



Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type E_7



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ABSTRACT

Let G be a connected reductive algebraic group defined over an algebraically closed field k . The aim of this paper is to present a method to find triples (G, M, H) with the following three properties. Property 1: G is simple and k has characteristic 2. Property 2: H and M are closed reductive subgroups of G such that $H < M < G$, and (G, M) is a reductive pair. Property 3: H is G -completely reducible, but not M -completely reducible. We exhibit our method by presenting a new example of such a triple in $G = E_7$. Then we consider a rationality problem and a problem concerning conjugacy classes as important applications of our construction.

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1. Introduction

Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic p . In [15, Sec. 3], J.P. Serre defined that a closed subgroup H of G is G -completely reducible (G -cr for short) if whenever H is contained in a parabolic subgroup P of G , H is contained in a Levi subgroup L of P . This is a faithful generalization

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of the notion of semisimplicity in representation theory since if $G = GL_n(k)$, a subgroup H of G is G -cr if and only if H acts completely reducibly on k^n [15, Ex. 3.2.2(a)]. It is known that if a closed subgroup H of G is G -cr, then H is reductive [15, Prop. 4.1]. Moreover, if $p = 0$, the converse holds [15, Prop. 4.2]. Therefore the notion of G -complete reducibility is not interesting if $p = 0$. In this paper, we assume that $p > 0$.

Completely reducible subgroups of connected reductive algebraic groups have been much studied [9,10,15]. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful [1–3]. In this paper, we see another application of GIT to complete reducibility (Proposition 3.6).

Here is the main problem we consider. Let H and M be closed reductive subgroups of G such that $H \leq M \leq G$. It is natural to ask whether H being M -cr implies that H is G -cr and vice versa. It is not difficult to find a counterexample for the forward direction. For example, take $H = M = PGL_2(k)$ and $G = SL_3(k)$ where $p = 2$ and H sits inside G via the adjoint representation. Another such example is [1, Ex. 3.45]. For many examples where H and M are connected with H being M -cr and M being G -cr, but not G -cr, even when each group is simple, see [18]. However, it is hard to get a counterexample for the reverse direction, and it necessarily involves a small p . In [3, Sec. 7], Bate et al. presented the only known counterexample for the reverse direction where $p = 2$, $H \cong S_3$, $M \cong A_1A_1$, and $G = G_2$, which we call “the G_2 example”. The aim of this paper is to prove the following.

Theorem 1.1. *Let G be a simple algebraic group of type E_7 defined over k of characteristic $p = 2$. Then there exist a connected reductive subgroup M of type A_7 of G and a reductive subgroup $H \cong D_{14}$ (the dihedral group of order 14) of M such that (G, M) is a reductive pair and H is G -cr but not M -cr.*

Our work is motivated by [3]. We recall a few relevant definitions and results here. We denote the Lie algebra of G by $\text{Lie } G = \mathfrak{g}$. From now on, by a subgroup of G , we always mean a closed subgroup of G .

Definition 1.2. Let H be a subgroup of G acting on G by inner automorphisms. Let H act on \mathfrak{g} by the corresponding adjoint action. Then H is called *separable* if $\text{Lie } C_G(H) = c_{\mathfrak{g}}(H)$.

Recall that we always have $\text{Lie } C_G(H) \subseteq c_{\mathfrak{g}}(H)$. In [3], Bate et al. investigated the relationship between G -complete reducibility and separability, and showed the following [3, Thm. 1.2, Thm. 1.4].

Proposition 1.3. *Suppose that p is very good for G . Then any subgroup of G is separable in G .*

Proposition 1.4. *Suppose that (G, M) is a reductive pair. Let H be a subgroup of M such that H is a separable subgroup of G . If H is G -cr, then it is also M -cr..*

Recall that a pair of reductive groups G and M is called a *reductive pair* if $\mathrm{Lie} M$ is an M -module direct summand of \mathfrak{g} . This is automatically satisfied if $p = 0$. Propositions 1.3 and 1.4 imply that the subgroup H in Theorem 1.1 must be non-separable, which is possible for small p only.

Now, we introduce the key notion of *separable action*, which is a slight generalization of the notion of a separable subgroup.

Definition 1.5. Let H and N be subgroups of G where H acts on N by group automorphisms. The action of H is called *separable* in N if $\mathrm{Lie} C_N(H) = \mathfrak{c}_{\mathrm{Lie} N}(H)$. Note that the condition means that the fixed points of H acting on N , taken with their natural scheme structure, are smooth.

Here is a brief sketch of our method. *Note that in our construction, p needs to be 2.*

1. Pick a parabolic subgroup P of G with a Levi subgroup L of P . Find a subgroup K of L such that K acts non-separably on the unipotent radical $R_u(P)$ of P . In our case, K is generated by elements corresponding to certain reflections in the Weyl group of G .
2. Conjugate K by a suitable element v of $R_u(P)$, and set $H = vKv^{-1}$. Then choose a connected reductive subgroup M of G such that H is not M -cr. Use a recent result from GIT (Proposition 2.4) to show that H is not M -cr. Note that K is M -cr in our case.
3. Prove that H is G -cr.

Remark 1.6. It can be shown using [17, Thm. 13.4.2] that K in Step 1 is a non-separable subgroup of G .

First of all, for Step 1, p cannot be very good for G by Propositions 1.3 and 1.4. It is known that 2 and 3 are bad for E_7 . We explain the reason why we choose $p = 2$, not $p = 3$ (Remark 2.9). Remember that the non-separable action on $R_u(P)$ was the key ingredient for the G_2 example to work. Since K is isomorphic to a subgroup of the Weyl group of G , we are able to turn a problem of non-separability into a purely combinatorial problem involving the root system of G (Section 3.1). Regarding Step 2, we explain the reason of our choice of v and M explicitly (Remarks 3.4, 3.5). Our use of Proposition 2.4 gives an improved way for checking G -complete reducibility (Remark 3.7). Finally, Step 3 is easy.

