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Computations of sheaves associated to the representation theory of $\mathfrak{sl}_2$

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ABSTRACT

We explicitly compute examples of sheaves over the projectivization of the spectrum of the cohomology of $\mathfrak{sl}_2$. In particular, we compute $\ker \Theta_M$ for every indecomposable M and we compute $\mathscr{F}_i(M)$ when M is an indecomposable Weyl module and $i \neq p$. We also give a brief review of the classification of $\mathfrak{sl}_2$-modules and of the general theory of such sheaves in the case of a restricted Lie algebra.

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0. Introduction

Let $\mathfrak{g}$ be a restricted Lie algebra over an algebraically closed field k of positive characteristic p. Suslin, Friedlander, and Bendel [6] have shown that the maximal spectrum of the cohomology of $\mathfrak{g}$ is isomorphic to the variety of p-nilpotent elements in $\mathfrak{g}$, i.e., the so called restricted nullcone $\mathcal{N}_p(\mathfrak{g})$. This variety has become an important invariant in representation theory; for example, it can be used to give a simple definition of the local Jordan type of a $\mathfrak{g}$-module M and consequently of the class of modules of constant Jordan type, a class first studied by Carlson, Friedlander, and Pevtsova [3] in the case of a finite group scheme. Friedlander and Pevtsova [4] have initiated what is, in the case of a Lie algebra $\mathfrak{g}$, the study of certain sheaves over the projectivization, $\mathbb{P}(\mathfrak{g})$, of $\mathcal{N}_p(\mathfrak{g})$. These sheaves are constructed from $\mathfrak{g}$-modules M so that representation theoretic information, such as projectivity of M, is encoded in their geometric properties. Explicit computations of these sheaves can be challenging due not only to their geometric nature but also to the inherent difficulty in describing representations of a general Lie algebra.

The purpose of this paper is to explicitly compute examples of these sheaves when $\mathfrak{g} = \mathfrak{sl}_2$. As $\mathfrak{sl}_2$ has tame representation type the category is rich enough to be interesting but the parameterized families allow for direct computations. We also note that the variety $\mathbb{P}(\mathfrak{sl}_2)$ over which we wish to compute these sheaves is isomorphic to $\mathbb{P}^1$. By a theorem of Grothendieck locally free sheaves are all sums of twists of the structure sheaf. This makes $\mathfrak{sl}_2$ uniquely suited for such computations.

We begin in Section 1 with the case of a general restricted Lie algebra $\mathfrak{g}$. We will review the definition of $\mathcal{N}_p(\mathfrak{g})$ and its projectivization $\mathbb{P}(\mathfrak{g})$. We use this to define the local Jordan type of a module M. We define the global operator Θ_M associated to a $\mathfrak{g}$-module M and use it to construct the sheaves we are interested in computing. We will review theorems which not only indicate the usefulness of these sheaves but are also needed for their computation.

In Section 2 we discuss the category of $\mathfrak{sl}_2$-modules. Our computations are based on having, for each indecomposable $\mathfrak{sl}_2$-module, an explicit basis and formulas for the $\mathfrak{sl}_2$ action. There are four families of $\mathfrak{sl}_2$-modules and for each family we specify an explicit basis and give formulas for the $\mathfrak{sl}_2$-action and the local Jordan type. For the Weyl modules $V(\lambda)$, dual Weyl modules $V(\lambda)^*$, and projective modules $Q(\lambda)$ this information was previously known but for the non-constant modules $\Phi_\xi(\lambda)$ we do not know if such an explicit description has previously been given. We also compute the Heller shifts $\Omega V(\lambda)$ of the Weyl modules for use in Section 4.

In Section 3 we digress from discussing Lie algebras and compute the kernels of four particular matrices over $k[s,t]$. These will represent the global operators of the four families of $\mathfrak{sl}_2$-modules, but in this section we do not work geometrically and instead consider these matrices to be maps of free $k[s,t]$-modules. This section contains the bulk of the computational effort of this paper.

In Section 4 we carry out the computations promised. Friedlander and Pevtsova have computed $\ker \Theta_{V(\lambda)}$ for the case $0 \leqslant \lambda \leqslant 2p-2$ [4]. We compute the sheaves $\ker \Theta_M$

for every indecomposable $\mathfrak{sl}_2$-module M. We also compute $\mathscr{F}_i(V(\lambda))$ for $i \neq p$ and $V(\lambda)$ indecomposable using an inductive argument.

1. Jordan type and global operators for Lie algebras

We review the definition of the restricted nullcone of a Lie algebra $\mathfrak{g}$, the global operator Θ_M and local Jordan type of a $\mathfrak{g}$-module M, and the sheaves associated to such an operator. Both Θ_M and local Jordan type can be defined for any infinitesimal group scheme of finite height. We give the definitions only for a restricted Lie algebra and take $\mathfrak{sl}_2$ as our only example. For the general case and additional examples see Friedlander and Pevtsova [4] or Stark [8].

Let $\mathfrak{g}$ be a restricted Lie algebra over an algebraically closed field k of positive characteristic p. This means $\mathfrak{g}$ is a Lie algebra equipped with a *p-operation* $(-)^{[p]}\colon \mathfrak{g} \to \mathfrak{g}$ satisfying certain axioms. Here we merely note that for the classical subalgebras of $\mathfrak{gl}_n$ the p-operation is given by raising a matrix to the pth power.

Definition 1.1. The restricted nullcone of $\mathfrak{g}$ is the set

$$\mathcal{N}_p(\mathfrak{g}) = \{x \mid x^{[p]} = 0\}$$

of p-nilpotent elements. This is a conical irreducible subvariety of the affine space $\mathfrak{g}$. We denote by $\mathbb{P}(\mathfrak{g})$ the projective variety whose points are lines through the origin in $\mathcal{N}_p(\mathfrak{g})$.

Example 1.2. Let $\mathfrak{g} = \mathfrak{sl}_2$ and take the usual basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\{x, y, z\}$ be the dual basis so that $\mathfrak{sl}_2$, as an affine space, can be identified with $\mathbb{A}^3$ and has coordinate ring $k[x, y, z]$. A 2×2 matrix over a field is nilpotent if and only if its square

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix}^2 = \left(xy + z^2\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is zero therefore, independent of p, we get that $\mathcal{N}_p(\mathfrak{sl}_2)$ is the zero locus of $xy + z^2$.

By definition $\mathbb{P}(\mathfrak{sl}_2)$ is the projective variety defined by the homogeneous polynomial $xy + z^2$. Let $\mathbb{P}^1$ have coordinate ring $k[s, t]$. One can check that the map $\iota\colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ defined by $[s, t] \mapsto [s^2 : -t^2 : st]$ is an isomorphism, thus $\mathbb{P}(\mathfrak{sl}_2) \simeq \mathbb{P}^1$.

Recall that a p-restricted partition is a weakly decreasing sequence of finitely many integers i satisfying $0 < i \leqslant p$. Let $\mathscr{P}_p$ denote the set of all p-restricted partitions. We write partitions using exponential notation.

If $A \in \mathbb{M}_n(k)$ is a p-nilpotent ($A^p = 0$) square matrix then the Jordan normal form of A is uniquely identified by listing the sizes of its blocks in weakly decreasing order. This yields a p-restricted partition called the *Jordan type*, $\mathrm{JType}(A)$, of the matrix A. The Jordan type of a p-nilpotent linear operator T is the Jordan type of any matrix representation of that operator. Note that $\mathrm{JType}(cT) = \mathrm{JType}(T)$ when $c \in k$ is a nonzero scalar.

Definition 1.3. Let M be a $\mathfrak{g}$-module and $v \in \mathbb{P}(\mathfrak{g})$. Set $\mathrm{JType}(v, M) = \mathrm{JType}(x)$ where $x \in \mathcal{N}_p(\mathfrak{g})$ is any non-zero point on the line v and its Jordan type is that of a p-nilpotent operator on M. The *local Jordan type* of M is the function

$$\mathrm{JType}(-, M): \mathbb{P}(\mathfrak{g}) \to \mathscr{P}_p$$

so defined.

Example 1.4. Assume $p > 2$ and consider the Weyl module $V(2)$, for $\mathfrak{sl}_2$. This is a 3-dimensional module where e, f, and h act via

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

respectively. The matrix

$$A = \begin{bmatrix} 2z & 2x & 0 \\ y & 0 & x \\ 0 & 2y & -2z \end{bmatrix}$$

describes the action of $xe + yf + zh \in \mathfrak{g}$ on $V(2)$. One can check that if $xy + z^2 = 0$ then the Jordan type of A is $[3]$. Thus $\mathrm{JType}(-, V(2))$ is the constant function $v \mapsto [3]$.

Definition 1.5. A $\mathfrak{g}$-module M has *constant Jordan type* if its local Jordan type is a constant function.

Modules of constant Jordan type will be significant for us for two reasons. The first is because of the following useful projectivity criterion.

