



On the coefficients of the Coxeter polynomial of an accessible algebra

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ABSTRACT

Let A be a finite dimensional algebra over an algebraically closed field k . Assume A is basic connected with n pairwise non-isomorphic simple modules. We consider the Coxeter polynomial $\chi_A(T)$ of a one-point extension algebra $A = B[M]$ and the polynomial of the extension $p(T) = \frac{1}{T}((1+T)\chi_B(T) - \chi_A(T))$. If M is exceptional then $p(T) = 1 + p_1T + \cdots + p_{n-3}T^{n-3} + T^{n-2}$. In that case, we call $s(A : B) = p_1$ the linear index of the extension $A = B[M]$. We give conditions for $s(A : B) \geq 0$. For a tower $\mathbb{T} = (k = A_1, A_2, \dots, A_n = A)$ of access to A , that is, A_i is a one-point (co-)extension of A_{i-1} by an exceptional module, the index $s(\mathbb{T}) = \sum_{i=2}^n s(A_i : A_{i-1}) = n - 1 - a_2$, is an invariant depending on the derived equivalence class of A , where a_2 is the quadratic coefficient of $\chi_A(T)$. We show that, in the case A is piecewise hereditary, then $a_2 = 1$ if and only if A is derived equivalent to a quiver algebra of type \mathbb{A}_n .

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1. Introduction

Assume throughout the paper that k is an algebraically closed field. Given a finite dimensional k -algebra A , an A -module M is called *exceptional* if $\text{End}_A(M)$ is trivial and $\text{Ext}_A^n(M, M) = 0$ for each $n \geq 1$. Recall that the *one-point extension* $A[M]$ and the *one-point coextension* $[M]A$ are the algebras given, respectively, by

$$\begin{bmatrix} A & 0 \\ M & k \end{bmatrix} \quad \text{and} \quad [M]A = \begin{bmatrix} A & DM \\ 0 & k \end{bmatrix}$$

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with the obvious matrix operations, where $D = \text{Hom}_k(-, k)$ is a duality functor. The Coxeter transformation ϕ_A is the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander–Reiten translation τ in the derived category $\text{Der}(\text{mod } A)$ of the module category $\text{mod } A$ of finite dimensional left A -modules. The characteristic polynomial of ϕ_A is called the *Coxeter polynomial* $\chi_A(t)$ of A and will play a relevant role in this paper.

For a one-point extension $A = B[M]$ we define the *polynomial of the extension* as $p(T) = \frac{1}{T}((1+T)\chi_B(T) - \chi_A(T))$. By [10], for an exceptional module M the polynomial $p(T)$ has the shape $p(T) = 1 + p_1T + \dots + p_{n-3}T^{n-3} + T^{n-2}$. In that case, we call $s(A : B) = p_1$ the *linear index* of the extension $A = B[M]$. We give conditions for $s(A : B) \geq 0$, for instance, we show that for M satisfying (a) $\text{p.dim.}_B M \leq 1$ and $\text{i.dim.}_B M \leq 1$ and (b) M is a preprojective B -module, then $s(A : B) \geq 1$ unless $B = k$ and then $s(A : B) = 0$. Following [14], we say that an exceptional B -module is *special* if the perpendicular category M^\perp of M in $\text{Der}(\text{mod } B)$ is derived equivalent to $\text{mod } C$ for some algebra C called the *perpendicular restriction* of B via M . In that case, the polynomial of the extension $A = B[M]$ is $p(T) = \chi_C(T)$. As an instance, if B is derived hereditary, then any exceptional module M is special and the corresponding perpendicular restriction C is piecewise hereditary. Then $1 \leq s(A : B)$ counts the number of connected components of C .

We shall consider the process of inductive construction of algebras applying the one-point extension procedure. Let A and B be two k -algebras. Following [13], we say that A is *accessible from* B if there is a sequence $B = B_1, B_2, \dots, B_s = A$ of algebras such that $B_{i+1} = B_i[M_i]$ or $[M_i]B_i$ for an exceptional B_i -module M_i . In particular, we say that A is *accessible* if A is accessible from the algebra k . We observe that the quiver of an accessible algebra A is connected and has no oriented cycles. For a tower $\mathbb{T} = (k = A_1, A_2, \dots, A_n = A)$ of access to A , that is, A_i is a one-point (co-)extension of A_{i-1} by an exceptional module, we introduce the *towering number* $s(\mathbb{T}) = \sum_{i=2}^n s(A_i : A_{i-1})$. We shall prove the following:

Theorem 1. *Let A be an accessible algebra with n vertices as above, then the towering number $s_{\mathbb{T}}(A) = \sum_{i=1}^{n-1} s_i$ of \mathbb{T} is a derived invariant, that is, depends only on the derived class of A . It is $s_{\mathbb{T}}(A) = n - 1 - a_2$, where a_2 is the coefficient of the quadratic term in the Coxeter polynomial of A .*

As shown in [13] many well-known algebras are accessible, often special accessible: hereditary algebras associated to quivers without cycles, more generally tree algebras, canonical algebras with three weights, poset algebras without crowns, and many extended canonical and supercanonical algebras. On the other hand, we observe that the canonical algebra A of type $(3, 3, 3)$ is accessible since $A = B[M]$ for a hereditary tree algebra B of type $[3, 3, 3]$ and an exceptional module M but the corresponding perpendicular restriction is a quiver algebra of type $\tilde{\mathbb{A}}_6$. We shall study properties of the towers of access to different classes of algebras and obtain applications. As an instance, an algebra A is derived hereditary of Dynkin type if there is a tower of access $A_1 = k, A_2, \dots, A_n = A$ such that the spectrum of $\chi_{A_i}(T)$ is contained in $\mathbb{S}^1 \setminus \{1\}$, for all $i = 1, \dots, n$. As another application we show the following theorem which generalizes a result of Happel, see [10].

