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Quasi-hereditary structure of twisted split category algebras revisited

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ABSTRACT

Let k be a field of characteristic 0, let C be a finite split category, let α be a 2-cocycle of C with values in the multiplicative group of k, and consider the resulting twisted category algebra $A := k_\alpha\mathsf{C}$. Several interesting algebras arise that way, for instance, the Brauer algebra. Moreover, the category of biset functors over k is equivalent to a module category over a condensed algebra $\varepsilon A\varepsilon$, for an idempotent ε of A. In [2] the authors proved that A is quasi-hereditary (with respect to an explicit partial order $\leqslant$ on the set of irreducible modules), and standard modules were given explicitly. Here, we improve the partial order $\leqslant$ by introducing a coarser order $\trianglelefteq$ leading to the same results on A, but which allows to pass the quasi-heredity result to the condensed algebra $\varepsilon A\varepsilon$ describing biset functors, thereby giving a different proof of a quasi-heredity result of Webb, see [21]. The new partial order $\trianglelefteq$ has not been considered before, even in the special cases, and we evaluate it explicitly for the case of biset functors and the Brauer algebra. It also puts further restrictions on the possible composition factors of standard modules.

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1. Introduction

Suppose that k is a field and that C is a finite category, that is, the morphisms in C form a finite set. Suppose further that α is a 2-cocycle of C with values in $k^\times$. Then the *twisted category algebra* $k_\alpha\mathsf{C}$ is the finite-dimensional k-algebra with the morphisms in C as a k-basis, and with multiplication induced by composition of morphisms, twisted by the 2-cocycle α; for a precise definition, see 3.1. In the case where the category has one object only this just recovers the notion of a twisted monoid algebra.

In recent years, (twisted) category algebras and (twisted) monoid algebras have been intensively studied by B. Steinberg et al., for instance in [10], as well as by Linckelmann and Stolorz, who, in particular, determined the isomorphism classes of simple $k_\alpha\mathsf{C}$-modules in [17]. As a consequence of [17, Theorem 1.2] the isomorphism classes of simple $k_\alpha\mathsf{C}$-modules can be parametrized by a set Λ of pairs whose first entry varies over certain finite groups related to C (called *maximal subgroups* of C), and whose second entry varies over the isomorphism classes of simple modules over twisted group algebras of these maximal subgroups.

For convenience, in the following we shall suppose that k has characteristic 0, but this condition can be relieved, as we shall see in Theorem 4.3. Moreover, we suppose that the category C is *split*, that is, every morphism in C is a split morphism, see 3.1(a). It has been shown by the authors in [2] and, independently, by Linckelmann and Stolorz in [18] that the resulting twisted category algebra $k_\alpha\mathsf{C}$ is quasi-hereditary in the sense of [6]. In [2] we also determined the standard modules of $k_\alpha\mathsf{C}$ with respect to a natural partial order $\leq$ on the labelling set Λ of isomorphism classes of simple modules, which depends only on the first entries of pairs in Λ and is explained in 3.6.

Since $k_\alpha\mathsf{C}$ is quasi-hereditary with respect to $(\Lambda, \leq)$, it is also quasi-hereditary with respect to any refinement of $\leq$, and the corresponding standard and costandard modules are the same as those with respect to $(\Lambda, \leq)$. This is an immediate consequence of Definition 2.1 below.

In this paper we introduce a new partial order $\trianglelefteq$ on Λ such that the partial order $\leq$ is a refinement of $\trianglelefteq$. We shall then show in Theorem 4.3 that the algebra $k_\alpha\mathsf{C}$ remains quasi-hereditary with respect to this new partial order. Furthermore, we shall show that the standard and costandard modules of $k_\alpha\mathsf{C}$ with respect to the two partial orders coincide, and we shall give explicit descriptions of these modules. The partial order $\trianglelefteq$ seems more natural than the initial one, since it depends on both entries of the pairs in Λ, and it allows to pass the hereditary structure to idempotent condensed algebras $\varepsilon \cdot k_\alpha\mathsf{C} \cdot \varepsilon$, for $\varepsilon^2 = \varepsilon \in k_\alpha\mathsf{C}$, in a particular case we are interested in and that is related to the category of biset functors, see Section 7. In Sections 5 and 6 we shall give a number of possible reformulations and simplifications of the defining properties of the partial order $\trianglelefteq$ that are particularly useful when considering concrete examples.

It is known, by work of Wilcox [22], that diagram algebras such as Brauer algebras, Temperley–Lieb algebras, partition algebras, and relatives of these arise naturally as twisted split category algebras and twisted regular monoid algebras; for a list of references, see the introductions to [2,18]. Our initial motivation for studying the structure of twisted category algebras comes from our results in [1], where we have shown that the double Burnside algebra of a finite group over k is isomorphic to a k-algebra that is obtained from a twisted split category algebra by idempotent condensation. The latter result is also valid for more general algebras related to the category of biset functors on a finite section-closed set of finite groups; see [1, Section 5] and Section 7 of this article. Therefore, in Section 7 and in Section 8 we shall apply our general results concerning the quasi-hereditary structure of twisted split category algebras to the algebra related to biset functors just mentioned and to the Brauer algebra, respectively. In doing so we shall, in particular, derive new information about decomposition numbers of the algebras under consideration, since our partial order $\trianglelefteq$ is in general strictly coarser than the well-known partial order $\leq$ on Λ.

Moreover, in Theorem 7.7 we recover and slightly improve a result of Webb in [21] stating that the category of biset functors over k is a highest weight category, when the underlying category of finite groups is finite. The key step towards the latter result will be showing that our newly introduced partial order $\trianglelefteq$ behaves particularly well with respect to this particular idempotent condensation, whereas the finer order $\leq$ does not; see Example 7.9.

Many known examples of twisted split category algebras $k_\alpha\mathsf{C}$ arise from categories equipped with a contravariant functor that is the identity on objects and that gives rise to a duality on the category of left $k_\alpha\mathsf{C}$-modules. Such a duality, in particular, sends standard modules to costandard modules, and allows for further simplifications of the partial order $\trianglelefteq$ on the set Λ. We shall analyze this duality in detail in Section 6, and apply these general results to our concrete examples in Section 7 and Section 8.

2. Quasi-hereditary algebras

In the following, k will denote an arbitrary field. We begin by briefly recalling the definition and some basic properties of a quasi-hereditary k-algebra needed in this article. For more details and background we refer the reader to [6] and [9, Appendix]. Unless specified otherwise, modules over any finite-dimensional k-algebra are understood to be finite-dimensional *left* modules.

2.1. Definition. (Cline, Parshall and Scott [6].) Let k be a field, and let A be a finite-dimensional k-algebra. Let further Λ be a finite set parametrizing the isomorphism classes of simple A-modules, and let $\leq$ be a partial order on Λ. For $\lambda \in \Lambda$, let D_λ be a simple A-module labelled by λ, let P_λ be a projective cover of D_λ, and let I_λ be an injective envelope of D_λ.

(a) For $\lambda \in \Lambda$, let Δ_λ be the unique maximal quotient module M of P_λ such that all composition factors of $\mathrm{Rad}(M)$ belong to the set $\{D_\mu \mid \mu < \lambda\}$. Then Δ_λ is called the *standard module* of A with respect to $(\Lambda, \leq)$ labelled by λ.

(b) For $\lambda \in \Lambda$, let ∇_λ be the unique maximal submodule N of I_λ such that all composition factors of $N/\mathrm{Soc}(N)$ belong to the set $\{D_\mu \mid \mu < \lambda\}$. Then ∇_λ is called the *costandard module* of A with respect to $(\Lambda, \leq)$ labelled by λ.

(c) The k-algebra A is called *quasi-hereditary* with respect to $(\Lambda, \leq)$ if, for each $\lambda \in \Lambda$, the projective module P_λ admits a filtration

$$0 = P_\lambda^{(0)} \subset P_\lambda^{(1)} \subset \cdots \subset P_\lambda^{(m_\lambda)} = P_\lambda$$

satisfying the following properties:

(i) $P_\lambda^{(m_\lambda)}/P_\lambda^{(m_\lambda-1)} \cong \Delta_\lambda$, and
(ii) if $1 \leq q < m_\lambda$ then $P_\lambda^{(q)}/P_\lambda^{(q-1)} \cong \Delta_\mu$, for some $\mu \in \Lambda$ with $\lambda < \mu$.

The notion of a quasi-hereditary algebra can be defined equivalently in terms of costandard modules:

2.2. Proposition. *(See [9, Definition A2.1, Lemma A3.5].) With the notation as in Definition 2.1, the k-algebra A is quasi-hereditary with respect to $(\Lambda, \leq)$ if and only if, for each $\lambda \in \Lambda$, the injective module I_λ admits a filtration*

$$0 = I_\lambda^{(0)} \subset I_\lambda^{(1)} \subset \cdots \subset I_\lambda^{(n_\lambda)} = I_\lambda$$

such that
(i) $I_\lambda^{(1)} \cong \nabla_\lambda$, *and*
(ii) *if $1 < q \leq n_\lambda$ then $I_\lambda^{(q)}/I_\lambda^{(q-1)} \cong \nabla_\mu$, for some $\mu \in \Lambda$ with $\lambda < \mu$.*

2.3. Dual modules. (a) Suppose again that A is a finite-dimensional k-algebra. For any left A-module M, we set $M^* := \mathrm{Hom}_k(M, k)$ and view M^* as a right A-module via $(\lambda \cdot a)(m) := \lambda(am)$, for $\lambda \in M^*$, $a \in A$, and $m \in M$. Similarly, if N is a right A-module, the k-linear dual N^* is a left A-module. If Λ labels the isomorphism classes of simple left A-modules then it also labels the isomorphism classes of simple right A-modules: if D_λ is the simple left A-modules labelled by $\lambda \in \Lambda$ then we choose D_λ^* to be the simple right A-module labelled by λ. The dual P_λ^* of the projective cover of D_λ is an injective envelope of D_λ^*, and the dual I_λ^* of the injective envelope of D_λ is a projective cover of D_λ^*. Moreover, if $\leq$ is a partial order on Λ and if Δ_λ (respectively, ∇_λ) is the corresponding standard (respectively, costandard) left A-module labelled by $\lambda \in \Lambda$, then ∇_λ^* (respectively, Δ_λ^*) is the corresponding standard (respectively, costandard) right A-module labelled by λ. Using Proposition 2.2 one sees that A is quasi-hereditary with respect to $\leq$ in the left module formulation if and only if A is quasi-hereditary in the similar right module formulation.