Theorem 1.6. *(See [7, 7.6].) A $\mathfrak{g}$-module M is projective if and only if its local Jordan type is a constant function of the form $v \mapsto [p]^n$.*

For the second note that when $\mathfrak{g}$ is the Lie algebra of an algebraic group G, the adjoint action of G on $\mathfrak{g}$ induces an action on each fiber of the local Jordan type. The adjoint action of SL_2 on $\mathbb{P}(\mathfrak{sl}_2)$ is transitive so we get the following.

Theorem 1.7. *(See [3, 2.5].) Every rational $\mathfrak{sl}_2$-module has constant Jordan type.*

Next we define the global operator associated to a $\mathfrak{g}$-module M and the sheaves associated to such an operator. Let $\{g_1, \ldots, g_n\}$ be a basis for $\mathfrak{g}$ with corresponding dual basis $\{x_1, \ldots, x_n\}$. We define $\Theta_{\mathfrak{g}}$ to be the operator

$$\Theta_{\mathfrak{g}} = x_1 \otimes g_1 + \cdots + x_n \otimes g_n.$$

As an element of $\mathfrak{g}^* \otimes_k \mathfrak{g} \simeq \operatorname{Hom}_k(\mathfrak{g}, \mathfrak{g})$ this is just the identity map and is therefore independent of the choice of basis. Now $\Theta_{\mathfrak{g}}$ acts on $k[\mathcal{N}_p(\mathfrak{g})] \otimes_k M \simeq k[\mathcal{N}_p(\mathfrak{g})]^{\dim M}$ as a degree 1 endomorphism of graded $k[\mathcal{N}_p(\mathfrak{g})]$-modules (where $\deg x_i = 1$). The map of sheaves corresponding to this homomorphism is the global operator.

Definition 1.8. Given a $\mathfrak{g}$-module M we define $\widetilde{M} = \mathcal{O}_{\mathbb{P}(\mathfrak{g})} \otimes_k M$. The *global operator* corresponding to M is the sheaf map

$$\Theta_M : \widetilde{M} \to \widetilde{M}(1)$$

induced by the action of $\Theta_{\mathfrak{g}}$.

Example 1.9. We have $\Theta_{\mathfrak{sl}_2} = x \otimes e + y \otimes f + z \otimes h$. Consider the Weyl module $V(2)$ from Example 1.4. The global operator corresponding to $V(2)$ is the sheaf map $\mathcal{O}^3_{\mathbb{P}(\mathfrak{sl}_2)} \to \mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1)^3$ defined by A. Taking the pullback through the map $\iota : \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ from Example 1.2 we get that $\Theta_{V(2)}$ is the sheaf map

$$\begin{bmatrix} 2st & 2s^2 & 0 \\ -t^2 & 0 & s^2 \\ 0 & -2t^2 & -2st \end{bmatrix} : \mathcal{O}^3_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2)^3.$$

The global operator Θ_M is not an endomorphism but we may still compose it with itself if we shift the degree of successive copies. Given $j \in \mathbb{N}$ we define

$$\begin{aligned} \ker \Theta^j_M &= \ker\left[\Theta_M(j-1) \circ \cdots \circ \Theta_M(1) \circ \Theta_M\right], \\ \operatorname{im} \Theta^j_M &= \operatorname{im}\left[\Theta_M(-1) \circ \cdots \circ \Theta_M(1-j) \circ \Theta_M(-j)\right], \\ \operatorname{coker} \Theta^j_M &= \operatorname{coker}\left[\Theta_M(-1) \circ \cdots \circ \Theta_M(1-j) \circ \Theta_M(-j)\right], \end{aligned}$$

so that $\ker \Theta^j_M$ and $\operatorname{im} \Theta^j_M$ are subsheafs of $\widetilde{M}$, and $\operatorname{coker} \Theta^j_M$ is a quotient of $\widetilde{M}$.

To see how these sheaves encode information about the Jordan type of M recall that the j-rank of a partition λ is the number of boxes in the Young diagram of λ that are not contained in the first j columns. If one knows the j-rank of a partition λ for all j, then one knows the size of each column in the Young diagram of λ and can therefore recover λ. Thus if one knows the local j-rank of a module M for all j then one knows its local Jordan type.

Definition 1.10. Let M be a $\mathfrak{g}$-module and let $v \in \mathbb{P}(\mathfrak{g})$. Set $\operatorname{rank}^j(v, M)$ equal to the j-rank of the partition $\operatorname{JType}(v, M)$. The *local j-rank* of M is the function

$$\operatorname{rank}^j(-, M): \mathbb{P}(\mathfrak{g}) \to \mathbb{N}_0$$

so defined.

Theorem 1.11. *(See [8, 3.2].) Let M be a $\mathfrak{g}$-module and $U \subseteq \mathbb{P}(\mathfrak{g})$ an open set. The local j-rank is constant on U if and only if the restriction $\operatorname{coker} \Theta_M^j|_U$ is a locally free sheaf. When this is the case $\ker \Theta_M^j|_U$ and $\operatorname{im} \Theta_M^j|_U$ are also locally free and $\operatorname{rank}^j(v, M) = \operatorname{rank} \operatorname{im} \Theta_M^j$ for all $v \in U$.*

We will also be interested in the sheaves $\mathscr{F}_i(M)$ for $1 \leqslant i \leqslant p$. These were first defined by Benson and Pevtsova [2] for kE-modules where E is an elementary abelian p-group.

Definition 1.12. Let M be a $\mathfrak{g}$-module and $1 \leqslant i \leqslant p$ an integer. Then

$$\mathscr{F}_i(M) = \frac{\ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1}}{\ker \Theta_M \cap \operatorname{im} \Theta_M^i}.$$

The following two theorems will be used in Section 4 when calculating $\mathscr{F}_i(M)$ where M is a Weyl module for $\mathfrak{sl}_2$. Both theorems were originally published by Benson and Pevtsova [2] but with minor errors. These errors have been corrected in the given reference.

Theorem 1.13. *(See [8, 3.7].) Let M be a $\mathfrak{g}$-module and $1 \leqslant i < p$ an integer. Then*

$$\mathscr{F}_i(M) \simeq \mathscr{F}_{p-i}(\Omega M)(p - i).$$

Theorem 1.14. *(See [8, 3.8].) Let $U \subseteq \mathbb{P}(\mathfrak{g})$ be open. The local Jordan type of a $\mathfrak{g}$-module M is constant on U if and only if the restrictions $\mathscr{F}_i(M)|_U$ are locally free for all $1 \leqslant i \leqslant p$. When this is the case and $a_i = \operatorname{rank} \mathscr{F}_i(M)$ we have $\operatorname{JType}(v, M) = [p]^{a_p}[p-1]^{a_{p-1}} \cdots [1]^{a_1}$ for all $v \in U$.*

2. The category of $\mathfrak{sl}_2$-modules

The calculations in Section 4 will be based on detailed information about the category of $\mathfrak{sl}_2$-modules, which we develop in this section. The indecomposable $\mathfrak{sl}_2$-modules have been classified, each is one of the following four types: a Weyl module $V(\lambda)$, its dual $V(\lambda)^*$, an indecomposable projective $Q(\lambda)$, or a non-constant module $\Phi_\xi(\lambda)$. Explicit bases for the first three types are known; we will remind the reader of these formulas and develop similar formulas for the $\Phi_\xi(\lambda)$. We will also calculate the local Jordan types of these modules and the Heller shifts $\Omega(V(\lambda))$.

We begin by stating the results for each of the four types and the classification theorem. Let λ be a non-negative integer and write $\lambda = rp + a$ where $0 \leqslant a < p$ is the remainder of λ modulo p. Each type is parametrized by the choice of λ, with the parametrization of $\Phi_\xi(\lambda)$ requiring also a choice of $\xi \in \mathbb{P}^1$. Also, see Stark [9] for a diagrammatic depiction of these modules.

- The **Weyl modules** $V(\lambda)$.

$$\begin{aligned}
\text{Basis:}\quad & \{v_0, v_1, \ldots, v_\lambda\} \\
\text{Action:}\quad & ev_i = (\lambda - i + 1)v_{i-1} \\
& fv_i = (i+1)v_{i+1} \\
& hv_i = (\lambda - 2i)v_i \\
\text{Local Jordan type:}\quad & \text{Constant Jordan type } [p]^r[a+1]
\end{aligned}$$

- The **dual Weyl modules** $V(\lambda)^*$.

$$\begin{aligned}
\text{Basis:}\quad & \{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_\lambda\} \\
\text{Action:}\quad & e\hat{v}_i = i\hat{v}_{i-1} \\
& f\hat{v}_i = (\lambda - i)\hat{v}_{i+1} \\
& h\hat{v}_i = (\lambda - 2i)\hat{v}_i \\
\text{Local Jordan type:}\quad & \text{Constant Jordan type } [p]^r[a+1]
\end{aligned}$$