Theorem 2. *Let A be a piecewise hereditary algebra with n vertices and let $\chi_A(T) = \sum_{i=0}^n a_i T^i$ be the Coxeter polynomial of A . Then $a_2 = 1$ if and only if A is derived equivalent to a quiver algebra of type $\tilde{\mathbb{A}}_n$.*

For general background on representation theory, the reader may see [1,6,12,13,15]. We thank Helmut Lenzing for fruitful discussions.

2. One-point extensions and accessible algebras

2.1. Let $\langle x, y \rangle$ be the bilinear form defined in the Grothendieck group $K_0(B)$ with the properties $\langle \dim X, \dim Y \rangle_B = \sum_{j=0}^{\infty} (-1)^j \dim_k \text{Ext}_B^j(X, Y)$, for any two B -modules X, Y , and such that $\langle x, \phi_B(y) \rangle_B = -\langle y, x \rangle_B$. In the case $A = B[M]$ we consider $K_0(B)$ as a sublattice of $K_0(A)$, that is, $K_0(A) = K_0(B) \oplus \mathbb{Z}$ and $\langle x, y \rangle_A = \langle x, y \rangle_B$ for elements $x, y \in K_0(B)$.

The Coxeter polynomial of A shall be written as $\chi_A(T) = 1 + a_1T + a_2T^2 + \dots + a_{n-2}T^{n-2} + a_{n-1}T^{n-1} + T^n$. Sometimes, for the sake of clarity, we write $a_i(A)$ for these coefficients.

2.2. Coxeter polynomial and Poincaré series

Let $\chi_B(T) = \sum_{i=0}^{n-1} b_i T^i$ be the characteristic polynomial of the Coxeter transformation of B and let $\chi_A(T) = \sum_{j=0}^n a_j T^j$ be the corresponding polynomial for the one-point extension $A = B[M]$. Calculating the coefficients of the Coxeter polynomial, it follows that $a_1 = b_1 + (1 - \langle m, m \rangle)$, where $m = \dim M$ is the dimension vector of M . In particular, if M is an exceptional module, $a_1 = b_1$.

More generally, in [10] it was shown that

$$a_i = b_i - (\langle m, m \rangle - 1)b_{i-1} - \sum_{k=2}^i \langle \phi_B^{k-1} m, m \rangle b_{i-k}.$$

2.3. For $A = B[M]$ with M as B -module, Happel's long exact sequence [8] relates the Hochschild cohomology groups $H^i(A)$ and $H^i(B)$ in the following way:

$$0 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow \text{End}_B(M)/k \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow \text{Ext}_B^1(M, M) \rightarrow H^2(A) \rightarrow \dots$$

Therefore, if M is exceptional we get $H^i(A) = H^i(B)$, for $i \geq 0$, and moreover, the cohomology rings $H^*(A)$ and $H^*(B)$ are isomorphic. More general, if A is accessible from B then $H^*(A) \cong H^*(B)$ as rings. In particular, if A is accessible then $H^i(A) = 0$ for $i > 0$ and $H^0(A) = k$.

An important expression for the linear term of $\chi_A(T)$ was shown in [9] as: $a_1 = \sum_{i=0}^n (-1)^i \dim_k H^i(A)$. In particular, for an accessible algebra A , we have $a_1 = a_{n-1} = 1$.

2.4. Representation-finite algebras

For an algebra A we denote by Γ_A the Auslander–Reiten quiver of A equipped with the translation τ . A preprojective component \mathcal{P} of Γ_A has a projective module in the τ -orbit of every module $X \in \mathcal{P}$. From [13] we recall the following:

Proposition. Let $A = kQ/I$ be a representation-finite connected algebra. Then A is accessible if and only if the following conditions hold:

- (i) $H^1(A) = 0$;
- (ii) Γ_A is a preprojective component.

In particular, if B a convex subcategory of A then B is accessible.

Proof. Assume (i) and (ii). There exists a source or a sink a in Q such that $Q \setminus \{a\}$ is connected. Say a is a source and $M = \text{rad } P_a$ is therefore indecomposable, since $H^1(A) = 0$ implies that A satisfies the separation condition [8]. Let $B = A/Ae_a A$ and write $A = B[M]$ as a one-point extension. Since M is preprojective, then it is exceptional and then $H^i(A) = H^i(B)$ for every $i \geq 0$. In particular, $H^1(B) = 0$ and Γ_B has a preprojective component by [15, (4.3.6)]. By induction hypothesis, B is accessible and therefore A is accessible. \square

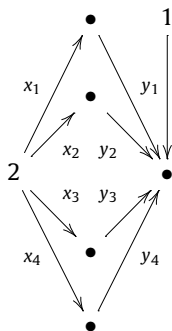
2.5. The polynomial of an extension

Consider a one-point extension $A = B[M]$ for some left B -module M . The polynomial of the extension is defined as

$$p(T) = \frac{1}{T}((1+T)\chi_B(T) - \chi_A(T)).$$

For instance, if M is an exceptional B -module and C is an algebra such that $\text{Der}(\text{mod } C)$ is derived equivalent to the perpendicular category M^\perp formed in $\text{Der}(\text{mod } B)$, then the polynomial of the extension $A = B[M]$ is $\chi_C(T)$. See (2.7).

