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Iterated twisted tensor products of nonlocal vertex algebras

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ABSTRACT

This is the second paper in a series devoted to studies of twisted tensor products of nonlocal vertex algebras. In this paper we introduce and study iterated twisted tensor products of nonlocal vertex algebras. Among the main results, we find conditions for constructing an iterated twisted tensor product of three factors, and prove that those conditions are enough for building an iterated twisted tensor product of any number of factors. And we also establish a universal property and give a characterization of an iterated twisted tensor product. Furthermore, we give an example of iterated twisted tensor product nonlocal vertex algebra.

1. Introduction

Let U and V be two nonlocal vertex algebras. Motivated by a recent study [LS2] on regular representations for Möbius quantum vertex algebras, a twisting operator in [LS1] was defined to be a linear map $R(x): V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ satisfying

$$R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v, \qquad R(x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1} \quad \text{for } u \in U, v \in V,$$
$$R(x_1)\big(1 \otimes Y(x_2)\big) = \big(Y(x_2) \otimes 1\big) R^{23}(x_1) R^{12}(x_1 + x_2),$$
$$R(x_1)\big(Y(x_2) \otimes 1\big) = \big(1 \otimes Y(x_2)\big) R^{12}(x_1 - x_2) R^{23}(x_1).$$

These conditions are stringy analogues of those for a twisting operator with associative algebras (see [CSV,VV]). The underlying space of the twisted tensor product $U \otimes_R V$ associated to $R(x)$ is $U \otimes V$, while the vacuum vector is $\mathbf{1}_U \otimes \mathbf{1}_V$ and the vertex operator map, denoted by Y_R, is given by

$$Y_R(u \otimes v, x)\big(u' \otimes v'\big) = \big(Y_U(x) \otimes Y_V(x)\big)\big(u \otimes R(x)\big(v \otimes u'\big) \otimes v'\big)$$

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for $u, u' \in U$, $v, v' \in V$. It was proved in [LS1] that $U \otimes_R V$ is a nonlocal vertex algebra, containing U and V canonically as subalgebras which satisfy a certain commutation relation. (If both U and V are weak quantum vertex algebras, it was proved that $U \otimes_R V$ is a weak quantum vertex algebra.) On the other hand, it was proved that if a nonlocal vertex algebra K, which is non-degenerate in the sense of [EK], contains subalgebras U and V satisfying a certain commutation relation, then K is isomorphic to the twisted tensor product $U \otimes_R V$ with respect to a twisting operator $R(x)$. Also established in that paper was a universal property for the twisted tensor product $U \otimes_R V$, similar to the one for the ordinary tensor product $U \otimes V$. The smash product $U \sharp V$, formulated in [L4], was also slightly generalized and realized as the twisted tensor product with respect to a canonical twisting operator.

For associative algebras, iterating twisted tensor products was studied in [MPPO]. For three given algebras A, B and C, and twisting maps $R_1 : B \otimes A \to A \otimes B$, $R_2 : C \otimes B \to B \otimes C$ and $R_3 : C \otimes A \to A \otimes C$, a sufficient condition for being able to define twisting maps $T_1 : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C$ and $T_2 : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C)$ associated to R_1, R_2 and R_3 and ensuring that the algebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ are equal, was given in terms of the twisting maps R_1, R_2 and R_3 only. That is, they have to satisfy the hexagon equation

$$R_2^{23} R_3^{12} R_1^{23} = R_1^{12} R_3^{23} R_2^{12}.$$

Furthermore, it was proved that they are enough for building an iterated twisted tensor product of any number of factors. And a universal property and the Coherence Theorem were proved in [MPPO]. It was showed that the noncommutative $2n$-planes defined by Connes and Dubois-Violette (see [CD]) could be seen as (iterated) twisted tensor products of commutative algebras.

In this paper, we study iterated twisted tensor products of nonlocal vertex algebras and of weak quantum vertex algebras. Let U, V and W be three nonlocal vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for the ordered pair (U, V), (V, W) and (U, W), respectively. We show that U, V and W can be iterated to be twisted tensor product with three factors, denote it by $U \otimes_{R_1} V \otimes_{R_2} W$, if $R_1(x)$, $R_2(x)$ and $R_3(x)$ satisfy the following condition

$$R_2^{23}(x_1 - x_2) R_3^{12}(x_1) R_1^{23}(x_2) = R_1^{12}(x_2) R_3^{23}(x_1) R_2^{12}(x_1 - x_2),$$

which is stringy analogues of that listed above for iterating twisted tensor products of associative algebras. This condition is closely related with the quantum Yang–Baxter equation (see [L2,L3]). And we find conditions for twisting operators $T_1(x) : W \otimes (U \otimes_{R_1} V) \to (U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x))$ and $T_2(x) : (V \otimes_{R_2} W) \otimes U \to U \otimes (V \otimes_{R_2} W) \otimes \mathbb{C}((x))$ can be split as a composition of two suitable twisting operators. If U, V and W are weak quantum vertex algebras, it is proved that $U \otimes_{R_1} V \otimes_{R_2} W$ is a weak quantum vertex algebra. Furthermore, we give out some iterated twisted tensor product nonlocal vertex algebras, isomorphic to nonlocal vertex algebra $U \otimes_{R_1} V \otimes_{R_2} W$, with some suitable conditions.

We prove that the iterated twisted tensor product of three factors can be lifted to that of any number of factors with compatible twisting operators. Also established in this paper is a universal property for the iterated twisted tensor product $V_1 \otimes_{R_{12}} V_2 \otimes_{R_{23}} \cdots \otimes_{R_{n-1,n}} V_n$, similar to the one for the ordinary tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. And at last we show that a toy example of weak quantum vertex algebra, noncommutative $2n$-planes defined by Connes and Dubois-Violette, can be realized as iterated twisted tensor product of some nonlocal vertex algebras with suitable twisting operators.

This paper is organized as follows: In Section 2, we present some basic notions and iterated twisted tensor product. In Section 3, we study the splitting and universal properties of iterated twisted tensor products of nonlocal vertex algebras. In Section 4, we present n-factor iterated twisted tensor products.

2. Iterated twisted tensor product

In this section, first we recall the notion of twisting operator and the twisted tensor product nonlocal vertex algebras. Then we establish a theorem to characterize the iterated twisted tensor products of nonlocal vertex algebras.

We start by recalling the notion of nonlocal vertex algebra. A *nonlocal vertex algebra* (see [L1], cf. [BK]) is a vector space V, equipped with a linear map

$$\begin{aligned} Y(\cdot,x): V &\to \mathrm{Hom}\big(V, V((x))\big) \subset (\mathrm{End}\, V)\big[\big[x,x^{-1}\big]\big] \\ v &\mapsto Y(v,x)=\sum_{n\in\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \mathrm{End}\, V), \end{aligned}$$

and equipped with a vector $\mathbf{1}\in V$, satisfying the conditions that for $v\in V$,

$$Y(\mathbf{1},x)v=v, \qquad Y(v,x)\mathbf{1}\in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v,x)\mathbf{1}=v, \tag{2.1}$$

and that for $u,v,w\in V$, there exists a nonnegative integer l such that

$$(x_0+x_2)^l Y(u,x_0+x_2)Y(v,x_2)w=(x_0+x_2)^l Y\big(Y(u,x_0)v,x_2\big)w \tag{2.2}$$

(*weak associativity*).

We sometimes denote a nonlocal vertex algebra by a triple $(V,Y,\mathbf{1})$, to emphasize the *vertex operator map* Y and the *vacuum vector* $\mathbf{1}$.

Let V be a nonlocal vertex algebra. Define a linear operator $\mathcal{D}$ on V by

$$\mathcal{D}(v)=v_{-2}\mathbf{1} \quad \text{for } v\in V. \tag{2.3}$$

Then

$$\big[\mathcal{D},Y(v,x)\big]=Y(\mathcal{D}v,x)=\frac{d}{dx}Y(v,x) \quad \text{for } v\in V. \tag{2.4}$$

Let U and V be two nonlocal vertex algebras. We have an (ordinary) tensor product nonlocal vertex algebra $U\otimes V$, where the vacuum vector is $\mathbf{1}\otimes\mathbf{1}$ and the vertex operator map is given by

$$Y(u\otimes v,x)\big(u'\otimes v'\big)=Y(u,x)u'\otimes Y(v,x)v' \quad \text{for } u,u'\in U,\ v,v'\in V.$$

That is,

$$Y_{U\otimes V}(x)=\big(Y_U(x)\otimes Y_V(x)\big)\sigma^{23},$$

where σ^{23} is the linear operator on $(U\otimes V)^{\otimes 2}$, defined by

$$\sigma^{23}\big(u\otimes v\otimes u'\otimes v'\big)=u\otimes u'\otimes v\otimes v'$$

for $u,u'\in U$, $v,v'\in V$.

Next, we recall from [LS1] the notion of twisting operator.

Definition 2.1. Let U and V be nonlocal vertex algebras. A *twisting operator* for the ordered pair (U, V) is a linear map

$$R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x)),$$

satisfying the following conditions:

$$R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v \quad \text{for } v \in V, \tag{2.5}$$

$$R(x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1} \quad \text{for } u \in U, \tag{2.6}$$

$$R(x_1)\big(1 \otimes Y(x_2)\big) = \big(Y(x_2) \otimes 1\big)R^{23}(x_1)R^{12}(x_1 + x_2), \tag{2.7}$$

$$R(x_1)\big(Y(x_2) \otimes 1\big) = \big(1 \otimes Y(x_2)\big)R^{12}(x_1 - x_2)R^{23}(x_1). \tag{2.8}$$

We say that a twisting operator $R(x)$ is *invertible* if $R(x)$, viewed as a $\mathbb{C}((x))$-linear map from $V \otimes U \otimes \mathbb{C}((x))$ to $U \otimes V \otimes \mathbb{C}((x))$, is invertible. The inverse of an invertible $R(x)$ is a $\mathbb{C}((x))$-linear map $R^{-1}(x)$ from $U \otimes V \otimes \mathbb{C}((x))$ to $V \otimes U \otimes \mathbb{C}((x))$. We often consider $R^{-1}(x)$ as a $\mathbb{C}$-linear map

$$R^{-1}(x) : U \otimes V \to V \otimes U \otimes \mathbb{C}((x)).$$

We shall need the following result of [LS1]:

Lemma 2.2. *If $R(x)$ is an invertible twisting operator for the ordered pair (U, V), then $R^{-1}(-x)$ is an invertible twisting operator for the ordered pair (V, U).*

In the following, we combine (2.7) and (2.8) into one condition.

Lemma 2.3. *In the definition of the notion of twisting operator, the conditions (2.7) and (2.8) can be replaced by*

$$\begin{aligned} &R(x_1)\big(Y(x_2) \otimes Y(x_2)\big) \\ &\quad = \big(Y(x_2) \otimes Y(x_2)\big)R^{23}(x_1 - x_2)R^{12}(x_1)R^{34}(x_1)R^{23}(x_1 + x_2). \end{aligned} \tag{2.9}$$

Proof. For $v \in V$, $u, u' \in U$, using (2.6) we have

$$\begin{gathered} R(x_1)\big(Y(x_2) \otimes Y(x_2)\big)\big(\mathbf{1} \otimes v \otimes u \otimes u'\big) = R(x_1)\big(v \otimes Y(u, x_2)u'\big), \\ \big(Y(x_2) \otimes Y(x_2)\big)R^{23}(x_1 - x_2)R^{12}(x_1)R^{34}(x_1)R^{23}(x_1 + x_2)\big(\mathbf{1} \otimes v \otimes u \otimes u'\big) \\ = \big(Y(x_2) \otimes 1\big)R^{23}(x_1)R^{12}(x_1 + x_2)\big(v \otimes u \otimes u'\big). \end{gathered}$$

Then we get (2.7). Similarly, for $v, v' \in V$, $u \in U$, using (2.5) we have

$$\begin{gathered} R(x_1)\big(Y(x_2) \otimes Y(x_2)\big)\big(v \otimes v' \otimes \mathbf{1} \otimes u\big) = R(x_1)\big(Y(v, x_2)v' \otimes u\big), \\ \big(Y(x_2) \otimes Y(x_2)\big)R^{23}(x_1 - x_2)R^{12}(x_1)R^{34}(x_1)R^{23}(x_1 + x_2)\big(v \otimes v' \otimes \mathbf{1} \otimes u\big) \\ = \big(1 \otimes Y(x_2)\big)R^{12}(x_1 - x_2)R^{23}(x_1)\big(v \otimes v' \otimes u\big). \end{gathered}$$

Then we obtain (2.8).