- The **projectives** $Q(\lambda)$.
  Define $Q(p-1) = V(p-1)$. For $0 \leqslant \lambda < p-1$ we define $Q(\lambda)$ via

$$\begin{aligned}
\text{Basis:}\quad & \{v_0, v_1, \ldots, v_{2p-\lambda-2}\} \cup \{w_{p-\lambda-1}, w_{p-\lambda}, \ldots, w_{p-1}\} \\
\text{Action:}\quad & ev_i = -(\lambda + i + 1)v_{i-1} \\
& fv_i = (i+1)v_{i+1} \\
& hv_i = -(\lambda + 2i + 2)v_i \\
& ew_i = -(\lambda + i + 1)w_{i-1} + \tfrac{1}{i}v_{i-1} \\
& fw_i = (i+1)w_{i+1} - \tfrac{1}{\lambda+1}\delta_{-1,i}v_p \\
& hw_i = -(\lambda + 2i + 2)w_i \\
\text{Local Jordan type:}\quad & \text{Constant Jordan type } [p]^2
\end{aligned}$$

- The **non-constant modules** $\Phi_\xi(\lambda)$.
  Assume $\lambda \geqslant p$ and let $\xi \in \mathbb{P}^1$. If $\xi = [1 : \varepsilon]$ then $\Phi_\xi(\lambda)$ is defined by

$$\begin{aligned}
\text{Basis:}\quad & \{w_{a+1}, w_{a+2}, \ldots, w_\lambda\} \\
\text{Action:}\quad & ew_i = (i+1)\left(w_{i+1} - \tbinom{d}{i}\varepsilon^{i-a}\delta_{\lambda,i}w_{a+1}\right) \\
& fw_i = (\lambda - i + 1)w_{i-1} \\
& hw_i = (2i - \lambda)w_i \\
\text{Local Jordan type:}\quad & [p]^{r-1}[p-a-1][a+1] \text{ at } \xi \text{ and } [p]^r \text{ elsewhere}
\end{aligned}$$

If $\xi = [0 : 1]$ then $\Phi_\xi(\lambda)$ is defined to be the submodule of $V(\lambda)$ spanned by the basis elements $\{v_{a+1}, v_{a+2}, \ldots, v_\lambda\}$. It has the same local Jordan type as above.

Theorem 2.1. *(See [5].) Each of the following modules is indecomposable:*

- *$V(\lambda)$ and $Q(\lambda)$ for $0 \leqslant \lambda < p$.*
- *$V(\lambda)$ and $V(\lambda)^*$ for $\lambda \geqslant p$ such that $p \nmid \lambda + 1$.*
- *$\Phi_\xi(\lambda)$ for $\xi \in \mathbb{P}^1$ and $\lambda \geqslant p$ such that $p \nmid \lambda + 1$.*

Moreover, these modules are pairwise non-isomorphic, save $Q(p-1) = V(p-1)$, and give a complete classification of the indecomposable restricted $\mathfrak{sl}_2$-modules.

See Benkart and Osborn [1] for the explicit bases of $V(\lambda)$, $V(\lambda)^*$, and $Q(\lambda)$. Theorem 1.7 gives that the local Jordan type of $V(\lambda)$ and $V(\lambda)^*$ is constant and it is easily computed at $ke \in \mathbb{P}(\mathfrak{sl}_2)$. Theorem 1.6 gives the local Jordan type of the $Q(\lambda)$ so we need only justify the description of $\Phi_\xi(\lambda)$ and its local Jordan type.

We recall the definition of $\Phi_\xi(\lambda)$. Let $B_2 \subseteq \mathrm{SL}_2$ be the Borel subgroup of upper triangular matrices and recall that the homogeneous space SL_2 / B_2 is isomorphic to $\mathbb{P}^1$ as a variety; the map $\phi: \mathbb{P}^1 \to \mathrm{SL}_2$ given by

$$[1 : \varepsilon] \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \quad \text{and} \quad [0 : 1] \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

induces an explicit isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathrm{SL}_2 / B_2$.

Definition 2.2. (See [5].) Let $\Phi(\lambda)$ be the $\mathfrak{sl}_2$-submodule of $V(\lambda)$ spanned by the vectors $\{v_{a+1}, v_{a+2}, \ldots, v_\lambda\}$. Given $\xi \in \mathbb{P}^1$ we define $\Phi_\xi(\lambda)$ to be the $\mathfrak{sl}_2$-module $\phi(\xi)\Phi(\lambda)$.

Observe first that $\Phi_{[0:1]}(\lambda) = \Phi(\lambda)$ so in this case we have the desired description. Now assume $\xi = [1 : \varepsilon]$ where $\varepsilon \in k$. As $\phi(\xi)$ is invertible multiplication by it is an isomorphism so $\Phi_\xi(\lambda)$ has basis $\{\phi(\xi)v_i\}$. Our basis for $\Phi_\xi(\lambda)$ will be obtained by essentially a row reduction of this basis, so to proceed we now compute the action of SL_2 on $V(\lambda)$.

Let $V = k^2$ be the standard representation of SL_2, then the dual V^* has basis $\{x, y\}$ (dual to the standard basis for V). The induced representation on the symmetric product $S(V^*)$ is degree preserving and $V(\lambda) = S^\lambda(V^*)^*$. Specifically, we let $v_i \in V(\lambda)$ be dual to $x^{\lambda-i}y^i$. Now observe:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} v_i\right)\left(x^{\lambda-j}y^j\right) = \sum \binom{\lambda-j}{s}\binom{j}{t} a^s b^{\lambda-j-s} c^t d^{j-t} v_j$$

where the sum is over pairs $(s,t) \in \mathbb{N}^2$ such that $0 \leqslant s \leqslant \lambda - j$, $0 \leqslant t \leqslant j$, and $s + t = \lambda - i$. Such pairs come in the form $(\lambda - i - t, t)$ where t ranges from $\max(0, j-i)$ to $\min(j, \lambda - i)$ therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} v_i = \sum_{j=0}^{r} \sum_{t=\max(0,j-i)}^{\min(j,\lambda-i)} \binom{\lambda-j}{\lambda-i-t} \binom{j}{t} a^{\lambda-i-t} b^{t+i-j} c^t d^{j-t} v_j.$$

For computing $\Phi_\xi(\lambda)$ we will need only the following special case:

$$\phi(\xi)v_i = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} v_i = \sum_{j=\lambda-i}^{\lambda} (-1)^j \binom{j}{\lambda-i} \varepsilon^{i+j-\lambda} v_j.$$

Proposition 2.3. *Given $i = qp + b$, $0 \leqslant b < p$, define*

$$w_i = \begin{cases} v_{\lambda-i} - \binom{r}{q} \varepsilon^{qp} v_{\lambda-b} & \text{if } b \leqslant a \\ v_{\lambda-i} & \text{if } b > a. \end{cases}$$

Then the vectors $w_{a+1}, w_{a+2}, \dots, w_\lambda$ form a basis of $\Phi_\xi(\lambda)$.

Proof. We will prove by induction that for all $a + 1 \leqslant i \leqslant \lambda$ the vector spaces spanned by $\{\phi(\xi)v_{a+1}, \dots, \phi(\xi)v_i\}$ and $\{w_{a+1}, \dots, w_i\}$ are equal. For the base case the formula above gives $\phi(\xi)v_{a+1} = (-1)^{rp-1} w_{a+1}$ so clearly the statement is true. For the inductive step we have that the spans of $\{\phi(\xi)v_{a+1}, \dots, \phi(\xi)v_i\}$ and $\{w_{a+1}, \dots, w_{i-1}, \phi(\xi)v_i\}$ are equal and we may replace $\phi(\xi)v_i$ with the vector

$$w' = (-1)^{\lambda-i}\phi(\xi)v_i - \sum_{j=a+1}^{i-1} (-1)^{i-j} \binom{\lambda-j}{\lambda-i} \varepsilon^{i-j} w_j$$

without changing the span so it suffices to prove that $w' = w_i$. This is done by writing w' and w_i as a linear combination of the v_i's, splitting into the cases $j < \lambda - i$, $j = \lambda - i$, $\lambda - i < j < rp$, and $rp \leqslant j \leqslant \lambda$, and checking that the coefficient of v_j is the same in each expression. $\square$

The formulas for the action of e, f, and h on $V(\lambda)$ then translate directly to formulas for their action on $\Phi_\xi(\lambda)$.

Proposition 2.4. *Let $i = qp + b$, $a + 1 \leqslant i \leqslant \lambda$. Then*

$$\begin{aligned} ew_i &= (i+1)\left(w_{i+1} - \delta_{ab}\binom{\lambda}{i}\varepsilon^{qp} w_{a+1}\right) \\ fw_i &= (\lambda - i + 1)w_{i-1} \\ hw_i &= (2i - \lambda)w_i \end{aligned}$$

where $w_a = w_{\lambda+1} = 0$ and δ_{ab} is the Kronecker delta.

