



Skew polynomial algebras with coefficients in Koszul Artin–Schelter regular algebras

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ARTICLE INFO

Article history:

Received 23 August 2012

Available online 17 June 2013

Communicated by J.T. Stafford

Keywords:

Koszul Artin–Schelter regular algebra

Skew polynomial algebra

Calabi–Yau algebra

Superpotential

PBW-deformation

ABSTRACT

Let A be a Koszul Artin–Schelter regular algebra with Nakayama automorphism ξ . We show that the Yoneda Ext-algebra of the skew polynomial algebra $A[z; \xi]$ is a trivial extension of a Frobenius algebra. Then we prove that $A[z; \xi]$ is Calabi–Yau; and hence each Koszul Artin–Schelter regular algebra is a subalgebra of a Koszul Calabi–Yau algebra. A superpotential \hat{w} is also constructed so that the Calabi–Yau algebra $A[z; \xi]$ is isomorphic to the derivation quotient of \hat{w} . The Calabi–Yau property of a skew polynomial algebra with coefficients in a PBW-deformation of a Koszul Artin–Schelter regular algebra is also discussed.

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Introduction

Let \mathbb{k} be a field of characteristic zero. Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a (\mathbb{Z}) -graded (\mathbb{k}) -algebra, and $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a graded left A -module. The n th shift of M is the graded A -module $M(n)$ whose k th component is: $M(n)_k = M_{n+k}$. If M is a graded A -bimodule and σ, φ are graded automorphisms of A , then ${}_{\sigma}M_{\varphi}$ is the graded A -bimodule whose left A -action is twisted by σ and right A -action is twisted by φ . A graded algebra A is called *Calabi–Yau* of dimension d , if (cf. [7]):

- (i) A is homologically smooth; that is, A has a bounded resolution of finitely generated graded projective A -bimodules;

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- (ii) $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ if $i \neq d$ and $\text{Ext}_{A^e}^d(A, A \otimes A) \cong A(l)$ for some integer l as A -bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A .

Calabi–Yau algebras are strongly related to Artin–Schelter (AS, for short) regular algebras. Recall that a connected graded algebra $A = \mathbb{k} \oplus A_1 \oplus A_2 \oplus \cdots$ is called an *AS-regular* algebra if (i) A has finite global dimension d ; (ii) $\text{Ext}_A^i(A \mathbb{k}, A) = 0$ if $i \neq d$, and $\dim \text{Ext}_A^d(A \mathbb{k}, A) = 1$. Here Ext is the derived functor of graded Hom (cf. [16]). If a graded Calabi–Yau algebra is also connected, then it must be AS-regular [3]. If an AS-regular algebra A of global dimension d is Noetherian or Koszul, then A differs from a Calabi–Yau algebra by an automorphism; more precisely, we have $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ for $i \neq d$, and $\text{Ext}_{A^e}^d(A, A \otimes A) \cong A_\xi(l)$ for some graded automorphism ξ of A [18,1]. We call the automorphism ξ the *Nakayama automorphism* of A .

Berger and Pichereau recently constructed in [2] an interesting class of Calabi–Yau algebras of dimension 3, which are related to deformations of Poisson algebras. Given an AS-regular algebra A of global dimension 2 (which must be Koszul), Dubois-Violette showed in [6] (also see [20]) that A is defined by an invertible matrix M , that is, $A \cong T(V)/(f)$ with V a finite dimensional vector space with a basis $\{x_1, \dots, x_n\}$ and $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$. Here the matrix multiplications should be regarded in $T(V)$. Given such an AS-regular algebra, Berger and Pichereau constructed a graded algebra $B(f)$ which is generated by $V \oplus \mathbb{k}z$, and whose generating relations are cyclic partial derivations of $w = fz$. They proved that $B(f)$ is a Calabi–Yau algebra of dimension 3, and gave the classification, up to isomorphisms, of the obtained Calabi–Yau algebras. They also showed that $B(f)$ is isomorphic to the skew polynomial algebra $A[z; \xi]$ for some automorphism ξ of A . We find that ξ is exactly the Nakayama automorphism of A , and the Calabi–Yau property of $A[z; \xi]$ holds for general Koszul AS-regular algebra A by inspecting the Yoneda Ext-algebra of $A[z; \xi]$. The main results of this paper are the following (cf. Theorem 3.3).

Theorem 0.1. *Let A be a Koszul AS-regular algebra of global dimension d , and ξ the Nakayama automorphism of A . Then the skew polynomial algebra $B = A[z; \xi]$ is a Calabi–Yau algebra of dimension $d + 1$.*

Since an AS-regular algebra of global dimension 2 is always Koszul, our main results provide a new proof of [2, Theorem 2.10] and our proof is totally different from that in [2].

It was shown in [6] and [5] that a Koszul AS-regular algebra A is determined by a twisted superpotential w . We show that the twisted superpotential w can be symmetrized into a superpotential \hat{w} by introducing a new indeterminate, so that the skew polynomial algebra $A[z; \xi]$ is isomorphic to the derivation quotient algebra obtained from the superpotential \hat{w} (see Theorem 4.4).

Let A be a Koszul AS-regular algebra, and ξ the Nakayama automorphism of A . A PBW-deformation of A is a filtered algebra U such that its associated graded algebra is isomorphic to A . For a PBW-deformation U of A , U has a filtration-preserving automorphism ζ such that $\text{gr}(\zeta) = \xi$, still called a Nakayama automorphism (in this case, ζ is not unique, see more details in Section 5). It is natural to ask whether $U[z; \zeta]$ is a Calabi–Yau algebra. Recall that a nongraded algebra U is *Calabi–Yau* of dimension d if (i) U is homologically smooth; (ii) $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ if $i \neq d$ and $\text{Ext}_{U^e}^d(U, U \otimes U) \cong U$ as U -bimodules.

Since a Nakayama automorphism ζ respects the filtration of U , we see that $U[z; \zeta]$ is in fact a PBW-deformation of $A[z; \xi]$, which is a Koszul algebra. Then we can use the techniques developed in [13,14] for PBW-deformations of Koszul algebras to discuss the Calabi–Yau property of $U[z; \zeta]$.

Now let A be a Koszul Calabi–Yau algebra, and U be a PBW-deformation of A . We may choose a specific Nakayama automorphism ζ of U (see Proposition 5.5) so that we have (cf. Theorem 5.8):

Theorem 0.2. *$U[z; \zeta]$ is Calabi–Yau.*

In the theorem above, if A is only an AS-regular algebra, then the result may fail. Counterexamples may be found in the case where A is AS-regular of global dimension 2. At the end of the paper, we provide a necessary and sufficient condition for $U[z; \zeta]$ to be Calabi–Yau with U a PBW-deformation of an AS-regular algebra of global dimension 2 (cf. Theorem 5.10).

1. Trivial extensions

Let $E = \mathbb{k} \oplus E_1 \oplus E_2 \oplus \cdots$ be a connected graded algebra, and M a graded E -bimodule. Recall that the trivial extension of E by M is the graded algebra $\Gamma(E, M) = E \oplus M$ with the product $(x_1, m_1) * (x_2, m_2) = (x_1 x_2, x_1 \cdot m_2 + m_1 \cdot x_2)$ for $x_i \in E$ and $m_i \in M$. If $M_i = 0$ for all $i \leq 0$, then $\Gamma(E, M)$ is a connected graded algebra with the i th component $\Gamma(E, M)_i = E_i \oplus M_i$.

We focus on trivial extensions of finite dimensional algebras. Let E be a finite dimensional connected graded algebra. We say that E is of length d if $E_d \neq 0$ and $E_i = 0$ for all $i > d$. Let E be a connected finite dimensional algebra of length d , and σ a graded automorphism of E . Let E^* be the dual vector space of E . Then E^* is a graded E -bimodule with the induced E -action. Let E_σ^* be the graded E -bimodule obtained from E^* with the right E -action twisted by σ . Given an integer $n > d$, consider the trivial extension of E by the bimodule $E_\sigma^*(-n)$: $\Gamma(E, E_\sigma^*(-n)) = E \oplus E_\sigma^*(-n)$. For simplicity, we write $\Gamma(E, \sigma, n)$ for $\Gamma(E, E_\sigma^*(-n))$. Now the product of $\Gamma(E, \sigma, n)$ is defined by: $(x_1, f_1) * (x_2, f_2) = (x_1 x_2, x_1 \cdot f_2 + f_1 \cdot \sigma(x_2))$ for $x_1, x_2 \in E$ and $f_1, f_2 \in E^*$.

For later discussions, we introduce first some terminology. A connected graded algebra E of length d is called a graded Frobenius algebra if there is an isomorphism of graded left A -modules $\Theta : E \cong E^*(-d)$, or equivalently, there is a nondegenerated graded bilinear form $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{k}(-d)$ such that $\langle x, yz \rangle = \langle xy, z \rangle$ for all $x, y, z \in E$. If E is graded Frobenius, then there is a unique graded automorphism φ of E such that $\Theta : E_\varphi \rightarrow E^*(-d)$ is an isomorphism of A -bimodules. The isomorphism φ is called the Nakayama automorphism of E . For the bilinear form, we have $\langle x, y \rangle = \langle y, \varphi(x) \rangle$. A graded Frobenius algebra E of length d is said to be graded symmetric, if $\langle x, y \rangle = (-1)^{i(d-i)} \langle y, x \rangle$ for all $x \in E_i$ and $y \in E_{d-i}$. In this case, $\varphi = \epsilon^{d-1}$, where $\epsilon : E \rightarrow E$ is defined by $\epsilon(x) = (-1)^i x$ for $x \in E_i$.

