $M(A[p])$ is the Dieudonné module of $A[p]$, then there is a natural isomorphism $M(A[p])^{(p)} \cong H_{\mathrm{DR}}^1(A)$. Denote by μ the given polarization on A_{-c} . The Verschiebung gives an isomorphism

$$\omega(A_{-c})/\mu(\omega((A_{-c})^\vee)) \xleftarrow{\sim} M(A_{-c}[p])/\mu((A_{-c})^\vee[p]) = M(\ker(\mu)).$$

Under this isomorphism the image $\omega(A)$ corresponds to the image of $M(A[p])$, to which x is sent in [10]. So the two constructions are equivalent.

Secondly, we get w^{-1} instead of w , because we use a different indexation. See [6, Section 2.4] for the different indexations for the EO-stratifications. In [10] the indexation of Moonen and Wedhorn is used, while in [5] the indexation of van der Geer is used.

Thirdly, here we don't need to divide out the action of an automorphism group. This action is trivial because of the level structures. \square

3.4. Comparison between supersingular EO strata and supersingular KR strata

By [3] and [6, Section 9] the projection $\pi: \mathcal{A}_l \rightarrow \mathcal{A}_g$ restricts to a finite étale surjective map $\mathcal{A}_{l, w\tau} \rightarrow \mathrm{EO}_w$ for $w \in W_{g, \mathrm{final}}$. In this section we show how this map relates to the above descriptions of supersingular KR- and EO-strata.

We have seen already that in both descriptions the same index set $\mathcal{A}_{\{c, g-c\}, \bar{\tau}}(k)$ is used. Next we look at the index sets of the stratifications.

Lemma 3.8. *We have:*

$$W_{g, \mathrm{final}} \cap W_{\{c, g-c\}} = W_{c, \mathrm{final}}.$$

Proof. Since $W_c \subset W_{\{c, g-c\}}$, the right-hand side is included in the left-hand side.

For the other inclusion, suppose that w is in $W_{g, \mathrm{final}} \cap W_{\{c, g-c\}}$. The set of simple reflections in a reduced expression of w is equal to $\{s_i, s_{i+1}, \dots, s_{g-1}, s_g\}$ for some i (see the proof of Lemma 7.1 in [10]). Since s_{g-c} is not in $W_{\{c, g-c\}}$, we must have $i > g-c$. Because $W_c \subset W_g$ is generated by all s_j with $j > g-c$, it contains w . \square

Finally we look at the Deligne–Lusztig varieties. Recall that we have identified

$$\bar{G}'_c \otimes k = \mathrm{Sp}_{2c, k} \times \mathrm{SL}_{g-2c, k} \times \mathrm{Sp}_{2c, k}.$$

It follows from the description of σ' in [5] that with respect to this decomposition

$$\sigma'(g_1, g_2, g_3) = (\sigma(g_3), \tilde{\sigma}(g_2), \sigma(g_1)),$$

where σ is the Frobenius of Sp_{2c} and $\tilde{\sigma}$ is the Frobenius of a SU_{g-2c} .

Anything related to \bar{G}'_c splits in a similar way. For instance, its absolute Weyl group splits as

$$W_{\{c, g-c\}} = W_c \times S_{g-2c} \times W_c, \quad (3.4.6)$$

where S_{g-2c} is the symmetric group, which is the Weyl group of SL_{g-2c} .

Proposition 3.9. *For $w \in W_c \subset W_g$, the projection*

$$\mathcal{B}(\bar{G}'_c) = \mathcal{B}(\mathrm{Sp}_{2c}) \times \mathcal{B}(\mathrm{SL}_{g-2c}) \times \mathcal{B}(\mathrm{Sp}_{2c}) \rightarrow \mathcal{B}(\mathrm{SL}_{g-2c}) \times \mathcal{B}(\mathrm{Sp}_{2c})$$

to the final two factors induces an isomorphism

$$X_{\bar{G}'_c}(w, \sigma') \cong X_{\mathrm{SL}_{g-2c}}(1, \tilde{\sigma}) \times X_{\mathrm{Sp}_{2c}}(w, \sigma^2).$$

Proof. The Deligne–Lusztig variety $X_{\bar{G}'_c}(w, \sigma')$ consists of triples (g_1, g_2, g_3) such that

$$\mathrm{inv}(g_1, \sigma(g_3)) = 1, \quad \mathrm{inv}(g_2, \tilde{\sigma}(g_2)) = 1, \quad \mathrm{inv}(g_3, \sigma(g_1)) = w'.$$

