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# On Rees algebras of linearly presented ideals



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## ABSTRACT

Let  $I$  be a height two perfect ideal with a linear presentation matrix in a polynomial ring  $R = k[x_1, \dots, x_d]$ . Assume that  $\mu(I) = d + 1$  and  $I$  satisfies the Artin–Nagata condition  $G_{d-1}$ . We determine the defining ideal of the Rees algebra  $\mathcal{R}(I)$  explicitly and we show that  $\mathcal{R}(I)$  is Cohen–Macaulay.

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## 1. Introduction

Let  $R$  be a Noetherian commutative ring and  $I = (f_1, \dots, f_n)$  an  $R$ -ideal. The Rees ring of the ideal  $I$  is the graded  $R$ -algebra  $\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j t^j$ , where  $t$  is an indeterminate. This algebra is related to the blowing-up of  $\text{Spec}(R)$  along the subscheme defined by the ideal  $I$ . A natural way to represent the Rees algebra  $\mathcal{R}(I)$  is to consider the map  $R[t_1, \dots, t_n] \rightarrow \mathcal{R}(I)$ ,  $t_i \mapsto f_i t$ . The kernel of this map called the defining ideal of the Rees algebra  $\mathcal{R}(I)$  is the set  $\mathcal{L}$  of all polynomials  $h \in R[t_1, \dots, t_n]$  such that  $h(f_1, \dots, f_n) = 0$ .

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In case  $R = k[x_1, \dots, x_d]$  is a polynomial ring over an algebraically closed field  $k$  and  $f_1, \dots, f_n$  are homogeneous polynomials of the same degree, we have a rational map  $\mathbb{P}_k^{d-1} \dashrightarrow \mathbb{P}_k^{n-1}$ :

$$(x_1 : \dots : x_d) \mapsto (f_1(x_1 : \dots : x_d) : \dots : f_n(x_1 : \dots : x_d)).$$

Its closed image is a subvariety of  $\mathbb{P}_k^{n-1}$ . Finding the defining equations of this subvariety is called the implicitization problem. This problem in low dimensions,  $d = 2$  or  $d = 3$ , and  $n = d + 1$  has significant applications in computer aided geometric design and modeling. It is motivated by the method of “implicitization by “moving curves and surfaces” introduced by Sederberg and Chen in 1995 [14]. In algebraic terms, the moving ideal of the rational map is exactly the defining ideal  $\mathcal{L}$  of the Rees algebra  $\mathcal{R}(I)$ .

The issue of finding the defining ideal of a Rees algebra is basically a problem in elimination theory. Theoretically, it can be solved using Gröbner basis techniques. However, even for small dimensions and small numbers of generators, this method may not be practical since it has high complexity and is computationally expensive. Although the implicitization problem has been studied for several decades in commutative algebra, algebraic geometry, and recently in geometric modeling, it is still open in most cases.

Another blow-up algebra which is closely related to the Rees algebra  $\mathcal{R}(I)$  is the symmetric algebra  $\mathcal{S}(I)$  of  $I$ . Again, as above, we have a polynomial presentation  $R[t_1, \dots, t_n] \rightarrow \mathcal{S}(I)$ ,  $t_i \mapsto f_i$ . The kernel  $\mathcal{J}$  of this map is generated by all homogeneous degree 1 polynomials  $h \in R[t_1, \dots, t_n]$  such that  $h(f_1, \dots, f_n) = 0$ . It is clear that  $\mathcal{J} \subseteq \mathcal{L}$ . One may try to get information on  $\mathcal{L}$  from  $\mathcal{J}$ . The advantage of this approach is that  $\mathcal{J}$  and sometimes even its entire free resolution are known. If  $\varphi$  is a presentation matrix for the ideal  $I = (f_1, \dots, f_n)$ , then  $\mathcal{J}$  is generated by the entries of the vector  $\underline{t} \cdot \varphi$ , where  $\underline{t} = (t_1, \dots, t_n)$ . But it is not obvious how to obtain the other generators of  $\mathcal{L}$  from  $\mathcal{J}$ .

When  $\mathcal{L} = \mathcal{J}$ , one says that  $I$  is of linear type. Micali [12] was the first to show that if the ideal  $I$  is generated by a regular sequence, then  $I$  is of linear type. Later, Huneke [8] introduced the notion of  $d$ -sequence, a generalization of regular sequence, and proved that any ideal generated by a  $d$ -sequence is of linear type. This result was also obtained independently by Valla [16]. Using the so-called approximation complexes, Herzog, Simis and Vasconcelos [7] were able to establish the linear type and proved the Cohen–Macaulayness of the Rees algebra for a large class of ideals which satisfy the condition  $\mu(I_\varphi) \leq \dim R_\varphi$  for every  $\varphi \in V(I)$ . This condition is called  $G_\infty$  and it is a necessary condition for an ideal to be of linear type.

One way to generalize the result of Herzog, Simis and Vasconcelos is to replace the condition  $G_\infty$  by the weaker condition  $G_s$  which means that, for an integer  $s$ ,  $\mu(I_\varphi) \leq \dim R_\varphi$  for every prime ideal  $\varphi \in V(I)$  with  $\dim R_\varphi \leq s - 1$ . Both conditions  $G_\infty$  and  $G_s$  were introduced by Artin and Nagata in [2]. Morey and Ulrich [13] are among the first authors who computed explicitly the defining ideal  $\mathcal{L}$  for a large class of ideals which are not of linear type. They consider linearly presented ideals in a polynomial ring

$R = k[x_1, \dots, x_d]$  over a field  $k$ . In this case, there is a unique matrix  $B(\varphi)$  with linear entries in  $k[t_1, \dots, t_n]$  such that  $\underline{t} \cdot \varphi = \underline{x} \cdot B(\varphi)$ , where  $\underline{x} = (x_1 \dots x_d)$ . The matrix  $B(\varphi)$  is called the *Jacobian dual* matrix of  $\varphi$ , [17]. Let  $I_d(B(\varphi))$  denote the ideal generated by  $d \times d$  minors of  $B(\varphi)$ .