Now we prove (2.9). For $v, v' \in V$, $u, u' \in U$, using (2.7) and (2.8) we get

$$\begin{aligned}
&R(x_1)\big(Y(x_2)\otimes Y(x_2)\big)\big(v\otimes v'\otimes u\otimes u'\big)\\
&\quad = R(x_1)\big(1\otimes Y(x_2)\big)\big(Y(v,x_2)v'\otimes u\otimes u'\big)\\
&\quad = \big(Y(x_2)\otimes 1\big)R^{23}(x_1)R^{12}(x_1+x_2)\big(Y(x_2)\otimes 1\otimes 1\big)\big(v\otimes v'\otimes u\otimes u'\big)\\
&\quad = \big(Y(x_2)\otimes 1\big)R^{23}(x_1)\big(1\otimes Y(x_2)\otimes 1\big)R^{12}(x_1)R^{23}(x_1+x_2)\big(v\otimes v'\otimes u\otimes u'\big)\\
&\quad = \big(Y(x_2)\otimes 1\big)\big(1\otimes 1\otimes Y(x_2)\big)R^{23}(x_1-x_2)R^{34}(x_1)R^{12}(x_1)R^{23}(x_1+x_2)\big(v\otimes v'\otimes u\otimes u'\big)\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\big)R^{23}(x_1-x_2)R^{12}(x_1)R^{34}(x_1)R^{23}(x_1+x_2)\big(v\otimes v'\otimes u\otimes u'\big).
\end{aligned}$$

This proves (2.9). □

We now present the twisted tensor product from [LS1].

Theorem 2.4. *Let U, V be nonlocal vertex algebras and let $R(x)$ be a twisting operator of the ordered pair (U, V). Set*

$$Y_R(x) = \big(Y(x)\otimes Y(x)\big)R^{23}(-x). \tag{2.10}$$

Then $(U\otimes V, Y_R, \mathbf{1}\otimes\mathbf{1})$, denoted by $U\otimes_R V$, carries the structure of a nonlocal vertex algebra, which contains U and V canonically as nonlocal vertex subalgebras.

Let U, V and W be nonlocal vertex algebras, let

$$\begin{aligned}
&R_1(x): V\otimes U \to U\otimes V\otimes \mathbb{C}((x)),\\
&R_2(x): W\otimes V \to V\otimes W\otimes \mathbb{C}((x)),\\
&R_3(x): W\otimes U \to U\otimes W\otimes \mathbb{C}((x))
\end{aligned}$$

be twisting operators for the ordered pairs (U, V), (V, W) and (U, W), respectively. Define the operators

$$\begin{aligned}
&T_1(x) = R_2^{23}(x)R_3^{12}(x): W\otimes (U\otimes_{R_1} V) \to (U\otimes_{R_1} V)\otimes W\otimes \mathbb{C}((x)),\\
&T_2(x) = R_1^{12}(x)R_3^{23}(x): (V\otimes_{R_2} W)\otimes U \to U\otimes (V\otimes_{R_2} W)\otimes \mathbb{C}((x)).
\end{aligned}$$

It is natural to ask if these operators are twisting operators. In the following theorem, we present necessary and sufficient conditions for making this happen.

The following is our main result of this section.

Theorem 2.5. *Let U, V and W be nonlocal vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for the ordered pairs (U, V), (V, W) and (U, W), respectively. Then the following conditions are equivalent:*

(1) *$T_1(x) = R_2^{23}(x)R_3^{12}(x)$ is a twisting operator.*
(2) *$T_2(x) = R_1^{12}(x)R_3^{23}(x)$ is a twisting operator.*
(3) *The twisting operators $R_1(x)$, $R_2(x)$ and $R_3(x)$ satisfy the following compatibility condition (called the hexagon equation):*

$$R_2^{23}(x_1-x_2)R_3^{12}(x_1)R_1^{23}(x_2) = R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1-x_2). \tag{2.11}$$

Furthermore, if all the three conditions are satisfied, then the nonlocal vertex algebras $U \otimes_{T_2} (V \otimes_{R_2} W)$ and $(U \otimes_{R_1} V) \otimes_{T_1} W$ are equal. In this case, we will denote this nonlocal vertex algebra by $U \otimes_{R_1} V \otimes_{R_2} W$, which contains U, V and W canonically as nonlocal vertex subalgebras.

Proof. We only prove the equivalence between (1) and (3). The proof of the equivalence between (2) and (3) is similar to the proof of the equivalence between (1) and (3), given below, so we omit it.

Suppose that the hexagon equation is satisfied. In order to prove that $T_1(x)$ is a twisting operator, we have to prove the following relations:

$$T_1(x)(w \otimes \mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes w \quad \text{for } w \in W, \tag{2.12}$$

$$T_1(x)(\mathbf{1} \otimes u \otimes v) = u \otimes v \otimes \mathbf{1} \quad \text{for } u \otimes v \in U \otimes_{R_1} V, \tag{2.13}$$

$$T_1(x_1)\big(1 \otimes Y_{R_1}(x_2)\big) = \big(Y_{R_1}(x_2) \otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1 + x_2), \tag{2.14}$$

$$T_1(x_1)\big(Y(x_2) \otimes 1 \otimes 1\big) = \big(1 \otimes 1 \otimes Y(x_2)\big)T_1^{12}(x_1 - x_2)T_1^{23}(x_1). \tag{2.15}$$

For (2.12), from (2.5) we have

$$\begin{aligned} T_1(x)(w \otimes \mathbf{1} \otimes \mathbf{1}) &= R_2^{23}(x)R_3^{12}(x)(w \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= R_2^{23}(x)(\mathbf{1} \otimes w \otimes \mathbf{1}) \\ &= \mathbf{1} \otimes \mathbf{1} \otimes w. \end{aligned}$$

For (2.13), from (2.6) we get

$$\begin{aligned} T_1(x)(\mathbf{1} \otimes u \otimes v) &= R_2^{23}(x)R_3^{12}(x)(\mathbf{1} \otimes u \otimes v) \\ &= R_2^{23}(x)(u \otimes \mathbf{1} \otimes v) \\ &= u \otimes v \otimes \mathbf{1}. \end{aligned}$$

To prove (2.14), for $u, u' \in U$, $v, v' \in V$, $w \in W$, using (2.7) we have

$$\begin{aligned} &T_1(x_1)\big(1 \otimes Y_{R_1}(x_2)\big)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = R_2^{23}(x_1)R_3^{12}(x_1)\big(1 \otimes Y(x_2) \otimes Y(x_2)\big)R_1^{34}(-x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = R_2^{23}(x_1)\big(Y(x_2) \otimes 1 \otimes Y(x_2)\big)R_3^{23}(x_1)R_3^{12}(x_1 + x_2)R_1^{34}(-x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = \big(Y(x_2) \otimes Y(x_2) \otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1 + x_2)R_3^{23}(x_1)R_3^{12}(x_1 + x_2)R_1^{34}(-x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = \big(Y(x_2) \otimes Y(x_2) \otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1 + x_2)R_3^{23}(x_1)R_1^{34}(-x_2)R_3^{12}(x_1 + x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big), \end{aligned}$$

$$\begin{aligned} &\big(Y_{R_1}(x_2) \otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1 + x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = \big(Y(x_2) \otimes Y(x_2) \otimes 1\big)R_1^{23}(-x_2)R_2^{45}(x_1)R_3^{34}(x_1)R_2^{23}(x_1 + x_2)R_3^{12}(x_1 + x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big) \\ &\quad = \big(Y(x_2) \otimes Y(x_2) \otimes 1\big)R_2^{45}(x_1)R_1^{23}(-x_2)R_3^{34}(x_1)R_2^{23}(x_1 + x_2)R_3^{12}(x_1 + x_2)\big(w \otimes u \otimes v \otimes u' \otimes v'\big). \end{aligned}$$

By the hexagon equation, we obtain (2.14).

Concerning (2.15), for $u \in U$, $v \in V$, $w, w' \in W$, using (2.8) we get

$$\begin{aligned}
&T_1(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes u\otimes v\big)\\
&\quad = R_2^{23}(x_1)R_3^{12}(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes u\otimes v\big)\\
&\quad = R_2^{23}(x_1)\big(1\otimes Y(x_2)\otimes 1\big)R_3^{12}(x_1-x_2)R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes v\big)\\
&\quad = \big(1\otimes 1\otimes Y(x_2)\big)R_2^{23}(x_1-x_2)R_2^{34}(x_1)R_3^{12}(x_1-x_2)R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes v\big),
\end{aligned}$$

$$\begin{aligned}
&\big(1\otimes 1\otimes Y(x_2)\big)T_1^{12}(x_1-x_2)T_1^{23}(x_1)\big(w\otimes w'\otimes u\otimes v\big)\\
&\quad = \big(1\otimes 1\otimes Y(x_2)\big)T_1^{12}(x_1-x_2)R_2^{34}(x_1)R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes v\big)\\
&\quad = \big(1\otimes 1\otimes Y(x_2)\big)R_2^{23}(x_1-x_2)R_3^{12}(x_1-x_2)R_2^{34}(x_1)R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes v\big).
\end{aligned}$$

It is evident that (2.15) holds.

Now we assume $T_1(x)$ is a twisting operator, satisfying the conditions (2.12), (2.13), (2.14) and (2.15). It is enough to apply (2.14) to an element of the form $w\otimes \mathbf{1}\otimes v\otimes u\otimes \mathbf{1}$ in order to recover the hexagon equation for a generic element $w\otimes v\otimes u$ of the tensor product $W\otimes V\otimes U$. For $u\in U$, $v\in V$, $w\in W$, we have

$$\begin{aligned}
&T_1(x_1)\big(1\otimes Y_{R_1}(x_2)\big)(w\otimes \mathbf{1}\otimes v\otimes u\otimes \mathbf{1})\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1+x_2)R_3^{23}(x_1)R_1^{34}(-x_2)R_3^{12}(x_1+x_2)(w\otimes \mathbf{1}\otimes v\otimes u\otimes \mathbf{1})\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1+x_2)R_3^{23}(x_1)R_1^{34}(-x_2)(\mathbf{1}\otimes w\otimes v\otimes u\otimes \mathbf{1}),
\end{aligned}$$

$$\begin{aligned}
&\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1+x_2)(w\otimes \mathbf{1}\otimes v\otimes u\otimes \mathbf{1})\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_1^{23}(-x_2)R_3^{34}(x_1)R_2^{23}(x_1+x_2)R_3^{12}(x_1+x_2)(w\otimes \mathbf{1}\otimes v\otimes u\otimes \mathbf{1})\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_1^{23}(-x_2)R_3^{34}(x_1)R_2^{23}(x_1+x_2)(\mathbf{1}\otimes w\otimes v\otimes u\otimes \mathbf{1}).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1+x_2)R_3^{23}(x_1)R_1^{34}(-x_2)(\mathbf{1}\otimes w\otimes v\otimes u\otimes \mathbf{1})\\
&\quad = \big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_1^{23}(-x_2)R_3^{34}(x_1)R_2^{23}(x_1+x_2)(\mathbf{1}\otimes w\otimes v\otimes u\otimes \mathbf{1}).
\end{aligned}$$

Thus by the injectivity of vertex operator and (2.5), we get

$$R_2^{23}(x_1-x_2)R_3^{12}(x_1)R_1^{23}(x_2)(w\otimes v\otimes u) = R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1-x_2)(w\otimes v\otimes u)$$

for any $u\in U$, $v\in V$ and $w\in W$.

To finish the proof, assume that the three equivalent conditions are satisfied. To see that the non-local vertex algebras $U\otimes_{T_2}(V\otimes_{R_2}W)$ and $(U\otimes_{R_1}V)\otimes_{T_1}W$ are equal, it is enough to expand the expression of the vertex operators

$$\begin{aligned}
Y_{T_2}(x) &= \big(Y(x)\otimes Y_{R_2}(x)\big)T_2^{23}(-x)\\
&= \big(Y(x)\otimes Y_{R_2}(x)\big)R_1^{23}(-x)R_3^{34}(-x)\\
&= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x),
\end{aligned}$$

$$\begin{aligned} Y_{T_1}(x) &= \big(Y_{R_1}(x)\otimes Y(x)\big)T_1^{23}(-x)\\ &= \big(Y_{R_1}(x)\otimes Y(x)\big)R_2^{45}(-x)R_3^{34}(-x)\\ &= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_1^{23}(-x)R_2^{45}(-x)R_3^{34}(-x), \end{aligned}$$

and realize that they are exactly the same application. Furthermore, for $u, u' \in U$, as $R_3(-x)(\mathbf{1}\otimes u') = u'\otimes \mathbf{1}$, $R_1(-x)(\mathbf{1}\otimes u') = u'\otimes \mathbf{1}$ and $R_2(-x)(\mathbf{1}\otimes \mathbf{1}) = \mathbf{1}\otimes \mathbf{1}$, we have

$$\begin{aligned} &Y_{T_2}(u\otimes \mathbf{1}\otimes \mathbf{1}, x)\big(u'\otimes \mathbf{1}\otimes \mathbf{1}\big)\\ &\quad= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x)\big(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}\otimes \mathbf{1}\big)\\ &\quad= Y(u,x)u'\otimes \mathbf{1}\otimes \mathbf{1}. \end{aligned}$$

It follows that the map $u\in U \mapsto u\otimes \mathbf{1}\otimes \mathbf{1}\in U\otimes V\otimes W$ is a one-to-one homomorphism of nonlocal vertex algebras. Similarly, the maps $v\in V\mapsto \mathbf{1}\otimes v\otimes \mathbf{1}\in U\otimes V\otimes W$ and $w\in W\mapsto \mathbf{1}\otimes \mathbf{1}\otimes w\in U\otimes V\otimes W$ are one-to-one homomorphisms of nonlocal vertex algebras. This concludes the proof. □

Remark 2.6. If we take U, V and W to be the same nonlocal vertex algebra V, and take $R_1(x)$, $R_2(x)$ and $R_3(x)$ to be the same twisting operator $R(x)$, the hexagon equation (2.11) becomes

$$R^{23}(x_1-x_2)R^{12}(x_1)R^{23}(x_2) = R^{12}(x_2)R^{23}(x_1)R^{12}(x_1-x_2). \tag{2.16}$$

It is well known that the relation (2.16) is equivalent to the quantum Yang–Baxter equation

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z) = \mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x), \tag{2.17}$$

with $\mathcal{S}(x) = R(x)\sigma$.

