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The stable rank of pullbacks

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ABSTRACT

We prove that the stable rank of any pullback (fibre product) does not exceed the stable ranks of factors. We obtain also another result bounding the stable rank of a ring by the stable ranks of its factor rings.

1. Statement of results

Following Bass [1], Vaserstein [5] defined the stable rank, $\mathrm{sr}(A)$ for any associative ring A. On Dec. 8, 2010 Yuanhang Zhang asked me whether $\mathrm{sr}(A) = \max(\mathrm{sr}(A_1), \mathrm{sr}(A_2))$ when $f : A \to C$ is the pullback of two surjective homomorphisms $f_j : A_j \to C = f_j(A_j)$ of associative rings with 1, also known as the fibre, fibred, fiber, or fibered product. Zhang was motivated by a result of Sheu [4, Corollary 3.16] involving topological stable ranks of C^*-algebras. By [3], the topological stable rank and the stable rank coincide for all C^*-algebras. Another special case was considered in [2, Lemma (44.25)].

In this note we obtain the affirmative answer in a more general setting. Namely, instead of two homomorphisms, we consider any collection of homomorphisms, and we work in a bigger category of all associative rings (without identities). As a consequence of our Theorem 1 below, we obtain

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Corollary 1. *Let $f_j : A_j \to C$ be any nonempty collection of morphisms in the category of associative rings such that $f_j(A_j) = C$ for all j. Let $f : A \to C$ be the pullback of all f_j. Then $\mathrm{sr}(A) = \sup(\mathrm{sr}(A_j))$.*

The morphism f can be defined as the (inverse or projective) limit of the diagram consisting of the morphisms $A_j \to C$. In set-theoretical terms, the ring A is the subring of the direct product $A' = \prod A_j$ consisting of the collections $a_i \in A_j$ with the common image $f_j(a_j) = f_k(a_k)$ in C. When $C = 0$ we have $A = A'$ and the theorem is trivial.

Now we recall the definition of the stable rank. For any associative ring A and any integer $n \geq 1$, let $Um_n A$ denote the set of all n-columns (b_i) over A such that there is an n-row (a_i) over A satisfying the condition $a_1 + b_1 + \sum_{i=1}^n a_i b_i = 0$. When A is a subring of a ring A_1 with 1, the last equality takes a more familiar form: $(1 + a_1)(1 + b_1) + \sum_{i=2}^n a_i b_i = 1$.

The Bass condition is:

(SR_n) for any $(b_i) \in Um_{n+1}A$ there are $c_1, \ldots, c_n \in A$ such that the column $(b_i + c_i b_{n+1})_{1 \leq i \leq n}$ belongs to $Um_n A$.

If no such n exists we set $\mathrm{sr}(A) = \infty$. Otherwise, we define $\mathrm{sr}(A)$ as the least n such that the condition (SR_n) holds. Then, by [5], (SR_m) holds for all $m \geq n$. Some properties of the stable rank, improving and generalizing results of Bass [1] are proven in [5] and [6]. In this paper, we will use that $\mathrm{sr}(A/J) \leq \mathrm{sr}(A)$ for any ideal J of any ring A and that $\mathrm{sr}(A) = \mathrm{sr}(A/\mathrm{rad}(A))$ where $\mathrm{rad}(A)$ is the Jacobson radical of A.

We will obtain Corollary 1 as an easy consequence of the following result.

Theorem 1. *Let A be a subring of A' and $f : A \to C$ be a ring morphism such that its kernel* $\ker(f)$ *is an ideal of A' and $f(A) = C$. Then $\mathrm{sr}(C) \leq \mathrm{sr}(A) \leq \mathrm{sr}(A')$.*

The following lemma will be used in the proof of Theorem 1.

Lemma 1. *Let B be an associative ring, $n \geq 1$, and $(b_i) = (b_1, \ldots, b_n, b_{n+1})^T \in Um_{n+1}B$. Let $d_i \in B$ and $y = d_1 + b_1 + d_1 b_1 + \cdots + d_n b_n$. Then there is $z \in yB$ such that $(b_1, \ldots, b_n, z b_{n+1})^T \in Um_{n+1}B$.*

Besides Theorem 1, the lemma can be used to obtain the following result.

Theorem 2. *Let A be an associative ring and $J_1, \ldots, J_m$ two-sided ideals of A such that $J_1 J_2 \cdots J_m \subset \mathrm{rad}(A)$. Then $\mathrm{sr}(A) \leq \max(\mathrm{sr}(A/J_j))$.*

Corollary 1 with finite collection of the rings A_j follows easily from Theorem 2 (namely, set J_j to be the kernel of the j-th projection $A \to C$; then the intersection of all J_j is 0).