(b) Suppose further that there is a k-algebra anti-involution $-^\circ : A \to A$, that is, $-^\circ$ is a k-linear isomorphism of order 2 that satisfies $(ab)^\circ = b^\circ a^\circ$, for all $a, b \in A$. Then, given any finite-dimensional left A-module M, its k-linear dual M^* also carries a left A-module structure via

$$(a \cdot f)(m) := f(a^\circ \cdot m) \quad (a \in A,\ f \in \mathrm{Hom}_k(M,k),\ m \in M)\,. \tag{1}$$

We denote the resulting left A-module by M°. Note that $(M^\circ)^\circ \cong M \cong (M^*)^*$ as left A-modules.

If A is the group algebra kG for a finite group G, then we have a canonical anti-involution induced by the map $G \to G,\ g \mapsto g^{-1}$. By abuse of notation, in this particular case we still write M^* for the left A-module M°. So, in this case, M^* can mean a left or a right kG-module, and we shall clarify this when necessary.

Let Λ be a set labelling the isomorphism classes of simple left A-modules, and let $\leq$ be a partial order on Λ. For $\lambda \in \Lambda$, the dual D_λ° of the simple left A-module D_λ is again a simple left A-module, so that there is some $\lambda^\circ \in \Lambda$ with $D_\lambda^\circ \cong D_{\lambda^\circ}$. Moreover, the dual P_λ° of the projective cover P_λ of D_λ then is an injective envelope of D_λ°. Imposing an additional assumption on the poset $(\Lambda, \leq)$, the duality also relates standard modules to costandard modules as follows.

2.4. Proposition. *Retain the assumptions from 2.3. Suppose further that, for all $\lambda, \mu \in \Lambda$, one has $\lambda \leq \mu$ if and only if $\lambda^\circ \leq \mu^\circ$. Then, for each $\lambda \in \Lambda$, one has A-module isomorphisms*

$$\Delta_\lambda^\circ \cong \nabla_{\lambda^\circ} \quad \text{and} \quad \nabla_\lambda^\circ \cong \Delta_{\lambda^\circ}\,.$$

Proof. Let $\lambda \in \Lambda$. By definition, Δ_λ is the largest quotient module of P_λ such that all composition factors of its radical belong to $\{D_\mu \mid \mu < \lambda\}$. Hence Δ_λ° is the largest submodule of $P_\lambda^\circ \cong I_{\lambda^\circ}$ such that all composition factors of its cosocle belong to the set $\{D_\mu^\circ \mid \mu < \lambda\} = \{D_{\mu^\circ} \mid \mu < \lambda\}$. Since, by our hypothesis, $\mu < \lambda$ if and only if $\mu^\circ < \lambda^\circ$, this shows that $\Delta_\lambda^\circ \cong \nabla_{\lambda^\circ}$.

The second isomorphism follows analogously. □

2.5. Remark. Keep the assumptions as in 2.3(b). In the case where, for every $\lambda \in \Lambda$, one has $D_\lambda^\circ \cong D_\lambda$, the quasi-hereditary algebra A is usually called a *BGG-algebra*, see for instance [7,13].

Idempotent condensation of a quasi-hereditary algebra results again in a quasi-hereditary algebra if the simple modules annihilated by the idempotent form a subset of Λ that is closed from above. The following proposition makes this more precise. It is an immediate consequence of Green's idempotent condensation theory, [12, Section 6.2], and [9, Proposition A.3.11]. We shall use it in Section 7 to show that, in certain situations, biset functor categories are equivalent to module categories of quasi-hereditary algebras.

2.6. Proposition. *Assume that A is a quasi-hereditary k-algebra with respect to $(\Lambda, \leqslant)$, and let e be an idempotent satisfying the following condition: If $\lambda \leqslant \mu$ are elements of Λ and if $eD_\lambda \neq \{0\}$ then also $eD_\mu \neq \{0\}$. Set $\Lambda' := \{\lambda \in \Lambda \mid eD_\lambda \neq \{0\}\}$ and set $A' := eAe$. Then, as λ varies over Λ', the A'-modules eD_λ form a complete set of representatives of the isomorphism classes of simple A'-modules, and A' is a quasi-hereditary algebra with respect to $(\Lambda', \leqslant)$. Moreover, for $\lambda \in \Lambda'$, the corresponding standard A'-module is given by $e\Delta(\lambda)$, and the corresponding costandard A'-module is given by $e\nabla(\lambda)$.*

3. Twisted category algebras

In the following we recall some properties of twisted category algebras from [2,17,18]. Throughout this paper we shall choose our notation in accordance with [2]. In particular, we shall use the following notation and situation repeatedly.

3.1. Notation. (a) Let C be a finite category, that is, the morphisms in C form a finite set $S := \mathrm{Mor}(\mathsf{C})$. We shall henceforth suppose that C is *split*, that is, for every morphism $s \in S$, there is some $t \in S$ with $s \circ t \circ s = s$.

Let k be a field. A *2-cocycle α of C with values in $k^\times$* assigns to each pair of morphisms $s, t \in S$ such that $s \circ t$ exists in S an element $\alpha(s,t) \in k^\times$ such that the following holds: whenever $s, t, u \in S$ are such that $u \circ t \circ s$ exists, one has $\alpha(u \circ t, s)\alpha(u,t) = \alpha(u, t \circ s)\alpha(t,s)$. The *twisted category algebra $k_\alpha\mathsf{C}$* is the k-vector space with k-basis S and with multiplication

$$t \cdot s := \begin{cases} \alpha(t,s) \cdot t \circ s & \text{if } t \circ s \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

for $s, t \in S$. For the remainder of this section, we set $A := k_\alpha\mathsf{C}$. The set of all 2-cocycles of C with values in $k^\times$ will be denoted by $Z^2(\mathsf{C}, k^\times)$. If C' is a skeleton of C and α' is the restriction of α to C' then $k_{\alpha'}\mathsf{C}'$ is Morita equivalent to $k_\alpha\mathsf{C}$.

(b) Following Green in [11], one has an equivalence relation $\mathscr{J}$ on S defined by

$$s \mathscr{J} t :\Leftrightarrow S \circ s \circ S = S \circ t \circ S \,,$$

for $s, t \in S$. The corresponding equivalence class of s is denoted by $\mathscr{J}(s)$, and is called a *$\mathscr{J}$-class of* C. One also has a partial order $\leq_{\mathscr{J}}$ on the set of $\mathscr{J}$-classes of C defined by

$$\mathscr{J}(s) \leq_{\mathscr{J}} \mathscr{J}(t) :\Leftrightarrow S \circ \mathscr{J}(s) \circ S \subseteq S \circ \mathscr{J}(t) \circ S \,, \tag{2}$$

for $s, t \in S$. Note also that $\mathscr{J}(s) \leq_{\mathscr{J}} \mathscr{J}(t)$ if and only if $S \circ s \circ S \subseteq S \circ t \circ S$. From now on, let $S_1, \ldots, S_n$ denote the $\mathscr{J}$-classes of C, ordered such that $S_i <_{\mathscr{J}} S_j$ implies $i < j$.

Since C is split, every $\mathscr{J}$-class S_i of C contains an idempotent endomorphism e_i, that is, $e_i \in \mathrm{End}_{\mathsf{C}}(X_i)$, for some $X_i \in \mathrm{Ob}(\mathsf{C})$, and $e_i \circ e_i = e_i$: one can, for instance, take $e_i := s_i \circ t_i$, for any $s_i \in S_i$ and any $t_i \in S$ such that $s_i \circ t_i \circ s_i = s_i$.

Note that

$$e'_i := \alpha(e_i, e_i)^{-1} \cdot e_i$$

is an idempotent in the algebra A, whereas e_i itself is in general not.

(c) For $i = 1, \ldots, n$, denote by Γ_{e_i} the group of units in the monoid $e_i \circ \mathrm{End}_{\mathsf{C}}(X_i) \circ e_i$, and set $J_{e_i} := (e_i \circ \mathrm{End}_{\mathsf{C}}(X_i) \circ e_i) \smallsetminus \Gamma_{e_i}$. The 2-cocycle α restricts to a 2-cocycle of the group Γ_{e_i}, so that one can regard the twisted group algebra $k_\alpha\Gamma_{e_i}$ as a (non-unitary) subalgebra of A. Moreover, for each $i = 1, \ldots, n$, one has the following k-vector space decomposition of the k-algebra $e'_i A e'_i$:

$$e'_i A e'_i = e_i A e_i = k_\alpha \Gamma_{e_i} \oplus k J_{e_i} \, ; \tag{3}$$

note that here $k_\alpha\Gamma_{e_i}$ is a unitary subalgebra and kJ_{e_i} is a two-sided ideal of $e'_i A e'_i$.