We identify each element u of U with the element $u\otimes \mathbf{1}\otimes \mathbf{1}$ of $U\otimes_{R_1} V\otimes_{R_2} W$, identify each element v of V with $\mathbf{1}\otimes v\otimes \mathbf{1}$ of $U\otimes_{R_1} V\otimes_{R_2} W$ and identify each element w of W with $\mathbf{1}\otimes \mathbf{1}\otimes w$ of $U\otimes_{R_1} V\otimes_{R_2} W$.

Lemma 2.7. *The $\mathcal{D}$-operator for $U\otimes_{R_1} V\otimes_{R_2} W$ is given by*

$$\mathcal{D}_T = \mathcal{D}\otimes 1\otimes 1 + 1\otimes \mathcal{D}\otimes 1 + 1\otimes 1\otimes \mathcal{D},$$

where the three $\mathcal{D}$'s denote the $\mathcal{D}$-operators of U, V and W, respectively. Furthermore, we have

$$Y_{T_2}(u,x)v,\ Y_{T_2}(v,x)w,\ Y_{T_2}(u,x)w \in (U\otimes_{R_1} V\otimes_{R_2} W)[[x]], \tag{2.18}$$

$$Y_{T_2}(v,x)u = e^{x\mathcal{D}_T}Y_{T_2}(-x)R_1(-x)(v\otimes u), \tag{2.19}$$

$$Y_{T_2}(w,x)v = e^{x\mathcal{D}_T}Y_{T_2}(-x)R_2(-x)(w\otimes v), \tag{2.20}$$

$$Y_{T_2}(w,x)u = e^{x\mathcal{D}_T}Y_{T_2}(-x)R_3(-x)(w\otimes u) \tag{2.21}$$

for $u\in U$, $v\in V$, $w\in W$.

Proof. Let $u \in U$, $v \in V$, $w \in W$. From definition we have

$$\begin{aligned}
&\mathrm{Res}_x x^{-2} Y_{T_2}(u \otimes v \otimes w, x)(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\
&\quad = \mathrm{Res}_x x^{-2} \big(Y(x) \otimes Y(x) \otimes Y(x)\big) R_2^{45}(-x) R_1^{23}(-x) R_3^{34}(-x)(u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\
&\quad = \mathrm{Res}_x x^{-2} \big(Y(u,x)\mathbf{1} \otimes Y(v,x)\mathbf{1} \otimes Y(w,x)\mathbf{1}\big) \\
&\quad = (\mathcal{D} \otimes 1 \otimes 1 + 1 \otimes \mathcal{D} \otimes 1 + 1 \otimes 1 \otimes \mathcal{D})(u \otimes v \otimes w).
\end{aligned} \tag{2.22}$$

For $u \in U$, $v \in V$, by $R_3(-x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1}$ and $R_2(-x)(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, we have

$$\begin{aligned}
Y_{T_2}(u,x)v &= \big(Y(x) \otimes Y(x) \otimes Y(x)\big) R_2^{45}(-x) R_1^{23}(-x) R_3^{34}(-x)(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes v \otimes \mathbf{1}) \\
&= Y(u,x)\mathbf{1} \otimes v \otimes \mathbf{1} \in (U \otimes_{R_1} V \otimes_{R_2} W)[[x]].
\end{aligned} \tag{2.23}$$

Similarly, we can prove $Y_{T_2}(v,x)w,\ Y_{T_2}(u,x)w \in (U \otimes_{R_1} V \otimes_{R_2} W)[[x]]$.

Let $u \in U$, $v, v' \in V$. As $R_3(x)(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $R_1(x)(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $R_2(x)(\mathbf{1} \otimes v') = v' \otimes \mathbf{1}$ and (2.23), we have

$$\begin{aligned}
&e^{x\mathcal{D}_T} Y_{T_2}(-x) R_1(-x)(v \otimes u) \\
&\quad = \big(e^{x\mathcal{D}} Y(-x) \otimes e^{x\mathcal{D}} \otimes e^{x\mathcal{D}}\big) R_1^{13}(-x)(v \otimes \mathbf{1} \otimes u \otimes \mathbf{1}) \\
&\quad = \big(1 \otimes Y(x) \otimes 1\big) R_1^{12}(-x)(v \otimes u \otimes \mathbf{1} \otimes \mathbf{1}) \\
&\quad = \big(Y(x) \otimes Y(x) \otimes Y(x)\big) R_2^{45}(-x) R_1^{23}(-x) R_3^{34}(-x)(\mathbf{1} \otimes v \otimes \mathbf{1} \otimes u \otimes \mathbf{1} \otimes \mathbf{1}) \\
&\quad = Y_{T_2}(v,x)u.
\end{aligned}$$

We can prove (2.20) and (2.21) in the same way. □

Next we study iterated twisted tensor product $U \otimes_{R_1} V \otimes_{R_2} W$ with U, V and W weak quantum vertex algebras.

First we recall the notion of weak quantum vertex algebra from [L2].

Definition 2.8. A *weak quantum vertex algebra* is a nonlocal vertex algebra V which satisfies $\mathcal{S}$*-locality* in the sense that for $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

(finitely many) such that

$$(x_1 - x_2)^k Y(u,x_1) Y(v,x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1) Y\big(v^{(i)}, x_2\big) Y\big(u^{(i)}, x_1\big) \tag{2.24}$$

for some nonnegative integer k.

The following basic facts can be found in [L2]:

Proposition 2.9. *Let V be a nonlocal vertex algebra and let*

$$u, v, u^{(i)}, v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r).$$

Then the $\mathcal{S}$-locality relation (2.24) is equivalent to

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\sum_{i=1}^{r} f_i(-x_0)Y\big(v^{(i)}, x_2\big)Y\big(u^{(i)}, x_1\big)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y\big(Y(u, x_0)v, x_2\big) \tag{2.25}$$

(*the $\mathcal{S}$*-Jacobi identity), *and it is also equivalent to*

$$Y(u, x)v = e^{x\mathcal{D}}\sum_{i=1}^{r} f_i(-x)Y\big(v^{(i)}, -x\big)u^{(i)} \tag{2.26}$$

(*the $\mathcal{S}$*-skew symmetry).

Remark 2.10. From definition, a nonlocal vertex algebra V is a weak quantum vertex algebra if and only if there exists a linear map

$$\mathcal{S}(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}\big((x)\big)$$

satisfying the condition that for $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y(x_1)\big(1 \otimes Y(x_2)\big)(u \otimes v \otimes w)$$
$$= (x_1 - x_2)^k Y(x_2)\big(1 \otimes Y(x_1)\big)\mathcal{S}^{12}(x_2 - x_1)(v \otimes u \otimes w) \tag{2.27}$$

for every $w \in V$, or equivalently,

$$Y(x)(u \otimes v) = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u). \tag{2.28}$$

The following is lifted from [LS1]:

Proposition 2.11. *Let U and V be weak quantum vertex algebras and let $R(x)$ be an invertible twisting operator for (U, V). Then $U \otimes_R V$ is a weak quantum vertex algebra.*

From Proposition 2.11, we immediately have:

Proposition 2.12. *Let U, V and W be weak quantum vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be invertible twisting operators for the ordered pairs (U, V), (V, W) and (U, W), respectively, such that*

$$R_2^{23}(x_1 - x_2)R_3^{12}(x_1)R_1^{23}(x_2) = R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1 - x_2).$$

Then $U \otimes_{T_2} (V \otimes_{R_2} W)$ and $(U \otimes_{R_1} V) \otimes_{T_1} W$ are isomorphic weak quantum vertex algebras, with twisting operators $T_2(x) = R_2^{23}(x)R_3^{12}(x)$ and $T_1(x) = R_1^{12}(x)R_3^{23}(x)$, respectively.

3. Splitting and universal properties

In this section we study the splitting property of twisting operators $T_1(x)$ and $T_2(x)$. Then we establish a universal property and give a characterization of the iterated twisted tensor product. Furthermore, we discuss the isomorphic relations of some iterated twisted tensor product nonlocal vertex algebras.

A natural question that arises is whether we have a twisting operator $T_1(x): W \otimes (U \otimes_{R_1} V) \to (U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x))$, it splits as a composition of two suitable twisting operators.

Theorem 3.1 *(Right splitting). Let U, V and W be nonlocal vertex algebras, let $R_1(x): V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ and $T_1(x): W \otimes (U \otimes_{R_1} V) \to (U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x))$ be twisting operators. The followings are equivalent:*

(1) *There exist twisting operators $R_2(x): W \otimes V \to V \otimes W \otimes \mathbb{C}((x))$ and $R_3(x): W \otimes U \to U \otimes W \otimes \mathbb{C}((x))$ such that*

$$T_1(x) = R_2^{23}(x) R_3^{12}(x). \tag{3.1}$$

(2) *The twisting operator $T_1(x)$ satisfies the (right) splitting conditions*:

$$T_1(x)\big(W \otimes (U \otimes \mathbf{1})\big) \subseteq (U \otimes \mathbf{1}) \otimes W \otimes \mathbb{C}((x)), \tag{3.2}$$

$$T_1(x)\big(W \otimes (\mathbf{1} \otimes V)\big) \subseteq (\mathbf{1} \otimes V) \otimes W \otimes \mathbb{C}((x)). \tag{3.3}$$

Proof. Using (2.5), (2.6) and the definition of $T_1(x)$, we have (3.2) and (3.3) directly.

Now we assume that the twisting operator $T_1(x)$ satisfies the right splitting conditions (3.2) and (3.3). By (3.3), the operator $R_2(x): W \otimes V \to V \otimes W \otimes \mathbb{C}((x))$ given as the unique $\mathbb{C}$-linear map such that

$$\big(Y(x) \otimes 1 \otimes 1\big) R_2^{34}(x) \sigma^{23} \sigma^{12} = T_1(x)\big(1 \otimes Y(x) \otimes 1\big)$$

is well defined, as

$$\begin{aligned} R_2^{23}(x)(\mathbf{1} \otimes w \otimes v) &= \big(Y(x) \otimes 1 \otimes 1\big) R_2^{34}(x)(\mathbf{1} \otimes \mathbf{1} \otimes w \otimes v) \\ &= \big(Y(x) \otimes 1 \otimes 1\big) R_2^{34}(x) \sigma^{23} \sigma^{12}(w \otimes \mathbf{1} \otimes \mathbf{1} \otimes v) \\ &= T_1(x)\big(1 \otimes Y(x) \otimes 1\big)(w \otimes \mathbf{1} \otimes \mathbf{1} \otimes v) \\ &= T_1(x)(w \otimes \mathbf{1} \otimes v) \in (\mathbf{1} \otimes V) \otimes W \otimes \mathbb{C}((x)) \end{aligned}$$

for $v \in V$, $w \in W$. Then we have

$$R_2(x)(w \otimes v) \in V \otimes W \otimes \mathbb{C}((x)).$$

From the fact that $T_1(x)$ is a twisting operator from $W \otimes (U \otimes_{R_1} V)$ to $(U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x))$, it is immediately deduced that also $R_2(x)$ is a twisting operator.