**Theorem 1.1.** (See [13, Theorem 1.3].) *Let  $I$  be a perfect  $R$ -ideal of a height 2 with a linear presentation matrix  $\varphi$ . Assume that  $\mu(I) = n > d$  and that  $I$  satisfies  $G_d$ . Then the Rees algebra  $\mathcal{R}(I)$  is Cohen–Macaulay and  $\mathcal{L} = (\underline{t} \cdot \varphi, I_d(B(\varphi)))$ .*

In this paper, we would like to replace the condition  $G_d$  in the above result by the weaker condition  $G_{d-1}$ . However, we need to restrict the number of generators of the ideal by requiring  $\mu(I) = d + 1$ . Our main result is the following.

**Theorem 1.2.** *Let  $k$  be an algebraically field  $k$ . Let  $I$  be a perfect  $R$ -ideal of height 2 with a linear presentation matrix  $\varphi$  with  $I_1(\varphi) = (x_1, \dots, x_d)$ . Assume that  $\mu(I) = d + 1$  and that  $I$  satisfies  $G_{d-1}$  but not  $G_d$ . Then the Rees algebra  $\mathcal{R}(I)$  is Cohen–Macaulay and  $\mathcal{L} = (\underline{t} \cdot \varphi, g)$ , where  $g$  is the degree  $d - 1$  divisor of  $\det B(\varphi)$ . Moreover, after a possible change of coordinates and elementary row and column operations on the matrix  $\varphi$ , the canonical module  $\omega_{\mathcal{R}(I)}$  is isomorphic to  $(x_1, \dots, x_{d-1}, x_d f_1 t) f_1 t \mathcal{R}(I)$  as a bigraded  $\mathcal{R}(I)$ -module, where  $f_1$  is the maximal minor of  $\varphi$  obtained by deleting the first row.*

If we do not assume  $I_1(\varphi) = (x_1, \dots, x_d)$ , then  $I_1(\varphi) = (x_1, \dots, x_{d-1})$  after a possible change of variables. Hence  $\varphi$  is a matrix with entries in  $k[x_1, \dots, x_{d-1}]$ , and we get back to the setting of Theorem 1.1.

The paper is organized as follows. In Section 2, we prepare some results on the Jacobian dual matrix and on linkage theory which we shall use in our investigation. Section 3 describes a special form of the representation matrix  $\varphi$  in the setting of Theorem 1.2. In Section 4, we show that the symmetric algebra is a complete intersection in this setting. This fact together with the special form of  $\varphi$  are used to establish the existence of a canonical non-linear relation  $g$  for the Rees algebra  $\mathcal{R}(I)$ . That relation together with the linear relations are shown to generate the defining ideal of the Rees algebra in Section 5. The proof of Theorem 1.2 will be completed there.

## 2. Preliminaries

Let  $R$  be a Noetherian ring and  $I$  an  $R$ -ideal. Recall that the ideal  $I$  satisfies the *condition*  $G_s$ ,  $s$  an integer, if  $\mu(I_\varphi) \leq \text{ht } \varphi$  for every prime ideal  $\varphi \in V(I)$  with  $\text{ht } \varphi \leq s - 1$ . The ideal  $I$  satisfies the *condition*  $G_\infty$  if it satisfies  $G_s$  for every  $s$ . When a presentation matrix  $\varphi$  of the ideal  $I$  is given, the condition  $G_s$  can be checked by computing the height of the ideals generated by minors of the matrix  $\varphi$ .

Let  $\varphi : R^m \rightarrow R^n$  be an  $R$ -linear map and  $M$  the cokernel of  $\varphi$ . For  $0 < i \leq \min\{m, n\}$ , we denote by  $I_i(\varphi)$  the ideal generated by all  $i \times i$  minors of the matrix  $\varphi$ .

For  $i \leq 0$ , let  $I_i(\varphi) = R$  and for  $i > \min\{m, n\}$ ,  $I_i(\varphi) = 0$ . Then one defines the  $i$ -th Fitting ideal  $F_i(M)$  of  $M$  to be the ideal  $I_{n-i}(\varphi)$ . The reason we shift the subindex is that the ideals of minors depend on the matrix  $\varphi$  while the Fitting ideals do not depend on the presentation matrix, see [4, (4.8) Lemma, p. 241; (4.10) Proposition, p. 242].

**Proposition 2.1.** (See [5, Proposition 20.6].) *When  $R$  is local, we have  $F_i(M) = R$  if and only if  $\mu(M) \leq i$ . In fact,  $V(F_i(M)) = \{\wp \in \text{Spec}(R) \mid \mu(M_\wp) > i\}$ .*

This proposition has a direct corollary which characterizes the condition  $G_s$  in terms of the height of the Fitting ideals.

**Corollary 2.2.** *The ideal  $I$  satisfies  $G_s$  if and only if  $\text{ht } F_i(I) \geq i + 1$  for all  $i \leq s - 1$ .*

**Proof.** First, suppose that the ideal  $I$  satisfies  $G_s$ . Assume that there exists an integer  $i_0$ ,  $i_0 \leq s - 1$ , such that  $\text{ht } F_{i_0}(I) \leq i_0$ . Then there exists a prime ideal  $\wp \in V(F_{i_0}(I))$  with  $\text{ht } \wp \leq i_0$ . On the other hand, by Proposition 2.1, we have  $\mu(I_\wp) > i_0$  and hence  $\mu(I_\wp) > \text{ht } \wp$ . This contradicts the fact that the ideal  $I$  satisfies  $G_s$ .

Now, assume that  $\text{ht } F_i(I) \geq i + 1$  for all  $i \leq s - 1$ . In particular, if  $\wp \in V(I)$  with  $\text{ht } \wp \leq s - 1$  then  $\wp \notin V(F_{\text{ht } \wp}(I))$ . Proposition 2.1 implies that  $\mu(I_\wp) \leq \text{ht } \wp$ . Therefore the ideal  $I$  satisfies  $G_s$ .  $\square$

**Example 2.3.** Let  $R = k[x, y, z]$  and  $I = I_3(\varphi)$  where

$$\varphi = \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \\ 0 & 0 & y \end{pmatrix}.$$

It is clear that  $\text{ht } F_1(\varphi) = \text{ht } F_2(\varphi) = 2$ . By Corollary 2.2, the ideal  $I$  satisfies  $G_2$  but it does not satisfy  $G_3$ .