Lastly we calculate that the Jordan type is as stated: $[p]^{r-1}[p-a-1][a+1]$ at ξ and $[p]^r$ elsewhere. First note that the result holds for $\Phi_{[0:1]}(\lambda)$ by Lemma 3.5; furthermore, that the point $[0:1] \in \mathbb{P}^1$ at which the Jordan type is $[p]^{r-1}[p-a-1][a+1]$ corresponds to the line through $f \in \mathcal{N}_p(\mathfrak{sl}_2)$ under the map ι from Example 1.2. Let $\mathrm{ad}\colon \mathrm{SL}_2 \to \mathrm{End}(\mathfrak{sl}_2)$ be the adjoint action of SL_2 on $\mathfrak{sl}_2$. As $V(\lambda)$ is a rational SL_2-module this satisfies $\mathrm{ad}(g)(E)\cdot m = g\cdot(E\cdot(g^{-1}\cdot m))$ for all $g \in \mathrm{SL}_2$, $E \in \mathfrak{sl}_2$, and $m \in V(\lambda)$. Along with $\Phi_\xi(\lambda) = \phi(\xi)\Phi_{[0:1]}(\lambda)$ we therefore get commutativity of the following diagram:

$$\begin{tikzcd} \Phi_{[0:1]}(\lambda) \arrow[r, "\phi(\xi)"] \arrow[d, "E"'] & \Phi_\xi(\lambda) \arrow[d, "\mathrm{ad}(\phi(\xi))(E)"] \\ \Phi_{[0:1]}(\lambda) \arrow[r, "\phi(\xi)"'] & \Phi_\xi(\lambda) \end{tikzcd}$$

Multiplication by $\phi(\xi)$ is an isomorphism, so letting E range over $\mathcal{N}_p(\mathfrak{sl}_2)$ we see that the module $\Phi_\xi(\lambda)$ has Jordan type $[p]^{r-1}[p-a-1][a+1]$ at $\mathrm{ad}(\phi(\xi))(f)$ and $[p]^r$ elsewhere. Then we simply calculate

$$\mathrm{ad}\big(\phi(\xi)\big)(f) = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix}^{-1} = \begin{bmatrix} -\varepsilon & -1 \\ \varepsilon^2 & \varepsilon \end{bmatrix}$$

and observe that, as an element of $\mathbb{P}(\mathfrak{sl}_2)$, this is $\iota([1:\varepsilon])$.

Thus we now have a complete description of the indecomposable $\mathfrak{sl}_2$-modules. We finish this section with one more computation that will be needed in Section 4: The computation of the Heller shifts $\Omega(V(\lambda))$ for indecomposable $V(\lambda)$. Note that $V(p-1) = Q(p-1)$ is projective so $\Omega(V(p-1)) = 0$. For other $V(\lambda)$ we have the following.

Proposition 2.5. *Let $\lambda = rp + a$ be a non-negative integer and $0 \leqslant a < p$ its remainder modulo p. If $a \neq p-1$ then $\Omega(V(\lambda)) = V((r+2)p - a - 2)$.*

Proof. This will be a direct computation. We will determine the projective cover $f\colon P \to V(\lambda)$ and then set $f(x) = 0$ for an arbitrary element $x \in P$. This will give us the relations determining $\ker f = \Omega(V(\lambda))$ which we will convert into a basis and identify with $V((r+2)p - a - 2)$.

The indecomposable summands of P are in bijective correspondence with the indecomposable summands (all simple) of the top of $V(\lambda)$, i.e. $V(\lambda)/\mathrm{rad}\, V(\lambda)$. This correspondence is given as follows, if $\pi_q\colon V(\lambda) \to V(a)$ is the projection onto a summand of the top of $V(\lambda)$ then the projective cover $\phi_a\colon Q(a) \to V(a)$ factors through π_q.

$$\begin{tikzcd} Q(a) \arrow[rr, "\phi_a"] \arrow[dr, dashed, "f_q"'] & & V(a) \\ & V(\lambda) \arrow[ur, "\pi_q"'] & \end{tikzcd}$$

The map $f_q\colon Q(a) \to V(\lambda)$ so defined is the restriction of $f\colon P \to V(\lambda)$ to the summand $Q(a)$ of P.

The module $V(\lambda)$ fits into a short exact sequence

$$0 \to V(p-a-2)^{\oplus r+1} \to V(\lambda) \xrightarrow{\pi} V(a)^{\oplus r+1} \to 0$$

where π has components π_q for $q = 0, 1, \dots, r$. Each $\pi_q\colon V(\lambda) \to V(a)$ is given by

$$v_i \mapsto \begin{cases} v_{i-qp} & \text{if } qp \leqslant i \leqslant qp+a \\ 0 & \text{otherwise.} \end{cases}$$

Hence the top of $V(\lambda)$ is $V(a)^{\oplus r+1}$ and $P = Q(a)^{\oplus r+1}$. The map f_q is uniquely determined up to a nonzero scalar and is given by

$$\begin{aligned}
& v_i \mapsto -(a+1)^2\binom{p-i-1}{a+1} v_{(q-1)p+a+i+1} && \text{if } 0 \leqslant i \leqslant p-a-2, \\
& w_i \mapsto (-1)^{i+a}\binom{a}{i+a+1-p}^{-1} v_{(q-1)p+a+i+1} && \text{if } p-a-1 \leqslant i \leqslant p-1, \\
& v_i \mapsto 0 && \text{if } p-a-1 \leqslant i \leqslant p-1, \\
& v_i \mapsto (-1)^{a+1}(a+1)^2\binom{i+a+1-p}{a+1} v_{(q-1)p+a+i+1} && \text{if } p \leqslant i \leqslant 2p-a-2.
\end{aligned}$$

This gives $f = [\, f_0 \quad f_1 \quad \cdots \quad f_r\,]$. To distinguish elements from different summands of $Q(a)^{\oplus r+1}$ let $\{v_{q,0}, v_{q,1}, \dots, v_{q,2p-a-2}\} \cup \{w_{q,p-a-1}, w_{q,p-a}, w_{q,p-1}\}$ be the basis of the qth summand of $Q(a)^{\oplus r+1}$. Then any element of the cover can be written in the form

$$x = \sum_{q=0}^{r}\left[\sum_{i=0}^{2p-a-2} c_{q,i}v_{q,i} + \sum_{i=p-a-1}^{p-1} d_{q,i}w_{q,i}\right]$$

for some $c_{q,i}, d_{q,i} \in k$. Applying f gives

$$\begin{aligned}
f(x) = \sum_{q=0}^{r}\Bigg[&-(a+1)^2\sum_{i=0}^{p-a-2}\binom{p-i-1}{a+1} c_{q,i}v_{(q-1)p+a+i+1} \\
&+ (-1)^{a+1}(a+1)^2\sum_{i=p}^{2p-a-2}\binom{i+a+1-p}{a+1} c_{q,i}v_{(q-1)p+a+i+1} \\
&+ \sum_{i=p-a-1}^{p-1}(-1)^{a+i}\binom{a}{i+a+1-p}^{-1} d_{q,i}v_{(q-1)p+a+i+1}\Bigg].
\end{aligned}$$

Observe that $0 \leqslant i \leqslant p-a-2$ and $p \leqslant i \leqslant 2p-a-2$ give $a+1 \leqslant a+i+1 \leqslant p-1$ and $p+a+1 \leqslant a+i+1 \leqslant 2p-1$ respectively, whereas $p-a-1 \leqslant i \leqslant p-1$ gives

$p \leqslant a+i+1 \leqslant p+a$. Looking modulo p we see that the basis elements $v_{(q-1)p+a+i+1}$, for $0 \leqslant q \leqslant r$ and $p-a-1 \leqslant i \leqslant p-1$, are linearly independent. Thus $f(x) = 0$ immediately yields $d_{q,i} = 0$ for all q and i.

Now rearranging we have

$$f(x) = \sum_{q=0}^{r-1} \sum_{i=0}^{p-a-2} \left[(-1)^a \binom{i+a+1}{i} c_{q,i+p} + \binom{p-i-1}{a+1} c_{q+1,i} \right] v_{qp+a+1+i}$$

so the kernel is defined by choosing $c_{q,i}$, for $0 \leqslant q \leqslant r-1$ and $0 \leqslant i \leqslant p-a-2$, such that $(-1)^a \binom{i+a+1}{i} c_{q,i+p} + \binom{p-i-1}{a+1} c_{q+1,i} = 0$. Note that $\frac{\binom{p-i-1}{a+1}}{\binom{i+a+1}{i}} = (-1)^{a+1}$ so the above equation simplifies to $c_{q,i+p} = c_{q+1,i}$. Thus for $0 \leqslant i \leqslant (r+2)p-a-2$ the vectors

$$v_i' = \begin{cases} v_{0,i} & \text{if } 0 \leqslant i < p, \\ v_{q,b} + v_{q-1,p+b} & \text{if } 1 \leqslant q \leqslant r,\ 0 \leqslant b \leqslant p-a-2, \\ v_{q,b} & \text{if } 1 \leqslant q \leqslant r,\ p-a-1 \leqslant b < p, \\ v_{r,b} & \text{if } q = r+1,\ 0 \leqslant b \leqslant p-a-2 \end{cases}$$

form a basis for the kernel, where $i = qp + b$ with $0 \leqslant b < p$ the remainder of i modulo p. It is now trivial to check that the $\mathfrak{sl}_2$-action on this basis is identical to that of $V((r+2)p-a-2)$. $\square$