Using (2.12), we have

$$R_2^{23}(x)(\mathbf{1} \otimes w \otimes \mathbf{1}) = T_1(x)(w \otimes \mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes w,$$

thus

$$R_2(x)(w\otimes\mathbf{1})=\mathbf{1}\otimes w.$$

Using (2.13), we get

$$R_2^{23}(x)(\mathbf{1}\otimes\mathbf{1}\otimes v)=T_1(x)(\mathbf{1}\otimes\mathbf{1}\otimes v)=\mathbf{1}\otimes v\otimes\mathbf{1},$$

then

$$R_2(x)(\mathbf{1}\otimes v)=v\otimes\mathbf{1}.$$

By $R_1(x)(v\otimes\mathbf{1})=\mathbf{1}\otimes v$, we have

$$\begin{aligned}
&T_1(x_1)\big(1\otimes Y_{R_1}(x_2)\big)\big(w\otimes\mathbf{1}\otimes v\otimes\mathbf{1}\otimes v'\big)\\
&\quad=T_1(x_1)\big(1\otimes Y(x_2)\otimes Y(x_2)\big)R_1^{34}(x_2)\big(w\otimes\mathbf{1}\otimes v\otimes\mathbf{1}\otimes v'\big)\\
&\quad=T_1(x_1)\big(1\otimes Y(x_2)\otimes Y(x_2)\big)\big(w\otimes\mathbf{1}\otimes\mathbf{1}\otimes v\otimes v'\big)\\
&\quad=T_1(x_1)\big(w\otimes\mathbf{1}\otimes Y(v,x_2)v'\big)\\
&\quad=R_2^{23}(x_1)\big(\mathbf{1}\otimes w\otimes Y(v,x_2)v'\big)\\
&\quad=R_2^{23}(x_1)\big(1\otimes 1\otimes Y(x_2)\big)\big(\mathbf{1}\otimes w\otimes v\otimes v'\big),
\end{aligned}$$

$$\begin{aligned}
&\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1+x_2)\big(w\otimes\mathbf{1}\otimes v\otimes\mathbf{1}\otimes v'\big)\\
&\quad=\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)R_2^{23}(x_1+x_2)\big(\mathbf{1}\otimes w\otimes v\otimes\mathbf{1}\otimes v'\big)\\
&\quad=\big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_2^{45}(x_1)R_2^{34}(x_1+x_2)\big(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v\otimes v'\big)\\
&\quad=\big(1\otimes Y(x_2)\otimes 1\big)R_2^{34}(x_1)R_2^{23}(x_1+x_2)\big(\mathbf{1}\otimes w\otimes v\otimes v'\big),
\end{aligned}$$

then

$$R_2(x_1)\big(1\otimes Y(x_2)\big)=\big(Y(x_2)\otimes 1\big)R_2^{23}(x_1)R_2^{12}(x_1+x_2).$$

And we have

$$\begin{aligned}
&T_1(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes\mathbf{1}\otimes v\big)\\
&\quad=T_1(x_1)\big(Y(w,x_2)w'\otimes\mathbf{1}\otimes v\big)\\
&\quad=R_2^{23}(x_1)\big(\mathbf{1}\otimes Y(w,x_2)w'\otimes v\big)\\
&\quad=R_2^{23}(x_1)\big(1\otimes Y(x_2)\otimes 1\big)\big(\mathbf{1}\otimes w\otimes w'\otimes v\big),
\end{aligned}$$

$$\begin{aligned}
&\big(1\otimes 1\otimes Y(x_2)\big)T_1^{12}(x_1-x_2)T_1^{23}(x_1)\big(w\otimes w'\otimes\mathbf{1}\otimes v\big)\\
&\quad=\big(1\otimes 1\otimes Y(x_2)\big)T_1^{12}(x_1-x_2)R_2^{34}(x_1)\big(w\otimes\mathbf{1}\otimes w'\otimes v\big)\\
&\quad=\big(1\otimes 1\otimes Y(x_2)\big)R_2^{23}(x_1-x_2)R_2^{34}(x_1)\big(\mathbf{1}\otimes w\otimes w'\otimes v\big),
\end{aligned}$$

thus

$$R_2(x_1)\big(Y(x_2)\otimes 1\big)=\big(1\otimes Y(x_2)\big)R_2^{12}(x_1-x_2)R_2^{23}(x_1).$$

Analogously, we can define $R_3(x): W\otimes U\to U\otimes W\otimes\mathbb{C}((x))$ as the unique $\mathbb{C}$-linear map such that

$$\sigma^{23}R_3^{12}(x)\big(1\otimes 1\otimes Y(x)\big)=T_1(x)\big(1\otimes 1\otimes Y(x)\big),$$

which is also a well defined twisting operator, by (3.2), as we have

$$\begin{aligned}\sigma R_3^{12}(x)(w\otimes u\otimes \mathbf{1})&=\sigma R_3^{12}(x)\big(1\otimes 1\otimes Y(x)\big)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1})\\&=T_1(x)\big(1\otimes 1\otimes Y(x)\big)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1})\\&=T_1(x)(w\otimes u\otimes \mathbf{1})\in(U\otimes \mathbf{1})\otimes W\otimes\mathbb{C}((x))\end{aligned}$$

for $u\in U$, $w\in W$. Then

$$R_3(x)(w\otimes u)\in U\otimes W\otimes\mathbb{C}((x)).$$

Using (2.12), we have

$$\mathbf{1}\otimes\mathbf{1}\otimes w=T_1(x)(w\otimes\mathbf{1}\otimes\mathbf{1})=\sigma^{23}R_3^{12}(x)(w\otimes\mathbf{1}\otimes\mathbf{1}),$$

that is, $R_3^{12}(x)(w\otimes\mathbf{1}\otimes\mathbf{1})=\mathbf{1}\otimes w\otimes\mathbf{1}$, then

$$R_3(x)(w\otimes\mathbf{1})=\mathbf{1}\otimes w.$$

By (2.13), we get

$$u\otimes\mathbf{1}\otimes\mathbf{1}=T_1(x)(\mathbf{1}\otimes u\otimes\mathbf{1})=\sigma^{23}R_3^{12}(\mathbf{1}\otimes u\otimes\mathbf{1}),$$

that is, $R_3^{12}(x)(\mathbf{1}\otimes u\otimes\mathbf{1})=u\otimes\mathbf{1}\otimes\mathbf{1}$, then

$$R_3(x)(\mathbf{1}\otimes u)=u\otimes\mathbf{1}.$$

By $R_1(x)(\mathbf{1}\otimes u)=u\otimes\mathbf{1}$, we have

$$\begin{aligned}&\sigma^{23}T_1(x_1)\big(1\otimes Y_{R_1}(x_2)\big)\big(w\otimes u\otimes\mathbf{1}\otimes u'\otimes\mathbf{1}\big)\\&\quad=\sigma^{23}T_1(x_1)\big(1\otimes Y(x_2)\otimes Y(x_2)\big)\big(w\otimes u\otimes u'\otimes\mathbf{1}\otimes\mathbf{1}\big)\\&\quad=\sigma^{23}T_1(x_1)\big(1\otimes Y(x_2)\otimes 1\big)\big(w\otimes u\otimes u'\otimes\mathbf{1}\big)\\&\quad=\sigma^{23}\sigma^{23}R_3^{12}(x_1)\big(1\otimes Y(x_2)\otimes 1\big)\big(w\otimes u\otimes u'\otimes\mathbf{1}\big)\\&\quad=R_3^{12}(x_1)\big(1\otimes Y(x_2)\otimes 1\big)\big(w\otimes u\otimes u'\otimes\mathbf{1}\big),\end{aligned}$$

$$\begin{aligned}
&\sigma^{23}\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1+x_2)\big(w\otimes u\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}\big)\\
&\quad=\sigma^{23}\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)\big(w\otimes u\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}\big)\\
&\quad=\sigma^{23}\big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_1^{23}(x_2)\sigma^{45}R_3^{34}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)\big(w\otimes u\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}\big)\\
&\quad=\big(Y(x_2)\otimes 1\otimes Y(x_2)\big)\sigma^{34}\sigma^{45}R_1^{23}(x_2)\sigma^{45}R_3^{34}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)\big(w\otimes u\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}\big)\\
&\quad=\big(Y(x_2)\otimes 1\otimes Y(x_2)\big)R_3^{23}(x_1)R_3^{12}(x_1+x_2)\big(w\otimes u\otimes u'\otimes \mathbf{1}\otimes \mathbf{1}\big)\\
&\quad=\big(Y(x_2)\otimes 1\otimes 1\big)R_3^{23}(x_1)R_3^{12}(x_1+x_2)\big(w\otimes u\otimes u'\otimes \mathbf{1}\big),
\end{aligned}$$

thus

$$R_3^{12}(x_1)\big(1\otimes Y(x_2)\big)=\big(Y(x_2)\otimes 1\big)R_3^{23}(x_1)R_3^{12}(x_1+x_2).$$

And we get

$$\begin{aligned}
&\sigma^{23}T_1(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big)\\
&\quad=\sigma^{23}\sigma^{23}R_3^{12}(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big)\\
&\quad=R_3^{12}(x_1)\big(Y(x_2)\otimes 1\otimes 1\big)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big),
\end{aligned}$$

$$\begin{aligned}
&\sigma^{23}\big(1\otimes 1\otimes Y(x_2)\big)T_1^{12}(x_1-x_2)T_1^{23}(x_1)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big)\\
&\quad=\big(1\otimes Y(x_2)\otimes 1\big)\sigma^{34}\sigma^{23}T_1^{12}(x_1-x_2)\sigma^{34}R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big)\\
&\quad=\big(1\otimes Y(x_2)\otimes 1\big)\sigma^{34}\sigma^{23}\sigma^{23}R_3^{12}(x_1-x_2)\sigma^{34}R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big)\\
&\quad=\big(1\otimes Y(x_2)\otimes 1\big)R_3^{12}(x_1-x_2)R_3^{23}(x_1)\big(w\otimes w'\otimes u\otimes \mathbf{1}\big),
\end{aligned}$$

then

$$R_3^{12}\big(Y(x_2)\otimes 1\big)=\big(1\otimes Y(x_2)\big)R_3^{12}(x_1-x_2)R_3^{23}(x_1).$$

Now we only have to check that $T_1(x)=R_2^{23}(x)R_3^{12}(x)$. Using $R_1(x)(\mathbf{1}\otimes\mathbf{1})=\mathbf{1}\otimes\mathbf{1}$ and the definitions of $R_2(x)$ and $R_3(x)$, we have

$$\begin{aligned}
&\lim_{x_2\to 0}T_1(x_1)\big(1\otimes Y_{R_1}(x_2)\big)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}T_1(x_1)\big(1\otimes Y(x_2)\otimes Y(x_2)\big)R_1^{34}(x_2)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}T_1(x_1)\big(1\otimes Y(x_2)\otimes Y(x_2)\big)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}T_1(x_1)\big(w\otimes Y(u,x_2)\mathbf{1}\otimes Y(\mathbf{1},x_2)v\big)\\
&\quad=\lim_{x_2\to 0}T_1(x_1)\big(w\otimes Y(u,x_2)\mathbf{1}\otimes v\big)\\
&\quad=T_1(x_1)(w\otimes u\otimes v),
\end{aligned}$$

$$\begin{aligned}
&\lim_{x_2\to 0}\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)T_1^{12}(x_1+x_2)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}\big(Y_{R_1}(x_2)\otimes 1\big)T_1^{23}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}\big(Y(x_2)\otimes Y(x_2)\otimes 1\big)R_1^{23}(x_2)T_1^{23}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=\lim_{x_2\to 0}\big(Y(x_2)\otimes Y(x_2)\otimes 1\big)\sigma^{23}(x_2)T_1^{23}(x_1)\sigma^{23}R_3^{12}(x_1+x_2)(w\otimes u\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\
&\quad=R_2^{23}(x_1)R_3^{12}(x_1)(w\otimes u\otimes v).
\end{aligned}$$

By (2.14), we get

$$T_1(x)=R_2^{23}(x)R_3^{12}(x).$$

This concludes the proof. □

Of course, there exists an analogous left splitting theorem, that we state for completeness, and whose proof is analogous to the former one.

Theorem 3.2 *(Left splitting). Let U, V and W be nonlocal vertex algebras, let $R_2(x): W\otimes V\to V\otimes W\otimes \mathbb{C}((x))$ and $T_2(x): (V\otimes_{R_2} W)\otimes U\to U\otimes (V\otimes_{R_2} W)\otimes \mathbb{C}((x))$ be twisting operators. The followings are equivalent:*

(1) *There exist twisting operators $R_1(x): V\otimes U\to U\otimes V\otimes \mathbb{C}((x))$ and $R_3(x): W\otimes U\to U\otimes W\otimes \mathbb{C}((x))$ such that $T_2(x)=R_1^{12}(x)R_3^{23}(x)$.*
(2) *The twisting operator $T_2(x)$ satisfies the (left) splitting conditions:*

$$T_2(x)\big((\mathbf{1}\otimes W)\otimes U\big)\subseteq U\otimes (\mathbf{1}\otimes W)\otimes \mathbb{C}((x)), \tag{3.4}$$

$$T_2(x)\big((V\otimes \mathbf{1})\otimes U\big)\subseteq U\otimes (V\otimes \mathbf{1})\otimes \mathbb{C}((x)). \tag{3.5}$$

The iterated twisted tensor product nonlocal vertex algebra $U\otimes_{R_1} V\otimes_{R_2} W$ has a universal property just as the ordinary iterated tensor product $U\otimes V\otimes W$ does (cf. [FHL,LL]).