(d) In accordance with [2], we also define

$$S_{\leq i} := \bigcup_{j \leq i} S_j \quad \text{and} \quad J_i := kS_{\leq i} \, , \tag{4}$$

for $i = 1, \ldots, n$, and we set $J_0 := \{0\}$. By [2, Proposition 3.3], this yields a chain $J_0 \subset J_1 \subset \cdots \subset J_n = A$ of two-sided ideals in A.

3.2. Remark. In the special case where C is a category with one object, one simply recovers the notion of a twisted monoid algebra. The property of being split is then usually called *regular*, see for instance [10].

3.3. Simple modules and standard modules. For each $i \in \{1, \ldots, n\}$, let $e_i \in S_i$ be an idempotent endomorphism, and let $T_{(i,1)}, \ldots, T_{(i,l_i)}$ be representatives of the isomorphism classes of simple $k_\alpha\Gamma_{e_i}$-modules. For $i \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, l_i\}$, denote by $\tilde{T}_{(i,r)}$ the inflation of $T_{(i,r)}$ to $e'_i A e'_i$ with respect to the ideal kJ_{e_i} and the decomposition (3). Consider the A-modules

$$\Delta_{(i,r)} := Ae'_i \otimes_{e'_i A e'_i} \tilde{T}_{(i,r)} \quad \text{and} \quad D_{(i,r)} := \mathrm{Hd}(\Delta_{(i,r)}) \, . \tag{5}$$

The isomorphism classes of $\Delta_{(i,r)}$ and $D_{(i,r)}$, respectively, are independent of the choice of the idempotent $e_i \in S_i$.

3.4. Theorem. *(See [17, Theorem 1.2].) The modules $D_{(i,r)}$ ($i = 1, \ldots, n$, $r = 1, \ldots, l_i$) form a set of representatives of the isomorphism classes of simple A-modules.*

3.5. Remark. Suppose again that $i \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, l_i\}$. By the general theory of idempotent condensation, see [12, Section 6.2], one has the following isomorphisms of $e'_i A e'_i$-modules:

$$e'_i \cdot D_{(i,r)} \cong \tilde{T}_{(i,r)} \cong e'_i \cdot \Delta_{(i,r)} \, ; \tag{6}$$

in particular, the idempotent e'_i of A annihilates every composition factor of $\mathrm{Rad}(\Delta_{(i,r)})$.

3.6. A partial order. By Theorem 3.4, the set $\Lambda := \{(i,r) \mid 1 \leq i \leq n, 1 \leq r \leq l_i\}$ parametrizes the isomorphism classes of simple A-modules. Moreover, one has a partial order $\leq$ on Λ that is defined as follows:

$$(i,r) < (j,s) :\Leftrightarrow S_j <_{\mathscr{J}} S_i\,, \tag{7}$$

for $(i,r),(j,s) \in \Lambda$. With this notation we recall the following result.

3.7. Theorem. *(See [2, Theorem 4.2].) Suppose that the group orders $|\Gamma_{e_1}|, \ldots, |\Gamma_{e_n}|$ are invertible in k. Then the k-algebra $A = k_\alpha \mathsf{C}$ is quasi-hereditary with respect to $(\Lambda, \leq)$. Moreover, for $(i,r) \in \Lambda$, the standard A-module labelled by (i,r) is isomorphic to $\Delta_{(i,r)}$.*

An independent proof of the quasi-heredity of $k_\alpha \mathsf{C}$ can be found in [18, Corollary 1.2].

4. Main theorem

Throughout this section, we retain the situation and notation from 3.1 and 3.3. Additionally, we assume that, for each $i = 1, \ldots, n$, the order of Γ_{e_i} is invertible in k, so that the twisted group algebra $k_\alpha \Gamma_{e_i}$ is semisimple. By Theorem 3.7, $A = k_\alpha \mathsf{C}$ is quasi-hereditary with simple modules $D_{(i,r)}$ and standard modules $\Delta_{(i,r)}$, $(i,r) \in \Lambda$, where Λ is endowed with the partial order $\leqslant$ from 3.6.

In the following we shall introduce a new partial order $\trianglelefteq$ on Λ such that the partial order $\leq$ defined in 3.6 is a refinement of $\trianglelefteq$. We aim to show that A is also quasi-hereditary with respect to $(\Lambda, \trianglelefteq)$, and that the standard and costandard modules do not change.

4.1. Definition. For $(i,r),(j,s) \in \Lambda$, let $f_{(i,r)} \in k_\alpha \Gamma_{e_i}$ and $f_{(j,s)} \in k_\alpha \Gamma_{e_j}$ be the block idempotents corresponding to the simple modules $T_{(i,r)}$ and $T_{(j,s)}$, respectively. We set

$$\begin{aligned}(i,r) \sqsubset (j,s) :\Leftrightarrow\ &\text{(i) } S_j <_{\mathscr{J}} S_i, \text{ and}\\ &\text{(ii) } f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1} \text{ or } f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}\,.\end{aligned}$$

Here $J_{j-1} \subset J_j$ are the two-sided ideals of A defined in (4). This defines a reflexive, anti-symmetric relation $\sqsubseteq$ on Λ. The transitive closure of the relation $\sqsubseteq$ on Λ will be denoted by $\trianglelefteq$. This is again a partial order on Λ.

4.2. Remark. (a) Note that the partial order $\leq$ on Λ defined in (7) is indeed a refinement of the new partial order $\trianglelefteq$.

(b) One can show that the partial order $\trianglelefteq$ on Λ does neither depend on the choice of the idempotent e_i in S_i nor on the chosen total order $S_1, S_2, \ldots, S_n$ of the $\mathscr{J}$-classes.

(c) We emphasize that the relation $\sqsubseteq$ in Definition 4.1 is in general not transitive; we shall give explicit examples later in Section 7 and Section 8. The condition in Definition 4.1(ii) can be reformulated, and can often be simplified; see Sections 5 and 6. In

Section 6 we shall, in particular, establish criteria for the conditions $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$ and $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$ to be equivalent.

We are now prepared to state and prove our main result:

4.3. Theorem. *Suppose that the group orders $|\Gamma_{e_1}|, \ldots, |\Gamma_{e_n}|$ are invertible in k. Then the twisted category algebra $A = k_\alpha \mathsf{C}$ is quasi-hereditary with respect to $(\Lambda, \trianglelefteq)$. Moreover, the A-modules $\Delta_{(i,r)}$ ($(i,r) \in \Lambda$), as defined in (5), are the corresponding standard modules.*

Proof. We shall use Definition 2.1 and proceed as in [2]. That is, we shall show that, for each $(i,r) \in \Lambda$, the A-module $\Delta_{(i,r)}$ satisfies (i) and (ii) below, and that the projective A-module $P_{(i,r)}$ admits a filtration

$$0 = P_{(i,r)}^{(0)} \subset \cdots \subset P_{(i,r)}^{(m_{ir})} = P_{(i,r)}$$

satisfying (iii) and (iv) below:

(i) $\mathrm{Hd}(\Delta_{(i,r)}) \cong D_{(i,r)}$;
(ii) $[\mathrm{Rad}(\Delta_{(i,r)}) : D_{(l,t)}] \neq 0 \Rightarrow (l,t) \lhd (i,r)$;
(iii) $P_{(i,r)}^{(m_{ir})} / P_{(i,r)}^{(m_{ir}-1)} \cong \Delta_{(i,r)}$;
(iv) $1 \leq q < m_{(i,r)} \Rightarrow P_{(i,r)}^{(q)} / P_{(i,r)}^{(q-1)} \cong \Delta_{(j,s)}$, for some $(j,s) \in \Lambda$ with $(i,r) \lhd (j,s)$.

Condition (i) has already been verified in the proof of [2, Theorem 4.2]. To show (ii), let $(i,r), (l,t) \in \Lambda$ be such $[\mathrm{Rad}(\Delta_{(i,r)}) : D_{(l,t)}] \neq 0$. Since $\Delta_{(i,r)}$ is the standard A-module (see Theorem 3.7) with respect to $(\Lambda, \leq)$ labelled by (i,r), we already know that $S_i <_{\mathscr{J}} S_l$. We shall show that $f_{(l,t)} \cdot J_i \cdot f_{(i,r)} \nsubseteq J_{i-1}$, so that $(l,t) \sqsubset (i,r)$, and thus $(l,t) \lhd (i,r)$.

Recall from (6) that $e'_l \cdot D_{(l,t)} \cong e'_l \cdot \Delta_{(l,t)} \cong \tilde{T}_{(l,t)}$ as $e'_l A e'_l$-modules. Moreover, note that $f_{(i,r)} \cdot \tilde{T}_{(i,r)} = \tilde{T}_{(i,r)}$, since $f_{(i,r)}$ is the block idempotent of $k_\alpha \Gamma_{e_i}$ corresponding to the simple module $T_{(i,r)}$ and $\tilde{T}_{(i,r)}$ is just the inflation of $T_{(i,r)}$ to $e'_i A e'_i$. Analogously, we have $f_{(l,t)} \cdot \tilde{T}_{(l,t)} = \tilde{T}_{(l,t)}$. This implies, in particular, that $f_{(l,t)} \cdot D_{(l,t)} = f_{(l,t)} e'_l \cdot D_{(l,t)} \neq \{0\}$, and thus also $f_{(l,t)} \cdot \Delta_{(i,r)} \neq \{0\}$, since we are assuming $[\Delta_{(i,r)} : D_{(l,t)}] \neq 0$ and since multiplication by $f_{(l,t)}$ is exact.