Now, we suppose that  $R = k[x_1, \dots, x_d]$  is a polynomial ring over a field  $k$  and that the ideal  $I$  has an  $n \times m$  presentation matrix  $\varphi$  whose entries are linear forms in  $R$ . Then there exists a unique  $d \times m$  matrix  $B(\varphi)$  with linear entries in  $k[t_1, \dots, t_n]$  so that

$$\underline{t} \cdot \varphi = \underline{x} \cdot B(\varphi),$$

where  $\underline{t} = (t_1 \dots t_n)$  and  $\underline{x} = (x_1 \dots x_d)$ . The matrix  $B(\varphi)$  is called the Jacobian dual matrix of  $\varphi$ . When there is no confusion, we write  $B$  instead of  $B(\varphi)$  for the Jacobian dual.

**Example 2.4.** For the matrix

$$\varphi = \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \\ 0 & 0 & y \end{pmatrix},$$

the Jacobian dual is

$$B(\varphi) = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ 0 & 0 & t_1 \end{pmatrix}.$$

As mentioned in the introduction, besides the degree 1 generators, it is not obvious how to obtain the other generators for the defining ideal  $\mathcal{L}$  of the Rees algebra  $\mathcal{R}(I)$ . However, it is possible to derive some higher degree generators from the Jacobian dual.

**Lemma 2.5.** *If  $m \geq d$ , then  $I_d(B(\varphi)) \subseteq \mathcal{L}$ .*

**Proof.** Let  $h$  be the maximal minor of  $B(\varphi)$  formed by the first  $d$  columns. We have

$$\underline{t} \cdot \widetilde{\varphi} = \underline{x} \cdot \widetilde{B(\varphi)}, \quad (2.5.1)$$

where  $\widetilde{\varphi}$ ,  $\widetilde{B(\varphi)}$  are formed from  $\varphi$  and  $B(\varphi)$  by keeping the first  $d$  columns. Multiplying to the right of both sides of (2.5.1) with the adjoint matrix  $\text{Adj}(\widetilde{B(\varphi)})$ , we see that

$$h \cdot (x_1, \dots, x_d) = \det \widetilde{B(\varphi)} \cdot (x_1, \dots, x_d) \subseteq \mathcal{J} \subseteq \mathcal{L}. \quad (2.5.2)$$

But since the ring  $R$  is a domain, the defining ideal  $\mathcal{L}$  is a prime ideal. From (2.5.2), we have  $h \in \mathcal{L}$  since  $x_i \notin \mathcal{L}$  for every  $i$ . Similarly, we can show that all maximal minors of  $B(\varphi)$  are in  $\mathcal{L}$ .  $\square$

**Example 2.6.** For the matrices  $\varphi$  and  $B(\varphi)$  of Example 2.4, from Lemma 2.5 we know that  $\det B(\varphi) = t_1(t_1t_3 - t_2^2) \in \mathcal{L}$ . Since  $\mathcal{L}$  is prime and  $t_1 \notin \mathcal{L}$ ,  $t_1t_3 - t_2^2 \in \mathcal{L}$ . Therefore, in this case we have

$$(\mathcal{J}, I_3(B(\varphi))) = (\mathcal{J}, t_1(t_1t_3 - t_2^2)) \subsetneq (\mathcal{J}, t_1t_3 - t_2^2) \subseteq \mathcal{L}.$$

The following result together with Lemma 2.5 say that when the dimension of the symmetric algebra  $\mathcal{S}(I)$  and the number of generators of the ideal  $I$  are equal, then the maximal minors of the Jacobian dual  $B(\varphi)$  contribute some non-trivial non-linear defining equations of the Rees algebra  $\mathcal{R}(I)$ .

**Lemma 2.7.** *If  $\dim \mathcal{S}(I) = \mu(I)$ , then  $I_d(B(\varphi)) \neq 0$ .*

**Proof.** Let  $M = \text{coker}(B(\varphi))$ . We have a presentation

$$k[\underline{t}]^m \xrightarrow{B(\varphi)} k[\underline{t}]^d \longrightarrow M \longrightarrow 0.$$

Then

$$\mathcal{S}_{k[\underline{t}]}(M) \cong k[\underline{t}][\underline{x}]/(\underline{x} \cdot B(\varphi)).$$

On the other hand, since  $\underline{t} \cdot \varphi = \underline{x} \cdot B(\varphi)$ , we can rewrite

$$\mathcal{S}(I) \cong k[\underline{x}][\underline{t}]/(\underline{t} \cdot \varphi) \cong k[\underline{t}][\underline{x}]/(\underline{x} \cdot B(\varphi)).$$

Therefore  $\mathcal{S}(I) \cong \mathcal{S}_{k[\underline{t}]}(M)$  and hence by Huneke–Rossi formula [10],

$$n = \dim S_{k[\underline{T}]}(M) \geq \dim k[\underline{T}] + \mu(M_{(0)}) = n + \mu(M_{(0)}).$$

It follows that  $\mu(M_{(0)}) = 0$  and hence by Proposition 2.1,  $(0) \notin V(F_0(M))$  or equivalently  $I_d(B(\varphi)) \not\subseteq (0)$ .  $\square$

Finally, we mention briefly the notion of linkage together with some results which we shall use later. We will assume that  $R$  is a Cohen–Macaulay local ring. Two non-trivial ideals  $I$  and  $J$  of  $R$  are *linked* if there exists an  $R$ -regular sequence  $\underline{\alpha} = \alpha_1, \dots, \alpha_g \subseteq I \cap J$  such that  $I = (\underline{\alpha}) : J$  and  $J = (\underline{\alpha}) : I$ . In this case, we write  $I \stackrel{(\underline{\alpha})}{\sim} J$  or  $I \sim J$  when  $\underline{\alpha}$  is clear in the context. When  $I$  and  $J$  are linked and  $\text{grade}(I + J) > g$ , we say  $I$  and  $J$  are *geometrically linked*. One can show that if  $I \stackrel{(\underline{\alpha})}{\sim} J$ , then  $I$  and  $J$  are geometrically linked if and only if  $I \cap J = (\underline{\alpha})$  [15, Remark 3(b)]. Two linked ideals share many common properties, see [5] or [17].

The following result gives a class of examples of linked ideals in the case  $R$  is a Gorenstein local ring.