3. Matrix theorems

In this section we determine the kernel of four particular maps between free $k[s,t]$-modules. The first map is given by the matrix $M_\varepsilon(\lambda) \in \mathbb{M}_{rp}(k[s,t])$ defined below. For convenience we index the rows and columns of this matrix using the integers $a+1, a+2, \ldots, \lambda$.

$$M_\varepsilon(\lambda)_{ij} = \begin{cases} is^2 & \text{if } i = j+1 \\ (2i-a)st & \text{if } i = j \\ (i-a)t^2 & \text{if } i = j-1 \\ -(a+1)\binom{r}{q}\varepsilon^{qp}s^2 & \text{if } (i,j) = (0, qp+a) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *The kernel of $M_\varepsilon(\lambda)$ is a free $k[s,t]$-module (ungraded) of rank r whose basis elements are homogeneous of degree $p-a-2$.*

Proof. The strategy is as follows: First we will determine the kernel of $M_\varepsilon(\lambda)$ when considered as a map of $k[s, \frac{1}{s}, t]$-modules. We do this by exhibiting a basis $H_1, \ldots, H_r$ via a direct calculation. Then by clearing the denominators from these basis elements we get a linearly independent set of vectors in the $k[s,t]$-kernel of $M_\varepsilon(\lambda)$. We conclude by arguing that these vectors in fact span, thus giving an explicit basis for the kernel of $M_\varepsilon(\lambda)$ considered as a map of $k[s,t]$-modules.

To begin, s is a unit in $k[s, \frac{1}{s}, t]$ so over this ring the kernel of $M_\varepsilon(\lambda)$ is equal to the kernel of the matrix $\frac{1}{s^2}M_\varepsilon(\lambda)$ with (i,j)th entry

$$\frac{1}{s^2}M_\varepsilon(\lambda)_{ij} = \begin{cases} i & \text{if } i = j+1, \\ (2i-a)x & \text{if } i = j, \\ (i-a)x^2 & \text{if } i = j-1, \\ -(a+1)\binom{r}{q}\varepsilon^{qp} & \text{if } (i,j) = (0, qp+a), \\ 0 & \text{otherwise,} \end{cases}$$

where $x = \frac{t}{s}$. Let $f = [\, f_{a+1} \quad f_{a+2} \quad \cdots \quad f_\lambda \,]^t$ be an arbitrary element of the kernel. Given $i = qp + b$ where $0 \leqslant b < p$ and $a + 1 \leqslant i \leqslant \lambda$ we claim that

$$f_i = (-1)^{\lambda-i}x^{\lambda-i}f_\lambda + (-1)^b\binom{p+a-b}{p-b-1}x^{p-b-1}h_{q+1} \tag{1}$$

for some choice of $h_1, \ldots, h_r \in k[s, \frac{1}{s}, t]$ and $h_{r+1} = 0$. Moreover, any such choice defines an element $f \in k[s, \frac{1}{s}, t]^{rp}$ such that $\frac{1}{s^2}M_\varepsilon(\lambda)f \subseteq [\, * \quad 0 \quad \cdots \quad 0\,]^t$ holds.

The proof of this claim is by completely elementary methods, we induct up the rows of $\frac{1}{s^2}M_\varepsilon(\lambda)$ observing that the condition imposed by each row in $\frac{1}{s^2}M_\varepsilon(\lambda)f = 0$ either determines the next f_i or is automatically satisfied allowing us to introduce a free parameter (the h_i).

For the base case plugging $i = \lambda$ into Eq. (1) gives $f_\lambda = f_\lambda$. The condition imposed by the last row in $\frac{1}{s^2}M_\varepsilon(\lambda)f = 0$ is $af_{\lambda-1} + axf_\lambda = 0$ so if $a \neq 0$ then $f_{\lambda-1} = -xf_\lambda$ and if $a = 0$ then this condition is automatically satisfied. The formula, when $a = 0$, gives $f_{rp-1} = -xf_\lambda + h_r$ so we take this as the definition of h_r.

Assume the formula holds for all f_j with $j > i \geqslant a + 1$ and that these f_j satisfy the conditions imposed by rows $i + 2, i + 3, \ldots, \lambda$ of $\frac{1}{s^2}M_\varepsilon(\lambda)f = 0$. First assume $i + 1 \neq 0$ in k or equivalently $b \neq p - 1$ where $i = qp + b$ and $0 \leqslant b < p$. Then the condition

$$(i+1)f_i + (2i-a+2)xf_{i+1} + (i-a+1)x^2f_{i+2} = 0$$

imposed by row $i + 1$ can be taken as the definition of f_i. Observe that

$$\frac{-1}{i+1}\big((-1)^{\lambda-i-1}(2i-a+2) + (-1)^{\lambda-i-2}(i-a+1)\big) = (-1)^{\lambda-i}$$

so $f_i = (-1)^{\lambda-i}x^{\lambda-i}f_\lambda + (\text{terms involving } h_j)$. For the h_j terms there are two cases. First assume $b < p - 2$. Then the expression

$$\frac{-1}{i+1}\left((-1)^{b+1}(2i-a+2)\binom{p+a-b-1}{p-b-2} + (-1)^{b+2}(i-a+1)\binom{p+a-b-2}{p-b-3}\right)$$

reduces to $(-1)^b\binom{p+a-b}{p-b-1}$. Putting these together,

$$f_i = (-1)^{\lambda-i} x^{\lambda-i} f_\lambda + (-1)^b \binom{p+a-b}{p-b-1} x^{p-b-1} h_{q+1}$$

as desired. Next assume $b = p-2$ so that $f_{i+2} = f_{(q+1)p}$. The coefficient of h_{q+2} in $f_{(q+1)p}$ involves the binomial $\binom{p+a}{p-1}$. As $0 \leqslant a < p-1$ there is a base p carry in $(p-1)+(a+1) = p+a$, thus this binomial is zero and the h_j terms of f_i are

$$\frac{(-1)^p}{i+1}(2i-a+2)\binom{a+1}{0} x h_{q+1} = (-1)^b \binom{a+2}{1} x h_{q+1}$$

as desired. Hence the induction continues when $i+1 \neq 0$ in k.

Now assume $i+1 = 0$ in k; equivalently, $b = p-1$. Then the condition imposed by row $i+1 = (q+1)p$ is $-axf_{(q+1)p} - ax^2 f_{(q+1)p+1} = 0$. If $a = 0$ then this is automatic. If $a > 0$ then there is a base p carry in $(p-2)+(a+1) = p+a-1$, hence $xf_{(q+1)p} + x^2 f_{(q+1)p+1} = 0$ (the f_λ terms cancel and the binomial is zero). So in either case the condition above is automatic. The formula for f_i when $i = qp + (p-1)$ is $f_i = (-1)^{(r-q-1)p+a+1} x^{(r-q-1)p+a+1} g + h_{q+1}$ so we take this as the definition of h_{q+1} and the induction is complete.

Now f must have the given form for some choice of $h_1, \dots, h_r$ and any such choice gives an element f such that $\frac{1}{s^2} M_\varepsilon(\lambda) f$ is zero in all coordinates save the top $(a+1)$. If $f \in \ker \frac{1}{s^2} M_\varepsilon(\lambda)$ then the $(a+1)$th coordinate of $\frac{1}{s^2} M_\varepsilon(\lambda) f$ is

$$(a+2)xf_{a+1} + x^2 f_{a+2} - (a+1)\sum_{q=1}^{r} \binom{r}{q} \varepsilon^{qp} f_{qp+a} = 0.$$

In $(a+2)xf_{a+1} + x^2 f_{a+2}$ the h_j terms are

$$(-1)^{a+1}\left((a+2)\binom{p-1}{p-a-2} - \binom{p-2}{p-a-3}\right) x^{p-a-1} h_1 = 0$$

and the coefficient of the h_j term in f_{qp+a} involves the binomial $\binom{p}{p-a-1}$ which is zero. Thus this imposes a condition only on f_λ, and this condition is

$$(-1)^{rp-1}(a+1)\left[\sum_{q=0}^{r} (-1)^{qp} \binom{r}{q} \varepsilon^{qp} x^{(r-q)p}\right] f_\lambda = 0.$$

Note that $x = \frac{t}{s}$ is algebraically independent over k in $k[s, \frac{1}{s}, t]$ and by hypothesis $a+1 \neq 0$ in k. As we are working in an integral domain we have $f_\lambda = 0$.

As the $h_1, \dots, h_r$ can be chosen arbitrarily this completes the determination of the kernel of $M_\varepsilon(\lambda)$, considered as a map of $k[s, \frac{1}{s}, t]$-modules. It is free of rank r and the basis elements are given by taking the coefficients of these h_q. Let H_q be the basis element that corresponds to h_q. I claim that $s^{p-a-2} H_q$, for $1 \leqslant q \leqslant r$, is a basis for the kernel of $M_\varepsilon(\lambda)$, considered as a map of $k[s,t]$-modules.