In the G_2 and E_7 examples, the G -cr and non- M -cr subgroups H are finite. The following is the only known example of a triple (G, M, H) with positive dimensional H such that H is G -cr but not M -cr. It is obtained by modifying [1, Ex. 3.45].

Example 1.7. Let $p = 2$, $m \geq 4$ be even, and $(G, M) = (GL_{2m}(k), Sp_{2m}(k))$. Let H be a copy of $Sp_m(k)$ diagonally embedded in $Sp_m(k) \times Sp_m(k)$. Then H is not M -cr by the

argument in [1, Ex. 3.45]. But H is G -cr since H is $GL_m(k) \times GL_m(k)$ -cr by [1, Lem. 2.12]. Also note that any subgroup of $GL(k)$ is separable in $GL(k)$ (cf. [1, Ex. 3.28]), so (G, M) is not a reductive pair by Proposition 1.4.

In view of this, it is natural to ask:

Open Problem 1.8. Is there a triple $H < M < G$ of connected reductive algebraic groups such that (G, M) is a reductive pair, H is non-separable in G , and H is G -cr but not M -cr?

Beyond its intrinsic interest, our E_7 example has some important consequences and applications. For example, in Section 4, we consider a rationality problem concerning complete reducibility. We need a definition first to explain our result there.

Definition 1.9. Let k_0 be a subfield of an algebraically closed field k . Let H be a k_0 -defined closed subgroup of a k_0 -defined reductive algebraic group G . Then H is called *G -cr over k_0* if whenever H is contained in a k_0 -defined parabolic subgroup P of G , it is contained in some k_0 -defined Levi subgroup of P .

Note that if k_0 is algebraically closed then G -cr over k_0 means G -cr in the usual sense. Here is the main result of Section 4.

Theorem 1.10. Let k_0 be a nonperfect field of characteristic $p = 2$, and let G be a k_0 -defined split simple algebraic group of type E_7 . Then there exists a k_0 -defined subgroup H of G such that H is G -cr over k , but not G -cr over k_0 .

As another application of the E_7 example, we consider a problem concerning conjugacy classes. Given $n \in \mathbb{N}$, we let G act on G^n by simultaneous conjugation:

$$g \cdot (g_1, g_2, \dots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \dots, gg_ng^{-1}).$$

In [16], Slodowy proved the following fundamental result applying Richardson's tangent space argument, [12, Sec. 3], [13, Lem. 3.1].

Proposition 1.11. Let M be a reductive subgroup of a reductive algebraic group G defined over k . Let $n \in \mathbb{N}$, let $(m_1, \dots, m_n) \in M^n$ and let H be the subgroup of M generated by m_1, \dots, m_n . Suppose that (G, M) is a reductive pair and that H is separable in G . Then the intersection $G \cdot (m_1, \dots, m_n) \cap M^n$ is a finite union of M -conjugacy classes.

Proposition 1.11 has many consequences. See [1, 16], and [19, Sec. 3] for example. In [3, Ex. 7.15], Bate et al. found a counterexample for $G = G_2$ showing that Proposition 1.11 fails without the separability hypothesis. In Section 5, we present a new counterexample to Proposition 1.11 without the separability hypothesis. Here is the main result of Section 5.

Theorem 1.12. *Let G be a simple algebraic group of type E_7 defined over an algebraically closed k of characteristic $p = 2$. Let M be the connected reductive subsystem subgroup of type A_7 . Then there exist $n \in \mathbb{N}$ and a tuple $\mathbf{m} \in M^n$ such that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of M -conjugacy classes. Note that (G, M) is a reductive pair in this case.*

Now, we give an outline of the paper. In Section 2, we fix our notation which follows [4,8,17]. Also, we recall some preliminary results, in particular, Proposition 2.4 from GIT. After that, in Section 3, we prove our main result, Theorem 1.1. Then in Section 4, we consider a rationality problem, and prove Theorem 1.10. Finally, in Section 5, we discuss a problem concerning conjugacy classes, and prove Theorem 1.12.

2. Preliminaries

2.1. Notation

Throughout the paper, we denote by k an algebraically closed field of positive characteristic p . We denote the multiplicative group of k by k^* . We use a capital roman letters, G, H, K , etc., to represent an algebraic group, and the corresponding lowercase gothic letters, $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$, etc., to represent its Lie algebra. We sometimes use another notation for Lie algebras: $\mathrm{Lie} G, \mathrm{Lie} H$, and $\mathrm{Lie} K$ are the Lie algebras of G, H , and K respectively.

We denote the identity component of G by G° . We write $[G, G]$ for the derived group of G . The unipotent radical of G is denoted by $R_u(G)$. An algebraic group G is *reductive* if $R_u(G) = \{1\}$. In particular, G is *simple* as an algebraic group if G is connected and all proper normal subgroups of G are finite.

In this paper, when a subgroup H of G acts on G , H always acts on G by inner automorphisms. The adjoint representation of G is denoted by $\mathrm{Ad}_{\mathfrak{g}}$ or just Ad if no confusion arises. We write $C_G(H)$ and $\mathfrak{c}_{\mathfrak{g}}(H)$ for the global and the infinitesimal centralizers of H in G and \mathfrak{g} respectively. We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of G respectively.

2.2. Complete reducibility and GIT

Let G be a connected reductive algebraic group. We recall Richardson's formalism [14, Secs. 2.1–2.3] for the characterization of a parabolic subgroup P of G , a Levi subgroup L of P , and the unipotent radical $R_u(P)$ of P in terms of a cocharacter of G and state a result from GIT (Proposition 2.4).

Definition 2.1. Let X be an affine variety. Let $\phi : k^* \rightarrow X$ be a morphism of algebraic varieties. We say that $\lim_{a \rightarrow 0} \phi(a)$ exists if there exists a morphism $\hat{\phi} : k \rightarrow X$ (necessarily unique) whose restriction to k^* is ϕ . If this limit exists, we set $\lim_{a \rightarrow 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let λ be a cocharacter of G . Define $P_\lambda := \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = 1\}$.