**Proposition 2.8.** (See [15, Proposition 4, p. 47].) *Assume that  $R$  is a local Gorenstein ring,  $I$  is an unmixed  $R$ -ideal of grade  $g$  (i.e.,  $\dim R_{\varphi} = g$  for every  $\varphi \in \text{Ass}(R/I)$ ),  $\underline{\alpha} = \alpha_1, \dots, \alpha_g$  an  $R$ -regular sequence,  $(\underline{\alpha}) \subsetneq I$ , and  $J = (\underline{\alpha}) : I$ . Then  $I \sim J$ .*

The ideal  $I$  is called in the *linkage class of a complete intersection*, or *licci*, if  $I$  is a complete intersection or there exists a chain  $I = I_0 \sim I_1 \sim \dots \sim I_n$ , where  $I_n$  is a complete intersection. For height 2 licci ideals there is the following simple characterization.

**Theorem 2.9.** (See [1, 6].) *Let  $R$  be a local Gorenstein ring and  $I$  an  $R$ -ideal of height 2. Then  $I$  is licci if and only if  $I$  is perfect.*

Recall that an ideal  $I = (f_1, \dots, f_n)$  is *strongly Cohen–Macaulay* if the Koszul homology module  $H_i(f_1, \dots, f_n)$  is Cohen–Macaulay for all  $0 \leq i \leq n$  [9]. The definition does not depend on the choice of a set of generators for  $I$ . Huneke proved that licci ideals are strongly Cohen–Macaulay.

**Theorem 2.10.** (See [9, Theorem 14].) *Let  $R$  be a local Gorenstein ring and  $I$  a licci  $R$ -ideal. Then  $I$  is strongly Cohen–Macaulay.*

### 3. The presentation matrix $\varphi$

In this section, we will describe the form of the presentation matrix of the ideals in the following setting.

**Setting 3.1.** *Let  $R = k[x_1, \dots, x_d]$  be a polynomial ring over an algebraically field  $k$  with the maximal homogeneous ideal  $\mathfrak{m} = (x_1, \dots, x_d)$ . Let  $I$  be a height 2 perfect homogeneous  $R$ -ideal with a linear presentation matrix  $\varphi$  with  $I_1(\varphi) = \mathfrak{m}$ . Assume that  $\mu(I) = n = d + 1$ , and  $I$  satisfies  $G_{d-1}$  but does not satisfies  $G_d$ .*

Note that we can apply Theorem 1.1 if  $I$  satisfies  $G_d$ . Obviously,  $G_d$  is satisfied if  $d = 2$ . Hence  $d \geq 3$  if  $I$  does not satisfy  $G_d$ .

The Artin–Nagata conditions help us to specify where  $I$  fails to be of linear type.

**Proposition 3.2.** *Let  $I$  and  $\varphi$  be as in the Setting 3.1. Then  $I_\varphi$  is of linear type for  $\varphi \in V(I) \setminus V(I_2(\varphi))$ .*

**Proof.** By Proposition 2.1,  $\{\varphi \in V(I) \mid \mu(I_\varphi) \geq d\} = V(I_2(\varphi))$  which implies that  $I_\varphi$  satisfies  $G_\infty$  for  $\varphi \in V(I) \setminus V(I_2(\varphi))$ . Since  $I$  is height 2 perfect, it is licci by Theorem 2.9. Therefore it is strongly Cohen–Macaulay by Theorem 2.10 and hence  $I_\varphi$  is strongly Cohen–Macaulay. Using [7, Theorem 9.1], we can conclude that  $I_\varphi$  is of linear type for every  $\varphi \in V(I) \setminus V(I_2(\varphi))$ .  $\square$

**Lemma 3.3.** *Let  $\varphi$  be as in Setting 3.1. After a possible change of coordinates and elementary row and column operations,  $\varphi$  has the form*

$$\varphi = \begin{pmatrix} * & \dots & * & x_d \\ & & & * \\ & & & \vdots \\ & \varphi' & & * \end{pmatrix}, \quad (3.3.1)$$

where  $\varphi'$  is a  $d$  by  $(d-1)$  matrix, and the entries of  $\varphi'$  and the “\*” entries are linear in  $k[x_1, \dots, x_{d-1}]$ .

**Proof.** Since  $I$  satisfies  $G_{d-1}$  but does not satisfy  $G_d$ ,  $\text{ht } I_2(\varphi) = d-1$  by Corollary 2.2. Let  $\varphi \in \text{Min}(I_2(\varphi))$ , then  $\varphi$  is a homogeneous prime ideal of height  $d-1$ . After a possible

change of coordinates, we can assume that  $\wp = (x_1, \dots, x_{d-1})$  since  $k$  is algebraically closed. On the other hand, by our assumption,  $\text{ht } I_1(\varphi) = d$ , so we can assume that  $a_{1d} \notin \wp = (x_1, \dots, x_{d-1})$ . We can assume furthermore that  $a_{1d} = x_d$ .

Let  $\varphi_1$  be the matrix with linear entries in  $k[x_d]$  that is obtained by reducing the matrix  $\varphi$  modulo the ideal  $\wp = (x_1, \dots, x_{d-1})$ . Write  $\varphi_1 = x_d \varphi_2$ , where  $\varphi_2$  has entries in  $k$ . Since  $\wp \in \text{Min}(I_2(\varphi))$ ,  $\mu(I_\wp) > d-1$  by Proposition 2.1. But by Nakayama's lemma,  $\mu(I_\wp) = \mu((I/\wp I)_\wp)$ . From linear algebra we know that

$$\mu((I/\wp I)_\wp) = (d+1) - \text{rank}_{k(x_d)} \varphi_1 = (d+1) - \text{rank}_{k(x_d)} \varphi_2 = (d+1) - \text{rank}_k \varphi_2.$$

Since we have  $\mu(I_\wp) > d-1$ ,  $\text{rank}_k \varphi_2 \leq 1$ . On the other hand,  $\text{rank}_k \varphi_2 \neq 0$  by the assumption that  $I_1(\varphi) = \mathfrak{m}$ . Therefore  $\text{rank}_k \varphi_2 = 1$ , which implies that  $\varphi$  has the asserted form.  $\square$

**Lemma 3.4.** *Let  $\varphi'$  be the matrix as in Lemma 3.3 and  $R' = k[x_1, \dots, x_{d-1}]$ . Then  $\text{ht}_{R'} I_2(\varphi') \geq d-2$ .*

**Proof.** Let  $\wp' \in V(I_2(\varphi')) \subseteq \text{Spec}(R')$ . First we notice that  $\wp'R \in \text{Spec}(R)$  and  $\text{ht}_R(\wp'R) = \text{ht}_{R'} \wp'$ .