Example. Consider the extended canonical algebra A given by the quiver



with relations $x_1y_1 + x_2y_2 = x_3y_3 = x_4y_4$. We get that

$$\chi_A(T) = (T-1)^2(T+1)^3(T^2+T+1).$$

Consider the algebras B_i obtained from A by deleting the vertex i , for $i = 1, 2$. Then $A = B_1[M_1]$ for an indecomposable projective B_1 module and

$$\chi_{B_1}(T) = (T-1)^2(T+1)^4 \quad \text{and} \quad p_1(T) = (T-1)^2(T+1)^3$$

where $p_1(T)$ is the polynomial of the extension. On the other hand, $A = B_2[M_2]$ for the indecomposable B_2 -module M_2 obtained as the restriction of the projective A -module P_2 . Observe that the module M_2 has self-extensions and therefore is not exceptional. Moreover

$$\chi_{B_2}(T) = (T+1)^4(T^2-3T+1) \quad \text{and} \quad p_2(T) = -4T(T+1)^3$$

where $p_2(T)$ is the polynomial of the extension.

Proposition. Let B be a connected algebra and $A = B[M]$ an extension with n vertices. Let $p(T) = p_0 + p_1T + \dots + p_dT^d$ be the polynomial of the extension $A = B[M]$. Then the following hold:

- (a) $d \leq n-2$;
- (b) $p_0 \neq 0$ if and only if $d = n-2$;
- (c) $p_0 = \langle m, m \rangle$, where $m = \dim M$; in particular, if M is exceptional then $p_0 = 1$.

Proof. As above, assume that $\chi_B(T) = \sum_{i=0}^{n-1} b_iT^i$ and $\chi_A(T) = \sum_{j=0}^n a_jT^j$ are the characteristic polynomials of the Coxeter transformations of B and A respectively. Since the leading coefficients are $b_{n-1} = 1 = a_n$ then $p_{n-1} = b_{n-1} - a_n = 0$. This shows (a).

For (b), observe that the self-reciprocity of Coxeter polynomials implies that

$$p_{n-2} = b_{n-1} + b_{n-2} - a_{n-1} = b_0 + b_1 - a_1 = p_0.$$

Moreover, $b_0 = 1$ and $a_1 = b_1 + 1 - \langle m, m \rangle$ yield $p_0 = \langle m, m \rangle$. \square

2.6. One-point extensions and perpendicular categories

Throughout this subsection we assume that A is a finite dimensional algebra of finite global dimension, implying that the bounded derived category $\text{Der}(\text{mod } A)$ of finite dimensional A -modules is homologically finite. Recall that a triangulated category \mathcal{T} is called *homologically finite* if for any two objects X and Y from \mathcal{T} the space $\text{Hom}_{\mathcal{T}}(X, Y[n])$ is non-zero only for finitely many n . Note that a module E is exceptional in $\text{mod } A$ if and only if it is exceptional as an object in the triangulated category $\text{Der}(\text{mod } A)$ under the standard embedding from $\text{mod } A$ to $\text{Der}(\text{mod } A)$.

Consider an exceptional object E in a triangulated category \mathcal{T} , then the *right perpendicular category* E^\perp of E consists of all objects X from \mathcal{T} satisfying the conditions $\text{Hom}_{\mathcal{T}}(E, X[n]) = 0$ for each integer n . Viewed as a full subcategory of \mathcal{T} , E^\perp is a triangulated category, and the exact inclusion of E^\perp in \mathcal{T} admits an exact left adjoint $e_\lambda : \mathcal{T} \rightarrow E^\perp$, see [5].

Forming one-point extensions is in some sense inverse to forming perpendicular categories (with respect to an exceptional object), as we recall in the following result. See [7, Proposition 4.11] and [11, Section 18.3] for a detailed discussion.

Proposition. Assume that $A = B[M]$ is the one-point extension of B by a B -module M , the following hold:

- (1) Let P denote the indecomposable projective right A -module $[M, k]$, then P^\perp is equivalent to $\text{Der}(\text{mod } B)$.
- (2) Assume B has finite global dimension and M is an exceptional module in $\text{mod } B$ satisfying either (S0): M is projective, or the following two conditions (S1): $\text{p.dim.}_B M = 1$ and (S1'): $\text{Hom}_B(M, B) = 0$.