Proposition 1.1. *Let E be a connected graded algebra of length d , and σ a graded automorphism of E . Then the trivial extension $\Gamma(E, \sigma, n)$ ($n > d$) is a graded Frobenius algebra of length n , and the Nakayama automorphism φ of $\Gamma(E, \sigma, n)$ is given by $\varphi(x, f) = (\sigma^{-1}(x), f \circ \sigma)$ for all $x \in E$ and $f \in E^*$.*

Proof. Define a bilinear form $\langle \cdot, \cdot \rangle : \Gamma(E, \sigma, n) \times \Gamma(E, \sigma, n) \rightarrow \mathbb{k}(-n)$ by $\langle (x_1, f_1), (x_2, f_2) \rangle = f_2(x_1) + f_1(\sigma(x_2))$ for $x_1, x_2 \in E$ and $f_1, f_2 \in E^*$. A straightforward verification shows that $\langle (x_1, f_1), (x_2, f_2) * (x_3, f_3) \rangle = \langle (x_1, f_1) * (x_2, f_2), (x_3, f_3) \rangle$. Obviously, the bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerated. Hence $\Gamma(E, \sigma, n)$ is a Frobenius algebra. Moreover, $\langle (x_1, f_1), (x_2, f_2) \rangle = f_2(x_1) + f_1(\sigma(x_2)) = f_2(\sigma \circ \sigma^{-1}(x_1)) + f_1 \circ \sigma(x_2) = \langle (x_2, f_2), (\sigma^{-1}(x_1), f_1 \circ \sigma) \rangle$. Hence the Nakayama automorphism φ of $\Gamma(E, \sigma, n)$ is defined by $\varphi(x, f) = (\sigma^{-1}(x), f \circ \sigma)$ for all $x \in E$ and $f \in E^*$. \square

Remark 1.2. In Proposition 1.1, if we choose $\sigma = \epsilon^{n-1}$, then the Nakayama automorphism φ is given as follows: for $x \in E_i$ and $f \in E_i^*$, $\varphi(x, f) = ((-1)^{(n-1)i} x, f \circ \epsilon^{(n-1)i}) = ((-1)^{(n-1)i} x, (-1)^{(n-1)i} f)$. Hence $\varphi = \epsilon^{n-1}$. Therefore, $\Gamma(E, \epsilon^{n-1}, n)$ is a graded symmetric algebra.

Recall that a (cochain) differential graded algebra (dga, for short) (E, δ_E) is a graded algebra $E = \bigoplus_{n \in \mathbb{Z}} E_n$ together with a derivation δ_E of degree 1 such that $\delta_E^2 = 0$. A left differential graded module ${}_E M$ is a left graded E -module together with a differential δ_M such that $\delta_M(xm) = \delta_E(x)m + (-1)^{|x|} x \delta_M(m)$ for all homogeneous elements $x \in E$ and $m \in M$, where $|x|$ denotes the degree of x . Similarly, one has right differential graded modules and differential graded bimodules.

A curved differential graded algebra (cdga, for short) (cf. [13,14]) is a triple (E, δ_E, θ_E) , where E is a graded algebra, δ_E is a derivation of degree 1 and θ_E is a special element in E_2 , such that $\delta_E(\theta_E) = 0$ and $\delta_E^2(x) = \theta_E x - x \theta_E$ for all homogeneous elements $x \in E$. The element θ_E is usually called the curvature element of E . Let $(E', \delta_{E'}, \theta_{E'})$ be another cdga. A cdga morphism $f : E \rightarrow E'$ is a graded algebra morphism such that $f(\theta_E) = \theta_{E'}$ and $f \delta_E = \delta_{E'} f$ (warning: our definition of cdga morphism given here is more restricted than that in [13]). A cdg E -bimodule is a graded E -bimodule M endowed with a differential δ_M which is compatible with the differential δ_E of E and satisfies the condition

$\delta_M^2(m) = \theta_E m - m\theta_E$. Note that if the curvature element is zero, then a cdga is just a usual dga, and a cdg bimodule is a usual dg bimodule.

Let (E, δ_E, θ_E) be a cdga, and let M be a cdg E -bimodule. The trivial extension of E by M is the cdga $(\Gamma_{cdg}(E, M), \delta_{\Gamma_{cdg}}, \theta_{\Gamma_{cdg}})$ defined as follows: as a graded algebra $\Gamma_{cdg}(E, M)$ is just the trivial extension $\Gamma(E^\sharp, M^\sharp)$, where E^\sharp is the underlying graded algebra by forgetting the derivation δ_E of E , and M^\sharp is the underlying graded bimodule of M ; the derivation $\delta_{\Gamma_{cdg}}$ is defined by

$$\delta_{\Gamma_{cdg}}(x, m) = (\delta_E(x), \delta_M(m))$$

for all $x \in E$ and $m \in M$; and the curvature element $\theta_{\Gamma_{cdg}} = (\theta_E, 0)$.

Let $M(n)$ be the n th shift of the graded E -bimodule M . Note that the differential $\delta_{M(n)}$ and the E -actions of $M(n)$ should be changed slightly so that $M(n)$ is also a cdg E -bimodule: For a homogeneous element $m \in M$, we denote by $m(n)$ the corresponding element in $M(n)$. Then $\delta_{M(n)}(m(n)) = (-1)^n \delta_M(m)(n)$. Let $x \in E$ be a homogeneous element. The left E -action on $M(n)$ is defined by $x \diamond (m(n)) = (-1)^{n|x|} (x \cdot m)(n)$ and the right E -action is defined by $(m(n)) \diamond x = (m \cdot x)(n)$, where $x \cdot m$ and $m \cdot x$ are E -actions on M .

Let $M^\vee = \bigoplus_{n \in \mathbb{Z}} M_n^*$ be the graded dual of M . Then M^\vee is a cdg E -bimodule with the differential δ_{M^\vee} and E -actions defined as follows: for homogeneous elements $f \in M^\vee$, $m \in M$ and $x \in E$, we have

$$\delta_{M^\vee}(f) = (-1)^{|f|+1} f \circ \delta_M, \quad (1)$$

$$(x \rightharpoonup f)(m) = (-1)^{|x|(|f|+|m|)} f(m \cdot x) \quad \text{and} \quad (f \leftharpoonup x)(m) = f(x \cdot m). \quad (2)$$

Now let $E = \mathbb{k} \oplus E_1 \oplus \cdots \oplus E_d$ ($E_d \neq 0$) be a finite dimensional cdga with differential δ_E . Then $E^* = E^\vee$ is a cdg E -bimodule. Hence the trivial extension $\Gamma_{cdg}(E, E^*(-d-1))$ is a cdga.

Let M be a cdg E -bimodule. Note that $(M(n))^\sharp$ is different from $M^\sharp(n)$ as graded E^\sharp -bimodules. We remark that the graded algebra $\Gamma_{cdg}(E, E^*(-d-1))^\sharp$ is different from the trivial extension $\Gamma(E^\sharp, (E^*)^\sharp(-d-1))$ of the graded algebra E^\sharp . However, we have the following result.

Lemma 1.3. *As a graded algebra, we have*

$$\Gamma_{cdg}(E, E^*(-d-1))^\sharp = \Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1)).$$

Proof. As before, we denote by $x \cdot f$ and $f \cdot x$ for $x \in E$ and $f \in E^*$ the E^\sharp -actions on the graded dual $(E^*)^\sharp$. Note that in the definition (2) of the E -actions on E^* , $f(x \cdot m)$ is zero unless $|x| + |m| = -|f|$. Hence we have $(x \rightharpoonup f) = (-1)^{|x|} x \cdot f$.

It suffices to verify the multiplication of the cdga $\Gamma_{cdg}(E, E^*(-d-1))$ is equal to the multiplication of $\Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1))$. Indeed, for homogeneous elements $x, y \in E$ and $f, g \in E^*$, we have

$$\begin{aligned} (x, f(d+1)) * (y, g(-d-1)) &= (xy, x \diamond (g(-d-1)) + (f(-d-1)) \diamond y) \\ &= (xy, (-1)^{|x|(d+1)}(x \rightharpoonup g)(-d-1) + (f \leftharpoonup y)(-d-1)) \\ &= (xy, (-1)^{|x|d}(x \cdot g)(-d-1) + (f \cdot y)(-d-1)) \\ &= (xy, ({}_{\epsilon^d}x) \cdot g(-d-1) + (f \cdot y)(-d-1)). \end{aligned}$$

Now it is easy to see that the last item in the identities above is exactly the multiplication of elements $(x, f(d+1))$ and $(y, g(-d-1))$ in the trivial extension $\Gamma(E^\sharp, {}_{\epsilon^d}(E^*)^\sharp(-d-1))$. \square

2. Yoneda algebras

In this section, we will compute the Yoneda products of a skew polynomial algebra with coefficients in a Koszul algebra. We first recall the definition of a Koszul algebra. Let V be a finite dimensional vector space. A *quadratic algebra* A is a connected graded algebra of form $A = T(V)/(R)$, where $R \subseteq V \otimes V$ and (R) is the two-sided ideal of $T(V)$ generated by R . The *quadratic dual* $A^!$ of a quadratic algebra A is defined to be $A^! = T(V^*)/(R^\perp)$, where $R^\perp \subseteq V^* \otimes V^*$ is the orthogonal complement of R . One easily sees that $(A^!)^! = A$. Let $\phi : A \rightarrow A$ be an automorphism of the quadratic algebra A . The restriction of ϕ to $A_1 = V$ induces a bijective linear map $f : V^* \rightarrow V^*$. Since A is quadratic, we see that f defines an automorphism $\phi^!$ of the quadratic dual algebra $A^!$. We call $\phi^!$ the *automorphism of $A^!$ dual to ϕ* . Since $(A^!)^! = A$, we have $(\phi^!)^! = \phi$.

A quadratic algebra A is called a *Koszul algebra* [15] if the trivial graded module ${}_A \mathbb{k}$ admits a graded projective resolution:

$$\cdots \rightarrow P^{-n} \rightarrow \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow {}_A \mathbb{k} \rightarrow 0,$$

such that the graded module P^{-n} is generated in degree n for all $n \geq 0$. Recall that if A is Koszul, then the Yoneda algebra $E(A) := \bigoplus_{i \geq 0} \text{Ext}_A^i({}_A \mathbb{k}, {}_A \mathbb{k}) \cong A^!$. Moreover, A is Koszul if and only if $A^!$ is Koszul [16]. We refer to [15] and [16] for further properties of Koszul algebras.