First note that H_q is supported in coordinates $(q+1)p-1$ through $qp+a+1$. These ranges are disjoint for different H_q therefore the $s^{p-a-2} H_q$ are clearly linearly independent. Let $f \in k[s,t]^{rp}$ be an element of the kernel of $M_\varepsilon(\lambda)$. Then as an element of $k[s,\frac{1}{s},t]$ we have that f is in the kernel of $\frac{1}{s^2}M_\varepsilon(\lambda)$ and can write $f = \sum_{q=1}^r c_q H_q$, where $c_q \in k[s,\frac{1}{s},t]$. The $((q+1)p-1)$th coordinate of f is c_q hence $c_q \in k[s,t]$. Also the $(qp+a+1)$th coordinate of f is $(-1)^{p-a-2}\binom{p-1}{p-a-2}c_q x^{p-a-2}$ and the binomial coefficient in that expression is nonzero in k so $c_q x^{p-a-2} \in k[s,t]$. In particular, s^{p-a-2} must divide c_q so write $c_q = s^{p-a-2}c'_q$ for some $c'_q \in k[s,t]$. We now have $f = \sum_{q=1}^r c'_q s^{p-a-2}H_q$ so the $s^{p-a-2}H_q$ span and are therefore a basis. Each H_q is homogeneous of degree 0 so each $s^{p-a-2}H_q$ is homogeneous of degree $p-a-2$. $\square$

The second map we wish to consider is given by the matrix $B(\lambda) \in \mathbb{M}_{\lambda+1}(k[s,t])$ defined by

$$B(\lambda)_{ij} = \begin{cases} -it^2 & \text{if } i = j+1 \\ (\lambda-2i)st & \text{if } i = j \\ (\lambda-i)s^2 & \text{if } i = j-1 \\ 0 & \text{otherwise,} \end{cases}$$

where the rows and columns of this matrix are indexed using the integers $0,1,\ldots,\lambda$.

Proposition 3.2. *The kernel of $B(\lambda)$ is a free $k[s,t]$-module of rank $r+1$. There is one basis element that is homogeneous of degree λ and the remaining are homogeneous of degree $p-a-2$.*

Proof. The proof is very similar to the proof of Proposition 3.1. We start by finding the kernel of the matrix $\frac{1}{s^2}B(\lambda)$ whose entries are given by

$$\frac{1}{s^2}B(\lambda)_{ij} = \begin{cases} -ix^2 & \text{if } i = j+1 \\ (\lambda-2i)x & \text{if } i = j \\ \lambda-i & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

with $x = \frac{t}{s}$. Let $f = [\, f_0 \quad f_1 \quad \cdots \quad f_\lambda \,]^t$ be an arbitrary element of the kernel. We induct down the rows of the matrix to show that if $i = qp+b$, where $0 \leqslant b < p$ then

$$f_{\lambda-i} = (-1)^{\lambda-i}x^{\lambda-i}g + (-1)^b\binom{p+a-b}{p-b-1}x^{p-b-1}h_q$$

where $h_r = 0$.

For the base case $i = \lambda$ in the formula gives $f_0 = g$ so we take this as the definition of g. The condition imposed by the first row is $axg + af_1 = 0$ so if $a \neq 0$ then $f_1 = -xg$. The formula gives $f_1 = -xg + (-1)^a - 1\binom{p+1}{p-a}x^{p-a}h_r = -xg$ so these agree. If $a = 0$ then the condition is automatically satisfied and the formula gives $f_1 = -xg + h_{r-1}$ so we take this as the definition of h_{r-1}.

For the inductive step assume the formula holds for $f_0, f_1, \dots, f_{i-1}$ and that these f_j satisfy the conditions imposed by rows $0, \dots, \lambda - i - 2$. The condition imposed by row $\lambda - i - 1$ is

$$(b-a+1)x^2 f_{\lambda-i-2} + (2b-a+2)x f_{\lambda-i-1} + (b+1) f_{\lambda-i} = 0.$$

If $b < p-2$ then we can solve this for $f_{\lambda-i}$ and we find that it agrees with the formula above (for the h_j terms the computation is identical to the one shown in Proposition 3.1). If $b = p-2$ we get $f_{\lambda-i} = (-1)^{\lambda-i} x^{\lambda-i} g - \binom{a+2}{1} x h_q$ as desired. Finally if $b = p-1$ then $b+1 = 0$ in k so the condition is $-ax^2 f_{\lambda-i-2} - ax f_{\lambda-i-1} = 0$ and this is automatically satisfied (the formulas are the same as in Proposition 3.1 again). Thus no condition is imposed on $f_{\lambda-i}$ so we take the formula $f_{\lambda-i} = (-1)^{\lambda-i} x^{\lambda-i} g + h_q$ as the definition of h_q. This completes the induction.

For the final row to be $\lambda - i - 1$ we must choose $i = -1$ and therefore are in the case where $b + 1 = 0$ and the condition is automatically satisfied. The rest of the proof goes as in Proposition 3.1, except that there is no final condition forcing $g = 0$. If we let G and $H_0, \dots, H_{r-1}$ be the basis vectors corresponding to g and $h_0, \dots, h_{r-1}$ then the H_q are linearly independent as before. The first (0th) coordinate of G is 1 while the first coordinate of each H_q is 0 therefore G is not in their span and adding it gives a basis for the kernel. The largest power of x in G is λ in the last coordinate and the largest power of x in H_q is $p - a - 2$ in the $(\lambda - qp - a + 1)$th coordinate. These basis vectors lift to basis vectors of the kernel as a $k[s,t]$-module and are in degrees λ and $p - a - 2$ as desired. □

Before we move on to the third map, let us first prove the following lemma which will be needed in Proposition 4.2.

Lemma 3.3. *Assume $0 \leqslant \lambda < p$. Then the (i,j)th entry of $B(\lambda)^\lambda$ is contained in the one dimensional space $ks^{\lambda+j-i}t^{\lambda-j+i}$.*

Proof. Let b_{ij} be the (i,j)th entry of $B(\lambda)$. The (i,j)th entry of $B(\lambda)^\lambda$ is given by

$$\bigl(B(\lambda)^\lambda\bigr)_{ij} = \sum_{n_1, n_2, \dots, n_{\lambda-1}} b_{in_1} b_{n_1 n_2} \cdots b_{n_{\lambda-1} j}$$

and from the definition of $B(\lambda)$ we have

$$\begin{aligned} &b_{ij} \in ks^2 \quad \text{if } j - i = 1, \\ &b_{ij} \in kst \quad \text{if } j - i = 0, \\ &b_{ij} \in kt^2 \quad \text{if } j - i = -1, \\ &b_{ij} = 0 \quad \text{otherwise.} \end{aligned}$$

So any given term $b_{in_1}b_{n_1n_2}\cdots b_{n_{\lambda-1}j}$ in the summation can be nonzero only if the $(\lambda+1)$-tuple $(n_0,n_1,\ldots,n_\lambda)$ is a *walk* from $n_0=i$ to $n_\lambda=j$, i.e. each successive term of the tuple must differ from the last by at most 1. One shows by induction that for such a walk we have $b_{n_0n_1}b_{n_1n_2}\cdots b_{n_{m-1}n_m}\in ks^{m+n_m-n_0}t^{m-n_m+n_0}$. The base case $m=1$ is given by the three cases above for $b_{n_0n_1}$ and one easily checks that the formula gives kt^2, kst, or ks^2 as needed. For the inductive step assume the statement holds for $m-1$ so that

$$b_{n_0n_1}\cdots b_{n_{m-2}n_{m-1}}b_{n_{m-1}n_m}\in ks^{m-1+n_{m-1}-n_0}t^{m-1-n_{m-1}+n_0}b_{n_{m-1}n_m}.$$

The three relevant cases to consider are $n_m=n_{m-1}+1, n_{m-1}, n_{m-1}-1$ and in each one easily verifies the statement. Now when $m=\lambda$ this gives

$$b_{n_0n_1}b_{n_1n_2}\cdots b_{n_{\lambda-1}n_\lambda}\in ks^{\lambda+n_\lambda-n_0}t^{\lambda-n_\lambda+n_0}=ks^{\lambda+j-i}t^{\lambda-j+i}$$

and completes the proof. □

The third map we wish to consider is $B'(\lambda)\in\mathbb{M}_{rp}(k[s,t])$ defined to be the rpth trailing principal minor of $B(\lambda)$, i.e., the minor of $B(\lambda)$ consisting of rows and columns $a+1,a+2,\ldots,\lambda$.

Proposition 3.4. *The kernel of $B'(\lambda)$ is a free $k[s,t]$-module (ungraded) of rank r whose basis elements are homogeneous of degree $p-a-2$.*

Proof. The induction from the proof of Proposition 3.2 applies giving

$$f_{\lambda-i}=(-1)^{\lambda-i}x^{\lambda-i}g+(-1)^b\binom{p+a-b}{p-b-1}x^{p-b-1}h_q$$

for $0\leqslant i<rp$. All that is left is the condition $-(a+2)xf_{a+1}-f_{a+2}=0$ from the first row of $\frac{1}{s^2}B'(\lambda)$. Substituting in the formulas we get $(-1)^{a+1}(a+1)x^{a+2}g=0$ which forces $g=0$. Thus as a basis for the kernel we get $H_0,\ldots,H_{r-1}$. □

Before we move on to the final map, let us first prove the following lemma which was needed in Section 2.