Theorem 3.3. *Let U, V, W be nonlocal vertex algebras and let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for (U,V), (V,W) and (U,W) respectively, $T_2(x)$ be a twisting operator for $(U, V\otimes_{R_2} W)$. Let K be a nonlocal vertex algebra and let $\psi_1: U\to K$, $\psi_2: V\to K$, $\psi_3: W\to K$ be any homomorphisms of nonlocal vertex algebras, satisfying the conditions that for $u\in U$, $v\in V$, $w\in W$,*

$$Y\big(\psi_2(v),x\big)\psi_3(w),\ Y\big(\psi_1(u),x\big)\big(\psi_2(v)_{-1}\psi_3(w)\big)\in K[[x]], \tag{3.6}$$

$$Y(x)(\psi_2\otimes\psi_1)=e^{x\mathcal{D}}Y(-x)(\psi_1\otimes\psi_2)R_1(-x), \tag{3.7}$$

$$Y(x)(\psi_3\otimes\psi_2)=e^{x\mathcal{D}}Y(-x)(\psi_2\otimes\psi_3)R_2(-x), \tag{3.8}$$

$$Y(x)(\psi_3\otimes\psi_1)=e^{x\mathcal{D}}Y(-x)(\psi_1\otimes\psi_3)R_3(-x). \tag{3.9}$$

Then the linear map $\psi: U\otimes_{T_2}(V\otimes_{R_2} W)\to K$, defined by

$$\psi(u\otimes v\otimes w)=\psi_1(u)_{-1}\psi_2(v)_{-1}\psi_3(w)\quad \text{for } u\in U,\ v\in V,\ w\in W$$

is a homomorphism of nonlocal vertex algebras, which extends ψ_1, ψ_2 and ψ_3 uniquely.

Proof. For $u \in U$, $v \in V$, $w \in W$, as $Y(\psi_2(v), x)\psi_3(w)$, $Y(\psi_1(u), x)(\psi_2(v)_{-1}\psi_3(w)) \in K[[x]]$ by assumption, we have

$$\begin{aligned}\psi(u \otimes v \otimes w) &= \operatorname{Res}_x x^{-1} Y\big(\psi_1(u), x\big)\big(\psi_2(v)_{-1}\psi_3(w)\big)\\ &= \lim_{x\to 0} Y\big(\psi_1(u), x\big)\big(\psi_2(v)_{-1}\psi_3(w)\big).\end{aligned}$$

It is clear that the linear map ψ extends ψ_1, ψ_2 and ψ_3. It is also clear that $\psi(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$. To prove that ψ is a homomorphism of nonlocal vertex algebras, we must prove

$$\psi\big(Y_{T_2}(u \otimes v \otimes w, x)\big(u' \otimes v' \otimes w'\big)\big) = Y\big(\psi(u \otimes v \otimes w), x\big)\psi\big(u' \otimes v' \otimes w'\big) \tag{3.10}$$

for $u, u' \in U$, $v, v' \in V$, $w, w' \in W$. Through the homomorphisms ψ_1, ψ_2 and ψ_3, K can be regarded as a U-module, a V-module and a W-module, respectively. Then, for $u', u \in U$, $v \in V$, $w \in W$, we have

$$\begin{aligned}&\psi\big(Y_{T_2}\big(u' \otimes \mathbf{1} \otimes \mathbf{1}, x_0\big)(u \otimes v \otimes w)\big)\\ &= \psi\big(\big(Y(x_0) \otimes Y_{R_2}(x_0)\big)T_2^{23}(-x_0)\big(u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x_0) \otimes Y_{R_2}(x_0)\big)\big(u' \otimes u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x_0) \otimes Y(x_0) \otimes Y(x_0)\big)R_2^{45}(x_0)\big(u' \otimes u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x_0) \otimes Y(x_0) \otimes Y(x_0)\big)\big(u' \otimes u \otimes \mathbf{1} \otimes v \otimes \mathbf{1} \otimes w\big)\big)\\ &= \psi\big(\big(Y(x_0) \otimes 1 \otimes 1\big)\big(u' \otimes u \otimes v \otimes w\big)\big)\\ &= \psi\big(Y\big(u', x_0\big)u \otimes v \otimes w\big)\\ &= \lim_{x\to 0} Y\big(\psi\big(Y\big(u', x_0\big)u\big), x\big)\psi_2(v)_{-1}\psi_3(w)\\ &= \lim_{x\to 0} Y\big(Y\big(u', x_0\big)u, x\big)\psi_2(v)_{-1}\psi_3(w)\\ &= \lim_{x\to 0} Y\big(u', x_0 + x\big)Y(u, x)\psi_2(v)_{-1}\psi_3(w)\\ &= \lim_{x\to 0} Y\big(\psi_1\big(u'\big), x_0 + x\big)Y\big(\psi_1(u), x\big)\psi_2(v)_{-1}\psi_3(w)\\ &= Y\big(\psi_1\big(u'\big), x_0\big)\psi_1(u)_{-1}\psi_2(v)_{-1}\psi_3(w).\end{aligned}$$

This shows that (3.10) holds with $u \otimes v \otimes w = u \otimes \mathbf{1} \otimes \mathbf{1}$.

We next show that (3.10) also holds with $u \otimes v \otimes w = \mathbf{1} \otimes v \otimes \mathbf{1}$. We have

$$\begin{aligned}&\psi\big(Y_{T_2}\big(\mathbf{1} \otimes v' \otimes \mathbf{1}, x\big)(u \otimes v \otimes w)\big)\\ &= \psi\big(\big(Y(x) \otimes Y_{R_2}(x)\big)T_2^{23}(-x)\big(\mathbf{1} \otimes v' \otimes \mathbf{1} \otimes u \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x) \otimes Y_{R_2}(x)\big)R_1^{23}(-x)R_3^{34}(-x)\big(\mathbf{1} \otimes v' \otimes \mathbf{1} \otimes u \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x) \otimes Y(x) \otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)\big(\mathbf{1} \otimes v' \otimes u \otimes \mathbf{1} \otimes v \otimes w\big)\big)\\ &= \psi\big(\big(Y(x) \otimes Y(x) \otimes Y(x)\big)R_1^{23}(-x)\big(\mathbf{1} \otimes v' \otimes u \otimes v \otimes \mathbf{1} \otimes w\big)\big)\\ &= \psi\big(\big(1 \otimes Y(x) \otimes 1\big)R_1^{12}(-x)\big(v' \otimes u \otimes v \otimes w\big)\big)\\ &= \lim_{x_2\to 0} Y(x_2)\big(1 \otimes Y(x)\big)(\psi_1 \otimes \psi_2 \otimes 1)R_1^{12}(-x)\big(v' \otimes u \otimes \big(\psi_2(v)_{-1}\psi_3(w)\big)\big).\end{aligned} \tag{3.11}$$

With the assumption (3.7), from [L2] (Proposition 5.2), there exists a nonnegative integer k such that

$$\begin{aligned}
&(x_2-x)^k Y\big(\psi_2\big(v'\big),x\big)Y\big(\psi_1(u),x_2\big)\big(\psi_2(v)_{-1}\psi_3(w)\big)\\
&\quad=(x_2-x)^k Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes 1)R_1^{12}(x_2-x)\big(v'\otimes u\otimes\big(\psi_2(v)_{-1}\psi_3(w)\big)\big).
\end{aligned}$$

Noticing that $R_1(x)(v'\otimes u)\in U\otimes V\otimes\mathbb{C}((x))$, we may replace k with a bigger integer so that $x^k R_1(x)(v'\otimes u)\in U\otimes V\otimes\mathbb{C}[[x]]$ also holds. Then

$$\begin{aligned}
&(-x)^k Y\big(\psi\big(\mathbf{1}\otimes v'\otimes\mathbf{1}\big),x\big)\psi(u\otimes v\otimes w)\\
&\quad=\lim_{x_2\to 0}(x_2-x)^k Y\big(\psi_2\big(v'\big),x\big)Y\big(\psi_1(u),x_2\big)\big(\psi_2(v)_{-1}\psi_3(w)\big)\\
&\quad=\lim_{x_2\to 0}(x_2-x)^k Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes 1)R_1^{12}(x_2-x)\big(v'\otimes u\otimes\big(\psi_2(v)_{-1}\psi_3(w)\big)\big)\\
&\quad=(-x)^k\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes 1)R_1^{12}(-x)\big(v'\otimes u\otimes\big(\psi_2(v)_{-1}\psi_3(w)\big)\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&Y\big(\psi\big(\mathbf{1}\otimes v'\otimes\mathbf{1}\big),x\big)\psi(u\otimes v\otimes w)\\
&\quad=\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes 1)R^{12}(-x)\big(v'\otimes u\otimes\big(\psi_2(v)_{-1}\psi_3(w)\big)\big).
\end{aligned}$$

Combining this with (3.11) we obtain

$$\psi\big(Y_{T_2}\big(\mathbf{1}\otimes v'\otimes\mathbf{1},x\big)(u\otimes v\otimes w)\big)=Y\big(\psi\big(\mathbf{1}\otimes v'\otimes\mathbf{1}\big),x\big)\psi(u\otimes v\otimes w),$$

proving that (3.10) holds with $u\otimes v\otimes w=\mathbf{1}\otimes v\otimes\mathbf{1}$.

At last we show that (3.10) also holds with $u\otimes v\otimes w=\mathbf{1}\otimes\mathbf{1}\otimes w$. We have

$$\begin{aligned}
&\psi\big(Y_{T_2}\big(\mathbf{1}\otimes\mathbf{1}\otimes w',x\big)(u\otimes v\otimes w)\big)\\
&\quad=\psi\big(\big(Y(x)\otimes Y_{R_2}(x)\big)T_2^{23}(-x)\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\otimes u\otimes v\otimes w\big)\big)\\
&\quad=\psi\big(\big(Y(x)\otimes Y_{R_2}(x)\big)R_1^{23}(-x)R_3^{34}(-x)\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\otimes u\otimes v\otimes w\big)\big)\\
&\quad=\psi\big(\big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x)\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\otimes u\otimes v\otimes w\big)\big)\\
&\quad=\psi\big(\big(1\otimes 1\otimes Y(x)\big)R_2^{23}(-x)R_3^{12}\big(w'\otimes u\otimes v\otimes w\big)\big)\\
&\quad=\lim_{x_1\to 0}\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x_1)\big)\big(1\otimes 1\otimes Y(x)\big)\\
&\qquad\cdot(\psi_1\otimes\psi_2\otimes\psi_3\otimes\psi_3)R_2^{23}(-x)R_3^{12}(-x)\big(w'\otimes u\otimes v\otimes w\big).
\end{aligned}\tag{3.12}$$

With the assumption (3.8) and (3.9), from [L2] (Proposition 5.2), there exists a nonnegative integer k such that

$$\begin{aligned}
&(x_1-x)^k(x_2-x)^k Y\big(\psi_3\big(w'\big),x\big)Y\big(\psi_1(u),x_2\big)Y\big(\psi_2(v),x_1\big)\psi_3(w)\\
&\quad=(x_1-x)^k(x_2-x)^k Y(x_2)\big(1\otimes Y(x_1)\big)\big(1\otimes 1\otimes Y(x)\big)\\
&\qquad\cdot(\psi_1\otimes\psi_2\otimes\psi_3\otimes\psi_3)R_2^{23}(x_1-x)R_3^{12}(x_2-x)\big(w'\otimes u\otimes v\otimes w\big).
\end{aligned}$$

Noticing that $R_3(x)(w'\otimes u)\in U\otimes W\otimes\mathbb{C}((x))$ and $R_2(x)(W\otimes V)\subseteq V\otimes W\otimes\mathbb{C}((x))$, we may replace k with a bigger integer so that $x^kR_3(x)(w\otimes u)\in U\otimes W\otimes\mathbb{C}[[x]]$ and $x^kR_2(x)(W\otimes V)\subseteq V\otimes W\otimes\mathbb{C}[[x]]$ also hold. Then

$$\begin{aligned}
&(-x)^{2k}Y\big(\psi\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\big),x\big)\psi(u\otimes v\otimes w)\\
&\quad=\lim_{x_1\to 0}\lim_{x_2\to 0}(x_1-x)^k(x_2-x)^kY\big(\psi_3\big(w'\big),x\big)Y\big(\psi_1(u),x_2\big)Y\big(\psi_2(v),x_1\big)\psi_3(w)\\
&\quad=\lim_{x_1\to 0}\lim_{x_2\to 0}(x_1-x)^k(x_2-x)^kY(x_2)\big(1\otimes Y(x_1)\big)\big(1\otimes 1\otimes Y(x)\big)\\
&\qquad\cdot(\psi_1\otimes\psi_2\otimes\psi_3\otimes\psi_3)R_2^{23}(x_1-x)R_3^{12}(x_2-x)\big(w'\otimes u\otimes v\otimes w\big)\\
&\quad=(-x)^{2k}\lim_{x_1\to 0}\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x_1)\big)\big(1\otimes 1\otimes Y(x)\big)\\
&\qquad\cdot(\psi_1\otimes\psi_2\otimes\psi_3\otimes\psi_3)R_2^{23}(-x)R_3^{12}(-x)\big(w'\otimes u\otimes v\otimes w\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&Y\big(\psi\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\big),x\big)\psi(u\otimes v\otimes w)\\
&\quad=\lim_{x_1\to 0}\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x_1)\big)\big(1\otimes 1\otimes Y(x)\big)\\
&\qquad\cdot(\psi_1\otimes\psi_2\otimes\psi_3\otimes\psi_3)R_2^{23}(-x)R_3^{12}(-x)\big(w'\otimes u\otimes v\otimes w\big).
\end{aligned}$$