Theorem 2 for commutative rings A with 1 is Lemma 7.6 of [7].

The condition $J_1 J_2 \cdots J_m \subset \operatorname{rad}(A)$ in Theorem 2 is equivalent to the condition $\bigcap J_j \subset \operatorname{rad}(A)$. The latter condition makes sense for an infinite set of ideals. However, the obvious extension of Theorem 2 to an infinite collection of ideals is false as the following example shows. Let A be the ring of integers and P an infinite set of maximal ideals of A. Then $\operatorname{sr}(A/J) = 1$ for all $J \in P$ and $\bigcap_{J\in P} J = 0$, while $\operatorname{sr}(A) = 2$.

2. Proofs

Here is how Corollary 1 follows from Theorem 1. The pullback A in Corollary 1 is a subring of the direct product A' of all A_j. The kernel of $f : A \to C = f(A)$ is the direct product of the kernels of $f_j : A_j \to C$, hence it is an ideal of A'. By Theorem 1, $\operatorname{sr}(A) \le \operatorname{sr}(A') = \sup(\operatorname{sr}(A_j))$. We have $\operatorname{sr}(C) \le \operatorname{sr}(A)$ because C is a factor ring of A.

On the other hand, $\operatorname{sr}(A_j) \le \operatorname{sr}(A)$ for all j because each A_j is a factor ring of A. So $\sup(\operatorname{sr}(A_j)) \le \operatorname{sr}(A)$. Thus, $\operatorname{sr}(A) = \sup(\operatorname{sr}(A_j))$. □

Proof of Lemma 1. Since $(b_i) \in Um_{n+1}B$, there are $a_i \in B$ such that $a_1 + b_1 + \sum_{i=1}^{n+1} a_i b_i = 0$. We multiply this equality by $-y$ on the left and add the result from the equality $y = d_1 + b_1 + d_1 b_1 + \cdots + d_n b_n$ to obtain

$$a'_1 + b_1 + \sum_{i=1}^{n+1} a'_i b_i = 0 \tag{1}$$

with

$$a'_1 = d_1 - y - ya_1 \in B, \qquad a'_i = d_i - ya_i \in B \quad (2 \le i \le n), \quad a'_{n+1} = -ya_{n+1} \in yB.$$

Next, we multiply (1) by $-b_1$ on the left and add the result to $b_1 = b_1$ to obtain

$$a''_1 + b_1 + \sum_{i=1}^{n+1} a''_i b_i = 0 \tag{2}$$

with

$$a''_1 = -b_1 - b_1 a'_1 \in B, \qquad a''_i = -b_1 a'_i \in B \quad (2 \le i \le n+1),$$
$$a''_{n+1} = b_1 y a_{n+1} \in ByB.$$

Thus, $(b_1, \ldots, b_n, zb_{n+1})^T \in Um_{n+1}B$ with $z = ya_{n+1} \in B$. □

Now we give a proof of Theorem 1. Since C is a factor ring of A we have $\operatorname{sr}(C) \le \operatorname{sr}(A)$. Next we have to prove that $\operatorname{sr}(A) \le \operatorname{sr}(A')$.

In other words, we assume that $n = \operatorname{sr}(A') < \infty$, and we have to prove the condition (SR_n).

Let $(b_i) \in Um_{n+1}A$. We want to prove that there exist $c_i \in A$ such that the n-column $(a_i + c_i a_{n+1})$ belongs to $Um_n A$.

Since $\mathrm{sr}(C) \le n$ and $f(A) = C$, there are $c_i', u_i \in A$ such that

$$y = u_1 + \big(b_1 + c_1' b_{n+1}\big) + \sum_{i=1}^{n} u_i \big(b_i + c_i' b_{n+1}\big) \in \ker(f).$$

By the lemma,

$$\big(b_1 + c_1' b_{n+1}, \ldots, b_n + c_n' b_{n+1}, z b_{n+1}\big)^T \in Um_{n+1}A \quad \text{for some } z = yA \in \ker(f).$$

Since $\mathrm{sr}(A') = n$, there are $c_i'' \in A'$ such that $(b_i + c_i' b_{n+1} + c_i'' z b_{n+1}) \in Um_n A'$.