Then the following hold:

- (i) M^\perp has a tilting object T' . Hence M^\perp is equivalent to $\text{Der}(\text{mod } C)$ for $C = \text{End}(T')$. Moreover, C has finite global dimension $\text{gl.dim } C \leq \text{gl.dim } B$ and there exists an epimorphism $B \rightarrow C$.
 - (ii) If F is an exceptional object from $\text{Der}(\text{mod } B)$, satisfying $\text{Hom}_B(T', F[n]) = 0$ for each $n \neq 0$, and $N = \text{Hom}_B(T', F)$, then B is derived equivalent to the one-point extension $C[F]$.
 - (iii) In the case (S0), $M = P_b$ an indecomposable projective B -module associated to a source b in the quiver of B , then the algebra C given in (i) is $C = B/(b)$ the quotient of B obtained by killing b .
- (3) Let M be an exceptional B -module and let C be an algebra such that $\text{Der}(\text{mod } C)$ is derived equivalent to the perpendicular category M^\perp formed in $\text{Der}(\text{mod } B)$. Then the Coxeter polynomial for the one-point extension $A = B[M]$ is given by

$$\chi_A(T) = (1 + T)\chi_B(T) - T\chi_C(T).$$

2.7. Special exceptional modules

Assume that $A = B[M]$ is a one-point extension of a connected algebra B by an exceptional B -module M and A has n vertices. Then we say that M is *special* if the perpendicular category M^\perp of M in $\text{Der}(\text{mod } B)$ is equivalent to $\text{Der}(\text{mod } C)$, for some algebra C . An algebra C as above is called a *perpendicular restriction* of B along M . In the case such a C exists, we say that M is *very special* if there exists an exceptional C -module F such that the one-point extension $C[F]$ is derived equivalent to B .

Remark. (i) Assume that M is a special exceptional B -module with M^\perp derived equivalent to $\text{Der}(\text{mod } C)$. Then C is well-defined only up to derived equivalence. Moreover, C has $n - 2$ vertices and the following holds:

$$\chi_A(T) = (1 + T)\chi_B(T) - T\chi_C(T),$$

that is, $p(T) = \chi_C(T)$ is the polynomial of the extension $A = B[M]$.

(ii) If moreover M is very special, then $H^i(C) = 0$, for $i > 0$. Indeed this follows from Happel's long exact sequence and the derived invariance of Hochschild cohomology.

(iii) Consider the canonical algebra A of weight type $(3, 3, 3)$ written as the extension $A = B[M]$ of a hereditary algebra B of extended Dynkin type $[3, 3, 3]$ by an exceptional module M in a tube of

rank 2. Then M is special and the perpendicular restriction of B via M is the hereditary algebra C of extended Dynkin type $[3, 3]$ with underlying graph of type \tilde{A}_6 . Moreover, M is not very special since it would imply that the Hochschild cohomology $H^1(C) = 0$. A contradiction.

In [14] the following was proved:

Theorem. *Let B be an accessible piecewise hereditary algebra then the following happen:*

- (i) *Exceptional B -modules are special;*
- (ii) *Assume that M is an exceptional B -module. Then $B[M]$ is accessible if and only if it is special accessible. Moreover if $\text{Der}(M^\perp) = \text{Der}(\text{mod } C)$ then C is derived hereditary.*

3. The towering index

3.1. The linear index of an extension

Let $A = B[M]$ be a one-point extension algebra such that B is an accessible algebra and M is an exceptional B -module. Consider the Coxeter polynomials:

$$\begin{aligned}\chi_A(T) &= 1 + T + a_2 T^2 + \cdots + a_{n-2} T^{n-2} + T^{n-1} + T^n; \\ \chi_B(T) &= 1 + T + b_2 T^2 + \cdots + b_{n-3} T^{n-3} + T^{n-2} + T^{n-1}\end{aligned}$$

and the polynomial of the extension $A = B[M]$,

$$p(T) = 1 + p_1 T + p_2 T^2 + \cdots + p_{n-4} T^{n-4} + p_{n-3} T^{n-3} + T^{n-2}.$$

We call

$$s(A : B) := p_1 = b_2 + 1 - a_2$$

the linear index of the extension $A = B[M]$. By (2.2),

$$s(A : B) = 1 + \langle \phi_B(\mathbf{dim} M), \mathbf{dim} M \rangle.$$

3.2. We are interested in finding conditions for an extension $A = B[M]$ to have $s(A : B) \geq 0$, as in the following proposition.

Proposition. *Let M be an exceptional B -module and $A = B[M]$ a one-point extension. Consider, as above, $s(A : B) = b_2 + 1 - a_2$ the linear index of the extension. Assume the following hold:*

- (a) $\text{p.dim.}_B M \leq 1$ and $\text{i.dim.}_B M \leq 1$;
- (b) M is a preprojective B -module.

Then $s(A : B) \geq 1$ unless $B = k$ and then $s(A : B) = 0$. In any case, $a_2 \leq b_2$.

Proof. According to [15, (2.4)], the conditions $\text{p.dim.}_B M \leq 2$ and $\text{i.dim.}_B M \leq 2$ imply

$$\phi_B(\mathbf{dim} M) = \mathbf{dim} \tau_B M - \mathbf{dim} I$$

for some injective B -module I . We shall prove, by induction on $\dim_k I$, that

$$\langle \phi_B(\mathbf{dim} M), \mathbf{dim} M \rangle \geq 0.$$

In the case $I = 0$, then $\langle \phi_B(\mathbf{dim} M), \mathbf{dim} M \rangle = \dim_k \operatorname{Hom}_B(\tau M, M) \geq 0$.