If  $d = 3$ , we need to show that  $\text{ht}_{R'} I_2(\varphi') \geq 1$ . Suppose that  $\text{ht}_{R'} I_2(\varphi') = 0$ . Then  $I_2(\varphi') = 0$  and hence the first generator  $f_1$  of the ideal  $I$  would be 0. This is a contradiction. Therefore  $\text{ht}_{R'} I_2(\varphi') \geq 1$ .

If  $d \geq 4$ , it is clear that  $I_4(\varphi) \subseteq I_2(\varphi')R \subseteq \wp'R$ . Since  $I$  satisfies condition  $G_{d-1}$ , it follows from Corollary 2.2 that  $\text{ht } I_4(\varphi) \geq d-2$ . Therefore we are done.  $\square$

In Proposition 3.2, we know where to look for the primes  $\wp$  such that  $I_\wp$  is not of linear type. The next result shows that we have exactly one such prime. Let  $\text{Min}(J)$  denote the set of minimal primes of  $J$ .

**Proposition 3.5.** *With assumptions as in Setting 3.1,  $\text{Min}(I_2(\varphi))$  has only one prime.*

**Proof.** Let  $\wp \in \text{Min}(I_2(\varphi))$ . We have  $\text{ht } \wp \geq d-1$ . If  $\text{ht } \wp = d$ , then  $\wp$  would be the homogeneous maximal ideal and would not be minimal. Therefore  $\text{ht } \wp = d-1$ .

From Lemma 3.4,  $d-2 \leq \text{ht}_{R'} I_2(\varphi') \leq d-1$ .

*Case 1:*  $\text{ht}_{R'} I_2(\varphi') = d-1$ . It is clear that  $(x_1, \dots, x_{d-1})R'$  is the only minimal prime of  $I_2(\varphi')$ . Therefore  $(x_1, \dots, x_{d-1})R \subseteq \wp$  as  $R$ -ideals and then we get the equality.

*Case 2:*  $\text{ht}_{R'} I_2(\varphi') = d-2$ . Let  $\wp' \in \text{Min}_{R'}(I_2(\varphi'))$ . If  $\text{ht } \wp' = d-1$ , then  $\wp' = (x_1, \dots, x_{d-1})R'$  and therefore  $\wp'$  would not be minimal over  $I_2(\varphi')$ . Therefore all the minimal primes of  $I_2(\varphi')$  in  $R'$  have height  $d-2$ .

We have  $I_2(\varphi')R' \subseteq \wp \cap R'$  and then  $\wp \cap R'$  contains a minimal prime of  $I_2(\varphi')$ . Therefore we can assume that  $(x_1, \dots, x_{d-2})R' \subseteq \wp \cap R'$  and hence  $(x_1, \dots, x_{d-2})R \subseteq \wp$ .

The entries of the matrix  $\varphi'$  in (3.3.1) generate the ideal  $(x_1, \dots, x_{d-1})R'$  since if not,  $I_3(\varphi)$  would lie in a prime ideal generated by  $d-2$  variables and therefore its height



would be at most  $d-2$ , which is a contradiction according to [Corollary 2.2](#). After possibly permuting the last  $d$  rows and the first  $d-1$  columns, and after a change of coordinates that does not change the ideals  $(x_1, \dots, x_{d-2})$ ,  $(x_1, \dots, x_{d-1})$ , the matrix  $\varphi$  has the form:

$$\varphi = \begin{pmatrix} * & \dots & * & \bullet & x_d \\ \bullet & \dots & \bullet & x_{d*1} & \bullet \\ * & \dots & * & \bullet & * \\ \vdots & \dots & \vdots & \vdots & \vdots \\ * & \dots & * & \bullet & * \end{pmatrix}, \quad (3.5.1)$$

where the “\*” entries are linear in  $x_1, \dots, x_{d-1}$  and the “•” entries are linear in  $x_1, \dots, x_{d-2}$ . Since  $I_3(\varphi) \not\subseteq (x_1, \dots, x_{d-2})$ , one of the “\*” entries in (3.5.1) is a linear combination of  $x_1, \dots, x_{d-1}$ , but not of  $x_1, \dots, x_{d-2}$ . Suppose that  $a_{3,d-2}$  is that element. Since  $a_{2,d-2}a_{3,d-1} - a_{2,d-1}a_{3,d-2} \in \wp$  and  $a_{2,d-2}, a_{3,d-1} \in (x_1, \dots, x_{d-2}) \subseteq \wp$ ,  $a_{2,d-1}a_{3,d-2} \in \wp$  and then  $x_{d-1} \in \wp$ . Therefore  $\wp = (x_1, \dots, x_{d-1})$ .  $\square$

We provide an example to illustrate for [Lemma 3.3](#) and [Proposition 3.5](#).

**Example 3.6.** Let

$$\varphi = \begin{pmatrix} x & 0 & 0 \\ y & x & z \\ 0 & x+y & x+z \\ 0 & x & -z \end{pmatrix}.$$

After elementary row operations we can replace the above matrix by the following matrix

$$\varphi = \begin{pmatrix} 0 & -x & z \\ x & 0 & 0 \\ y & 2x & 0 \\ 0 & 2x+y & x \end{pmatrix}.$$

It is clear that  $\text{Min}(I_3(\varphi)) = \{(x, y), (x, z)\}$ , thus  $\text{ht } I_3(\varphi) = 2$ . The Hilbert–Burch Theorem [[5](#), [Theorem 20.15](#)] implies that  $I := I_3(\varphi)$  is a height 2 perfect ideal. It is also clear that  $\text{Min}(I_2(\varphi)) = \{(x, y)\}$  and hence  $\text{ht } I_2(\varphi) = 2$ . Therefore  $I$  satisfies  $G_2$ , but not  $G_3$ .