Let $A = T(V)/(R)$ be a Koszul algebra, and let $C_0 = \mathbb{k}$, $C_{-1} = V$, $C_{-2} = R$ and $C_{-n} = \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{n-i-2}$ for $n \geq 3$. Then $C = \bigoplus_{n \geq 0} C_{-n}$ is a graded subcoalgebra of the tensor coalgebra $T(V)$. Moreover, as graded algebras, $E(A) \cong C^\vee = \bigoplus_{n \geq 0} C_{-n}^* \cong A^!$.

Consider the graded minimal projective resolution of the trivial module ${}_A \mathbb{k}$:

$$\cdots \rightarrow A \otimes C_{-n} \xrightarrow{\partial^{-n}} \cdots \xrightarrow{\partial^{-2}} A \otimes C_{-1} \xrightarrow{\partial^{-1}} A \rightarrow {}_A \mathbb{k} \rightarrow 0, \quad (3)$$

where the differential is given on pure tensors by

$$\partial^{-n}(a \otimes x_1 \otimes \cdots \otimes x_n) = ax_1 \otimes x_2 \otimes \cdots \otimes x_n,$$

for all $a \in A$ and $x_1, \dots, x_n \in V$.

Let σ be a graded automorphism of A . Since A is Koszul, σ induces an automorphism (also denoted by σ) of C in the obvious way. Let σ^\vee be the automorphism of graded algebra C^\vee induced by σ . Since $A^! \cong C^\vee$, we see that $\sigma^\vee = \sigma^!$. Let $B = A[z; \sigma]$ be the algebra of skew polynomials with coefficients in A . We assume that z is of degree 1. Then it is well known that B is also a Koszul algebra (cf. [17], for example). The elements of B are of the sums of the elements of the form az^i with $i \geq 0$ and $a \in A$, moreover $za = \sigma(a)z$. We want to construct a minimal projective resolution of the trivial module ${}_B \mathbb{k}$. The following construction is standard (cf. [8] or [12]).

Clearly, B is free both as a left A -module or as a right A -module. Applying the exact functor $B \otimes_A -$ to the projective resolution (3) of ${}_A \mathbb{k}$, we obtain the following complex:

$$\cdots \rightarrow B \otimes C_{-n} \rightarrow \cdots \rightarrow B \otimes C_{-1} \rightarrow B \rightarrow 0. \quad (4)$$

The complex is exact except at the final position. The cohomology at the final position is B/BA_+ . By abusing the notation, we also denote the differential of the complex (4) by ∂ .

For each $n \geq 1$, we define a homomorphism of left B -modules

$$f^{-n} : B \otimes C_{-n} \rightarrow B \otimes C_{-n}$$

by

$$f^{-n}(1 \otimes x_1 \otimes \cdots \otimes x_n) = z \otimes \sigma^{-1}(x_1) \otimes \cdots \otimes \sigma^{-1}(x_n)$$

for all $x_1, \dots, x_n \in V$. In addition, define a left B -module homomorphism $f^0 : B \rightarrow B$ by $f^0(1) = z$. It is easy to check that these f^{-n} are compatible with the differential of the complex (4). Hence $f = \prod_{n \geq 0} f^{-n}$ is a morphism of complexes. The mapping cone of f reads as follows:

$$\begin{aligned} \cdots \rightarrow B \otimes C_{-n} \oplus B \otimes C_{-n+1} &\xrightarrow{\delta^{-n}} \cdots \rightarrow B \otimes C_{-2} \oplus B \otimes C_{-1} \\ &\xrightarrow{\delta^{-2}} B \otimes C_{-1} \oplus B \xrightarrow{\delta^{-1}} B \rightarrow 0, \end{aligned} \quad (5)$$

where the differential is given by: $\delta^{-n} = \begin{pmatrix} \partial^{-n} & f^{-n+1} \\ 0 & -\partial^{-n+1} \end{pmatrix}$ for $n \geq 2$, and $\delta^{-1} = (\partial^{-1}, f_0)$ for $n = 1$.

A straightforward verification shows that the complex (5) is exact except at the final position, and the cohomology at the final position is \mathbb{k} . Hence we have the following lemma.

Lemma 2.1. *The complex (5) is a minimal projective resolution of the trivial module ${}_B \mathbb{k}$.*

Next we compute the Yoneda product of $E(B)$. Note that $\text{Hom}_B(B \otimes C_{-n} \oplus B \otimes C_{-n+1}, {}_B \mathbb{k}) \cong C_{-n}^* \oplus C_{-n+1}^*$. For $\alpha \in C_{-n}^*$ and $\beta \in C_{-n+1}^*$, we view (α, β) as a homomorphism from $B \otimes C_{-n} \oplus B \otimes C_{-n+1}$ to ${}_B \mathbb{k}$. Consider the following diagram:

$$\begin{array}{ccccccc} \cdots B \otimes C_{-n-k} \oplus B \otimes C_{-n-k+1} & \xrightarrow{\delta^{-n-k}} & \cdots & \longrightarrow & B \otimes C_{-n} \oplus B \otimes C_{-n+1} & \longrightarrow & \cdots \\ \downarrow g_k & & & & \downarrow g_0 & \searrow (\alpha, \beta) & \\ \cdots B \otimes C_{-k} \oplus B \otimes C_{-k+1} & \xrightarrow{\delta^{-k}} & \cdots & \longrightarrow & B & \xrightarrow{\delta^{-1}} & \mathbb{k} \longrightarrow 0, \end{array}$$

where the graded B -module homomorphisms g_k 's are defined as follows: for $k \geq 1$,

$$g_k(1 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+k}, 0) = (1 \otimes x_1 \otimes \cdots \otimes x_k \alpha(x_{k+1} \otimes \cdots \otimes x_{n+k}), 0),$$

and

$$\begin{aligned} &g_k(0, 1 \otimes x'_1 \otimes \cdots \otimes x'_k \otimes x'_{k+1} \otimes \cdots \otimes x'_{n+k-1}) \\ &= ((-1)^k 1 \otimes x'_1 \otimes \cdots \otimes x'_k \beta(x'_{k+1} \otimes \cdots \otimes x'_{n+k-1}), 1 \otimes x'_1 \otimes \cdots \\ &\quad \otimes x'_{k-1} \alpha(\sigma^{-1}(x'_k) \otimes \cdots \otimes \sigma^{-1}(x'_{n+k-1}))); \end{aligned}$$

for $k = 0$,

$$g_0(1 \otimes x_1 \otimes \cdots \otimes x_n, 1 \otimes x'_1 \otimes \cdots \otimes x'_{n-1}) = \alpha(x_1 \otimes \cdots \otimes x_n)1 + \beta(x'_1 \otimes \cdots \otimes x'_{n-1})1.$$

A direct verification shows that the above diagram is commutative.

Since the projective resolution (5) is minimal, we have $\text{Ext}_B^n({}_A \mathbb{k}, {}_A \mathbb{k}) \cong C_{-n}^* \oplus C_{-n+1}^*$ for all $n \geq 0$. Now assume $\alpha' \in C_{-k}^*$ and $\beta' \in C_{-k+1}^*$, we have

$$(\alpha', \beta') * (\alpha, \beta) = (\alpha', \beta') \circ g_k = (\alpha' \cdot \alpha, (-1)^k \alpha' \cdot \beta + \beta' \cdot (\sigma^{-1})^1(\alpha)). \quad (6)$$

Proposition 2.2. *Let A be a Koszul algebra, σ a graded automorphism of A , and $B = A[z; \sigma]$. Then $E(B) \cong \Gamma(A^1, {}_\epsilon A_\psi^1(-1))$, where $\psi = (\sigma^{-1})^1$ is the automorphism of A^1 dual to σ^{-1} .*

Proof. Note that $A^! \cong C^\vee$ as graded algebras. The lemma is a direct consequence of Eq. (6). \square

3. Skew polynomial algebras with coefficients in Koszul Artin–Schelter regular algebras

In this section, A is a Koszul Artin–Schelter regular algebra of global dimension d . The Artin–Schelter regularity of A implies that $E(A) \cong A^!$ is a Frobenius algebra of length d [16]. Let φ be the Nakayama automorphism of the Frobenius algebra $A^!$.

The following result was originally proved by Van den Bergh in [18] in the Noetherian case. The result for general Koszul algebras was proved by Berger and Marconnet in [1, proof of Theorem 6.3].

Lemma 3.1. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d . Let φ be the Nakayama automorphism of $A^!$, and $\phi := \varphi^!$ the automorphism of A dual to φ . Then $\text{Ext}_{A^e}^i(A, A \otimes A) = 0$ for $i \neq d$, and*

$$\text{Ext}_{A^e}^d(A, A \otimes A) \cong A_\xi(d),$$

where $\xi = \epsilon^{d+1}\phi^{-1}$.

The automorphism ξ in the lemma is called the *Nakayama automorphism* of A .

The above lemma implies the following result (also see [9]).

Lemma 3.2. *Let A be a Koszul algebra. Then A is Calabi–Yau if and only if $E(A)$ is a graded symmetric algebra.*

Now we may prove the main result of this section.