Now, writing $f_{(i,r)}$ as a sum of pairwise orthogonal primitive idempotents in $e'_i A e'_i$, there is a summand $\tilde{f}_{(i,r)}$ in this decomposition such that $\tilde{f}_{(i,r)} \cdot \tilde{T}_{(i,r)} \neq \{0\}$. By [20, Theorem 1.8.10], this in turn implies that the $e'_i A e'_i$-module $e'_i A e'_i \cdot \tilde{f}_{(i,r)}$ is a projective cover of $\tilde{T}_{(i,r)}$. Thus, as we have seen in the proof of [2, Theorem 4.2], the A-module $A \cdot \tilde{f}_{(i,r)}$ is a projective cover of $D_{(i,r)}$, so that we may from now on suppose that $A \cdot \tilde{f}_{(i,r)} = P_{(i,r)}$.

Next consider the following chain of A-modules from [2, (12)]:

$$\{0\} = J_0 \cdot \tilde{f}_{(i,r)} \subseteq J_1 \cdot \tilde{f}_{(i,r)} \subseteq \cdots \subseteq J_{i-1} \cdot \tilde{f}_{(i,r)} \subseteq J_i \cdot \tilde{f}_{(i,r)} = A \cdot \tilde{f}_{(i,r)} = P_{(i,r)} \,. \quad (8)$$

As we have shown in the proof of [2, Theorem 4.2], there is an A-module isomorphism

$$\Delta_{(i,r)} \cong J_i \cdot \tilde{f}_{(i,r)}/J_{i-1} \cdot \tilde{f}_{(i,r)} = (J_i/J_{i-1}) \cdot \tilde{f}_{(i,r)}\,.$$

Consequently, we have now proved that

$$\{0\} \neq f_{(l,t)} \cdot (J_i/J_{i-1}) \cdot \tilde{f}_{(i,r)} = f_{(l,t)} \cdot (J_i/J_{i-1}) \cdot f_{(i,r)} \cdot \tilde{f}_{(i,r)}\,,$$

and therefore $f_{(l,t)} \cdot J_i \cdot f_{(i,r)} \nsubseteq J_{i-1}$, implying $(l,t) \sqsubset (i,r)$, and thus also $(l,t) \lhd (i,r)$, as desired. This settles the proof of (ii).

It remains to verify conditions (iii) and (iv). To this end, we consider the chain (8) of A-modules again. Since we already know that $\Delta_{(i,r)} \cong J_i \cdot \tilde{f}_{(i,r)}/J_{i-1} \cdot \tilde{f}_{(i,r)}$, it suffices to show that (8) also satisfies (iv). We argue along the lines of the proof of [2, Theorem 4.2]: if $j \in \{1,\ldots,i-1\}$ is such that $S_j \not<_{\mathscr{J}} S_i$ then $J_j \cdot \tilde{f}_{(i,r)} = J_{j-1} \cdot \tilde{f}_{(i,r)}$. So suppose that $j \in \{1,\ldots,i-1\}$ is such that $S_j <_{\mathscr{J}} S_i$, and let M be an indecomposable direct summand of the A-module $J_j \cdot \tilde{f}_{(i,r)}/J_{j-1} \cdot \tilde{f}_{(i,r)}$. Then, by [2, Lemma 4.4], there is some $s \in \{1,\ldots,l_j\}$ with $M \cong \Delta_{(j,s)}$. Arguing as above, we deduce $f_{(j,s)} \cdot \tilde{T}_{(j,s)} = \tilde{T}_{(j,s)} \neq \{0\}$, and $e'_j \cdot \Delta_{(j,s)} \cong \tilde{T}_{(j,s)}$ as $e'_jAe'_j$-modules. Thus $f_{(j,s)} \cdot \Delta_{(j,s)} \neq \{0\}$, and

$$\{0\} \neq f_{(j,s)} \cdot (J_j/J_{j-1}) \cdot \tilde{f}_{(i,r)} = f_{(j,s)} \cdot (J_j/J_{j-1}) \cdot f_{(i,r)} \cdot \tilde{f}_{(i,r)}\,,$$

so that $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$. Since we are assuming $S_j <_{\mathscr{J}} S_i$, this implies $(i,r) \sqsubset (j,s)$, hence also $(i,r) \lhd (j,s)$, proving (iv). $\square$

4.4. Costandard modules. At the end of this section we should like to tie up some loose ends and comment on the costandard A-modules with respect to the partial orders $\leq$ and $\trianglelefteq$ on Λ.

To this end, first note that if B is any finite-dimensional k-algebra and f is a primitive idempotent of B then

$$\mathrm{Hd}(Bf)^* \cong \mathrm{Hd}(fB) \text{ as right } B\text{-modules and } \mathrm{Hd}(fB)^* \cong \mathrm{Hd}(Bf) \text{ as left } B\text{-modules.} \tag{9}$$

In other words, if D is a simple left B-module belonging to a particular Wedderburn component of $B/J(B)$ and if D' denotes the simple right B-module belonging to the same Wedderburn component, then $D' \cong D^*$ as right B-modules.

Next, observe that everything that was proven in Theorem 4.3 for left modules over the twisted split category algebra A can also be proved for right A-modules. More precisely, for $(i,r) \in \Lambda$, consider the simple right $k_\alpha\Gamma_{e_i}$-module $T'_{(i,r)}$ belonging to the same Wedderburn component as the simple left $k_\alpha\Gamma_{e_i}$-module $T_{(i,r)}$ and let $\tilde{T}'_{(i,r)}$ denote the inflation of $T'_{(i,r)}$ to $e'_iAe'_i$ with respect to the decomposition $e'_iAe'_i = k_\alpha\Gamma_{e_i} \oplus kJ_{e_i}$. Moreover, define the right A-modules

$$\Delta'_{(i,r)} := \tilde{T}'_{(i,r)} \otimes_{e'_iAe'_i} e'_iA \quad \text{and} \quad D'_{(i,r)} := \mathrm{Hd}(\Delta'_{(i,r)})\,. \tag{10}$$

Then, in analogy to [17, Theorem 1.2], the modules $D'_{(i,r)}$, $(i,r)\in\Lambda$, form a complete set of representatives of the isomorphism classes of simple right A-modules. Moreover, with respect to both the partial orders $\leq$ and $\trianglelefteq$ on Λ, the standard right A-module labelled by $(i,r)\in\Lambda$ is given by $\Delta'_{(i,r)}$. As well, A is quasi-hereditary with respect to these partial orders in the formulation for right modules, in a symmetric sense to Definition 2.1.

Note also that applying (9) to the algebras $B=k_\alpha\Gamma_{e_i}$ and to $B=e'_iAe'_i$, respectively, shows that

$$T'_{(i,r)}\cong T^*_{(i,r)} \text{ as right } k_\alpha\Gamma_{e_i}\text{-modules, and } \tilde{T}'_{(i,r)}\cong(\tilde{T}_{(i,r)})^* \text{ as right } e'_iAe'_i\text{-modules.} \tag{11}$$

Note also that $(\tilde{T}_{(i,r)})^*=\widetilde{T^*_{(i,r)}}$ as right $e'_iAe'_i$-module, so that we may simply denote this module by $\tilde{T}^*_{(i,r)}$, to avoid too many symbols.

Finally, assume that $\tilde{f}_{(i,r)}$ is a primitive idempotent of $e'_iAe'_i$ occurring in the decomposition of the block idempotent $f_{(i,r)}$ with $\tilde{f}_{(i,r)}\tilde{T}_{(i,r)}\neq\{0\}$. Then also $\widetilde{T'}_{(i,r)}\tilde{f}_{(i,r)}\neq\{0\}$. From [2, Theorem 4.2] we know that $A\tilde{f}_{(i,r)}$ is a projective cover of $D_{(i,r)}$, and, arguing completely analogously with right modules instead of left modules, we also deduce that $\tilde{f}_{(i,r)}A$ is a projective cover of $D'_{(i,r)}$. Thus, by (9), we obtain

$$D'_{(i,r)}\cong D^*_{(i,r)} \text{ as right } A\text{-modules.} \tag{12}$$

Consequently, we have:

4.5. Corollary. *Suppose that the group orders $|\Gamma_{e_1}|,\ \ldots,\ |\Gamma_{e_n}|$ are invertible in k. For $(i,r)\in\Lambda$, the corresponding standard right A-module, with respect to both $\leq$ and $\trianglelefteq$, is given by $\Delta'_{(i,r)}$ in (10). Furthermore, with respect to both $\leq$ and $\trianglelefteq$, the costandard left A-modules $\nabla_{(i,r)}$ and the costandard right A-modules $\nabla'_{(i,r)}$ are given by*

$$\nabla_{(i,r)}\cong\mathrm{Hom}_{e'_iAe'_i}(e'_iA,\tilde{T}_{(i,r)}) \quad \textit{and} \quad \nabla'_{(i,r)}\cong\mathrm{Hom}_{e'_iAe'_i}(Ae'_i,\tilde{T}^*_{(i,r)})\,.$$

Moreover,

$$\Delta'_{(i,r)}\cong\nabla^*_{(i,r)} \quad \textit{and} \quad \nabla'_{(i,r)}\cong\Delta^*_{(i,r)}$$

as right A-modules.