In the general situation, since $\text{p.dim. } M \leq 1$ then $0 \neq I = D(\operatorname{Hom}_B(M, B))$. Hence there is an indecomposable projective B -module P_x and a non-zero map $M \rightarrow P_x$. Since M lies in a preprojective component of Γ_B then either $M(x) = 0$ or $P_x = M$. We distinguish these cases.

If $M(x) = 0$ we may assume that x is a source of the quiver of B and write $B = C[N]$ where N is the radical of the projective B -module P'_x . Then M and $\tau_B M$ are C -modules and therefore $\tau_C M = \tau_B M$ and

$$\phi_C(\mathbf{dim} M) = \mathbf{dim} \tau_C M - \mathbf{dim} I'$$

where $I' = D(\operatorname{Hom}_C(M, C))$ and $I = I' \oplus S_x$. Moreover, the simple injective B -module S_x satisfies

$$\langle \mathbf{dim} S_x, \mathbf{dim} M \rangle = -\dim_k \operatorname{Ext}_B^1(S_x, M) \leq 0.$$

By induction hypothesis we get

$$\begin{aligned} \langle \phi_B(\mathbf{dim} M), \mathbf{dim} M \rangle &\geq \langle \mathbf{dim} \tau_C M, \mathbf{dim} M \rangle - \langle \mathbf{dim} I', \mathbf{dim} M \rangle \\ &= \langle \phi_C(\mathbf{dim} M), \mathbf{dim} M \rangle \geq 0. \end{aligned}$$

Finally, if $M = P_x$ then $\phi_B(\mathbf{dim} M) = -\mathbf{dim} I_x$ and since $\text{i.dim. } M \leq 1$ then

$$\langle \phi_B(\mathbf{dim} M), \mathbf{dim} M \rangle = -\langle \mathbf{dim} I_x, \mathbf{dim} P_x \rangle \geq -\dim_k \operatorname{Hom}_B(I_x, P_x)$$

which vanishes unless $P_x = I_x$ and then $B = k$. In the latter case, $b_2 = 0$ and $s(A : B) = 0$. \square

3.3. We recall that $\text{p.dim. } M \leq 1$ and $\operatorname{Hom}_A(M, A) = 0$ implies that $\phi_A(\mathbf{dim} M) = \mathbf{dim} \tau_A M$. In that case we know that M is special. A partial interpretation of the meaning of the linear index of an extension $s(A : B)$ is given by the following proposition.

Proposition. *Let $A = B[M]$ be a one-point extension algebra such that B is an accessible algebra and M is a very special exceptional B -module. Let C be a perpendicular restriction of B via M formed by s connected components. Then the following hold:*

- (a) $H^i(C) = 0$, for $i > 0$.
- (b) $s(A : B) = s$.
- (c) $a_2 \leq b_2$ and equality holds if and only if C is connected.

Proof. First recall that $p(T)$ is the Coxeter polynomial $\chi_C(T)$. Moreover, let F be an exceptional C -module such that $C[F]$ is derived equivalent to B .

(a) was observed above. Moreover,

$$p_1 = \sum_{i=0}^{n-2} (-1)^i \dim_k H^i(C) = \dim_k H^0(C) = s$$

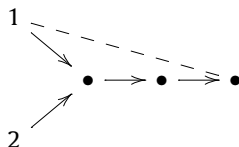
yields (b).

(c) follows from $a_2 - b_2 = 1 - s \leq 0$. \square

Corollary. *Let A be a piecewise hereditary algebra, then the quadratic coefficient a_2 of the Coxeter polynomial of A is ≤ 1 .*

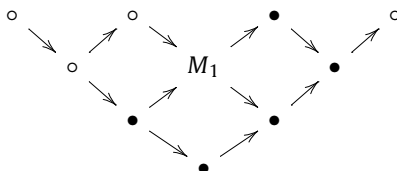
3.4. Examples

Let A be the algebra given by the following quiver with relations indicated by dashed line:



Consider the following two ways to access A : consider the algebras $B_i = A/(i)$ the quotients obtained by deleting the vertex $i = 1, 2$. Then $A = B_i[M_i]$, for very special exceptional modules M_i .

For $i = 1$, the algebra B_1 is hereditary and the perpendicular category M_1^\perp is generated by the modules marked by circles in the Auslander–Reiten quiver of B_1 , as follows:



Hence M_1^\perp is derived equivalent to a disconnected quiver $\bullet \rightarrow \bullet \quad \bullet$, that is $s(A : B_1) = 2$. On the other hand, for $A = B_2[M_2]$ we observe that M_2 is a projective B_2 -module and M_2^\perp is derived equivalent to the quiver algebra of type \mathbb{A}_3 , that is, $s(A : B_2) = 1$.

3.5. Towers

Consider a tower $\mathbb{T} = (A_1 = k, A_2, \dots, A_n = A)$ of access to A which satisfies that $A_{i+1} = A_i[M_i]$ or $[M_i]A_i$ for some exceptional A_i -module M_i . The *towering index* of \mathbb{T} is defined as

$$s(\mathbb{T}) = \sum_{i=1}^{n-1} s(A_{i+1} : A_i).$$

Theorem 1 states that $s(\mathbb{T})$ is an invariant depending only on the derived class of A . More precisely, we have:

Theorem. For any tower \mathbb{T} of access to A , we have $s(\mathbb{T}) = n - 1 - a_2$, where a_2 is the coefficient of quadratic order in the Coxeter polynomial $\chi_A(T)$.