Combining this with (3.12) we obtain

$$\psi\big(Y_{T_2}\big(\mathbf{1}\otimes\mathbf{1}\otimes w',x\big)(u\otimes v\otimes w)\big)=Y\big(\psi\big(\mathbf{1}\otimes\mathbf{1}\otimes w'\big),x\big)\psi(u\otimes v\otimes w),$$

proving that (3.10) holds with $u\otimes v\otimes w=\mathbf{1}\otimes\mathbf{1}\otimes w$. Since $U\otimes_{R_1}V\otimes_{R_2}W$ as a nonlocal vertex algebra is generated by the subset $U\cup V\cup W$, it follows that ψ is a homomorphism of nonlocal vertex algebras from $U\otimes_{R_1}V\otimes_{R_2}W$ to K. The uniqueness assertion is clear as $u\otimes v\otimes w=(u\otimes\mathbf{1}\otimes\mathbf{1})_{-1}(\mathbf{1}\otimes v\otimes\mathbf{1})_{-1}(\mathbf{1}\otimes\mathbf{1}\otimes w)$ for $u\in U$, $v\in V$, $w\in W$. $\square$

The following is a characterization of $U\otimes_{R_1}V\otimes_{R_2}W$ in terms of U, V, W, $R_1(x)$, $R_2(x)$ and $R_3(x)$. The proof is analogous to that of Proposition 2.9 in [LS1].

Proposition 3.4. *Let U, V, W, $R_1(x)$, $R_2(x)$ and $R_3(x)$ be given as in Theorem 3.3, and let K be a nonlocal vertex algebra which contains U, V and W as subalgebras, satisfying*

$$\begin{aligned}
&Y\big(\psi_2(v),x\big)\psi_3(w),\ Y\big(\psi_1(u),x\big)\big(\psi_2(v)_{-1}\psi_3(w)\big)\in K[[x]],\\
&Y(x)(\psi_2\otimes\psi_1)=e^{x\mathcal{D}}Y(-x)(\psi_1\otimes\psi_2)R_1(-x),\\
&Y(x)(\psi_3\otimes\psi_2)=e^{x\mathcal{D}}Y(-x)(\psi_2\otimes\psi_3)R_2(-x),\\
&Y(x)(\psi_3\otimes\psi_1)=e^{x\mathcal{D}}Y(-x)(\psi_1\otimes\psi_3)R_3(-x)
\end{aligned}$$

for $u\in U$, $v\in V$, $w\in W$. Assume that K as a nonlocal vertex algebra is generated by $U\cup V\cup W$ and that U as a U-module is irreducible and of countable dimension (over $\mathbb{C}$). Then the linear map $\theta: U\otimes_{R_1}V\otimes_{R_2}W\to K$, defined by $\theta(u\otimes v\otimes w)=u_{-1}v_{-1}w$ for $u\in U$, $v\in V$, $w\in W$, is a nonlocal vertex algebras isomorphism.

Recall from Lemma 2.2 that for any invertible twisting operator $R_2(x)$ for the ordered pair (V,W), then $R_2^{-1}(-x)$ is an invertible twisting operator for the ordered pair (W,V). Furthermore, we have:

Lemma 3.5. *Let $U \otimes_{R_1} V \otimes_{R_2} W$ be an iterated twisted tensor product nonlocal vertex algebra, let $R_2(x)$ be an invertible twisting operator for (V, W), such that*

$$R_2(x)(w \otimes v) \in V \otimes W \otimes \mathbb{C}[[x]] \tag{3.13}$$

for $v \in V$, $w \in W$. Then $U \otimes_{R_3} W \otimes_{R_2^{-1}(-x)} V$ carries the structure of an iterated twisted tensor product nonlocal vertex algebra.

Proof. Since $U \otimes_{R_1} V \otimes_{R_2} W$ is an iterated twisted tensor product nonlocal vertex algebra, from Theorem 2.5 we know that the hexagon equation holds:

$$R_2^{23}(y_1 - y_2)R_3^{12}(y_1)R_1^{23}(y_2) = R_1^{12}(y_2)R_3^{23}(y_1)R_2^{12}(y_1 - y_2).$$

Using the invertibility of $R_2(x)$, we have

$$R_3^{12}(y_1)R_1^{23}(y_2)\left(R_2^{-1}\right)^{12}(y_1 - y_2) = \left(R_2^{-1}\right)^{23}(y_1 - y_2)R_1^{12}(y_2)R_3^{23}(y_1). \tag{3.14}$$

Taking $y_1 = x_2$, $y_2 = x_1$ in (3.14) and by (3.13), we get

$$\left(R_2^{-1}\right)^{23}(-x_1 + x_2)R_1^{12}(x_1)R_3^{23}(x_2) = R_3^{12}(x_2)R_1^{23}(x_1)\left(R_2^{-1}\right)^{12}(-x_1 + x_2).$$

By Theorem 2.5, we know that $U \otimes_{R_3} W \otimes_{R_2^{-1}(-x)} V$ carries the structure of an iterated twisted tensor product nonlocal vertex algebra with twisting operators $T_1'(x) = (R_2^{-1})^{23}(-x)R_1^{12}(x)$ and $T_2'(x) = R_3^{12}(x)R_1^{23}(x)$. $\square$

The following result gives out the relation between two iterated twisted tensor product nonlocal vertex algebras $U \otimes_{R_1} V \otimes_{R_2} W$ and $U \otimes_{R_3} W \otimes_{R_2^{-1}(-x)} V$.

Proposition 3.6. *Let $U \otimes_{R_1} V \otimes_{R_2} W$ be an iterated twisted tensor product nonlocal vertex algebra, let $R_2(x)$ be an invertible twisting operator for (V, W) such that*

$$R_2(x)(w \otimes v) \in V \otimes W \otimes \mathbb{C}[[x]], \tag{3.15}$$

$$R_2^{-1}(x)(v \otimes w) \in W \otimes V \otimes \mathbb{C}[[x]] \tag{3.16}$$

for $v \in V$, $w \in W$. Then the linear map $\psi : U \otimes_{R_3} W \otimes_{R_2^{-1}(-x)} V \to U \otimes_{R_1} V \otimes_{R_2} W$, defined by

$$\psi(u \otimes w \otimes v) = u_{-1}w_{-1}v \quad (\text{in } U \otimes_{R_1} V \otimes_{R_2} W)$$

for $u \in U$, $v \in V$, $w \in W$, is a nonlocal vertex algebra isomorphism.

Proof. With the assumption (3.15), we get

$$\begin{aligned} Y_{T_2}(w, x)v &= \big(Y(x) \otimes Y(x) \otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes w \otimes \mathbf{1} \otimes v \otimes \mathbf{1}) \\ &= \big(1 \otimes 1 \otimes Y(x)\big)R_2^{23}(-x)(\mathbf{1} \otimes w \otimes v \otimes \mathbf{1}) \in (U \otimes_{R_1} V \otimes_{R_2} W)[[x]], \end{aligned}$$

and

$$\begin{aligned}
Y_{T_2}(u,x)(w_{-1}v) &= \lim_{x_2\to 0}\big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x)R_2^{56}(-x_2)\\
&\quad\cdot(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes w\otimes v)\\
&= \lim_{x_2\to 0}\big(Y(x)\otimes 1\otimes 1\big)R_2^{34}(-x_2)(u\otimes \mathbf{1}\otimes w\otimes v)\\
&\in (U\otimes_{R_1} V\otimes_{R_2} W)[[x]]
\end{aligned}$$

for $u\in U$, $v\in V$, $w\in W$.

From Lemma 2.7 we have

$$\begin{aligned}
Y_{T_2}(w,x)u &= e^{x\mathcal{D}_T}Y_{T_2}(-x)R_3(-x)(w\otimes u),\\
Y_{T_2}(v,x)u &= e^{x\mathcal{D}_T}Y_{T_2}(-x)R_1(-x)(v\otimes u).
\end{aligned}$$

And we can easily have

$$\begin{aligned}
Y_{T_2}(v,x)w &= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)R_2^{45}(-x)R_1^{23}(-x)R_3^{34}(-x)(\mathbf{1}\otimes v\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes w)\\
&= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)(\mathbf{1}\otimes \mathbf{1}\otimes v\otimes \mathbf{1}\otimes \mathbf{1}\otimes w)\\
&= \mathbf{1}\otimes Y(v,x)\mathbf{1}\otimes w\\
&= e^{x\mathcal{D}_T}\big(Y(-x)\otimes Y(-x)\otimes Y(-x)\big)(\mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v\otimes w\otimes \mathbf{1})\\
&= e^{x\mathcal{D}_T}Y_{T_2}(-x)R_2^{-1}(x)(v\otimes w).
\end{aligned}$$

By Theorem 3.3, ψ is a nonlocal vertex algebra homomorphism. Clearly, $\psi|_U=1$, $\psi|_V=1$ and $\psi|_W=1$. On the other hand, now we consider $U\otimes_{R_3} W\otimes_{R_2^{-1}(-x)} V$. With the assumption (3.16), we get

$$\begin{aligned}
Y_{T_2'}(v,x)w &= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)\big(R_2^{-1}\big)^{45}(-x)R_3^{23}(-x)R_1^{34}(-x)(\mathbf{1}\otimes \mathbf{1}\otimes v\otimes \mathbf{1}\otimes w\otimes \mathbf{1})\\
&= \big(1\otimes 1\otimes Y(x)\big)\big(R_2^{-1}\big)^{23}(x)(\mathbf{1}\otimes v\otimes w\otimes \mathbf{1})\in (U\otimes_{R_3} W\otimes_{R_2^{-1}(-x)} V)[[x]],
\end{aligned}$$

and

$$\begin{aligned}
Y_{T_2'}(u,x)(v_{-1}w) &= \lim_{x_2\to 0}\big(Y(x)\otimes Y(x)\otimes Y(x)\big)\big(R_2^{-1}\big)^{45}(x)R_3^{23}(-x)R_1^{34}(-x)\big(R_2^{-1}\big)^{56}(x_2)\\
&\quad\cdot(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v\otimes w)\\
&= \lim_{x_2\to 0}\big(Y(x)\otimes 1\otimes 1\big)\big(R_2^{-1}\big)^{34}(x_2)(u\otimes \mathbf{1}\otimes v\otimes w)\\
&\in (U\otimes_{R_3} W\otimes_{R_2^{-1}(-x)} V)[[x]]
\end{aligned}$$

for $u\in U$, $v\in V$, $w\in W$.

From Lemma 2.7 we also have

$$\begin{aligned}
Y_{T_2'}(v,x)u &= e^{x\mathcal{D}_T}Y_{T_2'}(-x)R_1(-x)(v\otimes u),\\
Y_{T_2'}(w,x)u &= e^{x\mathcal{D}_T}Y_{T_2'}(-x)R_3(-x)(w\otimes u).
\end{aligned}$$

And we also have

$$\begin{aligned}
Y_{T_2'}(w,x)v &= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)\big(R_2^{-1}\big)^{45}(x)R_3^{23}(-x)R_1^{34}(-x)(\mathbf{1}\otimes w\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes v)\\
&= \big(Y(x)\otimes Y(x)\otimes Y(x)\big)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes\mathbf{1}\otimes v)\\
&= \mathbf{1}\otimes Y(w,x)\mathbf{1}\otimes v\\
&= e^{x\mathcal{D}_T}\big(Y(-x)\otimes Y(-x)\otimes Y(-x)\big)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v\otimes\mathbf{1})\\
&= e^{x\mathcal{D}_T}Y_{T_2'}(-x)R_2(-x)(w\otimes v).
\end{aligned}$$

By Theorem 3.3, there is a nonlocal vertex algebra homomorphism $\phi : U\otimes_{R_1} V\otimes_{R_2} W\to U\otimes_{R_3} W\otimes_{R_2^{-1}(-x)} V$ such that $\phi(u\otimes v\otimes w)=u_{-1}v_{-1}w$ for $u\in U$, $v\in V$, $w\in W$. Because $\psi\circ\phi$ and $\phi\circ\psi$ are nonlocal vertex algebra homomorphisms preserving both U, V and W element-wise, it follows that $\psi\circ\phi=1$ and $\phi\circ\psi=1$. Therefore, ψ is a nonlocal vertex algebra isomorphism. $\square$

Next we will discuss the relation between two iterated twisted tensor product nonlocal vertex algebras $U\otimes_{R_1} V\otimes_{R_2} W$ and $V\otimes_{R_1^{-1}(-x)} U\otimes_{R_3} W$.