Note that P_λ is a parabolic subgroup of G , L_λ is a Levi subgroup of P_λ , and $R_u(P_\lambda)$ is a unipotent radical of P_λ [14, Secs. 2.1–2.3]. By [17, Prop. 8.4.5], any parabolic subgroup P of G , any Levi subgroup L of P , and any unipotent radical $R_u(P)$ of P can be expressed in this form. It is well known that $L_\lambda = C_G(\lambda(k^*))$.

Let M be a reductive subgroup of G . Then, there is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. Let $\lambda \in Y(M)$. We write $P_\lambda(G)$ or just P_λ for the parabolic subgroup of G corresponding to λ , and $P_\lambda(M)$ for the parabolic subgroup of M corresponding to λ . It is obvious that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

Definition 2.3. Let $\lambda \in Y(G)$. Define a map $c_\lambda : P_\lambda \rightarrow L_\lambda$ by $c_\lambda(g) := \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1}$.

Note that the map c_λ is the usual canonical projection from P_λ to $L_\lambda \cong P_\lambda/R_u(P_\lambda)$. Now, we state a result from GIT (see [1, Lem. 2.17, Thm. 3.1], [2, Thm. 3.3]).

Proposition 2.4. Let H be a subgroup of G . Let λ be a cocharacter of G with $H \subseteq P_\lambda$. If H is G -cr, there exists $v \in R_u(P_\lambda)$ such that $c_\lambda(h) = v h v^{-1}$ for every $h \in H$.

2.3. Root subgroups and root subspaces

Let G be a connected reductive algebraic group. Fix a maximal torus T of G . Let $\Psi(G, T)$ denote the set of roots of G with respect to T . We sometimes write $\Psi(G)$ for $\Psi(G, T)$. Fix a Borel subgroup B containing T . Then $\Psi(B, T) = \Psi^+(G)$ is the set of positive roots of G defined by B . Let $\Sigma(G, B) = \Sigma$ denote the set of simple roots of G defined by B . Let $\zeta \in \Psi(G)$. We write U_ζ for the corresponding root subgroup of G and \mathfrak{u}_ζ for the Lie algebra of U_ζ . We define $G_\zeta := \langle U_\zeta, U_{-\zeta} \rangle$.

Let H be a subgroup of G normalized by some maximal torus T of G . Consider the adjoint representation of T on \mathfrak{h} . The root spaces of \mathfrak{h} with respect to T are also root spaces of \mathfrak{g} with respect to T , and the set of roots of H relative to T , $\Psi(H, T) = \Psi(H) = \{\zeta \in \Psi(G) \mid \mathfrak{g}_\zeta \subseteq \mathfrak{h}\}$, is a subset of $\Psi(G)$.

Let $\zeta, \xi \in \Psi(G)$. Let ξ^\vee be the coroot corresponding to ξ . Then $\zeta \circ \xi^\vee : k^* \rightarrow k^*$ is a homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee \rangle := n$. Let s_ξ denote the reflection corresponding to ξ in the Weyl group of G . Each s_ξ acts on the set of roots $\Psi(G)$ by the following formula [17, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee \rangle \xi$. By [5, Prop. 6.4.2, Lem. 7.2.1], we can choose homomorphisms $\epsilon_\zeta : k \rightarrow U_\zeta$ so that

$$n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a), \quad \text{where } n_\xi = \epsilon_\xi(1) \epsilon_{-\xi}(-1) \epsilon_\xi(1). \quad (2.1)$$

We define $e_\zeta := \epsilon'_\zeta(0)$. Then we have

$$\mathrm{Ad}(n_\xi)e_\zeta = \pm e_{s_\xi \cdot \zeta}. \quad (2.2)$$

Now, we list four lemmas which we need in our calculations. The first one is [17, Prop. 8.2.1].

Lemma 2.5. *Let P be a parabolic subgroup of G . Any element u in $R_u(P)$ can be expressed uniquely as*

$$u = \prod_{i \in \Psi(R_u(P))} \epsilon_i(a_i), \quad \text{for some } a_i \in k,$$

where the product is taken with respect to a fixed ordering of $\Psi(R_u(P))$.

The next two lemmas [8, Lem. 32.5 and Lem. 33.3] are used to calculate $C_{R_u(P)}(K)$.

Lemma 2.6. *Let $\xi, \zeta \in \Psi(G)$. If no positive integral linear combination of ξ and ζ is a root of G , then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a).$$

Lemma 2.7. *Let Ψ be the root system of type A_2 spanned by roots ξ and ζ . Then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a)\epsilon_{\xi+\zeta}(\pm ab).$$

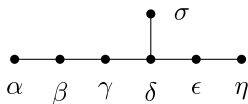
The last result is used to calculate $\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K)$.

Lemma 2.8. *Suppose that $p = 2$. Let W be a subgroup of G generated by all the n_ξ where $\xi \in \Psi(G)$ (the group W is isomorphic to the Weyl group of G). Let K be a subgroup of W . Let $\{O_i \mid i = 1 \cdots m\}$ be the set of orbits of the action of K on $\Psi(R_u(P))$. Then,*

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K) = \left\{ \sum_{i=1}^m a_i \sum_{\zeta \in O_i} e_\zeta \mid a_i \in k \right\}.$$

Proof. When $p = 2$, (2.2) yields $\mathrm{Ad}(n_\xi)e_\zeta = e_{n_\xi \cdot \zeta}$. Then an easy calculation gives the desired result. \square

Remark 2.9. Lemma 2.8 holds in $p = 2$ but fails in $p = 3$.

Fig. 1. Dynkin diagram of E_7 .

3. The E_7 example

3.1. Step 1

Let G be a simple algebraic group of type E_7 defined over k of characteristic 2. Fix a maximal torus T of G . Fix a Borel subgroup B of G containing T . Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ be the set of simple roots of G . Fig. 1 defines how each simple root of G corresponds to each node in the Dynkin diagram of E_7 .

From [6, Appendix, Table B], one knows the coefficients of all positive roots of G . We label all positive roots of G in Table 1 in Appendix A. Our ordering of roots is different from [6, Appendix, Table B], which will be convenient later on.