The first and last equations are equivalent to $g_1 = \sigma(g_3)$ and $\mathrm{inv}(g_3, \sigma^2(g_3)) = w'$, which is the equation for $X_{\mathrm{Sp}_{2c}}(w, \sigma^2)$. \square

Now let $J = \{1, \dots, c-1\}$, and let \mathcal{P}_J denote the corresponding partial flag variety for Sp_{2c} , i.e., \mathcal{P}_J is the Lagrangian Grassmannian. Since under the natural projection from the symplectic flag variety $\mathcal{B}(\mathrm{Sp}_{2c})$ to \mathcal{P}_J , $X_{\mathrm{Sp}_{2c}}(w, \sigma^2)$ is mapped to $X_{\mathrm{Sp}_{2c}}^J(w, \sigma^2)$, the proposition gives a morphism

$$f : X_{\bar{G}'_c}(w, \sigma') \rightarrow X_{\mathrm{Sp}_{2c}}(w, \sigma^2) \rightarrow X_{\mathrm{Sp}_{2c}}^J(w, \sigma^2).$$

Theorem 3.10. *For $w \in W_{c, \mathrm{final}}$ there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}_{I, w\tau} & \xrightarrow{\sim} & \mathcal{A}_{\{c, g-c\}, \bar{\tau}} \times X_{\bar{G}'_c}(w^{-1}, \sigma') \\ \pi \downarrow & & \downarrow \beta_c \times f \\ \mathrm{EO}_w & \xrightarrow{\sim} & \mathcal{A}_{\{c, g-c\}, \bar{\tau}} \times X_{\mathrm{Sp}_{2c}}^J(w^{-1}, \sigma^2) \end{array}$$

where the horizontal maps are the isomorphisms from 3.5 and 3.7.

Proof. Suppose $(A_i)_{i \in I}$ is an S -valued point in $\mathcal{A}_{I, w\tau}$. Lemma 3.6 gives us a unique point $(A_c \rightarrow A_{g-c})$ and a morphism $\rho : S \times A_{-c} \rightarrow A_0$. (Here we extend the chain $(A_c \rightarrow A_{g-c})$ to a periodic chain as usual.) Under the horizontal map in the upper row, $(A_i)_i$ is mapped to the flags of the images of the Hodge filtrations $\omega(A_i)$ inside $\omega(A_{g-c})/\omega(A_{g+c})$, $\omega(A_c)/\omega(A_{g-c})$, and $\omega(A_{-c})/\omega(A_c)$. See the proof of Proposition 6.1 in [5]. The projection considered in the previous proposition projects onto the latter two factors, and in particular, the map f we defined takes the flag $(\omega(A_i))_i$ to $\omega(A_0)$ which is the maximal totally isotropic member of the flag in the third factor. Since the map π which gives the left column maps our chain $(A_i)_i$ to A_0 , and the lower horizontal map maps A_0 to $((A_c \rightarrow A_{g-c}), \omega(A_0))$, we see that the diagram commutes. \square

Remark 3.11. For supersingular strata the isomorphism in diagram (2.3.3) corresponds to the identification of fine Deligne–Lusztig varieties with coarse Deligne–Lusztig varieties for a different parabolic subgroup, see [10, Corollary 2.7]. From [5, Theorem 6.3] we know that the fibers of π and of the horizontal map in the diagram are Deligne–Lusztig varieties.

4. Supersingular KR strata in the parahoric case

Let $J \subseteq I$ be the type of a standard parahoric subgroup. We want to describe supersingular Kottwitz–Rapoport strata in \mathcal{A}_J . As one would hope, there is a description completely analogous to the one in [5], and it can in fact be derived from the Iwahori version with relatively little additional work.

First let us generalize the definition of superspecial strata to the parahoric case.

Definition 4.1. Let $J \subseteq I$, and let $c \in I$. A stratum $\mathcal{A}_{J,x}$ is *c-superspecial*, if $\pi_{J,I}^{-1}(\mathcal{A}_{J,x})$ is a union of *c-superspecial* KR strata in \mathcal{A}_I , i.e., if

$$W_J x W_J \cap \text{Adm}(\mu) \subseteq W_{\{c, g-c\}} \tau.$$

A stratum $\mathcal{A}_{J,x}$ is *superspecial*, if it is *c-superspecial* for some $c \in I$.

Clearly every superspecial stratum is supersingular.