Proof. The first statement has already been derived in 4.4. Now let $(i,r)\in\Lambda$. Again, by the considerations in 4.4, the module $\Delta'_{(i,r)}$ is the right standard A-modules with respect to $\leq$ and $\trianglelefteq$. Let $I_{(i,r)}$ denote an injective envelope of the simple left A-module $D_{(i,r)}$. Then $I^*_{(i,r)}$ is a projective cover of the simple right A-module $D^*_{(i,r)}$, and thus of $D'_{(i,r)}$, by (12). Therefore, $\nabla^*_{(i,r)}$ is a quotient of $I^*_{(i,r)}$ such that each composition factor of $\mathrm{Rad}(\nabla^*_{(i,r)})$ is of the form $D^*_{(j,s)}\cong D'_{(j,s)}$ with $(j,s)\lhd(i,r)$, and $\nabla^*_{(i,r)}$ is the largest such quotient. For otherwise, by taking duals again, $I_{(i,r)}$ would have a submodule M strictly containing $\nabla_{(i,r)}$ such that the composition factors of $M/\mathrm{Soc}(M)$ are of the form $D_{(j,s)}$ with $(j,s) \lhd (i,r)$, which is not the case. Therefore, both $\Delta'_{(i,r)}$ and $\nabla^*_{(i,r)}$ are standard right A-modules with respect to $\leq$ and $\trianglelefteq$, with head isomorphic to $D'_{(i,r)} \cong D^*_{(i,r)}$. But this implies $\Delta'_{(i,r)} \cong \nabla^*_{(i,r)}$. Similarly, one shows that $\Delta_{(i,r)} = (\nabla'_{(i,r)})^*$.

Finally, by the usual adjunction isomorphism and (11), we obtain that

$$\begin{aligned}\nabla_{(i,r)} &\cong (\Delta'_{(i,r)})^* = \mathrm{Hom}_k(\tilde{T}'_{(i,r)} \otimes_{e'_iAe'_i} e'_iA, k) \cong \mathrm{Hom}_{e'_iAe'_i}(e'_iA, \mathrm{Hom}_k(\tilde{T}'_{(i,r)}, k)) \\ &\cong \mathrm{Hom}_{e'_iAe'_i}(e'_iA, \tilde{T}_{(i,r)})\end{aligned}$$

as left A-modules, and similarly one obtains an isomorphism $\nabla'_{(i,r)} \cong \mathrm{Hom}_{e'_iAe'_i}(Ae'_i, \tilde{T}^*_{(i,r)})$ of right A-modules. □

5. Reformulations of the relation $\sqsubseteq$

We retain the notation from Section 4. Thus, we assume the notation and situation from 3.1 and 3.3, and also assume that, for every idempotent endomorphism e in C, the group order $|\Gamma_e|$ is invertible in k. Thus, the corresponding twisted group algebra $k_\alpha\Gamma_e$ will then again be semisimple. For every $(i,r) \in \Lambda$, let $f_{(i,r)}$ denote the primitive central idempotent of $k_\alpha\Gamma_{e_i}$ satisfying $f_{(i,r)}T_{(i,r)} \neq \{0\}$, or equivalently, $T^*_{(i,r)}f_{(i,r)} \neq \{0\}$. The following proposition gives equivalent reformulations of one of the two conditions in Definition 4.1(ii), concerning the relation $\sqsubseteq$ and the resulting partial order $\trianglelefteq$ on Λ.

5.1. Proposition. *Let $i,j \in \{1,\ldots,n\}$ be such that $S_j <_{\mathscr{J}} S_i$. Then, for $(i,r),(j,s) \in \Lambda$, the following are equivalent:*

(i) $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$;
(ii) *there is some $t \in S_j \cap e_i \circ S \circ e_j$ with $f_{(i,r)} \cdot t \cdot f_{(j,s)} \neq 0$ in A;*
(iii) *there exists a non-zero $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule homomorphism from $f_{(i,r)}k_\alpha\Gamma_{e_i} \otimes f_{(j,s)}k_\alpha\Gamma_{e_j}$ to $e'_i(A/J_{j-1})e'_j$;*
(iv) *there exists a non-zero $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule homomorphism from $T_{(i,r)} \otimes T^*_{(j,s)}$ to $e'_i(A/J_{j-1})e'_j$;*
(v) *$T_{(i,r)} \otimes T^*_{(j,s)}$ is isomorphic to a direct summand of the $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule $e'_i(A/J_{j-1})e'_j$;*
(vi) $f_{(i,r)} \cdot (A/J_{j-1}) \cdot f_{(j,s)} \neq \{0\}$.

Proof. (i) $\iff$ (ii): Note first that we have

$$f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1} \Leftrightarrow \exists t \in S_{\leq j} : f_{(i,r)} \cdot t \cdot f_{(j,s)} \notin J_{j-1}\,.$$

So, suppose that $t \in S_{\leq j}$ is such that $f_{(i,r)} \cdot t \cdot f_{(j,s)} \notin J_{j-1}$. Then $t' := e_i \circ t \circ e_j$ is such that $t' \in e_i \circ S \circ e_j \cap S_{\leq j}$ and $f_{(i,r)} \cdot t' \cdot f_{(j,s)} = f_{(i,r)} \cdot t \cdot f_{(j,s)} \notin J_{j-1}$. But if t' was contained in some $\mathscr{J}$-class S_l with $l < j$ then we would get $f_{(i,r)} \cdot t' \cdot f_{(j,s)} \in J_{j-1}$, which is not the case. Thus $t' \in e_i \circ S \circ e_j \cap S_j$, which gives

$$f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1} \Leftrightarrow \exists\, t \in S_j \cap e_i \circ S \circ e_j \colon f_{(i,r)} \cdot t \cdot f_{(j,s)} \notin J_{j-1}\,. \tag{13}$$

Now let $t \in S_j \cap e_i \circ S \circ e_j$. Then $f_{(i,r)} \cdot t \cdot f_{(j,s)}$ is a k-linear combination of elements of the form $t_i \circ t \circ t_j$, for suitable $t_i \in \Gamma_{e_i}$ and $t_j \in \Gamma_{e_j}$. Fix such elements t_i and t_j. Then we have

$$\begin{aligned} S \circ t \circ S &= S \circ e_i \circ t \circ e_j \circ S \\ &= S \circ t_i^{-1} \circ t_i \circ t \circ t_j \circ t_j^{-1} \circ S \subseteq S \circ t_i \circ t \circ t_j \circ S \subseteq S \circ t \circ S, \end{aligned}$$

hence $\mathscr{J}(t_i \circ t \circ t_j) = \mathscr{J}(t) = S_j$, that is, $f_{(i,r)} \cdot t \cdot f_{(j,s)} \in kS_j$. Since $kS_j \cap J_{j-1} = \{0\}$, this finally yields

$$f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1} \Leftrightarrow \exists\, t \in S_j \cap e_i \circ S \circ e_j \colon f_{(i,r)} \cdot t \cdot f_{(j,s)} \neq 0\,,$$

proving the equivalence of (i) and (ii).

(i) $\Rightarrow$ (iii): Let $a \in f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \smallsetminus J_{j-1}$. Then the map

$$\varphi \colon f_{(i,r)} \cdot k_\alpha\Gamma_{e_i} \otimes f_{(j,s)} \cdot k_\alpha\Gamma_{e_j} \to f_{(i,r)} \cdot (A/J_{j-1}) \cdot f_{(j,s)} \subseteq e_i'(A/J_{j-1})e_j'$$

given by $\varphi(x \otimes y) = xay + J_{j-1}$, for $x \in f_{(i,r)} \cdot k_\alpha\Gamma_{e_i}$ and $y \in f_{(j,s)} \cdot k_\alpha\Gamma_{e_j} = k_\alpha\Gamma_{e_j} \cdot f_{(j,s)}$, is a $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule homomorphism with $\varphi(f_{(i,r)} \otimes f_{(j,s)}) = f_{(i,r)} \cdot a \cdot f_{(j,s)} + J_{j-1} = a + J_{j-1} \neq J_{j-1}$.

(iii) $\Rightarrow$ (i): Suppose that φ is a non-zero $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule homomorphism from $f_{(i,r)} \cdot k_\alpha\Gamma_{e_i} \otimes f_{(j,s)} \cdot k_\alpha\Gamma_{e_j}$ to $e_i'(A/J_{j-1})e_j'$. Then $0 \neq \varphi(f_{(i,r)} \otimes f_{(j,s)}) \in f_{(i,r)} \cdot (A/J_{j-1}) \cdot f_{(j,s)} \subseteq e_i'(A/J_{j-1})e_j'$. Note that $Af_{(j,s)} \subseteq Ae_j'f_{(j,s)} \subseteq J_jf_{(j,s)}$. Thus also $f_{(i,r)} \cdot A \cdot f_{(j,s)} = f_{(i,r)} \cdot J_j \cdot f_{(j,s)}$, and we obtain $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$.

(iii) $\iff$ (iv) $\iff$ (v): Since $k_\alpha\Gamma_{e_i}$ and $k_\alpha\Gamma_{e_j}$ are semisimple k-algebras, the category of $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodules is semisimple and the bimodule $f_{(i,r)}k_\alpha\Gamma_{e_i} \otimes f_{(j,s)}k_\alpha\Gamma_{e_j}$ is isomorphic to a direct sum of copies of $T_{(i,r)} \otimes T^*_{(j,s)}$. The assertions (iii)–(v) are now clearly equivalent.