Set $c_i = c_i' + c_i'' z \in A$ (we used that $\ker(f)$ is an ideal of A'). Then $(b_i + c_i b_{n+1}) \in Um_n A'$ and

$$y' = u_1 + (b_1 + c_1 b_{n+1}) + \sum_{i=1}^{n} u_i (b_i + c_i b_{n+1}) \in \ker(f). \tag{3}$$

Since $(b_i + c_i b_{n+1}) \in Um_n A'$ there are $v_i \in A'$ such that

$$0 = v_1 + (b_1 + c_1 b_{n+1}) + \sum_{i=1}^{n} v_i (b_i + c_i b_{n+1}). \tag{4}$$

Now we multiply (4) by y' on the left and subtract the result from (3):

$$0 = t_1 + (b_1 + c_1 b_{n+1}) + \sum_{i=1}^{n} t_i (b_i + c_i b_{n+1}) \tag{5}$$

with $t_1 = u_1 - y' v_1 - y' \in A$ and $t_i = u_i - y' v_i \in A$ $(2 \le i \le n)$. The equality (5) shows that the column $(b_i + c_i b_{n+1})$ is in $Um_n A$. $\square$

Now we prove Theorem 2.

Case 1: $m = 1$. Then $\mathrm{sr}(A) = \mathrm{sr}(A/J_1) = \mathrm{sr}(A/\mathrm{rad}(A))$.

Case 2: $m = 2$. Let $n = \max(\mathrm{sr}(A/J_1), \mathrm{sr}(A/J_2))$. Assuming that $n < \infty$, we want to check the condition (SR_n). Let $(b_i) \in Um_{n+1}(A)$. Since $\mathrm{sr}(A/J_2) \le n$, there are $u_i, c_i \in A$ such that

$$y := u_1 + (b_1 + c_1 b_{n+1}) + \sum_{i=1}^{n} u_i (b_i + c_i b_{n+1}) \in J_2.$$

By the lemma, there is $z \in yA \subset J_2$ such that

$$(b_1 + c_1 b_{n+1}, \ldots, b_n + c_n b_{n+1}, z b_{n+1})^T \in Um_{n+1}A.$$

Since $\mathrm{sr}(A/J_1) \le n$, there are $v_i, c'_i \in A$ such that

$$y' := v_1 + \left(b_1 + cb_{n+1} + c'_1 z b_{n+1}\right) + \sum_{i=1}^{n} v_i\left(b_i + c_i b_{n+1} + c'_i z b_{n+1}\right) \in J_1.$$

By the lemma, there is $z' \in y'A \subset J_1$ such that

$$\left(b_1 + c_1 b_{n+1} + c'_1 z b_{n+1}, \ldots, b_n + c_n b_{n+1} + c'_n z b_{n+1}, z' z b_{n+1}\right)^T \in Um_{n+1}A.$$

Since $z'z \in J_1 J_2 \subset \mathrm{rad}(A)$, we conclude that

$$\left(b_i + c''_i b_{n+1}\right) = \left(b_1 + c_1 b_{n+1} + c'_1 z b_{n+1}, \ldots, b_n + c_n b_{n+1} + c'_n z b_{n+1}\right)^T \in Um_n A$$

where $c'' = c_i + c'_i z \in A$.

General case. We proceed by induction on m. In view of Cases 1 and 2, we can assume that $m \ge 3$. Set $J' = J_2 \cdots J_m$. Then $J_1 J' = J_1 J_2 \cdots J_m \subset \mathrm{rad}(A)$ hence $\mathrm{sr}(A) = \max(\mathrm{sr}(A/J_1), \mathrm{sr}(A/J'))$ by Case 2. By the induction hypothesis, $\mathrm{sr}(A/J') = \max(\mathrm{sr}(A/J_2), \ldots, \mathrm{sr}(A/J_m))$. Thus, $\mathrm{sr}(A) \le \max(\mathrm{sr}(A/J_j))$. $\square$.

References

[1] H. Bass, K-theory and stable algebra, Inst. Hautes Études Sci. Publ. Math. 22 (1964) 5–60.
[2] C.W. Curtis, I. Reiner, Methods of Representation Theory, with Applications to Finite Groups and Orders, vol. II, Wiley, 1987.
[3] R. Herman, L.N. Vaserstein, The stable range of C^*-algebras, Invent. Math. 77 (3) (1984) 553–555.
[4] A. Sheu, A cancellation theorem for modules over the group C^*-algebras of certain nilpotent Lie groups, Canad. J. Math. 39 (2) (1987) 365–427.
[5] L.N. Vaserstein, K_1-theory and the congruence subgroup problem, Mat. Zametki 5 (2) (1969) 233–244; Math. Notes 5 (1969) 141–148.
[6] L.N. Vaserstein, The stable range of rings and the dimension of topological spaces, Funktsional. Anal. i Prilozhen. 5 (2) (1971) 17–27; Funct. Anal. Appl. 5 (1971) 102–110.
[7] L.N. Vaserstein, A.A. Suslin, Serre's problem on projective modules over polynomial rings and algebraic K-theory, Izv. Akad. Nauk Ser. Mat. 40 (5) (1976) 993–1054; Math. USSR-Izv. 10 (5) (1976) 937–1001.