Lemma 4.2. *There exists a c-superspecial stratum in \mathcal{A}_J if and only if c and $g - c$ are in J .*

Proof. If $c, g - c \in J$ then $W_J \tau W_J \subseteq W_{cgc} \tau$ which shows that the minimal KR stratum in \mathcal{A}_J is *c-superspecial*. To show the converse, assume that $c \notin J$ or $g - c \notin J$. In both cases $s_c \tau = \tau s_{g-c}$ is in $W_J \tau W_J$. It is easy to see that $s_c \tau$ is μ -permissible (see [5, Definition 2.4]) and hence μ -admissible. The union of all *c-superspecial* strata is closed, and a non-empty closed union of KR strata contains the minimal stratum because the closure relations are given by the Bruhat order (cf. [6, Proposition 2.6] and the references given there). Therefore it is enough to show that the minimal stratum in \mathcal{A}_J is not *c-superspecial*, i.e., that

$$W_J \tau W_J \cap \text{Adm}(\mu) \not\subseteq W_{\{c, g-c\}} \tau.$$

But $s_c \tau$ is not in $W_{\{c, g-c\}} \tau$, as $W_{\{c, g-c\}}$ is generated by s_i for $i \notin \{c, g - c\}$. \square

When c and $g - c$ are in J , a stratum is *c-superspecial* if and only if it satisfies condition (1) or (2) in Corollary 3.2 (with I replaced by J). When for instance c is not in J , the lemma and equation (2.1.2) tell us that for any $x \in \text{Adm}(\mu)$ and $c \notin J$, we can find a point $(A_j)_{j \in J} \in \mathcal{A}_{J,\bar{x}}$ which can be extended to a chain $(A_j)_{j \in I}$ which does not lie in the *c-superspecial* locus, i. e., where we do not have A_c, A_{g-c} superspecial and $A_c \rightarrow A_{g-c}^\vee$ the Frobenius.

We now describe *c-superspecial* KR-strata in terms of Deligne–Lusztig varieties. Suppose that J is a subset of I with $c, g - c \in J$ (so that *c-superspecial* strata exist in \mathcal{A}_J). Let $\mathcal{P}_{\sigma'(J)}(G'_{c,k})$ be the variety of parabolic subgroups of type $I \setminus \sigma'(J)$ in $G'_{c,k}$. Note that it is not defined over \mathbb{F}_p if J is not Frobenius invariant. There are coarse Deligne–Lusztig varieties

$$X_{G'_{c,k}, \sigma'(J)}(\bar{w}, \sigma') = \{P \in \mathcal{P}_{\sigma'(J)}(G'_{c,k}) \mid \text{inv}(P, \sigma'(P)) = \bar{w}\}$$

for all double cosets $\bar{w} \in W_{\sigma'(J)} \setminus W_{\{c, g-c\}} / W_J$.

Theorem 4.3. *Let $\mathcal{A}_{J,\bar{x}}$ be a c-superspecial stratum inside \mathcal{A}_J . Write $x = w\tau$, so that $w \in W_{\{c, g-c\}}$. There is an isomorphism*

$$\mathcal{A}_{J,\bar{x}} \cong \mathcal{A}_{\{c, g-c\}, \bar{\tau}} \times X_{G'_{c,k}, \sigma'(J)}(\bar{w}^{-1}, \sigma').$$

Proof. Taking into account the remarks above, the proof is the same as in the Iwahori case, see [5, Section 6]. \square

It is evident that every superspecial stratum is supersingular. We will show below that the converse is true, as well. We first prove a connectedness result in the parahoric case, analogous to [6, Theorem 7.3].

Proposition 4.4. *If a KR-stratum is not superspecial, then it is irreducible.*

The converse holds if the level structure is large enough. This follows from Theorem 4.3, because $\#\pi_{\{c, g-c\}, I}(\mathcal{A}_{I, \tau}) > 1$ except possibly for very small N ; cf. Corollary 6.6 in [5].

Proof. Let $x \in \text{Adm}(\mu)$ and suppose $\mathcal{A}_{J, \bar{x}}$ is not superspecial, i.e.

$$W_J x W_J \cap \text{Adm}(\mu) \not\subseteq W_{\{c, g-c\}} \tau, \quad \text{for all } c \in \{0, \dots, [g/2]\}.$$

Then there exists for each $c \in \{0, \dots, [g/2]\}$ an element in $W_J x W_J \cap \text{Adm}(\mu)$ larger than $s_c \tau$ or than $s_{g-c} \tau$. In fact, an element x is larger than $s_c \tau$ if and only if s_c occurs in any/every reduced expression of $x\tau^{-1}$ as a product of simple reflections, and similarly for s_{g-c} .