Lemma 3.7. *Let $U\otimes_{R_1} V\otimes_{R_2} W$ be an iterated twisted tensor product nonlocal vertex algebra, let $R_1(x)$ be an invertible twisting operator. Then $V\otimes_{R_1^{-1}(-x)} U\otimes_{R_3} W$ carries the structure of an iterated twisted tensor product nonlocal vertex algebra.*

Proof. Since $T_2(x)$ is a twisting operator, from Theorem 2.5 we have

$$R_2^{23}(y_1-y_2)R_3^{12}(y_1)R_1^{23}(y_2)=R_1^{12}(y_2)R_3^{23}(y_1)R_2^{12}(y_1-y_2).$$

Using the invertibility of $R_1(x)$, we get

$$R_3^{23}(y_1)R_2^{12}(y_1-y_2)\big(R_1^{-1}\big)^{23}(y_2)=\big(R_1^{-1}\big)^{12}(y_2)R_2^{23}(y_1-y_2)R_3^{12}(y_1).$$

Taking $y_1=x_1-x_2$ and $y_2=-x_2$, we have

$$R_3^{23}(x_1-x_2)R_2^{12}(x_1)\big(R_1^{-1}\big)^{23}(-x_2)=\big(R_1^{-1}\big)^{12}(-x_2)R_2^{23}(x_1)R_3^{12}(x_1-x_2).$$

By Theorem 2.5, we know that $V\otimes_{R_1^{-1}(-x)} U\otimes_{R_3} W$ carries the structure of an iterated twisted tensor product nonlocal vertex algebra with twisting operators $T_1''(x)=R_3^{23}(x)R_2^{12}(x)$ and $T_2''(x)=(R_1^{-1})^{12}(-x)R_2^{23}(x)$. $\square$

Remark 3.8. Compared with Lemma 3.5, here we do not need the following assumption

$$R_1(x)(v\otimes u)\in U\otimes V\otimes\mathbb{C}[[x]]$$

for $u\in U$, $v\in V$.

Similarly, we have:

Proposition 3.9. *Let $U \otimes_{R_1} V \otimes_{R_2} W$ be an iterated twisted tensor product nonlocal vertex algebra and let $R_1(x)$ be an invertible twisting operator for (U, V) such that*

$$R_1(x)(v \otimes u) \in U \otimes V \otimes \mathbb{C}[[x]], \tag{3.17}$$

$$R_1^{-1}(x)(u \otimes v) \in V \otimes U \otimes \mathbb{C}[[x]] \tag{3.18}$$

for $u \in U$, $v \in V$. Then the linear map $\varphi : V \otimes_{R_1^{-1}(-x)} U \otimes_{R_3} W \to U \otimes_{R_1} V \otimes_{R_2} W$, defined by

$$\varphi(v \otimes u \otimes w) = v_{-1}u_{-1}w \quad (\text{in } U \otimes_{R_1} V \otimes_{R_2} W)$$

for $u \in U$, $v \in V$, $w \in W$, is a nonlocal vertex algebra isomorphism.

From Propositions 3.6 and 3.9, we immediately have:

Corollary 3.10. *Let $U \otimes_{R_1} V \otimes_{R_2} W$ be an iterated twisted tensor product nonlocal vertex algebra, let $u \in U$, $v \in V$, $w \in W$.*

(1) *If $R_2(x)$ and $R_3(x)$ are invertible twisting operators, and satisfying (3.15), (3.16) and*

$$R_3(x)(w \otimes u) \in U \otimes W \otimes \mathbb{C}[[x]], \qquad R_3^{-1}(x)(u \otimes w) \in W \otimes U \otimes \mathbb{C}[[x]]. \tag{3.19}$$

Then $W \otimes_{R_3^{-1}(-x)} U \otimes_{R_1} V$ carries the structure of a nonlocal vertex algebra, which is isomorphic to $U \otimes_{R_1} V \otimes_{R_2} W$.

(2) *If $R_1(x)$ and $R_3(x)$ are invertible twisting operators, and satisfying (3.17), (3.18) and (3.19). Then $V \otimes_{R_2} W \otimes_{R_3^{-1}(-x)} U$ carries the structure of a nonlocal vertex algebra, which is isomorphic to $U \otimes_{R_1} V \otimes_{R_2} W$.*

(3) *If $R_1(x)$, $R_2(x)$ and $R_3(x)$ are invertible twisting operators, and satisfying (3.15)–(3.19). Then $W \otimes_{R_2^{-1}(-x)} V \otimes_{R_1^{-1}(-x)} U$ carries the structure of a nonlocal vertex algebra, which is isomorphic to $U \otimes_{R_1} V \otimes_{R_2} W$.*

4. N-factors iterated twisted tensor products

In this section, we construct a twisted tensor product of any number of factors. The way to prove this is mainly using induction. Then we state the universal property for the twisted tensor product of any number of factors.

Lemma 4.1. *Let $V_1, V_2, \dots, V_n$ be nonlocal vertex algebras, let $R_{ij}(x) : V_j \otimes V_i \to V_i \otimes V_j \otimes \mathbb{C}((x))$ be twisting operators for every $1 \leqslant i < j \leqslant n$, such that for any $i < j < k$ the twisting operators $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfying the hexagon equation, let $T^i_{n-1,n}(x) = R^{12}_{i,n-1}(x)R^{23}_{in}(x)$ be twisting operators defined by*

$$T^i_{n-1,n}(x) : (V_{n-1} \otimes_{R_{n-1,n}} V_n) \otimes V_i \to V_i \otimes (V_{n-1} \otimes_{R_{n-1,n}} V_n) \otimes \mathbb{C}((x))$$

for $i = 1, \dots, n-2$. Then for every $1 < i < j \leqslant n-2$, the twisting operators $R_{ij}(x)$, $T^j_{n-1,n}(x)$ and $T^i_{n-1,n}(x)$ satisfy the hexagon equation.

Proof. For $u^{n-1} \in V_{n-1}$, $u^n \in V_n$, $u^j \in V_j$, $u^i \in V_i$, $1 \leqslant i < j \leqslant n$, using the compatibilities of $R_{ij}(x)$, $R_{j,n-1}(x)$ and $R_{i,n-1}(x)$, and of $R_{ij}(x)$, $R_{jn}(x)$ and $R_{in}(x)$, we have

$$\begin{aligned}
&R_{ij}^{12}(x_2)\big(T_{n-1,n}^{i}\big)^{23}(x_1)\big(T_{n-1,n}^{j}\big)^{12}(x_1-x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{ij}^{12}(x_2)\big(T_{n-1,n}^{i}\big)^{23}(x_1)R_{j,n-1}^{12}(x_1-x_2)R_{jn}^{23}(x_1-x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{ij}^{12}(x_2)R_{i,n-1}^{23}(x_1)R_{in}^{34}(x_1)R_{j,n-1}^{12}(x_1-x_2)R_{jn}^{23}(x_1-x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{ij}^{12}(x_2)R_{i,n-1}^{23}(x_1)R_{j,n-1}^{12}(x_1-x_2)R_{in}^{34}(x_1)R_{jn}^{23}(x_1-x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{j,n-1}^{23}(x_1-x_2)R_{i,n-1}^{12}(x_1)R_{ij}^{23}(x_2)R_{in}^{34}(x_1)R_{jn}^{23}(x_1-x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{j,n-1}^{23}(x_1-x_2)R_{i,n-1}^{12}(x_1)R_{jn}^{34}(x_1-x_2)R_{in}^{23}(x_1)R_{ij}^{34}(x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= R_{j,n-1}^{23}(x_1-x_2)R_{jn}^{34}(x_1-x_2)R_{i,n-1}^{12}(x_1)R_{in}^{23}(x_1)R_{ij}^{34}(x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big)\\
&\quad= \big(T_{n-1,n}^{j}\big)^{23}(x_1-x_2)\big(T_{n-1,n}^{i}\big)^{12}(x_1)R_{ij}^{34}(x_2)\big(u^{n-1}\otimes u^n\otimes u^j\otimes u^i\big).
\end{aligned}$$

This concludes the proof. □

Now we present the main result of this section.

Theorem 4.2 *(Coherence Theorem). Let $V_1,\dots,V_n$ be nonlocal vertex algebras, let $R_{ij}(x): V_j\otimes V_i\to V_i\otimes V_j\otimes\mathbb{C}((x))$ be twisting operators for every $1\leqslant i<j\leqslant n$, such that for any $i<j<k$ the twisting operators $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfy the hexagon equation, and let $T_{j-1,j}^{i}(x)$ be twisting operators defined by*

$$\begin{aligned}
&T_{j-1,j}^{i}(x)=R_{i,j-1}^{12}(x)R_{ij}^{23}(x):(V_{j-1}\otimes_{R_{j-1,j}}V_j)\otimes V_i\to V_i\otimes(V_{j-1}\otimes_{R_{j-1,j}}V_j)\otimes\mathbb{C}((x))\\
&\quad\text{for } i\leqslant j-2,\\
&T_{j-1,j}^{i}(x)=R_{ji}^{23}(x)R_{j-1,i}^{12}(x):V_i\otimes(V_{j-1}\otimes_{R_{j-1,j}}V_j)\to(V_{j-1}\otimes_{R_{j-1,j}}V_j)\otimes V_i\otimes\mathbb{C}((x))\\
&\quad\text{for } i\geqslant j+1.
\end{aligned}$$

Then for every $i,k\notin\{j-1,j\}$ the twisting operators $R_{ik}(x)$, $T_{j-1,j}^{k}(x)$ and $T_{j-1,j}^{i}(x)$ satisfy the hexagon equation. Furthermore, for any $1\leqslant i\leqslant n$ the (inductively defined) twisted tensor product nonlocal vertex algebras

$$V_1\otimes_{R_{12}}\cdots\otimes_{R_{i-3,i-2}}V_{i-2}\otimes_{T_{i-1,i}^{i-2}}(V_{i-1}\otimes_{R_{i-1,i}}V_i)\otimes_{T_{i-1,i}^{i+1}}V_{i+1}\otimes_{R_{i+1,i+2}}\cdots\otimes_{R_{n-1,n}}V_n$$

are all equal.

Proof. As our hypothesis, we assume that we have $n-1$ nonlocal vertex algebras $U_1,\dots,U_{n-1}$, with twisting operators $Q_{ij}(x): U_j\otimes U_i\to U_i\otimes U_j\otimes\mathbb{C}((x))$ for every $1\leqslant i<j\leqslant n-1$, and such that for any $i<j<k$ the twisting operators $Q_{ij}(x)$, $Q_{jk}(x)$ and $Q_{ik}(x)$ satisfying the hexagon equation, then we can construct the iterated twisted tensor product $U_1\otimes_{Q_{12}}U_2\otimes_{Q_{23}}\cdots\otimes_{Q_{n-2,n-1}}U_{n-1}$.

From Theorem 2.5, we know that the map

$$T_{n-1,n}^{i}(x)=R_{i,n-1}^{12}(x)R_{in}^{23}(x):(V_{n-1}\otimes_{R_{n-1,n}}V_n)\otimes V_i\to V_i\otimes(V_{n-1}\otimes_{R_{n-1,n}}V_n)\otimes\mathbb{C}((x))$$

are twisting operators for $1\leqslant i\leqslant n-2$. By Lemma 4.1, we have $n-1$ nonlocal vertex algebras $V_1,\dots,V_{n-2}$ and $V_{n-1}\otimes_{R_{n-1,n}}V_n$ with $\frac{(n-2)(n-1)}{2}$ twisting operators $R_{ij}(x)$, $T_{n-1,n}^{j}(x)$ and $T_{n-1,n}^{i}(x)$

satisfying the hexagon equation for every $1 \leqslant i < j \leqslant n-2$. So we can apply the induction hypothesis and build the nonlocal vertex algebra

$$V_1 \otimes_{R_{12}} \cdots \otimes_{R_{n-1,n-2}} V_{n-2} \otimes_{T^{n-2}_{n-1,n}} (V_{n-1} \otimes_{R_{n-1,n}} V_n).$$

Using Theorem 2.5, we have

$$V_{n-2} \otimes_{T^{n-2}_{n-1,n}} (V_{n-1} \otimes_{R_{n-1,n}} V_n) = (V_{n-2} \otimes_{R_{n-2,n-1}} V_{n-1}) \otimes_{T^{n}_{n-2,n-1}} V_n,$$

thus we can group together any two consecutive factors. This concludes the proof. $\square$

As a consequence of this theorem, we can lift the universal property for the twisted tensor product of any number of factors, which is analogous to Theorem 3.3 and the proof is similar.