Proof. Consider a tower $\mathbb{T} = (A_1 = k, A_2, \dots, A_n = A)$ of access to A . Assume $A = A_n$ and $B = A_{n-1}$ such that $A = B[M]$ for M an exceptional B -module and assume B is accessible. Then $\chi_A(T) = (1 + T)\chi_B(T) - Tp(T)$ where $p(T)$ is the polynomial of the extension $A = B[M]$.

Write $\chi_B(T) = 1 + T + \sum_{i=2}^{n-3} b_i T^i + T^{n-2} + T^{n-1}$ and $p(T) = 1 + \sum_{i=1}^{n-3} p_i T^i + T^{n-2}$. Since $\mathbb{T}' = (A_0, A_1, \dots, A_{n-1} = B)$ is a tower of access to B , by induction hypothesis we may assume that $s(\mathbb{T}') = n - 2 - b_2$. On the other hand, the quadratic coefficient of $\chi_A(T)$ is $a_2 = b_2 + 1 - p_1$ with $p_1 = s(A : B)$. Then

$$s(\mathbb{T}) = s(\mathbb{T}') + s(A : B) = (n - 2 - b_2) + (b_2 + 1 - a_2) = n - 1 - a_2. \quad \square$$

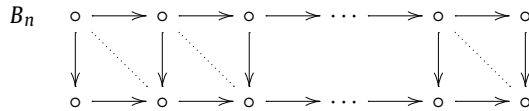
3.6. Special accessible algebras

A tower $\mathbb{T} = (A_1 = k, A_2, \dots, A_n = A)$ of access to A is *special* if $A_{i+1} = A_i[M_i]$ (resp. $A_{i+1} = [M_i]A_i$) for a special exceptional A_i -module M_i , for $i = 0, 1, \dots, n-1$. Assume that the perpendicular category M_i^\perp (resp. ${}^\perp M_i$) of M_i in $\text{Der}(\text{mod } A_i)$ is equivalent to $\text{Der}(\text{mod } C_{i-1})$. The existence of a special tower of access to A makes A a *special accessible algebra*.

In the above situation, as shown in [13], the Coxeter polynomial of A can be reconstructed from the Coxeter polynomials of the various C_i , $1 \leq i \leq n-2$ as:

$$\chi_A(T) = (1+T)^n - T \sum_{i=1}^{n-2} (1+T)^{n-2-i} \chi_{C_i}(T).$$

There are many examples of *special accessible algebras*. Typical examples are poset algebras, such as B_n given by the following quiver with commutativity relations:

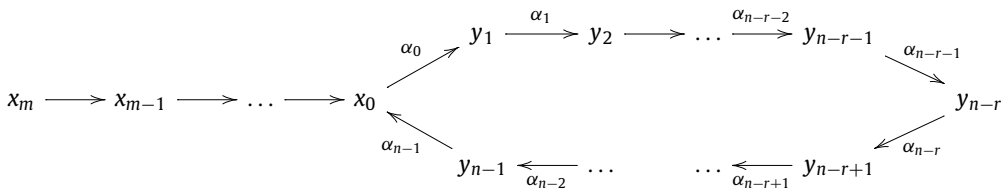


3.7. Derived discrete algebras

Let A be a k -algebra. Following [17], A is said to be *derived discrete* if for every vector $d = (d_i)_{i \in \mathbb{Z}}$ of natural numbers there are only finitely many isomorphism classes of indecomposable objects $X \in \text{Der}^b(\text{mod } A)$ with homology $(H_i X)_{i \in \mathbb{Z}}$ of dimension d . Clearly, A is derived discrete if and only if every algebra B which is derived equivalent to A is representation-finite. The classification of derived discrete algebras is known [17,4]:

(a) algebras A derived equivalent to $k[\tilde{\Delta}]$ such that Δ is a Dynkin diagram (algebras of derived Dynkin type);

(b) algebras A derived equivalent to $\Lambda(r, n, m)$ given by the following quiver and relations



with $\alpha_{n-1}\alpha_0 = 0, \alpha_{n-2}\alpha_{n-1} = 0, \dots, \alpha_{n-r}\alpha_{n-r+1} = 0$.

The Auslander–Reiten quiver of $\Lambda(r, n, m)$ has oriented cycles. Therefore an accessible algebra A is derived discrete if and only if A is of derived Dynkin type. In particular the Euler form q_A is positive definite and $\text{Spec } \varphi_A \subset \mathbb{S}^1 \setminus \{1\}$. In fact, the converse holds:

Theorem. Let A be an accessible algebra. The following are equivalent:

- (i) A is derived discrete;
- (ii) A is of derived Dynkin type;
- (iii) the Euler form q_A is positive definite;
- (iv) there is an accessible tower of algebras $A_1 = k, A_2, \dots, A_s = A$ all of derived Dynkin type;
- (v) there is an accessible tower of algebras $A_1 = k, A_2, \dots, A_s = A$ with $\text{Spec } \varphi_{A_i} \subset \mathbb{S}^1 \setminus \{1\}$ for all i .

Moreover, in that case, A is special accessible.

Proof. (i) \Leftrightarrow (ii) \Rightarrow (iii) were already shown. If $A = B[M]$ with B an accessible algebra and A derived discrete, then B is derived discrete. This shows that (i) \Rightarrow (iv) and (v). Moreover (iv) \Rightarrow (i) follows from (ii) \Rightarrow (i).