(i) $\iff$ (vi): This follows immediately from $f_{(i,r)} \cdot A \cdot f_{(j,s)} = f_{(i,r)} \cdot J_j \cdot f_{(j,s)}$. □

Similarly to the proof of the previous proposition one proves the following symmetric result:

5.2. Proposition. *Let again $i, j \in \{1, \ldots, n\}$ be such that $S_j <_{\mathscr{J}} S_i$ and let $(i,r), (j,s) \in \Lambda$. Then the following are equivalent:*

(i) $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$;
(ii) *there is some $t \in S_j \cap e_j \circ S \circ e_i$ with $f_{(j,s)} \cdot t \cdot f_{(i,r)} \neq 0$ in A;*

(iii) *there exists a non-zero* $(k_\alpha\Gamma_{e_j}, k_\alpha\Gamma_{e_i})$*-bimodule homomorphism from* $f_{(j,s)}k_\alpha\Gamma_{e_j} \otimes f_{(i,r)}k_\alpha\Gamma_{e_i}$ *to* $e'_j(A/J_{j-1})e'_i$;

(iv) *there exists a non-zero* $(k_\alpha\Gamma_{e_j}, k_\alpha\Gamma_{e_i})$*-bimodule homomorphism from* $T_{(j,s)} \otimes T^*_{(i,r)}$ *to* $e'_j(A/J_{j-1})e'_i$;

(v) $T_{(j,s)} \otimes T^*_{(i,r)}$ *is isomorphic to a direct summand of the* $(k_\alpha\Gamma_{e_j}, k_\alpha\Gamma_{e_i})$*-bimodule* $e'_j(A/J_{j-1})e'_i$;

(vi) $f_{(j,s)} \cdot (A/J_{j-1}) \cdot f_{(i,r)} \neq \{0\}$.

5.3. Remark. Suppose that $i \in \{1, \ldots, n\}$ and that the restriction of α to Γ_{e_i} is a coboundary, that is, that there exists a function $\mu: \Gamma_{e_i} \to k^\times$ such that $\alpha(t, t') = \mu(t)\mu(t')\mu(t \circ t')^{-1}$, for all $t, t' \in \Gamma_{e_i}$. Then we have a k-algebra isomorphism

$$k_\alpha\Gamma_{e_i} \xrightarrow{\sim} k\Gamma_{e_i}, \quad t \mapsto \mu(t) \cdot t, \tag{14}$$

where $t \in \Gamma_{e_i}$, and where $k\Gamma_{e_i}$ denotes the (untwisted) group algebra of Γ_{e_i} over k. In particular, this holds if the restriction of α to Γ_{e_i} is the constant function with some value $a \in k^\times$. In this case μ can also be chosen to be the constant function with value a.

If also $j \in \{1, \ldots, n\}$ and if the restriction of α to Γ_{e_j} is a coboundary then we have k-algebra isomorphisms

$$k_\alpha\Gamma_{e_i} \otimes k_\alpha\Gamma_{e_j} \cong k\Gamma_{e_i} \otimes k\Gamma_{e_j} \cong k[\Gamma_{e_i} \times \Gamma_{e_j}]. \tag{15}$$

We may and shall then identify every left $k_\alpha\Gamma_{e_i} \otimes k_\alpha\Gamma_{e_j}$-module with a $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module. Moreover, we shall always identify every $(k\Gamma_{e_i}, k\Gamma_{e_j})$-bimodule M with the left $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module M defined by $(x, y) \cdot m := x \cdot m \cdot y^{-1}$, for $m \in M$, $x \in \Gamma_{e_i}$, $y \in \Gamma_{e_j}$. Note that, under these identifications, the $(k_\alpha\Gamma_{e_i}, k_\alpha\Gamma_{e_j})$-bimodule $T_{(i,r)} \otimes T^*_{(j,s)}$ becomes the left $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module $T_{(i,r)} \otimes T^*_{(j,s)}$.

In our applications to biset functors and Brauer algebras we shall see that the restrictions of the relevant 2-cocycles to the maximal subgroups Γ_{e_i} of the respective categories are always 2-coboundaries (in fact, even constant). Before simplifying the first condition in Definition 4.1(ii) further in this situation, we introduce a last bit of notation.

5.4. Notation. Let $i, j \in \{1, \ldots, n\}$ be such that $S_j <_{\mathscr{J}} S_i$. Then the set $S_j \cap e_i \circ S \circ e_j$ carries a left $\Gamma_{e_i} \times \Gamma_{e_j}$-set structure via

$$(x, y) \cdot t := x \circ t \circ y^{-1} \quad (x \in \Gamma_{e_i},\, y \in \Gamma_{e_j},\, t \in S_j \cap e_i \circ S \circ e_j).$$

In fact, clearly $x \circ t \circ y^{-1} \in e_i \circ S \circ e_j$, and $x \circ t \circ y^{-1} \in S_j = \mathscr{J}(t)$, since $x \circ t \circ y^{-1} \in S \circ t \circ S$ and $t = x^{-1} \circ x \circ t \circ y^{-1} \circ y \in S \circ x \circ t \circ y^{-1} \circ S$. We denote the stabilizer of $t \in S_j \cap e_i \circ S \circ e_j$ by $\mathrm{stab}_{\Gamma_{e_i} \times \Gamma_{e_j}}(t)$, or simply by $\mathrm{stab}(t)$ when no confusion concerning the groups is possible.

Note that, analogously, the set $S_j \cap e_j \circ S \circ e_i$ carries a left $\Gamma_{e_j} \times \Gamma_{e_i}$-module structure.

5.5. Corollary. *Let $i, j \in \{1, \ldots, n\}$ be such that $S_j <_{\mathscr{J}} S_i$. Suppose that α restricts to constant 2-cocycles on Γ_{e_i} and on Γ_{e_j} with values a_i and a_j, respectively. Moreover, let $(i, r), (j, s) \in \Lambda$.*

(a) *Assume that $\alpha(x, t) = a_i$ and $\alpha(t, y) = a_j$ for all $x \in \Gamma_{e_i}$, $t \in S_j \cap e_i \circ S \circ e_j$, and $y \in \Gamma_{e_j}$. Then one has $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$ if and only if there is some $t \in S_j \cap e_i \circ S \circ e_j$ such that $T_{(i,r)} \otimes T^*_{(j,s)}$ is isomorphic to a direct summand of the permutation $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module $\mathrm{Ind}_{\mathrm{stab}(t)}^{\Gamma_{e_i} \times \Gamma_{e_j}}(k)$;*

(b) *Assume that $\alpha(y, t) = a_j$ and $\alpha(t, x) = a_i$ for all $y \in \Gamma_{e_j}$, $t \in S_j \cap e_j \circ S \circ e_i$, and $x \in \Gamma_{e_i}$. Then one has $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$ if and only if there is some $t \in S_j \cap e_j \circ S \circ e_i$ such that $T_{(j,s)} \otimes T^*_{(i,r)}$ is isomorphic to a direct summand of the permutation $k[\Gamma_{e_j} \times \Gamma_{e_i}]$-module $\mathrm{Ind}_{\mathrm{stab}(t)}^{\Gamma_{e_j} \times \Gamma_{e_i}}(k)$.*

Proof. We identify every $(k_\alpha \Gamma_{e_i}, k_\alpha \Gamma_{e_j})$-bimodule with a $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module as indicated in Remark 5.3. To prove (a), we first observe again that $J_j e'_j = A e'_j$, and we deduce that the cosets in A/J_{j-1} of the elements of $S_j \cap e_i \circ S \circ e_j$ form in fact a k-basis of $e'_i(A/J_{j-1})e'_j$. The hypothesis in (a) implies that under the resulting $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module structure, this k-basis is permuted by $\Gamma_{e_i} \times \Gamma_{e_j}$ in the same way as in 5.4. Thus, we obtain a $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module isomorphism

$$e'_i(A/J_{j-1})e'_j \cong \bigoplus_t \mathrm{Ind}_{\mathrm{stab}(t)}^{\Gamma_{e_i} \times \Gamma_{e_j}}(k)\,,$$

where t varies over a set of representatives of the $\Gamma_{e_i} \times \Gamma_{e_j}$-orbits on $S_j \cap e_i \circ S \circ e_j$. Now assertion (a) follows from Proposition 5.1(v).

Assertion (b) is proved analogously using Proposition 5.2. □

6. Duality in twisted category algebras

Again, we retain the notation from Section 4. Thus, we assume the notation and situation from 3.1 and 3.3, and also assume that, for every idempotent endomorphism e in C, the group order $|\Gamma_e|$ is invertible in k, so that Theorem 4.3 applies.

In this section we shall show that the partial orders $\leqslant$ and $\trianglelefteq$ defined in 3.6 and in Definition 4.1 behave well under a natural notion of duality introduced in Hypotheses 6.1. This will allow us to apply Proposition 2.4. If α restricts to particular coboundaries on the groups Γ_{e_i}, then we shall show that the two conditions in Definition 4.1(ii) are equivalent.

We shall need the following hypotheses, which will be satisfied in many instances, and, in particular, in the two applications we are interested in; see Sections 7 and 8.