Therefore by [6, Theorem 7.2] the closure of the union of all 1-dimensional strata in $\pi_{J, I}^{-1}(\mathcal{A}_{J, \bar{x}})$,

$$\overline{\bigcup_{v \in W_J x W_J \cap \text{Adm}(\mu)} \bigcup_{\substack{s \leq v \\ \ell(s)=1}} \mathcal{A}_{I, s}}$$

is connected. Now every connected component of

$$\overline{\bigcup_{v \in W_J x W_J \cap \text{Adm}(\mu)} \mathcal{A}_{I, v}},$$

meets the previous set (because every connected component of a KR stratum in \mathcal{A}_I has a point of the minimal stratum in its closure, [6, Theorem 6.2]). But this set is equal to the union of all KR strata $\mathcal{A}_{I, z}$ such that $z \leq v$ for some $v \in W_J x W_J$, or equivalently, such that $W_J z W_J \leq W_J x W_J$ (with respect to the Bruhat order of the double quotient by W_J). In other words, we have

$$\overline{\bigcup_{v \in W_J x W_J \cap \text{Adm}(\mu)} \mathcal{A}_{I, v}} = \pi_{J, I}^{-1}(\overline{\mathcal{A}_{J, \bar{x}}}),$$

and we have shown that this space is connected. But then its image $\overline{\mathcal{A}_{J, \bar{x}}}$ is connected, too. This closure is normal, because étale-locally KR strata are isomorphic to Schubert varieties in affine flag varieties, which are known to be normal; cf. [6, Proposition 2.6]. So connectedness implies irreducibility, and the proposition follows. \square

Theorem 4.5. *A KR-stratum is supersingular if and only if it is superspecial.*

Proof. We know that each superspecial stratum is supersingular. Now suppose that $\mathcal{A}_{J, \bar{x}}$ is supersingular. Its image in \mathcal{A}_g is a union of *supersingular* EO strata. We may assume that the level structure outside p is large enough, so that this union, and hence $\mathcal{A}_{J, \bar{x}}$, is reducible (cf. Proposition 2.5 (2) in [6]). By the previous proposition, this implies the desired statement. \square

Remark 4.6. Let us discuss the combinatorial meaning of the theorem. Start with $x \in \text{Adm}(\mu)$, such that $\mathcal{A}_{J,x}$ is supersingular, i.e. is contained in the supersingular locus $\mathcal{S}_J \subset \mathcal{A}_J$. The latter condition is equivalent to $\pi_{J,I}^{-1}(\mathcal{A}_{J,x})$ being contained in the supersingular locus or in other words to

$$W_J x W_J \cap \text{Adm}(\mu) \subseteq \bigcup_{c=0}^{\lfloor g/2 \rfloor} W_{\{c, g-c\}} \tau,$$

since we know that in the Iwahori case supersingular is the same as superspecial. By the theorem there exists $c \in J$ such that

$$W_J x W_J \cap \text{Adm}(\mu) \subseteq W_{\{c, g-c\}} \tau.$$

It seems hard to prove this statement combinatorially because it is not easy to understand what happens when one intersects the double coset $W_J x W_J$ with the admissible set.

5. Two open questions

Let us mention two open questions. First of all, which EO strata occur in the image $\pi(\mathcal{A}_x) \subseteq \mathcal{A}_g$ for $x \in \text{Adm}(\mu)$? The answer to this question could depend on p , but we don't expect that. Ekedahl and van der Geer first posed this question in [3]. They showed that if $x = w\tau$ for a final element w , then the image is just the single EO stratum corresponding to w , while in general, it is a union of EO strata. Our Theorem 2.8 says that for all $y \in W_J w \tau W_J \cap \text{Adm}(\mu)$, the image of \mathcal{A}_y under π is equal to EO_w . Hartwig [9] has computed this map in the case $g = 3$ using Dieudonné theory. In [11], the second named author investigates this question in much more detail, and gives an algorithmic answer in the general case.

Secondly, what is the dimension of the supersingular locus in moduli spaces with a parahoric level structure? For general g we only know this for $J = \{0\}$, or $= \{g\}$, by the work of Li and Oort [12], or as a consequence of the purity of the Newton stratification shown by de Jong and Oort, and for $J = I$ and g even (see [6], also for bounds in the case where g is odd). We expect that for $J \neq \{0\}, \{g\}$ the supersingular locus is usually not equi-dimensional. The lower bound on the dimension obtained from KR strata in the supersingular locus cannot be sharp in the general parahoric case. Note also that the union of supersingular EO strata achieves only approximately half the dimension of the supersingular locus in \mathcal{A}_g .

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