#### 4. The symmetric algebra

In this section we show that the symmetric algebra  $\mathcal{S}(I)$  is a complete intersection in [Setting 3.1](#). This fact will play a key role in the proof of [Theorem 1.2](#).

We write  $\varphi = (a_{ij})$  with  $1 \leq i \leq d+1$  and  $1 \leq j \leq d$ . The defining ideal  $\mathcal{J}$  of the symmetric algebra  $\mathcal{S}(I)$  is generated by  $d$  linear forms  $l_1, \dots, l_d$ , where

$$l_j = t_1 a_{1,j} + t_2 a_{2,j} + \dots + t_{d+1} a_{d+1,j}, 1 \leq j \leq d.$$

**Theorem 4.1.** *Let  $I$  be an ideal as in [Setting 3.1](#). Then  $\dim \mathcal{S}(I) = d + 1$ ,  $\text{ht } \mathcal{J} = d$  and the symmetric algebra  $\mathcal{S}(I)$  is a complete intersection.*

**Proof.** By Huneke–Rossi formula [[10, Theorem 2.6](#)], we have

$$\dim \mathcal{S}(I) = \sup_{\wp \in \text{Spec}(R)} \{ \dim R/\wp + \mu(I_\wp) \}.$$

If  $\wp = \mathfrak{m}$ , then

$$\dim R/\wp + \mu(I_\wp) = \dim R/\mathfrak{m} + \mu(I_\mathfrak{m}) = 0 + \mu(I) = n = d + 1.$$

If  $\wp \notin V(I)$  then  $\dim R/\wp + \mu(I_\wp) \leq d + 1$ . If  $\wp \in V(I)$  and  $\text{ht } \wp \leq d - 2$ , the condition  $G_{d-1}$  implies that

$$\dim R/\wp + \mu(I_\wp) \leq \dim R/\wp + \text{ht } \wp = d.$$

If  $\wp \in V(I)$  and  $\text{ht } \wp = d - 1$ , then

$$\dim R/\wp + \mu(I_\wp) \leq 1 + (d + 1) - 1 = d + 1$$

because  $I_1(\varphi) = \mathfrak{m}$  by our assumption. Therefore  $\dim \mathcal{S}(I) = d + 1$ . Since  $\mathcal{S}(I) \cong R[t_1, \dots, t_{d+1}]/(l_1, \dots, l_d)$ , it follows that  $\text{ht}(l_1, \dots, l_d) = d$ . Hence  $\mathcal{S}(I)$  is a complete intersection.  $\square$

As a consequence, we get a nonzero element in  $\mathcal{L}$  which is not in  $\mathcal{J}$ . That special element is  $\det(B(\varphi))$ . We shall see that  $\det(B(\varphi))$  is reducible and hence it has a non-trivial nonzero divisor which is also in  $\mathcal{L}$ . The proof uses the special form of the presentation matrix  $\varphi$ .

**Corollary 4.2.** *Let  $\varphi$  be the presentation matrix of  $I$  as in [Setting 3.1](#). Then  $0 \neq \det(B(\varphi)) = t_1 \cdot g \in \mathcal{L}$ , where  $g \neq 0$  is a homogeneous polynomial of degree  $d - 1$  in  $\mathcal{L} \cap k[t_1, \dots, t_n]$ .*

**Proof.** [Theorem 4.1](#), [Lemmas 2.7 and 2.5](#) imply directly that  $0 \neq \det(B(\varphi)) \in \mathcal{L}$ .

Now, according to [Lemma 3.3](#),  $\varphi$  has the form

$$\varphi = \begin{pmatrix} * & \dots & * & x_d \\ & & * & \\ & \varphi' & \vdots & \\ & & * & \end{pmatrix},$$

where  $\varphi'$  is a  $d$  by  $(d-1)$  matrix, and the entries of  $\varphi'$  and the “\*” entries are linear in  $k[x_1, \dots, x_{d-1}]$ . Therefore the Jacobian dual  $B(\varphi)$  has the form

$$B(\varphi) = \begin{pmatrix} b_{1,1} & \cdots & b_{1,d-1} & b_{1,d} \\ \cdots & \cdots & \cdots & \cdots \\ b_{d-1,1} & \cdots & b_{d-1,d-1} & b_{d-1,d} \\ 0 & \cdots & 0 & t_1 \end{pmatrix}. \quad (4.2.1)$$

Let

$$g = \begin{vmatrix} b_{1,1} & \cdots & b_{1,d-1} \\ \cdots & \cdots & \cdots \\ b_{d-1,1} & \cdots & b_{d-1,d-1} \end{vmatrix}. \quad (4.2.2)$$

Then  $g$  is a homogeneous polynomial of degree  $d-1$  in  $k[t_1, \dots, t_n]$ . From the facts that  $t_1 g = \det B(\varphi) \in \mathcal{L}$ ,  $\mathcal{L}$  is prime, and  $t_1 \notin \mathcal{L}$ , we get  $g \in \mathcal{L}$ .  $\square$

We can also use [Theorem 4.1](#) to show that  $g$  is irreducible. We need to recall the typical short exact sequence which is usually used to compare  $\mathcal{S}(I)$  and  $\mathcal{R}(I)$ :

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{S}(I) \xrightarrow{\alpha} \mathcal{R}(I) \longrightarrow 0.$$

This sequence is obtained by combining the polynomial presentations of  $\mathcal{S}(I)$  and  $\mathcal{R}(I)$ . In particular, we have  $\mathcal{A} \cong \mathcal{L}/\mathcal{J}$  and the ideal  $\mathcal{A}$  has a natural grading coming from  $t'_i$ s variables.