Theorem 3.3. *Let A be a Koszul AS-regular algebra of global dimension d with the Nakayama automorphism ξ . Then the skew polynomial algebra $B = A[z; \xi]$ is a Calabi–Yau algebra of dimension $d + 1$.*

Proof. Keep the notions as in Lemma 3.1. By Proposition 2.2, we have $E(B) \cong \Gamma(A^!, {}_\epsilon A_\psi^!(-1))$, where $\psi = (\xi^{-1})^!$. Note that $\xi^{-1} = \epsilon^{d+1}\phi$. Then $\psi = (\xi^{-1})^! = \epsilon^{d+1}\varphi$. Therefore, we have ${}_\epsilon A_\psi^! \cong A_{\epsilon^d\varphi}^!$. Since $A^!$ is graded Frobenius with Nakayama automorphism φ , we have an isomorphism of $A^!$ -bimodules $A_{\epsilon^d\varphi}^! \cong (A^!)^*(-d)$, which implies $A_{\epsilon^d\varphi}^! \cong (A^!)_{\epsilon^d}^*(-d)$ since $\epsilon\varphi = \varphi\epsilon$. Now we have

$$E(B) \cong \Gamma(A^!, {}_\epsilon A_\psi^!(-1)) \cong \Gamma(A^!, (A^!)_{\epsilon^d}^*(-d-1)) = \Gamma(A^!, \epsilon^d, d+1). \quad (7)$$

By Remark 1.2, $E(B)$ is a graded symmetric algebra. Lemma 3.2 implies that B is a Calabi–Yau algebra since B is a Koszul algebra. \square

Corollary 3.4. *If A is a Koszul Calabi–Yau algebra, so is $A[z]$.*

Let V be a vector space of dimension n . Fix a basis $\{x_1, \dots, x_n\}$ of V , and x_1^*, \dots, x_n^* the dual basis of V^* . Given an invertible $n \times n$ matrix M , let $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$, where the matrix multiplications are in the tensor algebra $T(V)$. The quadratic algebra A has the following properties [6,20]:

- (i) A is a Koszul AS-regular algebra of global dimension 2;
- (ii) A is a domain;
- (iii) the quadratic dual $A^!$ is defined by the matrix M in the following way: there is a basis ϖ of $A_2^!$, such that for $\alpha = a_1x_1^* + \dots + a_nx_n^* \in A_1^!$ and $\beta = b_1x_1^* + \dots + b_nx_n^* \in A_1^!$, $\alpha\beta = \mathbf{a}M\mathbf{b}^t\varpi$, where $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{k}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{k}^n$ (cf. [10]);

(iv) the Nakayama automorphism of A^1 is defined in the way: $\varphi(\varpi) = \varpi$ and

$$\varphi(x_1^*, \dots, x_n^*) = (x_1^*, \dots, x_n^*)M^{-1}M^t;$$

(v) the Nakayama automorphism ξ of A is defined in the following way:

$$\xi(x_1, \dots, x_n) = -(x_1, \dots, x_n)M^tM^{-1}.$$

Let us check the generating relations of the skew polynomial algebra $B = A[z; \xi]$. Since B is generated by x_1, \dots, x_n, z and is quadratic, B is defined by the relations $f = 0$, $zx_1 = \xi(x_1)z, \dots, zx_n = \xi(x_n)z$. Now $zx_i = -(x_1, \dots, x_n)M^tM^{-1}(0, \dots, 1, \dots, 0)^t z$. If we put zx_1, \dots, zx_n into a column, we obtain

$$M^t \begin{pmatrix} zx_1 \\ \vdots \\ zx_n \end{pmatrix} = -M \begin{pmatrix} x_1 z \\ \vdots \\ x_n z \end{pmatrix}. \quad (8)$$

Then we see that the skew algebra B above is isomorphic to the algebra $B(f)$ constructed in [2], which is the algebra generated by x_1, \dots, x_n, z with relations $f = 0$ and equations in (8). As a corollary, we recover [2, Theorem 2.10].

Theorem 3.5. (See [2].) *The graded algebra $B(f)$ is a Calabi–Yau algebra of dimension 3.*

Example 3.6. Recall that a Noetherian AS-regular algebra B of global dimension d is called a *quantum polynomial algebra* if B is a domain and has Hilbert series $H_B(t) = \frac{1}{(1-t)^d}$ (hence is Koszul). Let B be a Calabi–Yau quantum polynomial algebra of global dimension d . If B is \mathbb{Z}^2 -graded such that it is generated in degrees $(1, 0)$ and $(0, 1)$, and moreover $\dim B_{0,1} = 1$, then $B = A[z; \xi]$, where $A = \bigoplus_{n \geq 0} B_{n,0}$ is a quantum polynomial algebra of dimension $d - 1$ [11, Proposition 3.5].

Let A be a Koszul AS-regular algebra of global dimension 2. By [6,20], there is a finite dimensional vector space V with a fixed basis $\{x_1, \dots, x_n\}$ and an invertible $n \times n$ matrix M such that $A \cong T(V)/(f)$ where $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$. We already know that the Nakayama automorphism of A is defined by $\xi(x_1, \dots, x_n) = -(x_1, \dots, x_n)M^tM^{-1}$, and the Berger–Pichereau's algebra $B(f) \cong A[z; \xi]$. Let M' be another invertible $n \times n$ matrix, and $f' = (x_1, \dots, x_n)M'(x_1, \dots, x_n)^t$. Let $A' = T(V)/(f')$. Denote by ξ' the Nakayama automorphism of A' . The following result was proved in [2, Theorem 3.4] (indeed, Berger–Pichereau did not assume that M and M' are invertible).

Theorem 3.7. (See [2].) *$B(f) \cong B(f')$ as graded algebras if and only if M is congruent to a scalar multiple of M' ; that is, there is an invertible $n \times n$ matrix P and a scalar $k \in \mathbb{k}$ such that $M = kPM'P^t$. Moreover, if every element in \mathbb{k} is a square in \mathbb{k} then $B(f) \cong B(f')$ as graded algebras if and only if M and M' are congruent.*

However, we do not know whether there is a similar result for general Koszul AS-regular algebras.

4. Superpotentials

Let V be a finite dimensional vector space. For the discussions in this section, we need additional notation. Let $\tau : V \otimes V \rightarrow V \otimes V$ be the usual twisting map. For $d \geq 2$, we set a sequence of maps: $\tau_d^0 = 1^{\otimes d} : V^{\otimes d} \rightarrow V^{\otimes d}$, $\tau_d^1 = \tau \otimes 1^{\otimes d-2}, \dots, \tau_d^k = (1^{\otimes k-1} \otimes \tau \otimes 1^{\otimes d-k-1})\tau_d^{k-1}$ for all $k \geq 2$.

Let $\sigma : V \rightarrow V$ be a linear bijective map. Recall that an element $w \in V^{\otimes d}$ is called a *twisted superpotential of degree d* with respect to σ if

$$w = (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w). \quad (9)$$

If σ is the identity map, then w is called a *superpotential*.

Let $\psi : V \rightarrow \mathbb{k}$ be a linear map, and let $u \in V^{\otimes d}$. Following [5], we write

$$[\psi u] = (\psi \otimes 1^{\otimes d-1})(u), \quad \text{and} \quad [u\psi] = (1^{\otimes d-1} \otimes \psi)(u).$$

More generally, if $\Psi \in (V^*)^{\otimes k}$ ($k \leq d$), we have

$$[\Psi u] = (\Psi \otimes 1^{\otimes d-k})(u), \quad \text{and} \quad [u\Psi] = (1^{\otimes d-k} \otimes \Psi)(u).$$

One may check that an element $w \in V^{\otimes d}$ is a twisted superpotential with respect to σ if and only if, for all $\psi \in V^*$, $[\psi w] = (-1)^{d-1} [w(\psi \circ \sigma^{-1})]$.

For $\Psi \in V^* \otimes k$, define the partial derivation of a twisted superpotential w to be

$$\partial_\Psi(w) = [w\Psi].$$

Then $\partial_\Psi(w) \in V^{\otimes d-k}$. The *derivation quotient algebra* $A(w, k)$ of w is defined as follows [5]:

$$A(w, k) = T(V) / (\partial_\Psi(w) : \Psi \in V^* \otimes k).$$

Since in this paper we only discuss the quadratic derivation quotient algebra, we simply write $A(w)$ for $A(w, d-2)$ for a twisted superpotential w of degree d .

We now show that any twisted superpotential can be symmetrized into a superpotential by introducing an additional indeterminate. From Eq. (9), we have the following facts.

Lemma 4.1.

- (i) If $i \geq j \geq 1$, then $\tau_d^i \circ \tau_d^j = \tau_d^{j-1} \circ (1 \otimes \tau_{d-1}^{i-1})$, and $\underbrace{\tau_d^{d-1} \circ \dots \circ \tau_d^{d-1}}_{d \text{ factors}} = 1$;
 (ii) Let $w \in V^{\otimes d}$ be a twisted superpotential with respect to a bijection σ of V . Then we have

$$w = \sigma^{\otimes d}(w).$$

Proof. (i) is trivial. For the statement (ii), we have

$$\begin{aligned} w &= (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w) \\ &= (-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})((-1)^{d-1} \tau_d^{d-1} \circ (\sigma \otimes 1^{\otimes d-1})(w)) \\ &= (-1)^{2(d-1)} \tau_d^{d-1} \circ \tau_d^{d-1} \circ (\sigma^{\otimes 2} \otimes 1^{\otimes d-1})(w) \\ &\vdots \\ &= (-1)^{d(d-1)} \underbrace{\tau_d^{d-1} \circ \dots \circ \tau_d^{d-1}}_{d \text{ factors}} \circ \sigma^{\otimes d}(w) \\ &= \sigma^{\otimes d}(w). \quad \square \end{aligned}$$

Proposition 4.2. Assume that $w \in V^{\otimes d}$ is a twisted superpotential with respect to a bijection σ of V . We construct an element $\hat{w} := \hat{w}(w, \sigma) \in (V \oplus \mathbb{k}z)^{\otimes d+1}$ as follows:

$$\hat{w} := \hat{w}(w, \sigma) = \sum_{i=0}^d (-1)^i \tau_{d+1}^i (1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w).$$

Then \hat{w} is a superpotential of degree $d+1$.