6.1. Hypotheses. For the remainder of this section, suppose that there is a contravariant functor $-^\circ : \mathsf{C} \to \mathsf{C}$ satisfying the following properties:

(i) $X^\circ = X$, for every $X \in \mathrm{Ob}(\mathsf{C})$;
(ii) $(s^\circ)^\circ = s$, for every $s \in \mathrm{Mor}(\mathsf{C}) = S$;
(iii) $s \circ s^\circ \circ s = s$, for every $s \in \mathrm{Mor}(\mathsf{C}) = S$;
(iv) $\alpha(s,t) = \alpha(t^\circ, s^\circ)$, for all $s,t \in S$;
(v) $e_i^\circ = e_i$, for $i = 1,\ldots,n$.

Given any subset M of S, we set $M^\circ := \{s^\circ : s \in M\}$.

6.2. Remark. Note that, given a contravariant functor $-^\circ : \mathsf{C} \to \mathsf{C}$ with properties (i)–(iv) above, we can always choose the idempotents $e_1,\ldots,e_n$ such that they satisfy (v), by setting $e_i := s_i \circ s_i^\circ$, for any $s_i \in S_i$.

As immediate consequences of Hypotheses 6.1 we obtain the following, which will be used repeatedly throughout this article.

6.3. Lemma. (a) *The functor $-^\circ : \mathsf{C} \to \mathsf{C}$ induces a k-algebra anti-involution*

$$-^\circ : A \to A, \quad \sum_{s\in S} a_s s \mapsto \sum_{s\in S} a_s s^\circ ,$$

where $a_s \in k$, for $s \in S$.

(b) *For $i \in \{1,\ldots,n\}$, one has $S_i = S_i^\circ$, and thus $J_i = J_i^\circ$, where J_i is the ideal in A defined in (4).*

(c) *For $i \in \{1,\ldots,n\}$ and $x \in \Gamma_{e_i}$, one has $x^\circ = x^{-1}$.*

Proof. Part (a) is a straightforward calculation. For (b) note that Hypothesis 6.1(iii) implies $\mathscr{J}(s^\circ) \leq_{\mathscr{J}} \mathscr{J}(s)$, and (ii) then implies $\mathscr{J}(s) \leq_{\mathscr{J}} \mathscr{J}(s^\circ)$. Thus $\mathscr{J}(s) = \mathscr{J}(s^\circ)$, and both assertions in (b) are immediate from this.

For (c), let $i \in \{1,\ldots,n\}$, and let $X_i \in \mathrm{Ob}(\mathsf{C})$ be such that $e_i \in \mathrm{End}_{\mathsf{C}}(X_i)$. Then, for $x \in \Gamma_{e_i}$, we have $x^\circ = (e_i \circ x \circ e_i)^\circ = e_i^\circ \circ x^\circ \circ e_i^\circ = e_i \circ x^\circ \circ e_i$, thus $x^\circ \in e_i \circ \mathrm{End}_{\mathsf{C}}(X_i) \circ e_i$. Since $x \in \Gamma_{e_i}$, there is some $y \in \Gamma_{e_i}$ with $x \circ y = e_i = y \circ x$. Hence

$$y^\circ \circ x^\circ = e_i^\circ = x^\circ \circ y^\circ ,$$

so that also $x^\circ \in \Gamma_{e_i}$, since $e_i^\circ = e_i$. Moreover, by Hypothesis 6.1(iii), $x \circ x^\circ$ is then an idempotent in the group Γ_{e_i}, implying $x \circ x^\circ = e_i$, thus $x^\circ = x^{-1}$. $\square$

6.4. Dual A-modules. (a) We apply the conventions from 2.3 to the k-algebra involution $-^\circ : A \to A$ from Lemma 6.3(a). Thus, whenever M is a left A-module, its k-linear dual becomes a left A-module M°, via (1).

(b) Suppose that B is a (not necessarily unitary) k-subalgebra of A such that $B^\circ = B$. Then the k-algebra anti-involution $-^\circ : A \to A$ restricts to a k-algebra anti-involution of B, and thus also the k-linear dual of every left B-module N becomes a left B-module, which we again denote by N°.

By our Hypothesis 6.1(v) and Lemma 6.3(b), this is, in particular, satisfied if B is one of the algebras $e_i'Ae_i'$ or $k_\alpha\Gamma_{e_i}$, for $i = 1, \ldots, n$.

(c) Again suppose that B is a k-subalgebra of A such that $B^\circ = B$. Let M be a left B-module, and let f be a central idempotent in B. Then f° is also a central idempotent in B, and one easily checks that the restriction map

$$f^\circ \cdot M^\circ \to (f \cdot M)^\circ\,, \quad \varphi \mapsto \varphi|_{f\cdot M}\,, \tag{16}$$

defines a left B-module isomorphism.

So, in particular, if $B = k_\alpha\Gamma_{e_i}$, for some $i \in \{1, \ldots, n\}$, and if $f_{(i,r)}$ is the block idempotent of $k_\alpha\Gamma_{e_i}$ corresponding to the simple module $T_{(i,r)}$ then $f^\circ_{(i,r)}$ is the block idempotent of $k_\alpha\Gamma_{e_i}$ corresponding to the simple module $T^\circ_{(i,r)}$.

(d) Suppose now that $f \in A$ is an idempotent such that $f^\circ = f$, let $B := fAf$, and let M be a left A-module. Then the restriction map

$$f \cdot M^\circ \mapsto (f \cdot M)^\circ\,, \quad \varphi \mapsto \varphi|_{f\cdot M}\,, \tag{17}$$

is a left B-module isomorphism.

6.5. Notation. As before, for each $(j, s) \in \Lambda$, we denote by $\Delta_{(j,s)}$ and $\nabla_{(j,s)}$ the standard A-module and the costandard A-module, respectively, labelled by (j, s) with respect to $(\Lambda, \leq)$ and $(\Lambda, \trianglelefteq)$, as defined in (5) and determined in Corollary 4.5. In accordance with 2.3, for $(i, r) \in \Lambda$, we denote by $(i, r)^\circ \in \Lambda$ the label of the simple A-module $D^\circ_{(i,r)}$. Analogously, let $r^\circ \in \{1, \ldots, l_i\}$ be such that $T_{(i,r^\circ)} \cong T^\circ_{(i,r)}$ as $k_\alpha\Gamma_{e_i}$-modules.

With this, we now have:

6.6. Proposition. *For $(i, r), (j, s) \in \Lambda$, one has*

(a) $\Delta_{(i,r)^\circ} \cong Ae_i' \otimes_{e_i'Ae_i'} \tilde{T}^\circ_{(i,r)} \cong \Delta_{(i,r^\circ)}$, *thus* $(i, r)^\circ = (i, r^\circ)$;
(b) $(i, r) \leq (j, s)$ *if and only if* $(i, r)^\circ \leq (j, s)^\circ$;
(c) $(i, r) \trianglelefteq (j, s)$ *if and only if* $(i, r)^\circ \trianglelefteq (j, s)^\circ$;
(d) $\Delta_{(i,r)^\circ} \cong \nabla^\circ_{(i,r)}$ *and* $\nabla_{(i,r)^\circ} \cong \Delta^\circ_{(i,r)}$.

Proof. Since $T^\circ_{(i,r)} \cong T_{(i,r^\circ)}$ as $k_\alpha\Gamma_{e_i}$-modules, we have $M := Ae_i' \otimes_{e_i'Ae_i'} \tilde{T}^\circ_{(i,r)} \cong \Delta_{(i,r^\circ)}$ as A-modules. In order to show that $(i, r^\circ) = (i, r)^\circ$, recall that every standard module is determined by its head, and the isomorphism class of $D_{(i,r^\circ)} = \mathrm{Hd}(M)$ is determined by the property

$$e_i' \cdot M \cong \tilde{T}^\circ_{(i,r)} \cong e_i' \cdot \mathrm{Hd}(M)$$

as $e_i'Ae_i'$-modules. Since, by (6), $e_i' \cdot D_{(i,r)} \cong e_i' \cdot \Delta_{(i,r)} \cong \tilde{T}_{(i,r)}$ as $e_i'Ae_i'$-modules, we also have

$$\tilde{T}_{(i,r^\circ)} \cong \tilde{T}^\circ_{(i,r)} \cong (e_i' \cdot D_{(i,r)})^\circ \cong e_i' \cdot D^\circ_{(i,r)} \cong e_i' \cdot D_{(i,r)^\circ}$$

as $e'_iAe'_i$-modules. Note that here we applied (17) with $f = e'_i$ to derive the third isomorphism. So, altogether, this implies $D_{(i,r)^\circ} \cong D_{(i,r^\circ)}$ and $\Delta_{(i,r)^\circ} \cong \Delta_{(i,r^\circ)} \cong M$, proving (a). From this, assertion (b) follows immediately.

From (a), 6.4(c), and Lemma 6.3(b) we now obtain

$$\begin{aligned}(i,r) \sqsubset (j,s) &\Leftrightarrow S_j <_{\mathscr{J}} S_i \text{ and } (f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1} \text{ or } f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1})\\ &\Leftrightarrow S_j <_{\mathscr{J}} S_i \text{ and } (f^\circ_{(j,s)} \cdot J_j \cdot f^\circ_{(i,r)} \nsubseteq J_{j-1} \text{ or } f^\circ_{(i,r)} \cdot J_j \cdot f^\circ_{(j,s)} \nsubseteq J_{j-1})\\ &\Leftrightarrow (i,r^\circ) \sqsubset (j,s^\circ) \Leftrightarrow (i,r)^\circ \sqsubset (j,s)^\circ\,,\end{aligned}$$

which proves (c).