Lemma 3.5. *Let $s,t\in k$ so that $B'(\lambda)\in\mathbb{M}_{rp}(k)$.*

$$\operatorname{JType}\bigl(B'(\lambda)\bigr)=\begin{cases}[1]^{rp} & \text{if } s=t=0,\\ [p]^{r-1}[p-a-1][a+1] & \text{if } s=0,\ t\neq 0,\\ [p]^r & \text{if } s\neq 0.\end{cases}$$

Proof. If $(s,t)=(0,0)$ then $B'(\lambda)$ is the zero matrix. If $s=0$ and $t\neq0$ then $B'(\lambda)$ only has non-zero entries on the sub-diagonal and we need only read the block sizes. Using the row numbering from $B(\lambda)$ the zeros on the sub-diagonal occur at rows $p,2p,\ldots,rp$.

Thus the first block is size $p-a-1$, followed by $r-1$ blocks of size p, and the last block is size $a+1$.

Now assume $s \neq 0$. There are exactly $r(p-1)$ non-zero entries on the super-diagonal and no non-zero entries above the super-diagonal therefore $\operatorname{rank} B'(\lambda) \geqslant r(p-1)$. But this is the maximal rank that a nilpotent matrix can achieve and such a matrix has Jordan type $[p]^r$. $\square$

The final map we wish to consider is given by the matrix $C(\lambda) \in \mathbb{M}_{\lambda+1}(k[s,t])$ defined by

$$C(\lambda)_{ij} = \begin{cases} (i-\lambda-1)t^2 & \text{if } i = j+1 \\ (\lambda-2i)st & \text{if } i = j \\ (i+1)s^2 & \text{if } i = j-1 \\ 0 & \text{otherwise,} \end{cases}$$

where the rows and columns of this matrix are indexed using the integers $0, 1, \ldots, \lambda$.

Proposition 3.6. *The kernel of $C(\lambda)$ is a free $k[s,t]$-module (ungraded) of rank $r+1$ whose basis elements are homogeneous of degree a.*

Proof. Let $f = [\, f_0 \quad f_1 \quad \cdots \quad f_\lambda \,]^t$ be an arbitrary element of the kernel of $\frac{1}{s^2}C(\lambda)$ whose entries are given by

$$\frac{1}{s^2}C(\lambda)_{ij} = \begin{cases} (i-\lambda-1)x^2 & \text{if } i = j+1 \\ (\lambda-2i)x & \text{if } i = j \\ i+1 & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

with $x = \frac{t}{s}$. We show by induction that if $i = qp + b$ and $0 \leqslant b < p$ then $f_i = (-1)^b \binom{\lambda}{b} x^b h_q$. For the base case the formula gives $f_0 = h_0$ so we take this as the definition of h_0. The condition imposed by row 1 is $-\lambda x f_0 + f_1 = 0$ which gives $f_1 = -x h_0$ as desired.

For the inductive step assume the formula holds for indices less then i and the condition imposed by all rows of index less than $i-1$ is satisfied. The condition imposed by row $i-1$ is $(i-\lambda-2)x^2 f_{i-2} + (\lambda-2i+2)x f_{i-1} + i f_i = 0$. If $i \neq 0, 1$ in k then one checks directly that $f_i = (-1)^b \binom{\lambda}{b} x^b h_q$ as desired. If $i = 0$ in k then

$$(i-\lambda-2)x^2 f_{i-2} + (\lambda-2i+2)x f_{i-1} = \left[\binom{\lambda}{p-2} + \binom{\lambda}{p-1}\right](\lambda+2)x^p h_{q-1}.$$

But $a+1 \neq 0$ so $\binom{\lambda}{p-1} = 0$ and if $\binom{\lambda}{p-2} \neq 0$ then $\lambda + 2 = 0$. In any case the above expression is 0 so the condition is automatically satisfied. The formula gives $f_i = h_q$ so we take this as definition. Finally assume $i = 1$ in k. Then we have $f_i = (\lambda + 1)\binom{\lambda}{p-1} x^{p+1} h_{q-1} - \lambda x h_q = -\binom{\lambda}{1} x h_q$ as desired. This completes the induction.

We know that the given formulas for f_i satisfy the conditions imposed by all rows save the last, whose condition is $-x^2 f_{\lambda-1} - \lambda x f_\lambda = 0$. We have $\lambda x f_\lambda = (-1)^a \lambda x^{a+1} h_r$. If $a = 0$ then $x^2 f_{\lambda-1} = (-1)^{p-1}\binom{\lambda}{p-1} x^{p+1} h_{r-1} = 0$ and $\lambda = 0$ so this conditions is satisfied. If $a \neq 0$ then $x^2 f_{\lambda-1} = (-1)^{a-1} a x^{a-1} h_r$ so $x^2 f_{\lambda-1} + \lambda x f_\lambda = (-1)^a(\lambda - a)x^{a+1} h_r = 0$ and the condition is again satisfied so we have found a basis. If H_q is the basis vector associated to h_q then the smallest and largest powers of x in H_q are 0 in coefficient qp and a in coefficient $qp + a$. By the usual arguments the H_q lift to a basis for the kernel of $C(\lambda)$ that is homogeneous of degree a. □

The final map we want to consider is parametrized by $0 \leqslant a < p-1$. Given such an a, let $D(a) \in \mathbb{M}_{2p}(k[s,t])$ be the block matrix

$$D(a) = \begin{bmatrix} B(2p-a-2) & D'(a) \\ 0 & B(a)^\dagger \end{bmatrix}$$

where $D'(a)$ and $B(a)^\dagger$ are as follows. The matrix $D'(a)$ is a $(2p-a-1) \times (a+1)$ matrix whose (i,j)th entry is

$$D'(a)_{ij} = \begin{cases} \frac{1}{i+1}s^2 & \text{if } i - j = p - a - 2 \\ \frac{1}{a+1}t^2 & \text{if } (i,j) = (p,a) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B(a)^\dagger$ is produced from $B(a)$ by taking the transpose and then swapping the variables s and t.

Proposition 3.7. *The inclusion of $k[s,t]^{2p-a-1}$ into $k[s,t]^{2p}$ as the top $2p-a-1$ coordinates of a column vector induces an isomorphism* $\ker B(2p-a-2) \simeq \ker D(a)$.

Proof. As $D(a)$ is block upper-triangular with $B(2p-a-2)$ the top most block on the diagonal it suffices to show that every element of $\ker D(a)$ is of the form $\left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right]$ with respect to this block decomposition. That is, we must show that if $f = [\, f_0 \quad f_1 \quad \cdots \quad f_{2p-1}\,]^t$ is an element of $\ker D(a)$ then $f_i = 0$ for all $2p - a - 1 \leqslant i \leqslant 2p - 1$. Obviously it suffices to prove this for $\frac{1}{t^2} D(a)$ over $k[s,t,\frac{1}{t}]$ so let $x = \frac{s}{t}$.

We start by proving that $f_{2p-1} = 0$. There are two cases, first assume that $a + 2 = 0$ in k. Then row p of $\frac{1}{t^2} D(a)$ has only one nonzero entry, a $\frac{1}{a+1}$ in column $2p-1$, thus $f_{2p-1} = 0$. Next assume that $a + 2 < p$. Then the induction from Proposition 3.2 applies to rows $p+1, \dots, 2p-a-2$ and gives $f_i = (-1)^{a+i} x^{2p-a-2-i} f_{2p-a-2}$ for $p \leqslant i \leqslant 2p-a-2$. The condition imposed by row p is $-(a+2)x f_p - (a+2)x^2 f_{p+1} + \frac{1}{a+1} f_{2p-1} = 0$. But note that the induction gave us $f_p = -x f_{p+1}$ so this simplifies to $\frac{1}{a+1} f_{2p-1} = 0$ and again we have $f_{2p-1} = 0$.

Now the condition imposed by the last row of $D(a)$ gives $f_{2p-2} = a x f_{2p-1} = 0$. By induction the ith row gives $-i f_{i-1} = (2i + a + 2)x f_i + (i + a + 2)x^2 f_{i+1} = 0$, hence $f_{i-1} = 0$, for $p - a \leqslant i \leqslant 2p - 2$ and this completes the proof. □

4. Explicit computation of $\ker \Theta_M$ and $\mathscr{F}_i(V(\lambda))$

In this final section we carry out the explicit computations of the sheaves $\ker \Theta_M$, for every indecomposable $\mathfrak{sl}_2$-module M, and $\mathscr{F}_i(V(\lambda))$ for $i \neq p$. Friedlander and Pevtsova [4, Proposition 5.9] have calculated the sheaves $\ker \Theta_{V(\lambda)}$ for Weyl modules $V(\lambda)$ such that $0 \leqslant \lambda \leqslant 2p-2$. Using the explicit descriptions of these modules found in Section 2 we can do the calculation for the remaining indecomposable modules in the category.