Proof. We need to show the identity: $\hat{w} = (-1)^d \tau_{d+1}^d(\hat{w})$. This follows from the following computations:

$$\begin{aligned} \tau_{d+1}^d(\hat{w}) &= \tau_{d+1}^d \left(\sum_{i=0}^d (-1)^i \tau_{d+1}^i (1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w) \right) \\ &= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \tau_d^{d-1})(1 \otimes \sigma^{\otimes i} \otimes 1^{\otimes d-i})(z \otimes w) \\ &= \tau_{d+1}^d(z \otimes w) \\ &\quad + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \tau_d^{d-1})(1 \otimes 1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i})(1 \otimes \sigma \otimes 1^{\otimes d-1})(z \otimes w) \\ &= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes \tau_d^{d-1}(\sigma \otimes 1^{\otimes d-1})(w)) \\ &= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^i \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes (-1)^{d-1} w) \\ &= \tau_{d+1}^d(z \otimes w) + \sum_{i=1}^d (-1)^{d+i-1} \tau_{d+1}^{i-1} (1 \otimes \sigma^{\otimes i-1} \otimes 1^{\otimes d-i+1})(z \otimes w) \\ &= \tau_{d+1}^d(z \otimes w) + \sum_{j=0}^{d-1} (-1)^{d+j} \tau_{d+1}^j (1 \otimes \sigma^{\otimes j} \otimes 1^{\otimes d-j})(z \otimes w) \\ &= \tau_{d+1}^d(z \otimes \sigma^{\otimes d}(w)) + \sum_{j=0}^{d-1} (-1)^{d+j} \tau_{d+1}^j (1 \otimes \sigma^{\otimes j} \otimes 1^{\otimes d-j})(z \otimes w) \\ &= (-1)^d \hat{w}. \quad \square \end{aligned}$$

Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d . It is established in [5] and [6] that A is defined by a twisted superpotential; that is, $A \cong A(w)$ for some twisted superpotential w of degree d with respect to a suitable bijection $\sigma : V \rightarrow V$. We give a “visualized” description of the bijection σ and the twisted superpotential w .

Assume $\dim V = n$ and fix a basis $\{x_1, \dots, x_n\}$ of V . We use the definitions and notations as in Section 2. Since A is of global dimension d , we have $\dim C_{-d} = 1$ and $\dim C_{-d+1} = n$. Choose a nonzero element $w \in C_{-d}$. Since $C_{-d} = \bigcap_{i=0}^{d-2} V^{\otimes i} \otimes R \otimes V^{\otimes d-i-2}$, it follows that $w \in V \otimes C_{-d+1} \cap C_{-d+1} \otimes V$. We fix a basis of C_{-d+1} , say, $\{\theta_1, \dots, \theta_n\}$. Since $w \in V \otimes C_{-d+1}$, we may write w as $w = (x_1, \dots, x_n)M(\theta_1, \dots, \theta_n)^t$ for some $n \times n$ matrix M with entries in \mathbb{k} . On the

other hand, $w \in C_{-d+1} \otimes V$ implies that $w = (\theta_1, \dots, \theta_n)N(x_1, \dots, x_n)^t$ for some $n \times n$ matrix N . Let $\{x_1^*, \dots, x_n^*\}$ be the dual basis of V^* , and $\theta_1^*, \dots, \theta_n^*$ be the dual basis of C_{-d+1}^* . Since A is AS-regular, $A^! \cong C^\vee$ is a graded Frobenius algebra of length d . For $\alpha = (x_1^*, \dots, x_n^*)(a_1, \dots, a_n)^t \in V^* = A_1^!$ and $\beta = (\theta_1^*, \dots, \theta_n^*)(b_1, \dots, b_n)^t \in C_{-d+1}^* = A_{d-1}^!$, it is easy to see that the Yoneda product is given by

$$\alpha * \beta = (a_1, \dots, a_n)M \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} w^*,$$

and

$$\beta * \alpha = (b_1, \dots, b_n)N \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} w^*.$$

Now the Frobenius property of $A^!$ implies that both M and N are invertible matrices. Let φ be the Nakayama automorphism of $A^!$. Then from the Yoneda product above, we see that

$$\varphi(x_1^*, \dots, x_n^*) = (x_1^*, \dots, x_n^*)N^{-1}M^t.$$

Let $\phi := \varphi^!$ be the automorphism of A dual to φ . Then the Nakayama automorphism of A is $\xi = \epsilon^{d+1}\phi^{-1}$, which acts on $A_1 = V$ as follows:

$$\xi(x_1, \dots, x_n) = (-1)^{d+1}(x_1, \dots, x_n)N^tM^{-1}.$$

We rewrite the element w in terms of the Nakayama automorphism ξ as follows:

$$\begin{aligned} w &= (\theta_1, \dots, \theta_n)N \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\theta_1, \dots, \theta_n)M^t(M^{-1})^tN \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= (-1)^{d-1}(\theta_1, \dots, \theta_n)M^t\xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

That is, $w = (-1)^{d-1}\tau_d^{d-1}(\xi \otimes 1^{\otimes d-1})(w)$.

Summarizing the above arguments, we obtain the following lemma.

Lemma 4.3. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of dimension d with ξ the Nakayama automorphism. Then w is a twisted superpotential with respect to $\xi|_V$.*

Theorem 4.4. *Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension $d \geq 2$ with ξ the Nakayama automorphism. Then*

- (i) $A \cong A(w)$, where w is a nonzero element in $\bigcap_{i=1}^{d-2} V^{\otimes i} \otimes R \otimes V^{d-2-i}$;
- (ii) $A[z; \xi] \cong A(\hat{w})$, where the superpotential $\hat{w} = \hat{w}(w, \xi)$ is formed in [Proposition 4.2](#).

Proof. The statement (i) is essentially proved in [6] and [5]. We include here our own proof. Assume $\dim R = m$. Since A^1 is Frobenius and $A_2^1 = R^*$, we have that $A_{d-2}^1 \cong C_{-d+2}^*$ is of dimension m . Fix a basis $\{r_1, \dots, r_m\}$ of R , and a basis $\{\vartheta_1, \dots, \vartheta_m\}$ of C_{-d+2}^* . As before, we let $\{r_1^*, \dots, r_m^*\}$ and $\{\vartheta_1^*, \dots, \vartheta_m^*\}$ be the dual bases of R^* and C_{-d+2}^* respectively. Note that we also have $w \in R \otimes C_{-d+2}^*$. Hence there is an $m \times m$ matrix L such that

$$w = (r_1, \dots, r_m)L(\vartheta_1, \dots, \vartheta_m)^t. \quad (10)$$

For $\alpha = (r_1^*, \dots, r_m^*)(a_1, \dots, a_m)^t$ and $\beta = (\vartheta_1^*, \dots, \vartheta_m^*)(b_1, \dots, b_m)^t$, we have

$$\alpha * \beta = (a_1, \dots, a_m)L(b_1, \dots, b_m)^t w^*.$$

By the Frobenius property of A^1 , we have that L is invertible. Then from the expression of w as in (10), we see that $R = \{\partial_\psi(w) \mid \psi \in (V^*)^{\otimes d-2}\}$. Therefore $A \cong A(w)$.

(ii) Since w is a twisted superpotential with respect to ξ , \hat{w} is a superpotential of degree $d+1$ by Proposition 4.2. Let $U = V \oplus \mathbb{k}z$. Then $\{x_0^* = z^*, x_1^*, \dots, x_n^*\}$ is a basis of U^* . Let us check the following facts:

- (a) $\{\partial_\psi(\hat{w}) \mid \psi \in (V^*)^{\otimes d-1}\} = \text{span}\{z \otimes x_i - \xi(x_i) \otimes z \mid i = 1, \dots, n\}$;
- (b) $R = \text{span}\{\partial_\psi(\hat{w}) \mid \psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-1}}^* \text{ at least one of } i_1, \dots, i_{d-1} \text{ is zero}\}$.

For $\psi \in (V^*)^{\otimes d-1}$, we have

$$\partial_\psi(\hat{w}) = (1 \otimes 1 \otimes \psi)[(z \otimes w) - \tau_{d+1}^1 \circ (1 \otimes \xi \otimes 1^{\otimes d-1})(z \otimes w)].$$

Recall that

$$w = (x_1, \dots, x_n)M \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}.$$

So, if we write $w = \sum_{i=1}^n x_i \otimes y_i$, then $(y_1, \dots, y_n) = (\theta_1, \dots, \theta_n)M^t$. Since M is invertible, we obtain that y_1, \dots, y_n are linear independent in $V^{\otimes d-1}$. Now we have

$$\partial_\psi(\hat{w}) = \sum_{i=1}^n (z \otimes x_i - \xi(x_i) \otimes z) \psi(y_i).$$

Thus (a) follows.

For the identity (b), we choose $\psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-2}}^* \otimes z^*$. Then $\partial_\psi(\hat{w}) = (-1)^d (1 \otimes 1 \otimes \psi)(\xi^{\otimes d}(w) \otimes z)$. On the other hand, as we have seen that we may write w in the form (10). Again since L is invertible, we have

$$\text{span}\{\partial_\psi(\hat{w}) \mid \psi \text{ is of the form } x_{i_1}^* \otimes \dots \otimes x_{i_{d-2}}^* \otimes z^*\} = \text{span}\{(\xi \otimes \xi)(r_i) \mid i = 1, \dots, m\}.$$

As A is Koszul and ξ is the Nakayama automorphism of A , we have $\text{span}\{(\xi \otimes \xi)(r_i) \mid i = 1, \dots, m\} = R$. Since we obviously have $R \supseteq \text{span}\{\partial_\psi(\hat{w}) \mid \psi = x_{i_1}^* \otimes \dots \otimes x_{i_{d-1}}^* \text{ at least one of } i_1, \dots, i_{d-1} \text{ is zero}\}$, (b) follows.