The set of positive roots is $\Psi^+(G) = \{1, 2, \dots, 63\}$. Note that $\{1, \dots, 35\}$ and $\{36, \dots, 42\}$ are precisely the roots of G such that the coefficient of σ is 1 and 2 respectively. We call the roots of the first type *weight-1 roots*, and the second type *weight-2 roots*. Define

$$L_{\alpha\beta\gamma\delta\epsilon\eta} := \langle T, G_{43}, \dots, G_{63} \rangle, \quad P_{\alpha\beta\gamma\delta\epsilon\eta} := \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, U_1, \dots, U_{42} \rangle.$$

Then $P_{\alpha\beta\gamma\delta\epsilon\eta}$ is a parabolic subgroup of G , and $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is a Levi subgroup of $P_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is of type A_6 . We have $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})) = \{1, \dots, 42\}$. Define

$$q_1 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta, \quad q_2 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta n_\eta n_\delta n_\beta, \quad K := \langle q_1, q_2 \rangle.$$

Let ζ_1, ζ_2 be simple roots of G . From the Cartan matrix of E_7 [7, Sec. 11.4] we have

$$\langle \zeta_1, \zeta_2 \rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2, \\ -1, & \text{if } \zeta_1 \text{ is adjacent to } \zeta_2 \text{ in the Dynkin diagram,} \\ 0, & \text{otherwise.} \end{cases}$$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee \rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ and for all $\zeta \in \Sigma$. These calculations show how $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon$, and n_η act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$. Let $\pi : \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \rangle \rightarrow \text{Sym}(\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))) \cong S_{42}$ be the corresponding homomorphism. Then we have

$$\begin{aligned} \pi(q_1) &= (1\ 2)(3\ 6)(4\ 7)(9\ 10)(11\ 12)(13\ 14)(15\ 20)(16\ 17)(18\ 21)(19\ 23)(22\ 25)(24\ 26) \\ &\quad (27\ 28)(29\ 32)(31\ 33)(34\ 35)(36\ 38)(37\ 39)(40\ 41), \\ \pi(q_2) &= (1\ 6\ 7\ 5\ 4\ 3\ 2)(8\ 10\ 12\ 14\ 13\ 11\ 9)(15\ 16\ 21\ 23\ 26\ 27\ 22)(17\ 20\ 25\ 28\ 24\ 19\ 18) \\ &\quad (29\ 30\ 32\ 33\ 35\ 34\ 31)(36\ 38\ 39\ 41\ 42\ 40\ 37). \end{aligned}$$

It is easy to see that $K \cong D_{14}$. The orbits of K in $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ are

$$\begin{aligned} O_1 &= \{1, \dots, 7\}, & O_8 &= \{8, \dots, 14\}, & O_{15} &= \{15, \dots, 28\}, \\ O_{29} &= \{29, \dots, 35\}, & O_{36} &= \{36, \dots, 42\}. \end{aligned}$$

Thus [Lemma 2.8](#) yields

Proposition 3.1.

$$\begin{aligned} \mathfrak{c}_{\text{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))(K)} &= \left\{ a \left(\sum_{\lambda \in O_1} e_\lambda \right) + b \left(\sum_{\lambda \in O_8} e_\lambda \right) + c \left(\sum_{\lambda \in O_{15}} e_\lambda \right) + d \left(\sum_{\lambda \in O_{29}} e_\lambda \right) \right. \\ &\quad \left. + m \left(\sum_{\lambda \in O_{36}} e_\lambda \right) \mid a, b, c, d, m \in k \right\}. \end{aligned}$$

The following is the most important technical result in this paper.

Proposition 3.2. *Let $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))(K)$. Then u must have the form,*

$$u = \prod_{i=1}^7 \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i) \quad \text{for some } a, b, c, a_i \in k.$$

Proof. By [Lemma 2.5](#), u can be expressed uniquely as $u = \prod_{i=1}^{42} \epsilon_i(b_i)$ for some $b_i \in k$. By (2.1), we have $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(a)$ for any $a \in k$ and $\xi, \zeta \in \Psi(G)$. Thus we have

$$\begin{aligned} q_1 u q_1^{-1} &= q_1 \left(\prod_{i=1}^{42} \epsilon_i(b_i) \right) q_1^{-1} \\ &= \left(\prod_{i=1}^7 \epsilon_{q_1 \cdot i}(b_i) \right) \left(\prod_{i=8}^{14} \epsilon_{q_1 \cdot i}(b_i) \right) \left(\prod_{i=15}^{28} \epsilon_{q_1 \cdot i}(b_i) \right) \left(\prod_{i=29}^{35} \epsilon_{q_1 \cdot i}(b_i) \right) \\ &\quad \left(\prod_{i=36}^{42} \epsilon_{q_1 \cdot i}(b_i) \right). \end{aligned} \tag{3.1}$$

A calculation using the commutator relations ([Lemma 2.6](#) and [Lemma 2.7](#)) shows that

$$\begin{aligned} q_1 u q_1^{-1} &= \epsilon_1(b_2) \epsilon_2(b_1) \epsilon_3(b_6) \epsilon_4(b_7) \epsilon_5(b_5) \epsilon_6(b_3) \epsilon_7(b_4) \epsilon_8(b_8) \epsilon_9(b_{10}) \epsilon_{10}(b_9) \epsilon_{11}(b_{12}) \epsilon_{12}(b_{11}) \\ &\quad \epsilon_{13}(b_{14}) \epsilon_{14}(b_{13}) \epsilon_{15}(b_{20}) \epsilon_{16}(b_{17}) \epsilon_{17}(b_{16}) \epsilon_{18}(b_{21}) \epsilon_{19}(b_{23}) \epsilon_{20}(b_{15}) \epsilon_{21}(b_{18}) \\ &\quad \epsilon_{22}(b_{25}) \epsilon_{23}(b_{19}) \epsilon_{24}(b_{26}) \epsilon_{25}(b_{22}) \epsilon_{26}(b_{24}) \epsilon_{27}(b_{28}) \epsilon_{28}(b_{27}) \epsilon_{29}(b_{32}) \epsilon_{30}(b_{30}) \\ &\quad \epsilon_{31}(b_{33}) \epsilon_{32}(b_{29}) \epsilon_{33}(b_{31}) \epsilon_{34}(b_{35}) \epsilon_{35}(b_{34}) \\ &\quad \left(\prod_{i=36}^{41} \epsilon_i(a_i) \right) \epsilon_{42}(b_4 b_7 + b_{11} b_{12} + b_{22} b_{25} + b_{34} b_{35} + b_{42}) \quad \text{for some } a_i \in k. \end{aligned} \tag{3.2}$$