(iii) \Rightarrow (i): Since $\text{gl dim } A$ is finite, $\text{Der}(\text{mod } A) = \underline{\text{mod}} \hat{A}$ the stable category of modules over the repetitive algebra \hat{A} . By [2], to show that \hat{A} is locally representation-finite it is enough to check that all reflections $S_{i_r}^+ \dots S_{i_1}^+ A$ are representation-finite.

We recall that the reflection $S_i^+ A$ of A , at the sink i of the quiver of A , is the quotient of the one-point extension $A(I_i)$ by the ideal generated by e_i . Then A and $S_i^+ A$ are derived equivalent. It is enough to show that for representation-finite accessible algebra B any reflection $S_i^+ B$ is again so. It was shown in [3] that $S_i^+ B$ satisfies the separation condition and $q_{S_i^+ B} = q_B$ is positive. Hence $S_i^+ B$ is representation-finite and $H^1(S_i^+ B) = 0$ by [16]. Then Proposition (2.4) shows the claim.

(v) \Rightarrow (i): Let $A_1 = k, A_2, \dots, A_s = A$ be an accessible tower with either $A_{i+1} = A_i[M_i]$ or $[M_i]A_i$ for some exceptional module M_i and $\text{Spec } \varphi_{A_i} \subset \mathbb{S}^1 \setminus \{1\}$. We proceed by induction on s , the case $s = 1$ being clear. Suppose $A = B[M]$ with B accessible of derived Dynkin type and $\text{Spec } \varphi_A \subset \mathbb{S}^1 \setminus \{1\}$. As shown in [13] (see also 3.8), A is derived equivalent to a hereditary algebra $H = k[\Delta]$ of some tree graph Δ . If Δ is of wild type then $\rho_A > 1$; if Δ is extended Dynkin then $1 \in \text{Spec } \varphi_A$. Hence Δ is of Dynkin type.

To show that (i) implies that A is special accessible, we may assume that $A = B[M]$ is a one-point extension of a special accessible algebra B by an indecomposable module M and A is of derived Dynkin type. We have to show that M is a special exceptional module and $M^\perp = \text{Der}(\text{mod } C)$ for an algebra C . Indeed, we may suppose that $F : \text{Der}(\text{mod } B) \rightarrow \text{Der}(\text{mod } H)$ is a triangular equivalence for a hereditary algebra H of Dynkin type. Without loss of generality $F(M)$ is a H -module and $F : M^\perp \rightarrow F(M)^\perp$ is an equivalence and $\text{Der}(F(M)^\perp) = \text{Der}(\text{mod } H')$ for a hereditary algebra H' , necessarily of Dynkin type. \square

3.8. Remarks

(a) It is not true that condition (i) above is equivalent to $\text{Spec } \varphi_A \subset \mathbb{S}^1 \setminus \{1\}$, as the extended canonical algebra B of type (2, 3, 7) shows. Indeed, B is wild but $\chi_A = \Phi_{42}$ is a cyclotomic polynomial whose roots lie in $\mathbb{S}^1 \setminus \{1\}$. See [13].

(b) The following observation shown in [13] is useful.

Lemma. *Let B be an algebra derived equivalent to the path algebra of a Dynkin quiver $\vec{\Delta}$. Let A be a one-point extension or coextension by an indecomposable B -module. Then A is derived equivalent to the path algebra of a quiver $\vec{\Delta}'$ obtained from $\vec{\Delta}$ by adding a new vertex and a new arrow.*

In fact, the above lemma provides another approach to the implication (iii) \Rightarrow (i) of Theorem 3.7: Assume A is an accessible algebra with non-negative Euler form q_A , we want to show that A is derived discrete. Let $A_1 = k, \dots, A_s = A$ be an accessible tower with $A_{i+1} = A_i[M_i]$ or $[M_i]A_i$ for some exceptional module M_i , $i = 1, \dots, s - 1$. In the case A is not derived discrete, there is a last A_r derived Dynkin in the above list, $r < s$. Hence $A_{r+1} = A_r[M_r]$ or $[M_r]A_r$ is derived equivalent to a hereditary algebra $H = k[\vec{\Gamma}]$ with $\vec{\Gamma}$ not a Dynkin graph. Moreover, A is derived equivalent to an algebra C containing H as a convex subcategory (proceed by extensions and coextensions from A_{r+1} to $A_s = A$). Since H is not representation-finite, neither is C . Hence A is not derived discrete. \square

4. On the quadratic coefficient of the Coxeter polynomial of an algebra

4.1. The following expression of the quadratic coefficient of the Coxeter polynomial is useful.

Proposition. *Let A be an algebra and let a_2 be the quadratic coefficient of the Coxeter matrix $\chi_A(T)$. Then*

$$a_2 = \frac{1}{2}(\text{tr}(\phi_A^2) - (\text{tr}(\phi_A))^2).$$

Proof. Let $\chi_A(T) = T^n + a_1 T^{n-1} + a_2 T^{n-2} + \cdots + a_{n-1} T + 1$ with roots $\lambda_1, \dots, \lambda_n$, the eigenvalues of ϕ_A . Let s_k be the sum of the k -th powers of the eigenvalues of ϕ_A . The first Newton's formulas read:

$$s_1 + a_1 = 0 \quad \text{and} \quad s_2 + a_1 s_1 + 2a_2 = 0.$$

Therefore

$$a_2 = -\frac{1}{2}(s_2 - s_1^2) = -\frac{1}{2}(\text{tr}(\phi_A^2) - (\text{tr}(\phi_A))^2)$$

yields the result. \square

4.2. Similarly to (3.1) we have the following:

Proposition. Let $A = B[M]$ be a one-point extension satisfying:

- (a) B is special accessible and M is a special exceptional B -module;
- (b) $\text{gl.dim. } B \leq 2$;
- (c) $H^1(C) = 0$.