Finally, since $R + \text{span}\{z \otimes x_i - \xi(x_i) \otimes z \mid i = 1, \dots, n\}$ is exactly the generating relations of $A[z; \xi]$, we have $A[z; \xi] \cong A(\hat{w})$. \square

5. PBW-deformations

Let $A = A_0 \oplus A_1 \oplus \cdots$ be a positively graded algebra. Recall that a PBW-deformation of A is a filtered algebra U with an ascending filtration $0 \subseteq F_0 U \subseteq F_1 U \subseteq F_2 U \subseteq \cdots$ such that the associated graded algebra $\text{gr}(U)$ is isomorphic to A . If $A = T(V)/(R)$ is a Koszul algebra, then a PBW-deformation U of A is determined by two linear maps $\nu : R \rightarrow V$ and $\theta : R \rightarrow \mathbb{k}$ in the sense that $U \cong T(V)/(r - \nu(r) - \theta(r) : r \in R)$ [4,13]. If $\theta = 0$, then we call U an *augmented* PBW-deformation of A . The dual map ν^* of the linear map $\nu : R \rightarrow V$ induces a derivation δ_{A^\dagger} on the dual algebra A^\dagger of A . If we view the linear map $\theta : R \rightarrow \mathbb{k}$ as an element in A_2^\dagger , then $(A^\dagger, \delta_{A^\dagger}, \theta)$ is a cdga. We call $(A^\dagger, \delta_{A^\dagger}, \theta)$ the *dual cdga* of U . Conversely, if there is a curved differential graded structure $(A^\dagger, \delta_{A^\dagger}, \theta)$, then the dual map of the linear map $\delta_{A^\dagger}|_{V^*} : V^* \rightarrow R^*$ and $\theta \in A_2^\dagger = R^*$ define a PBW-deformation of A [13].

Now let A be a Koszul AS-regular algebra of global dimension d , and let U be a PBW-deformation of A . Assume that ξ is the Nakayama automorphism of A . The following lemma was proved by Yekutieli [19] when A is Noetherian. For the general case, see [10].

Lemma 5.1. *We have $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ for $i \neq d$ and $\text{Ext}_{U^e}^d(U, U \otimes U) \cong U_\zeta$ as U -bimodules, where ζ is a filtration-preserving automorphism of U such that $\text{gr}(\zeta) = \xi$.*

The automorphism ζ in Lemma 5.1 is not unique. If ζ' is another automorphism of U such that the conditions in the lemma hold, then ζ' differs from ζ by an inner automorphism, that is, there is a unit $u \in U$ such that for all $a \in U$, $\zeta'(a) = u\zeta(a)u^{-1}$. Hence ζ is unique up to inner automorphisms. We call an automorphism ζ satisfying the conditions in Lemma 5.1 a *Nakayama automorphism* of U . Note that if A is a domain, then there is a unique automorphism satisfies the condition in Lemma 5.1. Hence in this case, we may say “the” Nakayama automorphism of U .

Next we discuss the Calabi–Yau property of the skew polynomial algebra $U[z; \zeta]$ with ζ a Nakayama automorphism of U .

The skew polynomial algebra $U[z; \zeta]$ is also a filtered algebra with filtration: $F_0 U[z; \zeta] = \mathbb{k}$, $F_n U[z; \zeta] = \sum_{i+j=n} F_i U z^j$ for all $n > 0$ and $i, j \geq 0$. It is easy to see that $\text{gr}(U[z; \zeta]) \cong A[z; \xi]$. Hence we obtain:

Lemma 5.2. *$U[z; \zeta]$ is a PBW-deformation of $A[z; \xi]$.*

The following result was proved in [10].

Lemma 5.3. *Let B be a Koszul Calabi–Yau algebra of dimension d , and let $U = T(V)/(r - \nu(r) - \theta(r) : r \in R)$ be a PBW-deformation of B , and let $(B^\dagger, \delta_{B^\dagger}, \theta)$ be the cdga dual to U . If $\delta_{B^\dagger}(B_{d-1}^\dagger) = 0$, then U is a Calabi–Yau algebra.*

Conversely, if B is a domain and U is Calabi–Yau, then $\delta_{B^\dagger}(B_{d-1}^\dagger) = 0$.

Let $B = A[z; \xi]$. As we have already proved in Section 3, B is a Koszul Calabi–Yau algebra of dimension $d + 1$. To see whether $U[z; \zeta]$ is a Calabi–Yau algebra, it is sufficient to see whether the condition $\delta_{B^\dagger}(B_d^\dagger) = 0$ holds. By Proposition 2.2, $B^\dagger \cong \Gamma(A^\dagger, {}_\epsilon A_\psi^\dagger(-1))$ where $\psi = (\xi^{-1})^\dagger$. We need to write out the differential on $\Gamma(A^\dagger, {}_\epsilon A_\psi^\dagger(-1))$ induced by δ_{B^\dagger} through the previous isomorphism. The following lemma is trivial.

Lemma 5.4. *Let D and D' be quadratic algebras. If there are invertible linear maps $f : D_1 \rightarrow D'_1$ and $g : D_2 \rightarrow D'_2$ such that the following diagram commutes*

$$\begin{array}{ccc} D_1 \otimes D_1 & \xrightarrow{f \otimes f} & D'_1 \otimes D'_1 \\ \downarrow \mu_D & & \downarrow \mu_{D'} \\ D_2 & \xrightarrow{g} & D'_2, \end{array}$$

where the vertical maps are multiplications of D and D' respectively, then the linear map f defines an isomorphism $\Phi : D \rightarrow D'$ in the following way: for any $x_1, \dots, x_n \in D_1$, $\Phi(x_1 x_2 \cdots x_n) = f(x_1) f(x_2) \cdots f(x_n)$.

Let us write down $B_1^!$ and $B_2^!$ of $B^!$ explicitly. Write $\widehat{V} = V \oplus \mathbb{k}z$. As before, we fix a basis $\{x_1, \dots, x_n\}$ for V , and let $\{x_1^*, \dots, x_n^*\}$ be the dual basis of V^* . Let $\tilde{r}_i = z \otimes \xi^{-1}(x_i) - x_i \otimes z$ for $i = 1, \dots, n$, and $\tilde{R} = \text{span}\{\tilde{r}_1, \dots, \tilde{r}_n\} \subseteq \widehat{V} \otimes \widehat{V}$. Then $B = T(\widehat{V})/(\tilde{R})$, where $\widehat{R} = R \oplus \tilde{R}$. Let $\{\tilde{r}_1^*, \dots, \tilde{r}_n^*\}$ be the dual basis of \tilde{R} and z^* be the element in \widehat{V}^* such that $z^*(z) = 1$ and $z^*(V) = 0$. We have $B_1^! = \widehat{V}^*$ and $B^! = \widehat{R}^* = R^* \oplus \tilde{R}^*$, equivalently $B_1^! = A_1^! \oplus \mathbb{k}z^*$ and $B_2^! = A_2^! \oplus \tilde{R}^*$.

Assume that the automorphism ξ of A acts on $A_1 = V$ as

$$\xi(x_1, \dots, x_n) = (x_1, \dots, x_n)P \quad (11)$$

where $P = (p_{ij})$ is an invertible $n \times n$ matrix. Assume further $P^{-1} = (l_{ij})$. Then it is not hard to see that the product of two elements in $B_1^!$ is given as follows: for $x_i^*, x_j^* \in A_1^!$, the product $x_i^* \cdot x_j^*$ is just the product in $A^!$; $z^* \cdot z^* = 0$;

$$x_i^* \cdot z^* = -\tilde{r}_i^* \in \tilde{R}^* \subseteq B_2^! \quad \text{and} \quad z^* \cdot x_i^* = \sum_{j=1}^n l_{ij} \tilde{r}_j^*.$$

Recall that the graded algebra $B^!$ is isomorphic to $\Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))$ with $\psi = (\xi^{-1})^!$ (Proposition 2.2). We construct an isomorphism from $\Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))$ to $B^!$ in detail. Note that $\Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))_1 = A_1^! \oplus \mathbb{k}$ and $\Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))_2 = A_2^! \oplus V^*$. We define linear maps $f : \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))_1 \rightarrow B_1^!$ and $g : \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1))_2 \rightarrow B_2^!$ as follows: $f(x_i^*, 0) = x_i^*$ for all i and $f(0, 1) = z^*$; $g(\alpha, 0) = \alpha$ for all $\alpha \in A_2^!$ and $g(0, x_i^*) = \tilde{r}_i^*$ for all $i = 1, \dots, n$. Now one may easily check that the conditions of Lemma 5.4 above hold for f and g . Therefore f defines an isomorphism

$$\Phi : \Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1)) \rightarrow B^! \quad (12)$$

since both algebras are Koszul.

As before, let φ be the Nakayama automorphism of $A^!$, and let $\phi = \varphi^!$ be the automorphism of A dual to φ . Then $\xi = \epsilon^{d+1} \phi^{-1}$ by Lemma 3.1. Hence $\varphi(x_i^*) = (-1)^{d+1} (\xi^{-1})^!(x_i^*)$ for all $i = 1, \dots, n$. Since $A^!$ is Frobenius, there is an isomorphism of graded $A^!$ -bimodules $\Theta : A_{\phi}^! \rightarrow A^{!*}(-d)$. Let $\varpi \in A_d^!$ be the element such that $\Theta(1)(\varpi) = 1$. Then ϖ is a basis of $A_d^!$. By the Frobenius property of $A^!$ again, we may choose elements $\omega_1, \dots, \omega_n$ in $A_{d-1}^!$ such that $x_i^* \omega_j = \delta_j^i \varpi$, where δ is the Kronecker delta function. Clearly, $\{\omega_1, \dots, \omega_n\}$ is a basis of $A_{d-1}^!$. Let ϖ^* and $\{\omega_1^*, \dots, \omega_n^*\}$ be the dual basis of the space $(A_d^!)^*$ and $(A_{d-1}^!)^*$ respectively. Consider the composition of the following isomorphisms:

$$h : {}_{\epsilon}A_{\psi}^! \xrightarrow{\epsilon^{d+1}} {}_{\epsilon}A_{\phi}^! \xrightarrow{\Theta} {}_{\epsilon}A^{!*}(-d).$$

We have $h(1) = \varpi^*$ and $h(x_i^*) = \sum_{j=1}^n p_{ij} \omega_j^*$. The isomorphism h induces an isomorphism $\Gamma(A^!, {}_{\epsilon}A_{\psi}^!(-1)) \rightarrow \Gamma(A^!, {}_{\epsilon}A^{!*}(-d-1))$. Combining this isomorphism with the inverse of Φ constructed in previous paragraph, we get an isomorphism of graded algebras:

$$\Psi : B^! \rightarrow \Gamma(A^!, {}_{\epsilon}A^{!*}(-d-1)). \quad (13)$$

Now we have $\Psi(\alpha) = (\alpha, 0)$ for all $\alpha \in A^!$, and

$$\Psi(z^*) = (0, \varpi^*) \quad \text{and} \quad \Psi(\tilde{r}_i^*) = \left(0, \sum_{j=1}^n p_{ij} \omega_j^*\right) \quad (14)$$

for all $i = 1, \dots, n$.