Since q_1 and q_2 centralize u , we have $b_1 = \cdots = b_7$, $b_8 = \cdots = b_{14}$, $b_{15} = \cdots = b_{28}$, $b_{29} = \cdots = b_{35}$. Set $b_1 = a$, $b_8 = b$, $b_{15} = c$, $b_{29} = d$. Then (3.2) simplifies to

$$q_1 u q_1^{-1} = \prod_{i=1}^7 \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(d) \left(\prod_{i=36}^{41} \epsilon_i(a_i) \right) \epsilon_{42}(a^2 + b^2 + c^2 + d^2 + b_{42}).$$

Since q_1 centralizes u , comparing the arguments of the ϵ_{42} term on both sides, we must have

$$b_{42} = a^2 + b^2 + c^2 + d^2 + b_{42},$$

which is equivalent to $a + b + c + d = 0$. Then we obtain the desired result. \square

Proposition 3.3. *K acts non-separably on $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$.*

Proof. In view of Proposition 3.1, it suffices to show that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \notin \text{Lie } C_{R_u(P_\lambda)}(K)$. Suppose the contrary. Since by [17, Cor. 14.2.7] $C_{R_u(P_\lambda)}(K)^\circ$ is isomorphic as a variety to k^n for some $n \in \mathbb{N}$, there exists a morphism of varieties $v : k \rightarrow C_{R_u(P_\lambda)}(K)^\circ$ such that $v(0) = 1$ and $v'(0) = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$. By Lemma 2.5, $v(a)$ can be expressed uniquely as $v(a) = \prod_{i=1}^{42} \epsilon_i(f_i(a))$ for some $f_i \in k[X]$. Differentiating the last equation, and evaluating at $a = 0$, we obtain $v'(0) = \sum_{i \in \{1, \dots, 42\}} (f_i)'(0) e_i$. Since $v'(0) = \sum_{i \in O_1} e_i$, we have

$$(f_i)'(0) = \begin{cases} 1 & \text{if } i \in O_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$f_i(a) = \begin{cases} a + g_i(a) & \text{if } i \in O_1, \\ g_i(a) & \text{otherwise,} \end{cases}$$

where $g_i \in k[X]$ has no constant or linear term.

Then from Proposition 3.2, we obtain $(a + g_1(a)) + g_8(a) + g_{15}(a) = g_{29}(a)$. This is a contradiction. \square

3.2. Step 2

Let $C_1 := \{\prod_{i=1}^7 \epsilon_i(a) \mid a \in k\}$, pick any $a \in k^*$, and let $v(a) := \prod_{i=1}^7 \epsilon_i(a)$. Now, set

$$H := v(a) K v(a)^{-1} = \langle q_1 \epsilon_{40}(a^2) \epsilon_{41}(a^2) \epsilon_{42}(a^2), q_2 \epsilon_{36}(a^2) \epsilon_{39}(a^2) \rangle,$$

$$M := \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, G_{36}, \dots, G_{42} \rangle.$$

Remark 3.4. By Proposition 3.1 and Proposition 3.2, the tangent space of C_1 at the identity, $T_1(C_1)$, is contained in $\mathfrak{c}_{\text{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))(K)}$ but not contained in $\text{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K))$. The element $v(a)$ can be any non-trivial element in C_1 .

Remark 3.5. In this case σ is the unique simple root not contained in $\Psi(L_{\alpha\beta\gamma\delta\epsilon\eta})$. M was chosen so that M is generated by a Levi subgroup $L_{\alpha\beta\gamma\delta\epsilon\eta}$ containing K and all root subgroups of σ -weight 2.

We have $H \subset M, H \not\subset L_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $\Psi(M) = \{\pm 36, \dots, \pm 63\}$. Since M is generated by all root subgroups of even σ -weight, it is easy to see that $\Psi(M)$ is a closed subsystem of $\Psi(G)$, thus M is reductive by [3, Lem. 3.9]. Note that M is of type A_7 .

Proposition 3.6. H is not M -cr.

Proof. Let $\lambda = 3\alpha^\vee + 6\beta^\vee + 9\gamma^\vee + 12\delta^\vee + 8\epsilon^\vee + 4\eta^\vee + 7\sigma^\vee$. We have

$$\begin{aligned} \langle \alpha, \lambda \rangle &= 0, & \langle \beta, \lambda \rangle &= 0, & \langle \gamma, \lambda \rangle &= 0, & \langle \delta, \lambda \rangle &= 0, \\ \langle \epsilon, \lambda \rangle &= 0, & \langle \eta, \lambda \rangle &= 0, & \langle \sigma, \lambda \rangle &= 2. \end{aligned}$$

So $L_{\alpha\beta\gamma\delta\epsilon\eta} = L_\lambda, P_{\alpha\beta\gamma\delta\epsilon\eta} = P_\lambda$.

It is easy to see that L_λ is of type A_6 , so $[L_\lambda, L_\lambda]$ is isomorphic to either SL_7 or PGL_7 . We rule out the latter. Pick $x \in k^*$ such that $x \neq 1, x^7 = 1$. Then $\lambda(x) \neq 1$ since $\sigma(\lambda(x)) = x^2 \neq 1$. Also, we have $\lambda(x) \in Z([L_\lambda, L_\lambda])$. Therefore $[L_\lambda, L_\lambda] \cong SL_7$. It is easy to check that the map $k^* \times [L_\lambda, L_\lambda] \rightarrow L_\lambda$ is separable, so we have $L_\lambda \cong GL_7$.