Consider a_2 (resp. b_2) the quadratic coefficient of $\chi_A(T)$ (resp. $\chi_B(T)$). Then $a_2 \leq b_2 \leq 1$. Moreover, equality $a_2 = b_2$ holds if and only if C is connected with vanishing Hochschild cohomology.

Proof. Let C be the perpendicular restriction of B by M . Then $p(T) = \chi_C(T)$ is the polynomial of the extension $A = B[M]$. The linear coefficient of $p(T)$ is $p_1 = b_2 + 1 - a_2$. By induction hypothesis, $b_2 \leq 1$ and $a_2 \leq 2 - p_1$.

Associated to the full exact embedding $\text{Der}(\text{mod } C) \rightarrow \text{Der}(\text{mod } B)$ there is a homological epimorphism $B \rightarrow C$, see [7, (4.4)]. In particular, $\text{gl.dim. } C \leq 2$ and $H^i(C) = 0$, for $i \geq 3$. Therefore

$$p_1 = \sum_{i=0}^2 (-1)^i \dim_k H^i(C) = \dim_k H^0(C) + \dim_k H^2(C) \geq 1.$$

Moreover, $a_2 = b_2$ if and only if $p_1 = 1$ if and only if C is connected and $H^i(C) = 0$, for $i > 1$. \square

4.3. Recall that an *extended canonical algebra* of weight type $\langle p_1, \dots, p_t \rangle$ is a one-point extension of the canonical algebra of weight type $[p_1, \dots, p_t]$ by an indecomposable projective module. As in (1.3), the extended canonical algebras of type $\langle p_1, p_2, p_3 \rangle$ is special accessible. Moreover, the extended canonical algebra A of type $\langle 3, 4, 5 \rangle$ (with 12 points) has Coxeter polynomial $1 + T + T^2 + \cdots + T^{12}$ which is also the Coxeter polynomial of a linear hereditary algebra H with 12 vertices. Clearly A and H are not derived equivalent.

Theorem. Let A be an accessible algebra with n vertices and let $\chi_A(T) = \sum_{i=0}^n a_i(A) T^i$ be the Coxeter polynomial of A . The following are equivalent:

- (i) for any tower $\mathbb{T} = (A_1 = k, \dots, A_{n-1}, A_n = A)$ of access to A the coefficients $a_2(A_i) = 1$, for all $1 \leq i \leq n$;
- (ii) for any tower $\mathbb{T} = (A_1 = k, \dots, A_{n-1}, A_n = A)$ of access to A the coefficients $a_2(A_i) > 0$, for all $1 \leq i \leq n$;
- (iii) A is derived equivalent to a quiver algebra of type \mathbb{A}_n .

Proof. We know that an algebra A derived equivalent to a quiver algebra of type \mathbb{A}_n has $\chi_A(T) = \sum_{i=0}^n T^i$, in particular, $a_2(A) = 1$. Hence only (ii) implies (iii) deserves a proof.

Assume that A is accessible and $A = B[M]$ with M exceptional. By induction hypothesis, B is derived equivalent to \mathbb{A}_{n-1} and $a_2 = a_2(A) > 0$. In particular, $b_2 = a_2(B) = 1$ and B is representation-finite with a preprojective component \mathcal{P} such that the orbit graph $\mathcal{O}(\mathcal{P})^\tau$ is of type \mathbb{A}_{n-1} (recall that the orbit graph has vertices as the τ -orbits in the quiver \mathcal{P} with Auslander–Reiten translation τ and there is an edge between the orbit of X and the orbit of Y if there is some numbers a, b and an irreducible morphism $\tau^a X \rightarrow \tau^b Y$). Observe that for any X in $\text{Der}(\text{mod } A)$ not in the orbit of M , there is some translation $\tau^a X$ belonging to M^\perp , implying that in the case M^τ has two neighbours in the orbit graph then M^\perp is not connected, that is $s(A : B) > 1$ and $a_2 = b_2 + 1 - s(A : B) \leq 0$, a contradiction. Therefore, M^τ has just one neighbour in $\mathcal{O}(\mathcal{P})^\tau$, hence A is derived of type \mathbb{A}_n . \square

4.4. The following generalizes a result of Happel who considers the case of derived hereditary algebras [10].

Corollary. *Let A be a piecewise hereditary algebra with $a_2(A) > 0$, then A is derived equivalent to \mathbb{A}_n .*

Proof. By (4.2), $A = B[M]$ for some exceptional B -module M and a piecewise hereditary algebra B with $a_2(B) \geq a_2(A) > 0$. By induction hypothesis, A satisfies the conditions of (4.3). \square

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