Theorem 4.1. *Let $\lambda = rp + a$ with $0 \leqslant a < p$ the remainder of λ modulo p. The kernel bundles associated to the indecomposable $\mathfrak{sl}_2$-modules from Theorem 2.1 are*

$$\begin{aligned}
\ker \Theta_{\Phi_\xi(\lambda)} &\simeq \mathcal{O}_{\mathbb{P}^1}(a+2-p)^{\oplus r} \\
\ker \Theta_{V(\lambda)} &\simeq \mathcal{O}_{\mathbb{P}^1}(-\lambda) \oplus \mathcal{O}_{\mathbb{P}^1}(a+2-p)^{\oplus r} \\
\ker \Theta_{V(\lambda)^*} &\simeq \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus r+1} \\
\ker \Theta_{Q(a)} &\simeq \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+2-2p)
\end{aligned}$$

Proof. Assume first that $\xi = [1 : \varepsilon]$. Then using the basis from Section 2 we get that the matrix defining $\Theta_{\Phi_\xi(\lambda)}$ has entries

$$(\Theta_{\Phi_\xi(\lambda)})_{ij} = \begin{cases} ix & \text{if } i = j+1 \\ (2i-a)z & \text{if } i = j \\ (a-i)y & \text{if } i = j-1 \\ -(a+1)\binom{r}{q}\varepsilon^{qp}x & \text{if } (i,j) = (0, qp+a) \\ 0 & \text{otherwise.} \end{cases}$$

Pulling back along the map $\iota : \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ from Example 1.2 corresponds with extending scalars through the homomorphism

$$\frac{k[x,y,z]}{(xy+z^2)^n} \to k[s,t] \qquad (x,y,z) \mapsto \left(s^2, -t^2, st\right).$$

Thus the matrix of $\Theta_{\Phi_\xi(\lambda)}$ becomes the matrix $M_\varepsilon(\lambda)$ from Proposition 3.1 and we see that the kernel is free. A basis element, homogeneous of degree m, spans a summand of the kernel isomorphic to $k[s,t][-m]$. By definition the $\mathcal{O}_{\mathbb{P}^1}$-module corresponding to $k[s,t][-m]$ is $\mathcal{O}_{\mathbb{P}^1}(-m)$ so the description of the kernel translates directly to the description of the sheaf above.

The remaining cases are all identical. The modules $V(\lambda)$, $\Phi_{[0:1]}(\lambda)$, $V(\lambda)^*$, and $Q(a)$ give the matrices $B(\lambda)$, $B'(\lambda)$, $C(\lambda)$, and $D(a)$ whose kernels are calculated in Propositions 3.2, 3.4, 3.6, and 3.7 respectively. □

Next we compute $\mathscr{F}_i(V(\lambda))$ for any $i \neq p$ and any indecomposable $V(\lambda)$. The proof is by induction on r in the expression $\lambda = rp + a$. For the base case we start with

$V(\lambda)$ a simple module, i.e., $r = 0$. Note that for the base case we do indeed determine $\mathscr{F}_p(V(\lambda))$, it is during the inductive step that we lose $i = p$.

Proposition 4.2. *If $0 \leqslant \lambda < p$ then*

$$\mathscr{F}_i\big(V(\lambda)\big) = \begin{cases} \ker \Theta_{V(\lambda)} & \textit{if } i = \lambda + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First note that $V(\lambda)$ has constant Jordan type $[\lambda + 1]$ so Theorem 1.14 tells us that when $i \neq \lambda + 1$ the sheaf $\mathscr{F}_i(V(\lambda))$ is zero.

For $i = \lambda + 1$ recall from the previous proof that the map $\Theta_{V(\lambda)}$ of sheaves is given in the category of $k[s,t]$-modules by the matrix $B(\lambda)$. The $(\lambda + 1)$th power of a matrix of Jordan type $[\lambda + 1]$ is zero so the entries of $B(\lambda)^{\lambda+1}$ are polynomials representing the zero function. As k is algebraically closed $B(\lambda)^{\lambda+1} = 0$ and therefore $\Theta^{\lambda+1}_{V(\lambda)} = 0$. In particular $\operatorname{im} \Theta^{\lambda}_{V(\lambda)} \subseteq \ker \Theta_{V(\lambda)}$ so the definition of $\mathscr{F}_{\lambda+1}(V(\lambda))$ gives

$$\mathscr{F}_{\lambda+1}\big(V(\lambda)\big) = \frac{\ker \Theta_{V(\lambda)} \cap \operatorname{im} \Theta^{\lambda}_{V(\lambda)}}{\ker \Theta_{V(\lambda)} \cap \operatorname{im} \Theta^{\lambda+1}_{V(\lambda)}} = \operatorname{im} \Theta^{\lambda}_{V(\lambda)}. \tag{2}$$

We have a short exact sequence of $k[s,t]$-modules

$$0 \to \operatorname{im} B(\lambda)^{\lambda} \to \ker B(\lambda) \to \frac{\ker B(\lambda)}{\operatorname{im} B(\lambda)^{\lambda}} \to 0.$$

If we show that the quotient $\ker B(\lambda)/\operatorname{im} B(\lambda)^{\lambda}$ is finite dimensional then by Serre's theorem and Eq. (2) this gives a short exact sequence of sheaves

$$0 \to \mathscr{F}_i\big(V(\lambda)\big) \to \ker \Theta_{V(\lambda)} \to 0 \to 0$$

and completes the proof.

To show that $\ker B(\lambda)/\operatorname{im} B(\lambda)^{\lambda}$ is a finite dimensional module note that from $B(\lambda)^{\lambda+1} = 0$ we get that the columns of $B(\lambda)^{\lambda}$ are contained in the kernel of $B(\lambda)$ which, in Proposition 3.2 we determined is a free $k[s,t]$-module with basis element $G = [\, s^{\lambda} \quad -s^{\lambda-1}t \quad \cdots \quad (-1)^{\lambda}t^{\lambda}\,]^t$. We also know by Lemma 3.3 that the first entry in the jth column of $B(\lambda)^{\lambda}$ is $c_j s^{\lambda+j}t^{\lambda-j}$ for some $c_j \in k$, so the jth column must therefore be $c_j s^j t^{\lambda-j} G$. The columns of $B(\lambda)^{\lambda}$ range from $j = 0$ to $j = \lambda$ so this shows that G times any monomial of degree λ is contained in the image of $B(\lambda)^{\lambda}$. Thus the quotient $\ker B(\lambda)/\operatorname{im} B(\lambda)^{\lambda}$ is spanned, as a vector space, by the set of vectors of the form G times a monomial of degree strictly less than λ. There are only finitely many such monomials therefore $\ker B(\lambda)/\operatorname{im} B(\lambda)^{\lambda}$ is finite dimensional and the proof is complete. $\square$

Now for the inductive step we will make use of Theorem 1.13, but in a slightly different form. Note that the shift in Theorem 1.13 is given by tensoring with the sheaf $\mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1)$

associated to the shifted module $\frac{k[x,y,z]}{xy+z^2}[1]$. Likewise we consider $\mathcal{O}_{\mathbb{P}^1}(1)$ to be the sheaf associated to $k[s,t][1]$. Pullback through the isomorphism $\iota\colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ of Example 1.2 yields $\iota^*\mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1) = \mathcal{O}_{\mathbb{P}^1}(2)$. Consequently, Theorem 1.13 has the following corollary.

Corollary 4.3. *Let M be an $\mathfrak{sl}_2$-module and $1 \leqslant i < p$. With twist coming from $\mathbb{P}^1$ we have*

$$\mathscr{F}_i(M) \simeq \mathscr{F}_{p-i}(\Omega M)(2p-2i).$$

Observe that $i \neq p$ in the theorem; this is why our calculation of $\mathscr{F}_p(V(\lambda))$ for $\lambda < p$ does not induce a calculation of $\mathscr{F}_p(V(\lambda))$ when $\lambda \geqslant p$.

Theorem 4.4. *If $V(\lambda)$ is indecomposable and $i \neq p$ then*

$$\mathscr{F}_i\big(V(\lambda)\big) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-\lambda) & \text{if } i = \lambda + 1 \pmod p \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda = rp+a$ where $0 \leqslant a < p$ is the remainder of λ modulo p. We prove the result by induction on r. The base case $r = 0$ follows from Theorem 4.1 and Proposition 4.2. For the inductive step assume $r \geqslant 1$. By hypothesis the formula holds for $rp - a - 2$ and by Proposition 2.5 we have $\Omega V(rp - a - 2) = V(\lambda)$. Applying Corollary 4.3 we get $\mathscr{F}_i(V(\lambda)) = \mathscr{F}_{p-i}(V(rp-a-2))(-2i)$. If $i = a+1$ then $\mathscr{F}_{a+1}(V(\lambda)) = \mathscr{F}_{p-a-1}(V(rp-a-2))(-2a-2) = \mathcal{O}_{\mathbb{P}^1}(-\lambda)$ whereas if $i \neq a+1$ then $p - i \neq p - a - 1$ so $\mathscr{F}_{p-i}(V(rp-a-2)) = 0$. This completes the proof. □

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