Let $c_\lambda : P_\lambda \rightarrow L_\lambda$ be the homomorphism as in Definition 2.3. In order to prove that H is not M -cr, by Theorem 2.4 it suffices to find a tuple $(h_1, h_2) \in H^2$ which is not $R_u(P_\lambda(M))$ -conjugate to $c_\lambda((h_1, h_2))$. Set $h_1 := v(a)q_1v(a)^{-1}, h_2 := v(a)q_2v(a)^{-1}$. Then

$$\begin{aligned} c_\lambda((h_1, h_2)) &= \lim_{x \rightarrow 0} (\lambda(x)q_1\epsilon_{40}(a^2)\epsilon_{41}(a^2)\epsilon_{42}(a^2)\lambda(x)^{-1}, \lambda(x)q_2\epsilon_{36}(a^2)\epsilon_{39}(a^2)\lambda(x)^{-1}) \\ &= (q_1, q_2). \end{aligned}$$

Now suppose that (h_1, h_2) is $R_u(P_\lambda(M))$ -conjugate to $c_\lambda((h_1, h_2))$. Then there exists $m \in R_u(P_\lambda(M))$ such that

$$mv(a)q_1v(a)^{-1}m^{-1} = q_1, \quad mv(a)q_2v(a)^{-1}m^{-1} = q_2.$$

Thus we have $mv(a) \in C_{R_u(P_\lambda)}(K)$. Note that $\Psi(R_u(P_\lambda(M))) = \{36, \dots, 42\}$. So, by Lemma 2.5, m can be expressed uniquely as $m := \prod_{i=36}^{42} \epsilon_i(a_i)$ for some $a_i \in k$. Then we have

$$mv(a) = \epsilon_1(a)\epsilon_2(a)\epsilon_3(a)\epsilon_4(a)\epsilon_5(a)\epsilon_6(a)\epsilon_7(a) \left(\prod_{i=36}^{42} \epsilon_i(a_i) \right) \in C_{R_u(P_\lambda)}(K).$$

This contradicts Proposition 3.2. \square

Remark 3.7. In [3, Sec. 7, Prop. 7.17], Bate et al. used [1, Lem. 2.17, Thm. 3.1] to turn a problem on M -complete reducibility into a problem involving M -conjugacy. We have used Proposition 2.4 to turn the same problem into a problem involving $R_u(P \cap M)$ -conjugacy, which is easier.

Remark 3.8. Instead of using C_1 to define $v(a)$, we can take $C_8 := \{\prod_{i=8}^{14} \epsilon_i(a) \mid a \in k\}$, $C_{15} := \{\prod_{i=15}^{28} \epsilon_i(a) \mid a \in k\}$, or $C_{29} := \{\prod_{i=29}^{35} \epsilon_i(a) \mid a \in k\}$. In each case, a similar argument goes through and gives rise to a different example with the desired property.

3.3. Step 3

Proposition 3.9. H is G -cr.

Proof. First note that H is conjugate to K , so H is G -cr if and only if K is G -cr. Then, by [1, Lem. 2.12, Cor. 3.22], it suffices to show that K is $[L_\lambda, L_\lambda]$ -cr. We can identify K with the image of the corresponding subgroup of S_7 under the permutation representation $\pi_1 : S_7 \rightarrow SL_7(k)$. It is easy to see that $K \cong D_{14}$. A quick calculation shows that this representation of D_{14} is a direct sum of a trivial 1-dimensional and 3 irreducible 2-dimensional subrepresentations. Therefore K is $[L_\lambda, L_\lambda]$ -cr. \square

4. A rationality problem

We prove Theorem 1.10. The key here is again the existence of a 1-dimensional curve C_1 such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\text{Lie}(R_u(P_\lambda))}(K)$ but not contained in $\text{Lie}(C_{R_u(P_\lambda)}(K))$. The same phenomenon was seen in the G_2 example.

Proof of Theorem 1.10. Let k_0 , k , and G be as in the hypothesis. We choose a k_0 -defined k_0 -split maximal torus T such that for each $\zeta \in \Psi(G)$ the corresponding root ζ , coroot ζ^\vee , and homomorphism ϵ_ζ are defined over k_0 . Since k_0 is not perfect, there exists $\tilde{a} \in k \setminus k_0$ such that $\tilde{a}^2 \in k_0$. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. Let

$$\begin{aligned} H &= \langle v(\tilde{a})q_1v(\tilde{a})^{-1}, v(\tilde{a})q_2v(\tilde{a})^{-1} \rangle \\ &= \langle q_1\epsilon_{40}(\tilde{a}^2)\epsilon_{41}(\tilde{a}^2)\epsilon_{42}(\tilde{a}^2), q_2\epsilon_{36}(\tilde{a}^2)\epsilon_{39}(\tilde{a}^2) \rangle. \end{aligned}$$

Now it is obvious that H is k_0 -defined. We already know that H is G -cr by Proposition 3.9. Since G and T are k_0 -split, P_λ and L_λ are k_0 -defined by [4, V.20.4, V.20.5]. Suppose that there exists a k_0 -Levi subgroup L' of P_λ such that L' contains H . Then there exists $w \in R_u(P_\lambda)(k_0)$ such that $L' = wL_\lambda w^{-1}$ by [4, V.20.5]. Then $w^{-1}Hw \subseteq L_\lambda$ and $v(\tilde{a})^{-1}Hv(\tilde{a}) \subseteq L_\lambda$. So we have $c_\lambda(w^{-1}hw) = w^{-1}hw$ and $c_\lambda(v(\tilde{a})^{-1}hv(\tilde{a})) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. We also have $c_\lambda(w) = c_\lambda(v(\tilde{a})) = 1$ since $w, v(\tilde{a}) \in R_u(P_\lambda)(k)$. Therefore we obtain $w^{-1}hw = c_\lambda(w^{-1}hw) = c_\lambda(h) = c_\lambda(v(\tilde{a})^{-1}hv(\tilde{a})) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. So we have $w = v(\tilde{a})z$ for some $z \in C_{R_u(P_\lambda)}(K)(k)$. By Proposition 3.2, z must have the form

$$z = \prod_{i=1}^7 \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i) \quad \text{for some } a, b, c, a_i \in k.$$