**Proposition 4.3.** *With assumptions as in [Setting 3.1](#),  $\mathcal{A}_t = 0$  if and only if  $t \leq d-2$ .*

**Proof.** We will show that  $\mathcal{S}_t(I)$  is torsion free for  $t \leq d-2$ . Then  $\mathcal{A}_t = 0$  for  $t \leq d-2$  since  $\mathcal{A}$  is the  $R$ -torsion of  $\mathcal{S}(I)$ . It is enough to show that  $\mathcal{S}_t(I)$  satisfies the Serre condition  $S_1$  for  $t \leq d-2$ . Let  $C = R[t_1, \dots, t_{d+1}]$  be the standard graded polynomial ring over  $R$  in the variables  $t_1, \dots, t_{d+1}$ . By [Theorem 4.1](#),  $\mathcal{S}(I)$  is a complete intersection,  $\mathcal{S}(I) \cong C/(l_1, \dots, l_d)$  where  $l_i = a_{1,i}t_1 + \dots + a_{d+1,i}t_{d+1}$  are a regular sequence of linear forms. Therefore the Koszul complex of  $l_1, \dots, l_d$  is the resolution of  $\mathcal{S}(I)$ :

$$0 \longrightarrow C(-d) \longrightarrow C(-d+1)^d \longrightarrow C(-d+2)^{d(d-1)/2} \longrightarrow \dots \longrightarrow C(-1)^d \longrightarrow C. \quad (4.3.1)$$

Hence the resolution of  $\mathcal{S}_t(I)$  for  $t \leq d-2$  is

$$0 \longrightarrow C_{t-d+2}^{d(d-1)/2} \longrightarrow \dots \longrightarrow C_{t-1}^d \longrightarrow C_t \longrightarrow \mathcal{S}_t(I) \longrightarrow 0.$$

For every  $\wp \in \text{Spec}(R)$ , the  $R_\wp$ -modules  $C_{i\wp}$  are free and hence  $\text{depth } C_{i\wp} \geq \dim R_\wp$ , for all  $i$ . Then the depth lemma shows that  $\text{depth } \mathcal{S}_t(I)_\wp \geq \dim R_\wp - d + 2$  whenever

$t \leq d - 2$ . In particular,  $\text{depth } S_t(I_\varphi) \geq 1$  whenever  $\dim R_\varphi \geq d - 1$  and  $t \leq d - 2$ . On the other hand,  $S_t(I_\varphi)$  is torsion free if  $\dim R_\varphi \leq d - 2$  according to Proposition 3.2. So we have proved that  $S_t(I)$  satisfies the Serre condition  $S_1$  and hence is torsion free for  $t \leq d - 2$ .

On the other hand,  $g \in \mathcal{L}$  is homogeneous polynomial of degree  $d - 1$  in  $k[t]$ . Therefore,  $\mathcal{A}_{d-1} \neq 0$ .  $\square$

**Corollary 4.4.** *The polynomial  $g$  in (4.2.2) is irreducible.*

**Proof.** Suppose that  $g = g_1 g_2$ , where  $g_i$  is a homogeneous polynomial in  $k[t]$  with  $r_i := \deg g_i < d - 1$  for  $i = 1, 2$ . Since  $g \in \mathcal{L}$  and  $\mathcal{L}$  is prime, we can suppose that  $g_1 \in \mathcal{L}$ . It is clear that  $g_1 \notin \mathcal{J}$ . Combining all the facts, we get  $0 \neq \overline{g_1} \in (\mathcal{L}/\mathcal{J})_{r_1} \cong \mathcal{A}_{r_1}$ . But this contradicts Proposition 4.3 since  $r_1 < d - 1$ .  $\square$

## 5. The Rees algebra

In this section, we compute the defining ideal  $\mathcal{L}$  of the Rees algebra  $\mathcal{R}(I)$  explicitly and show that it is Cohen–Macaulay in Setting 3.1. Furthermore, we compute the canonical module of the Rees algebra.

We want to show that  $\mathcal{L} = (\mathcal{J}, g)$ , where  $g$  is defined by (4.2.2). For that we shall need the following properties of  $(\mathcal{J}, g)$ .

**Theorem 5.1.** *With the assumptions as in Setting 3.1,  $(\mathcal{J}, g)$  is a Cohen–Macaulay ideal of height  $n - 1$ .*

**Proof.** From the special form of the Jacobian dual  $B$  (4.2.1), we can rewrite  $(\underline{t}) \cdot \varphi$  in a slightly different way  $(\underline{t}) \cdot \varphi = (x_1 x_2 \dots x_d t_1) \cdot B'$ , where

$$B' = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,d-1} & b_{1,d} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,d-1} & b_{2,d} \\ \cdots & & \cdots & & \cdots \\ b_{d-1,1} & b_{d-1,2} & \cdots & b_{d-1,d-1} & b_{d-1,d} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Therefore  $\mathcal{J} \subseteq (x_1, x_2, \dots, x_d t_1) =: \mathcal{K}$ . They are both complete intersections of height  $d$ . Then, [18, Theorem 2] implies that  $\mathcal{J} : \mathcal{K} = (\mathcal{J}, \det B')$ . But it is clear that  $\det B' = g$  and hence  $\mathcal{J} : \mathcal{K} = (\mathcal{J}, g)$ . Now,  $(\mathcal{J}, g)$  is linked to  $\mathcal{K}$  by Proposition 2.8. The Cohen–Macaulayness of  $(\mathcal{J}, g)$  follows from [5, Theorem 21.23].

To show that  $\text{ht}(\mathcal{J}, g) = n - 1$ , it is enough to show that  $\text{ht}(\mathcal{J}, g) \geq n - 1$  since  $(\mathcal{J}, g) \subseteq \mathcal{L}$  and  $\text{ht } \mathcal{L} = n - 1$ . By Theorem 4.1,  $\text{ht } \mathcal{J} = d$  and hence  $\text{ht}(\mathcal{J}, g) \geq \text{ht } \mathcal{J} = d = n - 1$ .  $\square$

**Theorem 5.2.** *With the assumptions as in [Setting 3.1](#),  $\mathcal{L} = (\mathcal{J}, g)$ .*

**Proof.** It is enough to check that  $(\mathcal{J}, g)_P = \mathcal{L}_P$  for every  $P \in \text{Ass}(\mathcal{J}, g)$ . If we can show that every associated prime of  $(\mathcal{J}, g)$  does not contain the minimal prime  $\mathfrak{q} = (x_1, \dots, x_{d-1})$  over  $I_2(\varphi)$ , which is unique according to [Theorem 3.5](#), then we are done by [Proposition 3.2](#).