Since $U[z; \zeta]$ is a PBW-deformation of $B = A[z; \xi]$, to study the curved differential structure of $B^!$, we need to pick a specific Nakayama automorphism ζ . The following result was proved in [10].

Proposition 5.5. *Let $A = T(V)/(R)$ be a Koszul AS-Gorenstein algebra of global dimension d , and let $A^!$ be its dual algebra. Assume that $\{x_1, \dots, x_n\}$ is a basis of V , and $\{x_1^*, \dots, x_n^*\}$ is the dual basis of V^* .*

Let $U = T(V)/(r - v(r) - \theta(r): r \in R)$ be a PBW-deformation of A , and let $(A^!, \delta_{A^!}, \theta)$ be the cdga dual to U . Choose a basis ϖ of $A_{d'}^!$, and assume that $\{\omega_1, \dots, \omega_n\}$ is the basis of $A_{d-1}^!$ such that $x_i^ \omega_j = \delta_j^i \varpi$. Assume further $\delta_{A^!}(\omega_i) = \lambda_i \varpi$ for all $i = 1, \dots, n$. Then $\text{Ext}_{U^e}^i(U, U \otimes U) = 0$ for $i \neq d$, and*

$$\text{Ext}_{U^e}^d(U, U \otimes U) \cong U_\zeta,$$

where the automorphism ζ acts on the generator as follows:

$$\zeta(x_i) = \xi(x_i) + \lambda_i.$$

Convention. From now on, ζ is the Nakayama automorphism of U as defined in Proposition 5.5.

Note that in Proposition 5.5, we view $V \oplus \mathbb{k}$ as a subspace of U through the obvious injective map. The scalars $\lambda_1, \dots, \lambda_n$ are independent of the choice of the basis ϖ . In fact, if we choose another element ϖ' as a basis of $A_{d'}^!$, then $\varpi' = k\varpi$ for some $k(\neq 0) \in \mathbb{k}$. Hence $x_i^*(k\omega_j) = \delta_j^i \varpi'$ for all $i, j = 1, \dots, n$. Set $\omega'_i = k\omega_i$ for $i = 1, \dots, n$. Then $\{\omega'_1, \dots, \omega'_n\}$ is the basis of $A_{d-1}^!$ satisfying the condition in the proposition. Clearly, we have $\delta_{A^!}(\omega'_i) = k\lambda_i \varpi = \lambda_i \varpi'$.

Now we can write down the linear maps that determine the PBW-deformation $U[z; \zeta]$ of $A[z; \xi]$. Recall that $A[z; \xi] \cong T(\widehat{V})/(\widehat{R})$ with $\widehat{V} = V \oplus \mathbb{k}z$ and $\widehat{R} = R \oplus \widehat{R}$.

Lemma 5.6. $\widehat{U} := U[z; \zeta]$ viewed as a PBW-deformation of $B = A[z; \xi]$ is determined by the following linear maps:

$$\hat{v}: R \oplus \widetilde{R} \rightarrow V \oplus \mathbb{k}z, \quad \hat{v}(r) = v(r) \quad \text{for all } r \in R, \quad \text{and} \quad \hat{v}(\tilde{r}_i) = \lambda_i z \quad (i = 1, \dots, n);$$

$$\hat{\theta}: R \oplus \widetilde{R} \rightarrow \mathbb{k}, \quad \hat{\theta}(r) = \theta(r) \quad \text{for all } r \in R, \quad \text{and} \quad \hat{\theta}(\tilde{r}_i) = 0 \quad (i = 1, \dots, n).$$

That is, $\widehat{U} \cong T(\widehat{V})/(\hat{r} - \hat{v}(\hat{r}) - \hat{\theta}(\hat{r}): \hat{r} \in \widehat{R})$.

Proof. The lemma is clear since $z\xi^{-1}(x_i) - \lambda_i z = x_i z$ by Proposition 5.5, or equivalently, $z\xi^{-1}(x_i) - x_i z - \lambda_i z = 0$ in \widehat{U} . \square

Let us check the linear dual maps of \hat{v} and $\hat{\theta}$. We have $\hat{v}^*: \widehat{V}^* \rightarrow \widehat{R}^*$, $\hat{v}^*|_{V^*} = v^*$, and $\hat{v}^*(z^*) = \sum_{i=1}^n \lambda_i \tilde{r}_i^*$; $\hat{\theta}^* = \theta^*: \mathbb{k} \rightarrow R^* \subseteq \widehat{R}^*$. Let $(A^!, \delta_{A^!}, \theta_{A^!})$ be the cdga dual to the PBW-deformation U of A . Since in the cdga $(B^!, \delta_{B^!}, \theta_{B^!})$, the differential $\delta_{B^!}$ is determined by \hat{v}^* and the curvature element $\theta_{B^!} = \hat{\theta}$, we have

$$\delta_{B^!}(\alpha) = \delta_{A^!}(\alpha) \quad \text{for all } \alpha \in V^*$$

and

$$\delta_{B^!}(z^*) = \sum_{i=1}^n \lambda_i \tilde{r}_i^*.$$

The cdga structure on $B^!$ induces a cdga structure on the graded algebra $\Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$ through the isomorphism ψ as in (13) and (14). Denote by $\Gamma := \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$. Let $(\Gamma, \delta_\Gamma, \theta_\Gamma)$ be the cdga induced by $(B^!, \delta_{B^!}, \theta_{B^!})$. Then we have

$$\theta_\Gamma = (\theta_{A^!}, 0); \quad (15)$$

$$\delta_\Gamma(\alpha, 0) = (\delta_{A^!}(\alpha), 0) \quad \text{for all } \alpha \in A^!; \quad (16)$$

$$\delta_\Gamma(0, \varpi^*) = \left(0, \sum_{j=1}^n \lambda_j \left(\sum_{i=1}^n p_{ji} \omega_i^* \right) \right). \quad (17)$$

Proposition 5.7. Let $A = T(V)/(R)$ be a Koszul AS-regular algebra of global dimension d with ξ the Nakayama automorphism, and $A^!$ be its quadratic dual algebra with φ the Nakayama automorphism. Let U be a PBW-deformation of A , and $(A^!, \delta_{A^!}, \theta_{A^!})$ be the cdga dual to U . If the composition $\epsilon^{d+1}\varphi$ is an automorphism of $\text{cdga}(A^!, \delta_{A^!}, \theta_{A^!})$, then

- (i) The trivial extension $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$ of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ is the dual cdga of the PBW-deformation $U[z; \zeta]$ of $A[z; \xi]$, where ζ is the Nakayama automorphism of U as in Proposition 5.5;
- (ii) $U[z; \zeta]$ is a Calabi–Yau algebra.

Proof. Keep the notions as before except we now choose the basis ϖ in Proposition 5.5 such that $\Theta(1)(\varpi) = 1$. Let us check the differential $\delta_{A^!}^*$ on $(A^!)^*$. Recall that $\delta_{A^!}(\omega_i) = \lambda_i \varpi$ for all $i = 1, \dots, n$ by assumption. Thus we have

$$\delta_{A^!}^*(\varpi^*) = \sum_{i=1}^n \lambda_i \omega_i^*.$$

Note that $\varphi = \epsilon^{d+1}(\xi^{-1})^!$, and ξ is represented as in (11). An easy computation shows that $\varphi(\varpi) = \varpi$ and that

$$\varphi(\omega_i) = (-1)^{d+1} \sum_{j=1}^n p_{ji} \omega_j.$$

Now by assumption of the proposition $\delta_{A^!} \circ (\epsilon^{d+1}\varphi) = (\epsilon^{d+1}\varphi) \circ \delta_{A^!}$, we have $\delta_{A^!}^* \circ (\epsilon^{d+1}\varphi)^* = (\epsilon^{d+1}\varphi)^* \circ \delta_{A^!}^*$. Applying these morphisms to ϖ^* , we obtain

$$(\epsilon^{d+1}\varphi)^* \circ \delta_{A^!}^*(\varpi^*) = (\epsilon^{d+1}\varphi)^* \left(\sum_{i=1}^n \lambda_i \omega_i^* \right) = \sum_{i=1}^n \lambda_i \sum_{j=1}^n p_{ij} \omega_j^*;$$

and

$$\delta_{A^!}^* \circ (\epsilon^{d+1}\varphi)^*(\varpi^*) = \delta_{A^!}^*(\varpi^*) = \sum_{i=1}^n \lambda_i \omega_i^*.$$

Hence we arrive at

$$\sum_{i=1}^n \lambda_i \omega_i^* = \sum_{i=1}^n \lambda_i \sum_{j=1}^n p_{ij} \omega_j^*. \quad (18)$$

Comparing Eqs. (17) and (18), we see that the differential δ_Γ on $\Gamma := \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$, induced from the cdga $(B^!, \delta_{B^!}, \theta_{B^!})$, acts on the elements of degree 1 as

$$\delta_\Gamma(\alpha, 0) = (\delta_{A^!}(\alpha), 0) \quad \text{for } \alpha \in A_1^!, \quad (19)$$

$$\delta_\Gamma(0, \varpi^*) = \left(0, \sum_{i=1}^n \lambda_i \omega_i^*\right). \quad (20)$$

Since Γ is a quadratic algebra, the differential δ_Γ is determined by its action on the elements of degree 1. By Lemma 1.3, the underlying graded algebra of the trivial extension $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$ of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ is exactly the graded algebra $\Gamma = \Gamma(A^!, \epsilon^d(A^!)^*(-d-1))$. Let $\delta_{\Gamma_{\text{cdg}}}$ be the differential of $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$. By a straightforward check we have $\delta_{\Gamma_{\text{cdg}}}(\alpha, 0) = (\delta_{A^!}(\alpha), 0)$, for $\alpha \in A_1^!$ and $\delta_{\Gamma_{\text{cdg}}}(0, z^*) = (0, \sum_{i=1}^n \lambda_i \omega_i^*)$. Comparing these equations with (19) and (20), we see that the cdga $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$ is isomorphic to the cdga $(\Gamma, \delta_\Gamma, \theta_\Gamma)$. Hence the statement (i) follows.