Then

$$\begin{aligned} w &= \left(\prod_{i=1}^7 \epsilon_i(\tilde{a}) \right) \prod_{i=1}^7 \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i) \\ &= \prod_{i=1}^7 \epsilon_i(\tilde{a}+a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(b_i) \quad \text{for some } b_i \in k. \end{aligned}$$

Since w is a k_0 -point, b , c , and $a+b+c$ all belong to k_0 , so $a \in k_0$. But $a+\tilde{a}$ belongs to k_0 as well, so $\tilde{a} \in k_0$. This is a contradiction. \square

Remark 4.1. As in Section 3, we can take $v(\tilde{a})$ from C_8 , C_{15} , or C_{29} . In each case, a similar argument goes through, and gives rise to a different example.

Remark 4.2. [1, Ex. 5.11] shows that there is a k_0 -defined subgroup of G of type A_n which is not G -cr over k even though it is G -cr over k_0 . Note that this example works for any $p > 0$.

5. A problem of conjugacy classes

We prove Theorem 1.11. Here, the key is again the existence of a 1-dimensional curve C_1 such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\text{Lie}(R_u(P_\lambda))}(K)$ but not contained in $\text{Lie}(C_{R_u(P_\lambda)}(K))$ as in the G_2 example. Let G , M , k be as in the hypotheses of the theorem. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. A calculation using the commutator relations (Lemma 2.6) shows that

$$Z(R_u(P_\lambda)) = \langle U_{36}, U_{37}, U_{38}, U_{39}, U_{40}, U_{41}, U_{42} \rangle.$$

Let $K_0 := \langle K, Z(R_u(P_\lambda)) \rangle$. It is standard that there exists a finite subset $F = \{z_1, z_2, \dots, z_{n'}\}$ of $Z(R_u(P))$ such that $C_{P_\lambda}(\langle K, F \rangle) = C_{P_\lambda}(K_0)$. Let $\mathbf{m} := (q_1, q_2, z_1, \dots, z_{n'})$. Let $n := n' + 2$. For every $x \in k^*$, define $\mathbf{m}(x) := v(x) \cdot \mathbf{m} \in P_\lambda(M)^n$.

Lemma 5.1. $C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0)$.

Proof. It is obvious that $C_{R_u(P_\lambda)}(K_0) \subseteq C_{P_\lambda}(K_0)$. We prove the converse. Let $lu \in C_{P_\lambda}(K_0)$ for some $l \in L_\lambda$ and $u \in R_u(P_\lambda)$. Then lu centralizes $Z(R_u(P_\lambda))$, so l centralizes $Z(R_u(P_\lambda))$, since u does. It suffices to show that $l = 1$. Let $l = t\tilde{l}$ where $t \in Z(L_\lambda)^\circ = \lambda(k^*)$ and $\tilde{l} \in [L_\lambda, L_\lambda]$. We have

$$\langle i, \lambda \rangle = 4 \quad \text{for any } i \in \{36, \dots, 42\}. \quad (5.1)$$

So for any $z \in Z(R_u(P_\lambda))$, there exists $\alpha \in k^*$ such that $t \cdot z = \alpha z$. Then we have $\tilde{l} \cdot z = \alpha^{-1}z$. Now define $A := \{\tilde{l} \in [L_\lambda, L_\lambda] \mid \tilde{l} \text{ acts on } Z(R_u(P_\lambda)) \text{ by multiplication by a scalar}\}$. Then it is easy to see that $A \trianglelefteq [L_\lambda, L_\lambda]$. Since $[L_\lambda, L_\lambda] \cong SL_7$ and $L_\lambda \cong GL_7$, we have $A = Z([L_\lambda, L_\lambda])$. Therefore we obtain $\tilde{l} \in A = Z([L_\lambda, L_\lambda]) \subseteq \lambda(k^*)$. So we have $l = c\tilde{l} \in \lambda(k^*)$. Then we obtain $l \in C_{\lambda(k^*)}(Z(R_u(P_\lambda)))$. By (5.1) this implies $l = 1$. \square

Lemma 5.2. $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$ -conjugacy classes.

Proof. Fix $a' \in k^*$. By Lemma 5.1, we have $C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0) \subseteq C_{R_u(P_\lambda)}(K)$. Then we obtain

$$C_{P_\lambda}(v(a')K_0v(a')^{-1}) = v(a')C_{P_\lambda}(K_0)v(a')^{-1} \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}. \quad (5.2)$$

Choose $b' \in k^*$ such that $\mathbf{m}(a')$ is $P_\lambda(M)$ -conjugate to $\mathbf{m}(b')$. Then there exists $m \in P_\lambda(M)$ such that $m \cdot \mathbf{m}(b') = \mathbf{m}(a')$. By (5.2), we have

$$mv(b')v(a')^{-1} \in C_{P_\lambda}(v(a')K_0v(a')^{-1}) \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}.$$

By Proposition 3.2, we have

$$v(a')^{-1}mv(b') = \prod_{i=1}^7 \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i),$$

for some $a, b, c, a_i \in k$.

This yields

$$m = \prod_{i=1}^7 \epsilon_i(a+a'+b') \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(b_i),$$

for some $a, b, c, b_i \in k$.

But $m \in P_\lambda(M)$, so $a+a'+b'=0$, $b=0$, $c=0$, $a+b+c=0$. Hence we have $a'=b'$. Thus we have shown that if $a' \neq b'$, then $\mathbf{m}(a')$ is not $P_\lambda(M)$ -conjugate to $\mathbf{m}(b')$. So, in particular, $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$ -conjugacy classes. \square

We need the next result [11, Lem. 4.4]. We include the proof to make this paper self-contained.