Theorem 4.3. *Let $V_1, \ldots, V_n$ be nonlocal vertex algebras, let $R_{ij}(x)$ be twisting operators for (V_i, V_j) for $1 \leqslant i < j \leqslant n$, such that for any $i < j < k$ the twisting operators $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfy the hexagon equation. Let K be any nonlocal vertex algebra and let $\psi_i : V_i \to K$ be any homomorphisms, satisfying the conditions that for $v^i \in V_i$, $1 \leqslant i < j \leqslant n$,*

$$\begin{gathered}
Y\big(\psi_{n-1}\big(v^{n-1}\big), x\big)\psi_n\big(v^n\big),\ Y\big(\psi_{n-2}\big(v^{n-2}\big), x\big)\big(\psi_{n-1}\big(v^{n-1}\big)_{-1}\psi_n\big(v^n\big)\big), \\
\ldots,\ Y\big(\psi_1\big(v^1\big), x\big)\big(\psi_2\big(v^2\big)_{-1} \cdots \psi_{n-1}\big(v^{n-1}\big)_{-1}\psi_n\big(v^n\big)\big) \in K[[x]], \\
Y(x)(\psi_j \otimes \psi_i) = e^{x\mathcal{D}} Y(-x)(\psi_i \otimes \psi_j) R_{ij}(-x).
\end{gathered}$$

Then the linear map $\psi : V_1 \otimes_{R_{12}} \cdots \otimes_{R_{n-2,n-1}} V_{n-1} \otimes_{R_{n-1,n}} V_n \to K$, defined by

$$\psi\big(v^1 \otimes v^2 \otimes \cdots \otimes v^n\big) = \psi_1\big(v^1\big)_{-1}\psi_2\big(v^2\big)_{-1} \cdots \psi_n\big(v^n\big) \quad \text{for } v^i \in V_i,\ 1 \leqslant i \leqslant n$$

is a homomorphism of nonlocal vertex algebras, which extends ψ_i uniquely, for $1 \leqslant i \leqslant n$.

Next, using the rest of this section we present a toy example. We show that the weak quantum vertex algebras associated with the noncommutative $2n$-planes defined by Connes and Dubois-Violette in [CD] can be realized as iterated twisted tensor products of nonlocal vertex algebras. And this weak quantum vertex algebra is very similar with the quantum vertex algebra of Zamolodchikov–Faddev type studied in [KL].

Definition 4.4. Let l be a positive integer and let $\mathbf{Q} = (q_{ij})_{i,j=1}^{l}$ be a complex matrix such that

$$q_{ii} = q_{ij}q_{ji} = 1 \quad \text{for } 1 \leqslant i, j \leqslant l. \tag{4.1}$$

Define $\mathcal{A}_{\mathbf{Q}}$ to be the associative algebra with identity (over $\mathbb{C}$) with generators

$$X_{i,n}, \qquad Y_{i,n} \quad (i = 1, \ldots, l,\ n \in \mathbb{Z}),$$

subject to relations

$$X_{i,m}X_{j,n} = q_{ij}X_{j,n}X_{i,m}, \qquad Y_{i,m}Y_{j,n} = q_{ij}Y_{j,n}Y_{i,m}, \qquad X_{i,m}Y_{j,n} = q_{ji}Y_{j,n}X_{i,m} \tag{4.2}$$

for $i, j = 1, \ldots, l$, $m, n \in \mathbb{Z}$.

Let $\{e_1, \ldots, e_l\}$ denote the standard $\mathbb{Z}$-basis of $\mathbb{Z}^l$. It is straightforward to see that $\mathcal{A}_{\mathbf{Q}}$ is a $\mathbb{Z}^l$-graded algebra with the grading defined by

$$\deg X_{i,m} = e_i, \qquad \deg Y_{i,m} = -e_i, \quad \text{for } 1 \leqslant i \leqslant l,\ m \in \mathbb{Z}. \tag{4.3}$$

Set

$$\mathcal{A}_{\mathbf{Q}}^{+} = \langle X_{i,m},\ Y_{j,n} \mid i, j = 1, \ldots, l,\ m, n \geqslant 0\rangle,$$
$$\mathcal{A}_{\mathbf{Q}}^{-} = \langle X_{i,m},\ Y_{j,n} \mid i, j = 1, \ldots, l,\ m, n < 0\rangle,$$

which are $\mathbb{Z}^l$-graded subalgebras of $\mathcal{A}_{\mathbf{Q}}$.

A vector w in an $\mathcal{A}_{\mathbf{Q}}$-module W is called a *vacuum vector* if $\mathcal{A}_{\mathbf{Q}}^{+} w = 0$, and an $\mathcal{A}_{\mathbf{Q}}$-module W equipped with a vacuum vector which generates W is called a *vacuum $\mathcal{A}_{\mathbf{Q}}$-module*.

Set

$$V_{\mathbf{Q}} = \mathcal{A}_{\mathbf{Q}} / \big(\mathcal{A}_{\mathbf{Q}} \mathcal{A}_{\mathbf{Q}}^{+}\big), \tag{4.4}$$

a left $\mathcal{A}_{\mathbf{Q}}$-module, and set

$$\mathbf{1} = 1 + \big(\mathcal{A}_{\mathbf{Q}} \mathcal{A}_{\mathbf{Q}}^{+}\big) \in V_{\mathbf{Q}}.$$

Clearly, $\mathbf{1}$ is a vacuum vector and $V_{\mathbf{Q}}$ equipped with $\mathbf{1}$ is a vacuum $\mathcal{A}_{\mathbf{Q}}$-module.

For $1 \leqslant i \leqslant l$, set

$$u^{(i)} = X_{i,-1}\mathbf{1}, \qquad v^{(i)} = Y_{i,-1}\mathbf{1} \in V_{\mathbf{Q}} \tag{4.5}$$

and set

$$X_i(x) = \sum_{n \in \mathbb{Z}} X_{i,n} x^{-n-1}, \qquad Y_i(x) = \sum_{n \in \mathbb{Z}} Y_{i,n} x^{-n-1} \in \mathcal{A}_{\mathbf{Q}}\big[\big[x, x^{-1}\big]\big]. \tag{4.6}$$

Now we endow $V_{\mathbf{Q}}$ with the structure of a weak quantum vertex algebra (cf. [KL]).

Theorem 4.5. *Let $\mathbf{Q} = (q_{ij})_{1 \leqslant i, j \leqslant l}$ be a complex matrix such that $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$ for $1 \leqslant i, j \leqslant l$, let $\mathcal{A}_{\mathbf{Q}}$ be the associative algebra associated with $\mathbf{Q}$ and let $V_{\mathbf{Q}}$ be the universal vacuum $\mathcal{A}_{\mathbf{Q}}$-module. There exists a (unique) weak quantum vertex algebra structure on $V_{\mathbf{Q}}$ with $\mathbf{1}$ as the vacuum vector such that*

$$Y\big(u^{(i)}, x\big) = X_i(x), \qquad Y\big(v^{(i)}, x\big) = Y_i(x) \quad \text{for } i = 1, \ldots, l.$$

Let W be any $\mathcal{A}_{\mathbf{Q}}$-module satisfying the condition that for any $w \in W$, $X_{i,m} w = Y_{i,m} w = 0$ for $1 \leqslant i \leqslant l$ and for m sufficiently large. Then there exists a (unique) $V_{\mathbf{Q}}$-module structure on W with

$$Y_W\big(u^{(i)}, x\big) = X_i(x), \qquad Y_W\big(v^{(i)}, x\big) = Y_i(x) \quad \text{for } i = 1, \ldots, l.$$

Conversely, any $V_{\mathbf{Q}}$-module W is an $\mathcal{A}_{\mathbf{Q}}$-module with

$$X_i(x) = Y_W\big(u^{(i)}, x\big), \qquad Y_i(x) = Y_W\big(v^{(i)}, x\big) \quad \text{for } i = 1, \ldots, l.$$

Furthermore, similar with Proposition 3.8 in [KL], we have:

Proposition 4.6. *Let V be any nonlocal vertex algebra and let ψ be any map from $\{u^{(i)}, v^{(i)} \mid i = 1, \dots, l\}$ to V such that*

$$Y\big(\psi\big(u^{(i)}\big), x_1\big)Y\big(\psi\big(u^{(j)}\big), x_2\big) = q_{ij}Y\big(\psi\big(u^{(j)}\big), x_2\big)Y\big(\psi\big(u^{(i)}\big), x_1\big),$$
$$Y\big(\psi\big(v^{(i)}\big), x_1\big)Y\big(\psi\big(v^{(j)}\big), x_2\big) = q_{ij}Y\big(\psi\big(v^{(j)}\big), x_2\big)Y\big(\psi\big(v^{(i)}\big), x_1\big),$$
$$Y\big(\psi\big(u^{(i)}\big), x_1\big)Y\big(\psi\big(v^{(j)}\big), x_2\big) = q_{ji}Y\big(\psi\big(v^{(j)}\big), x_2\big)Y\big(\psi\big(u^{(i)}\big), x_1\big)$$

for $1 \leqslant i, j \leqslant l$. Then there exists a unique nonlocal vertex algebra homomorphism from $V_{\mathbf{Q}}$ to V, extending ψ.

For each $1 \leqslant i \leqslant l$, $n \in \mathbb{Z}$, the algebra $\mathcal{A}_{q_{ii}}$ (associated with 1×1 matrix q_{ii}) generated by the elements $X_{i,n}$ and $Y_{i,n}$ is commutative and is isomorphic to $\mathbb{C}[X_{i,n}, Y_{i,n}]$, which is a polynomial algebra. Let V_{ii} be the universal vacuum module constructed from $\mathcal{A}_{q_{ii}}$. From Theorem 4.5, we know that V_{ii} is a nonlocal vertex subalgebra of $V_{\mathbf{Q}}$. In the following we show that $V_{\mathbf{Q}}$ can be realized as the iterated twisted tensor products of V_{ii}, by suitable twisting operators $R_{ij}(x)$ for the ordered pair (V_{ii}, V_{jj}), for $1 \leqslant i < j \leqslant l$.

The following is straightforward:

Lemma 4.7. *Let V_{ii}, nonlocal vertex subalgebras of $V_{\mathbf{Q}}$, be the universal vacuum $\mathcal{A}_{q_{ii}}$-module for $1 \leqslant i \leqslant l$. Let $R_{ij}(x)$ be a linear map defined by*

$$\begin{aligned} R_{ij}(x) : V_{jj} \otimes V_{ii} \to V_{ii} \otimes V_{jj} u^{(j)}_{m_1} \cdots u^{(j)}_{m_l} v^{(j)}_{n_1} \cdots v^{(j)}_{n_k} \mathbf{1} \otimes u^{(i)}_{r_1} \cdots u^{(i)}_{r_p} v^{(i)}_{s_1} \cdots v^{(i)}_{s_q} \mathbf{1} \\ \mapsto q_{ij}^{(k-l)(p-q)} u^{(i)}_{r_1} \cdots u^{(i)}_{r_p} v^{(i)}_{s_1} \cdots v^{(i)}_{s_q} \mathbf{1} \otimes u^{(j)}_{m_1} \cdots u^{(j)}_{m_l} v^{(j)}_{n_1} \cdots v^{(j)}_{n_k} \mathbf{1} \end{aligned} \tag{4.7}$$

for $u^{(i)}, v^{(i)} \in V_{ii}$, $u^{(j)}, v^{(j)} \in V_{jj}$, $1 \leqslant i < j \leqslant l$. Then $R_{ij}(x)$ is an invertible twisting operator for the ordered pair (V_{ii}, V_{jj}), for $1 \leqslant i < j \leqslant l$.

From Theorem 2.4 we have:

Proposition 4.8. *Let $R_{ij}(x)$ be the twisting operator of the ordered pair (V_{ii}, V_{jj}). Set*

$$Y_{R_{ij}}(x) = \big(Y(x) \otimes Y(x)\big) R^{23}_{ij}(-x) \tag{4.8}$$

Then $(V_{ii} \otimes_{R_{ij}} V_{jj}, Y_{R_{ij}}, \mathbf{1} \otimes \mathbf{1})$ carries the structure of a nonlocal vertex algebra, for $1 \leqslant i < j \leqslant l$.

Let V_{ii}, V_{jj} and V_{kk} be nonlocal vertex subalgebras of $V_{\mathbf{Q}}$, let $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ be the twisting operators for (V_{ii}, V_{jj}), (V_{jj}, V_{kk}) and (V_{ii}, V_{kk}) respectively, defined in Lemma 4.7, for $1 \leqslant i < j < k \leqslant l$. We can directly check that $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfy the hexagon equation. From the Coherence Theorem we get:

Proposition 4.9. *The nonlocal vertex algebra $V_{\mathbf{Q}}$ is isomorphic to the iterated twisted tensor product nonlocal vertex algebra $V_{11} \otimes_{R_{12}} V_{22} \otimes_{R_{23}} \cdots \otimes_{R_{n-1,n}} V_{nn}$.*

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