Write $\widehat{\Gamma} := \Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$. Then $\widehat{\Gamma}_d = A_d^! \oplus (A_1^!)^*$ and $\widehat{\Gamma}_{d+1} = \mathbb{k}$. Now it is clear that $\delta_{\Gamma_{\text{cdg}}}(\widehat{\Gamma}_d) = 0$. By Theorem 3.3, $A[z; \xi]$ is Calabi–Yau. Thus the statement (ii) follows from Lemma 5.3. \square

As a special case of Proposition 5.7, we obtain the following theorem.

Theorem 5.8. *Let A be a Koszul Calabi–Yau algebra of global dimension d , and let U be an arbitrary PBW-deformation of A . Assume that ζ is the Nakayama automorphism of U as in Proposition 5.5. Then $U[z; \zeta]$ is Calabi–Yau.*

Proof. Since A is Calabi–Yau, then the quadratic dual $A^!$ is graded symmetric; that is, the Nakayama automorphism of $A^!$ is $\varphi = \epsilon^{d+1}$. Then $\epsilon^{d+1}\varphi = \text{id}$, which is certainly an automorphism of the dual cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ of the PBW-deformation U . \square

If U is an augmented PBW-deformation, then the ground field \mathbb{k} is a left U -module through the augmentation map. Let $E(U) := \bigoplus_{i \geq 0} \text{Ext}_{U^e}^i(U \otimes \mathbb{k}, U \otimes \mathbb{k})$. Note that the curvature element of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$ dual to U is zero. Thus $(A^!, \delta_{A^!})$ is a usual dga; that is, $\delta_{A^!}^2 = 0$. So, the cohomology $HA^!$ of $(A^!, \delta_{A^!})$ is a graded algebra.

Proposition 5.9. *Let A be a Koszul Calabi–Yau algebra, and U an augmented PBW-deformation of A . Assume that ζ is the Nakayama automorphism of U as in Proposition 5.5. Then*

$$E(U[z; \zeta]) \cong \Gamma(H(A^!), \epsilon^{d+1}H(A^!)^*(-d-1)).$$

Proof. If U is an augmented PBW-deformation of A , then $U[z; \zeta]$ is an augmented PBW-deformation of $A[z]$. Hence $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$ is a dga. By [13, Chapter 5, Proposition 6.1] and Proposition 5.7, $E(U[z; \zeta])$ is isomorphic to the cohomology algebra of the dga $\Gamma_{\text{cdg}}(A^!, (A^!)^*(-d-1))$. Now Lemma 1.3 implies the desired isomorphism. \square

In Proposition 5.7, we need the condition that the composition $\epsilon^{d+1}\varphi$ is an automorphism of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$. Certainly, there is no reason to expect that $\epsilon^{d+1}\varphi$ is always compatible with the cdga structure on $A^!$. For example, if A is an AS-regular algebra of global dimension 2, then $A^!$ is of length 2. Hence any linear map $\delta: A_1^! \rightarrow A_2^!$ and any element $\theta \in A_2^!$ form a cdga $(A^!, \delta, \theta)$. Below, we show that the condition that $\epsilon^{d+1}\varphi$ is compatible with the cdga structure on $A^!$ is necessary in case that A is an AS-regular algebra of global dimension 2.

From now on, we assume that A is an AS-regular algebra of global dimension 2. Then $A \cong T(V)/(f)$ where V is an n -dimensional vector space with a fixed basis $\{x_1, \dots, x_n\}$, and $f = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$ with M an invertible $n \times n$ matrix [20,6]. Some properties of A has been listed below Corollary 3.4. In the following discussions, we keep the same notions as those in the items listed below Corollary 3.4. We chose a new basis $\{\omega_1, \dots, \omega_n\}$ of $A_1^!$ as

$$(\omega_1, \dots, \omega_n) = (x_1^*, \dots, x_n^*)M^{-1}.$$

Then we have $x_i^* \omega_j = \delta_j^i \varpi$, where ϖ is the basis of $A_2^!$ as in the item (iii) below Corollary 3.4.

Let U be a PBW-deformation of A . Since A is a domain, as we pointed out earlier, there is a unique Nakayama automorphism. Hence we may say “the” Nakayama automorphism of U .

Theorem 5.10. *Let U be a PBW-deformation of A with ζ the Nakayama automorphism of U , and let $(A^!, \delta_{A^!}, \theta_{A^!})$ be the dual cdga of U . Assume $\delta_{A^!}(\omega_1, \dots, \omega_n) = (\lambda_1 \varpi, \dots, \lambda_n \varpi)$. The following are equivalent:*

- (i) $\epsilon\varphi$ is an automorphism of the cdga $(A^!, \delta_{A^!}, \theta_{A^!})$;
- (ii) $U[z; \zeta]$ is Calabi–Yau;
- (iii) $(\lambda_1, \dots, \lambda_n)M = -(\lambda_1, \dots, \lambda_n)M^t$.

Proof. That (i) implies (ii) follows from Proposition 5.7.

(ii) \Rightarrow (iii). Let $B = A[z; \xi]$, $\widehat{U} = U[z; \zeta]$, and $(B^!, \delta_{B^!}, \theta_{B^!})$ the dual cdga of \widehat{U} . Let $\Gamma := \Gamma(A^!, (A^!)^*(-3))$, and $(\Gamma, \delta_\Gamma, \theta_\Gamma)$ the cdga induced by $(B^!, \delta_{B^!}, \theta_{B^!})$ through the isomorphism Ψ given in (13) and (14). Eq. (17) in the present case reads as follows:

$$\delta_\Gamma(0, \varpi^*) = (0, X), \quad (21)$$

where $X = -(\omega_1^*, \dots, \omega_n^*)(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$. Now, since \widehat{U} is Calabi–Yau and B is obviously a domain, we have $\delta_\Gamma(\Gamma_2) = 0$ by Lemma 5.3. Hence we have

$$\begin{aligned} 0 &= \delta_\Gamma((\omega_i, 0) * (0, \varpi^*)) \\ &= \delta_\Gamma(\omega_i, 0) * (0, \varpi^*) - (\omega_i, 0) * \delta_\Gamma(0, \varpi^*) \\ &= (\lambda_i \varpi, 0) * (0, \varpi^*) - (\omega_i, 0) * (0, X) \\ &= (0, \lambda_i \varpi \cdot \varpi^*) - (0, \omega_i \cdot X) \\ &= (0, \lambda_i) - (0, \omega_i \cdot X), \end{aligned}$$

where the notion “ \cdot ” in $\varpi \cdot \varpi^*$ and $\omega_i \cdot X$ is the left $A^!$ -module action on $(A^!)^*$, and in the last identity, we identify \mathbb{k} with $(A_0^!)^*$. Thus we obtain

$$\lambda_i = \omega_i \cdot X \quad (22)$$

for all $i = 1, \dots, n$. Note that $\omega_i \cdot X = -(0, \dots, 0, 1, 0, \dots, 0)(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$. From Eq. (22), we obtain $(\lambda_1, \dots, \lambda_n)^t = -(M^{-1})^t M(\lambda_1, \dots, \lambda_n)^t$, and hence (iii) follows.

(iii) \Rightarrow (i). We have

$$\begin{aligned}(\epsilon\varphi)\delta_{A^!}(\omega_1^*, \dots, \omega_n^*) &= \epsilon\varphi(\lambda_1\varpi^*, \dots, \lambda_n\varpi^*) \\ &= (\lambda_1\varpi^*, \dots, \lambda_n\varpi^*);\end{aligned}$$

and

$$\begin{aligned}\delta_{A^!}(\epsilon\varphi)(\omega_1^*, \dots, \omega_n^*) &= -\delta_{A^!}\varphi(x_1^*, \dots, x_n^*)M^{-1} \\ &= -\delta_{A^!}(x_1^*, \dots, x_n^*)M^{-1}M^tM^{-1} \\ &= -(\lambda_1\varpi, \dots, \lambda_n\varpi)M^tM^{-1}.\end{aligned}$$

Now the condition (iii) insures that $(\epsilon\varphi)\delta_{A^!} = \delta_{A^!}(\epsilon\varphi)$. Therefore $\epsilon\varphi$ is an automorphism of $\text{cdga}(A^!, \delta_{A^!}, \theta_{A^!})$ since $A^!$ is of length 2. \square

Acknowledgments

The authors thank the referee for his/her valuable comments. The work is supported by an FWO grant and grants from NSFC (No. 11171067), ZJNSF (No. LY12A01013), Science and Technology Department of Zhejiang Province (No. 2011R10051), and SRF for ROCS, SEM.

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