Lemma 5.3. $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of M -conjugacy classes if and only if it is a finite union of $P_\lambda(M)$ -conjugacy classes.

Proof. Pick $\mathbf{m}_1, \mathbf{m}_2 \in G \cdot \mathbf{m} \cap P_\lambda(M)^n$ such that \mathbf{m}_1 and \mathbf{m}_2 are in the same M -conjugacy class of $G \cdot \mathbf{m} \cap P_\lambda(M)^n$. Then there exists $m \in M$ such that $m \cdot \mathbf{m}_1 = \mathbf{m}_2$. Let $Q = m^{-1}P_\lambda(M)m$. Then we have $\mathbf{m}_1 \in (P_\lambda(M) \cap Q)^n$. Now let S be a maximal torus

of M contained in $P_\lambda(M) \cap Q$. Since S and $m^{-1}Sm$ are maximal tori of Q , they must be Q -conjugate. So there exists $q \in Q$ such that

$$qSq^{-1} = m^{-1}Sm. \quad (5.3)$$

Since $Q = m^{-1}P_\lambda(M)m$, there exists $p \in P_\lambda(M)$ such that $q = m^{-1}pm$. Then from (5.3), we obtain $pmSm^{-1}p^{-1} = S$. This implies $m^{-1}p^{-1} \in N_M(S)$. Fix a finite set $N \subseteq N_M(S)$ of coset representatives for the Weyl group $W = N_M(S)/S$. Then we have

$$m^{-1}p^{-1} = ns \quad \text{for some } n \in N, s \in S.$$

So we obtain $\mathbf{m}_1 = m^{-1} \cdot \mathbf{m}_2 = (nsp) \cdot \mathbf{m}_2 \in (nP_\lambda(M)) \cdot \mathbf{m}_2$. Since N is a finite set, this shows that an M -conjugacy class in $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of $P_\lambda(M)$ -conjugacy classes. The converse is obvious. \square

Proof of Theorem 1.12. By Lemma 5.2 and Lemma 5.3, we conclude that $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of M -conjugacy classes. Now it is evident that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of M -conjugacy classes. \square

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Appendix A

Table 1
The set of positive roots of $G = E_7$.

① 0 0 1 1 1 0	② 1 1 1 1 0 0	③ 0 1 1 2 1 1	④ 0 0 1 2 2 1
⑤ 1 1 2 2 1 0	⑥ 0 1 1 2 2 1	⑦ 1 2 2 2 1 1	⑧ 0 0 0 0 0 0
⑨ 0 0 0 1 0 0	⑩ 0 1 1 1 1 0	⑪ 0 0 1 2 1 1	⑫ 1 2 2 2 2 1
⑬ 1 1 2 3 2 1	⑭ 1 2 3 3 2 1	⑮ 0 0 1 1 0 0	⑯ 0 0 0 1 1 0
⑰ 0 1 1 1 0 0	⑱ 0 0 0 1 1 1	⑲ 0 0 1 2 1 0	⑳ 1 1 1 1 1 0
㉑ 0 1 1 1 1 1	㉒ 1 1 1 2 1 1	㉓ 1 2 2 2 1 0	㉔ 1 1 2 2 1 1

(continued on next page)

Table 1 (continued)

(25) $\begin{smallmatrix} & & & 1 \\ 0 & 1 & 2 & 2 & 2 & 1 \end{smallmatrix}$	(26) $\begin{smallmatrix} & & & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 \end{smallmatrix}$	(27) $\begin{smallmatrix} & & & 1 \\ 0 & 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$	(28) $\begin{smallmatrix} & & & 1 \\ 1 & 2 & 2 & 3 & 2 & 1 \end{smallmatrix}$
(29) $\begin{smallmatrix} & & & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$	(30) $\begin{smallmatrix} & & & 1 \\ 0 & 1 & 1 & 2 & 1 & 0 \end{smallmatrix}$	(31) $\begin{smallmatrix} & & & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 \end{smallmatrix}$	(32) $\begin{smallmatrix} & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$
(33) $\begin{smallmatrix} & & & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{smallmatrix}$	(34) $\begin{smallmatrix} & & & 1 \\ 0 & 1 & 2 & 2 & 1 & 1 \end{smallmatrix}$	(35) $\begin{smallmatrix} & & & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 \end{smallmatrix}$	(36) $\begin{smallmatrix} & & & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$
(37) $\begin{smallmatrix} & & & 2 \\ 1 & 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$	(38) $\begin{smallmatrix} & & & 2 \\ 1 & 2 & 2 & 3 & 2 & 1 \end{smallmatrix}$	(39) $\begin{smallmatrix} & & & 2 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{smallmatrix}$	(40) $\begin{smallmatrix} & & & 2 \\ 1 & 2 & 3 & 4 & 2 & 1 \end{smallmatrix}$
(41) $\begin{smallmatrix} & & & 2 \\ 1 & 2 & 3 & 4 & 3 & 1 \end{smallmatrix}$	(42) $\begin{smallmatrix} & & & 2 \\ 1 & 2 & 3 & 4 & 3 & 2 \end{smallmatrix}$	(43) $\begin{smallmatrix} & & & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	(44) $\begin{smallmatrix} & & & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix}$
(45) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{smallmatrix}$	(46) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{smallmatrix}$	(47) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{smallmatrix}$	(48) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{smallmatrix}$
(49) $\begin{smallmatrix} & & & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix}$	(50) $\begin{smallmatrix} & & & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	(51) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(52) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$
(53) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{smallmatrix}$	(54) $\begin{smallmatrix} & & & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	(55) $\begin{smallmatrix} & & & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(56) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$
(57) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	(58) $\begin{smallmatrix} & & & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(59) $\begin{smallmatrix} & & & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	(60) $\begin{smallmatrix} & & & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$
(61) $\begin{smallmatrix} & & & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	(62) $\begin{smallmatrix} & & & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	(63) $\begin{smallmatrix} & & & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	

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