Let  $P \in \text{Ass}(\mathcal{J}, g)$ . By [Theorem 5.1](#), we know that  $\text{ht } P = n - 1$ . We will show that  $\text{ht}(\mathfrak{q} + P) \geq n$ . This will imply that  $\mathfrak{q} \not\subseteq P$  and we are done. It is clear that  $(x_1, \dots, x_{d-1}, x_d t_1, g) \subseteq (x_1, \dots, x_{d-1}, P)$ . So it is enough to show that  $\text{ht}(x_1, \dots, x_{d-1}, x_d t_1, g) \geq n$  or equivalently to  $\text{ht}(t_1, g) \geq n - d + 1 = 2$ . But from [Proposition 4.3](#), we know that  $g$  is irreducible. Notice that  $g \neq t_1$  since  $g$  has degree  $d - 1$  which is greater than 1. Now, it is clear that  $\text{ht}(t_1, g) = 2$ .  $\square$

From the obtained form of  $\mathcal{L}$ , we can show that the fiber ring  $\mathcal{F}(I)$  has maximal dimension. Recall that  $\mathcal{F}(I) = \mathcal{R}(I) \otimes_R k \cong \bigoplus_{j \geq 0} I^j / \mathfrak{m} I^j$ . The Krull dimension of  $\mathcal{F}(I)$ , denoted by  $\ell(I)$ , is called the *analytic spread* of  $I$ . It is known that  $\ell(I) \leq \dim R$  [[17, Corollary 5.1.4](#)] and when equality holds, one says that  $I$  has *maximal analytic spread*. A polynomial presentation of  $\mathcal{F}(I)$  can be obtained by tensoring the polynomial presentation  $R[t_1, \dots, t_n] \twoheadrightarrow \mathcal{R}(I)$ ,  $t_i \mapsto f_i t$  with  $k$  over  $R$ ,

$$k[t_1, \dots, t_n] = R[t_1, \dots, t_n] \otimes_R k \twoheadrightarrow \mathcal{R}(I) \otimes_R k = \mathcal{F}(I).$$

The kernel of this map, denoted by  $\mathcal{N}$ , is called the defining ideal of the special fiber ring  $\mathcal{F}(I)$ .

**Corollary 5.3.** *With the assumptions of [Setting 3.1](#), the ideal  $I$  has maximal analytic spread.*

**Proof.** Since  $\mathcal{L} = (\mathcal{J}, g)$ , where  $g$  is a homogeneous polynomial in  $k[t_1, \dots, t_{d+1}]$ , it is clear that the defining ideal  $\mathcal{N}$  of the fiber ring  $\mathcal{F}(I)$  is generated by  $g$ . Therefore  $\mathcal{N}$  is an ideal of height 1 in  $k[t_1, \dots, t_{d+1}]$ . This implies that  $\ell(I) = d$ .  $\square$

By [Theorem 5.1](#) and [Theorem 5.2](#), the Rees algebra  $\mathcal{R}(I)$  is Cohen–Macaulay. Therefore, we can define its canonical module as  $\omega_{\mathcal{R}(I)} := \text{Ext}_C^n(\mathcal{R}(I), \omega_C)$ , where  $\omega_C$  is the canonical module of the polynomial ring  $C = k[x_1, \dots, x_d][t_1, \dots, t_{d+1}]$  [[3, Proposition 3.6.9\(b\), Proposition 3.6.12\(b\)](#)]. The canonical module  $\omega_{\mathcal{R}(I)}$  is a bigraded  $\mathcal{R}(I)$ -module with  $\deg(x_i) = (1, 0)$  and  $\deg(t_j) = (0, 1)$ . The canonical module encodes some important information on the structure of the base ring such as Gorensteinness. From the bigrading of the polynomial ring  $C$ , we can see that  $\omega_C \cong C(-d, -d - 1)$  [[3, Example 3.6.10](#)].

In our setting,  $\mathcal{L}$  is linked to a complete intersection  $\mathcal{K}$ . This fact allows us to compute the canonical module  $\omega_{\mathcal{R}(I)}$  explicitly.

**Theorem 5.4.** *With assumptions as in [Setting 3.1](#),*

$$\omega_{\mathcal{R}(I)} \cong (x_1, \dots, x_{d-1}, x_d f_1 t) f_1 t \mathcal{R}(I),$$

where  $f_1$  is the maximal minor of  $\varphi$  in [\(3.3.1\)](#) obtained by deleting the first row.

**Proof.** By [\[11, the proof of Lemma 2.3\]](#), we have  $\omega_{C/\mathcal{L}} \cong (\mathcal{K}\omega_C/\mathcal{J}\omega_C)(d, d)$ . But  $\omega_C \cong C(-d, -d-1)$  [\[3, Example 3.6.10\]](#). Therefore,

$$\omega_{C/\mathcal{L}} \cong (\mathcal{K}C/\mathcal{J})(-d+d, -d-1+d) = (\mathcal{K}C/\mathcal{J})(0, -1). \quad (5.4.1)$$

By [Theorem 5.2](#) and [Proposition 2.8](#)  $\mathcal{L}$  and  $\mathcal{K}$  are linked. Since  $\text{ht}_{\mathcal{R}(I)} \mathcal{K} > 0$ ,  $\mathcal{L}$  and  $\mathcal{K}$  are geometrically linked. Hence  $\mathcal{K} \cap \mathcal{L} = \mathcal{J}$  by [\[15, Remark 3\(b\)\]](#). We have

$$\mathcal{K}\mathcal{R}(I) \cong (\mathcal{K} + \mathcal{L})/\mathcal{L} \cong \mathcal{K}C/\mathcal{K} \cap \mathcal{L} = \mathcal{K}C/\mathcal{J}.$$

This and the isomorphism [\(5.4.1\)](#) imply that  $\omega_{C/\mathcal{L}} \cong \mathcal{K}\mathcal{R}(I)(0, -1)$ . Therefore

$$\omega_{\mathcal{R}(I)} = \omega_{C/\mathcal{L}} \cong \mathcal{K}f_1 t \mathcal{R}(I) \cong (x_1, \dots, x_{d-1}, x_d f_1 t) f_1 t \mathcal{R}(I). \quad \square$$

Now, one can see that  $\omega_{\mathcal{R}(I)} \not\cong \mathcal{R}(I)$  hence  $\mathcal{R}(I)$  is not Gorenstein [\[3, Proposition 3.6.11\]](#).

Finally, putting [Theorems 5.1, 5.2 and 5.4](#) together, we will obtain [Theorem 1.2](#).

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