Assertion (d) follows from (a), (b), and Proposition 2.4. □

6.7. Corollary. *Let $i,j \in \{1,\ldots,n\}$ be such that $S_j <_{\mathscr{J}} S_i$ and suppose that α restricts to constant 2-cocycles on Γ_{e_i} and on Γ_{e_j} with values a_i and a_j, respectively. Assume further that one has $\alpha(x,t) = a_i$ and $\alpha(t,y) = a_j$ for all $x \in \Gamma_{e_i}$, $t \in S_j \cap e_i \circ S \circ e_j$, and $y \in \Gamma_{e_j}$. Then, for $(i,r),(j,s) \in \Lambda$, one has $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$ if and only if $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$.*

Proof. Recall that, by Lemma 6.3(b), we have

$$f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1} \Leftrightarrow f^\circ_{(i,r)} \cdot J_j \cdot f^\circ_{(j,s)} \nsubseteq J_{j-1}\,. \tag{18}$$

By Lemma 6.3(c), the simple left $k\Gamma_{e_i}$-module $T^\circ_{(i,r)}$ associated with $f^\circ_{(i,r)}$ is equal to $T^*_{(i,r)}$. Thus, by Proposition 5.1 and Remark 5.3, we obtain

$$\begin{aligned}f^\circ_{(i,r)} \cdot J_j \cdot f^\circ_{(j,s)} \nsubseteq J_{j-1} &\Leftrightarrow \mathrm{Hom}_{k[\Gamma_{e_i}\times\Gamma_{e_j}]}(T^*_{(i,r)} \otimes T_{(j,s)}, e'_i(A/J_{j-1})e'_j) \neq \{0\}\\ &\Leftrightarrow \mathrm{Hom}_{k[\Gamma_{e_i}\times\Gamma_{e_j}]}(T_{(i,r)} \otimes T^*_{(j,s)}, (e'_i(A/J_{j-1})e'_j)^*) \neq \{0\}\,.\end{aligned}$$

The last equivalence holds because $k[\Gamma_{e_i} \times \Gamma_{e_j}]$ is semisimple.

But, as in the proof of Corollary 5.5, the $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-module $e'_i(A/J_{j-1})e'_j$ is a permutation module, and thus self-dual. Hence $e'_i(A/J_{j-1})e'_j \cong (e'_i(A/J_{j-1})e'_j)^*$ as $k[\Gamma_{e_i} \times \Gamma_{e_j}]$-modules. Altogether, this implies

$$\begin{aligned}f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1} &\Leftrightarrow \mathrm{Hom}_{k[\Gamma_{e_i}\times\Gamma_{e_j}]}(T_{(i,r)} \otimes T^*_{(j,s)}, e'_i(A/J_{j-1})e'_j) \neq \{0\}\\ &\Leftrightarrow f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}\,,\end{aligned}$$

where the last equivalence again follows from Proposition 5.1. □

7. Application I: biset functors

In this section we shall apply our results from Sections 4, 5 and 6 to the case where the twisted category algebra is the one introduced in [1, Example 5.15]. This category algebra is closely related to the category of biset functors, as observed in [1]. The goal of this section is to reprove, via a different approach, a result due to Webb (see [21, Theorem 7.2]) stating that the category of biset functors over a field of characteristic zero is a highest weight category, if one disregards a finiteness condition on injective objects. We shall also give an improvement on the relevant partial order on the set Λ of isomorphism classes of simple modules. Here we only deal with the case that the underlying category of finite groups has finitely many objects. In this case the finiteness condition on injective objects is clearly satisfied. This situation is sufficient for many purposes, as established in [21].

We begin by recalling the relevant notation as well as some results from [1] about the category C we need to consider. The connection to biset functors will be given in more detail in Remark 7.2. From now on we suppose that k is a field of characteristic 0.

7.1. Notation. (a) Given finite groups G and H, we denote by p_1 and p_2 the canonical projections $G \times H \to G$ and $G \times H \to H$, respectively. Moreover, for every $L \leq G \times H$, we set $k_1(L) := \{g \in G \mid (g, 1) \in L\}$ and $k_2(L) := \{h \in H \mid (1, h) \in L\}$, so that $k_i(L) \trianglelefteq p_i(L)$, for $i = 1, 2$.

Note that, by Goursat's Lemma, we may and shall from now on identify every subgroup L of $G \times H$ with the quintuple $(p_1(L), k_1(L), \eta_L, p_2(L), k_2(L))$, where η_L is the group isomorphism given by

$$\eta_L : p_2(L)/k_2(L) \xrightarrow{\sim} p_1(L)/k_1(L)\,, \quad hk_2(L) \mapsto gk_1(L)\,,$$

whenever $(g, h) \in L$. The common isomorphism class of $p_1(L)/k_1(L)$ and $p_2(L)/k_2(L)$ will be denoted by $q(L)$.

Furthermore, in the case where $p_1(L) = G = H = p_2(L)$ and $k_1(L) = 1 = k_2(L)$, we have $\eta_L = \alpha$ for some automorphism α of G, and we also denote the group L by $\Delta_\alpha(G)$; in particular, for $\alpha = \mathrm{id}_G$, this gives $\Delta_\alpha(G) = \Delta(G) := \{(g, g) \mid g \in G\}$.

If $g \in G$ then the corresponding inner automorphism $G \to G$, $x \mapsto gxg^{-1}$, will be denoted by c_g, and we also set $\Delta_g(G) := \Delta_{c_g}(G)$.

By a *section* of a finite group G we understand a pair (P, K) such that $K \trianglelefteq P \leq G$.

(b) Let C be a category with the following properties: the objects of C form a finite set of pairwise non-isomorphic finite groups that is *section-closed*, that is, whenever $G \in \mathrm{Ob}(\mathsf{C})$ and (P, K) is a section of G then there is some $H \in \mathrm{Ob}(\mathsf{C})$ such that $P/K \cong H$. The morphism set, for $G, H \in \mathrm{Ob}(\mathsf{C})$, is defined by

$$\mathrm{Hom}_{\mathsf{C}}(H, G) := \mathsf{C}_{G,H} := \{L \mid L \leq G \times H\}\,,$$

and the composition of morphisms in C is given by

$$L \circ M := L * M := \{(g, k) \in G \times K \mid \exists h \in H : (g, h) \in L,\, (h, k) \in M\},$$

for $G, H, K \in \mathrm{Ob}(\mathsf{C})$, $L \in \mathsf{C}_{G,H}$, $M \in \mathsf{C}_{H,K}$. For $L \in \mathsf{C}_{G,H}$, let $L^\circ := \{(h, g) \in H \times G \mid (g, h) \in L\} \in \mathsf{C}_{H,G}$. The category C is finite by construction, and split since $L * L^\circ * L = L$, for any $G, H \in \mathrm{Ob}(\mathsf{C})$ and $L \in \mathsf{C}_{G,H}$; see [1, Proposition 2.7(ii)]. Note that, by the last statement in 3.1(a), the assumption that the objects of C are pairwise non-isomorphic groups is not a significant restriction.

By [1, Proposition 3.5], we have a 2-cocycle $\kappa \in Z^2(\mathsf{C}, k^\times)$ defined by

$$\kappa(L, M) := \frac{|k_2(L) \cap k_1(M)|}{|H|}, \tag{19}$$

for $G, H, K \in \mathrm{Ob}(\mathsf{C})$, $L \in \mathsf{C}_{G,H}$, $M \in \mathsf{C}_{H,K}$. The resulting twisted category algebra $k_\kappa\mathsf{C}$ will be denoted by A, for the remainder of this section. Moreover, we denote the objects of C by $G_1, \ldots, G_n$ such that $|G_i| \leqslant |G_{i+1}|$, for $i = 1, \ldots, n-1$, and we set $S := \mathrm{Mor}(\mathsf{C})$. Note that mapping $L \in \mathsf{C}_{G,H}$ to $L^\circ \in \mathsf{C}_{H,G}$ gives rise to a contravariant functor $-^\circ \colon \mathsf{C} \to \mathsf{C}$ satisfying the properties (i)–(v) in Hypotheses 6.1 with respect to the 2-cocycle κ of C. Concrete idempotents $e_1, \ldots, e_n$ will be determined in Proposition 7.3 below.

7.2. Remark. Biset functors on C over k are related to the twisted category algebra $A = k_\kappa\mathsf{C}$ as follows. For each $i = 1, \ldots, n$, we set $\varepsilon_i := \sum_{g \in G_i} \Delta_g(G_i) = |Z(G_i)| \cdot \sum_{\alpha \in \mathrm{Inn}(G_i)} \Delta_\alpha(G_i) \in e_i' A e_i'$, and we set $\varepsilon := \varepsilon_{\mathsf{C}} := \sum_{i=1}^n \varepsilon_i$. Then $\varepsilon_1, \ldots, \varepsilon_n$ are pairwise orthogonal idempotents of A, ε is an idempotent of A, and the left module category of the k-algebra $\varepsilon A \varepsilon$ is equivalent to the category of biset functors on C over k; see [1, Example 5.15(c)] for more detailed explanations. By Theorem 4.3, we know that A is quasi-hereditary with respect to $(\Lambda, \trianglelefteq)$, using the notation from Sections 3 and 4.

Our goal is to show that also the condensed k-algebra $\varepsilon A \varepsilon$ is quasi-hereditary. Recall from Green's idempotent condensation theory (see [12, Section 6.2]) that the simple modules of $\varepsilon A \varepsilon$ are of the form $\varepsilon \cdot D_{(i,r)}$, with $(i,r) \in \Lambda$ such that $\varepsilon \cdot D_{(i,r)} \neq \{0\}$ and that any two distinct such indices (i,r) result in non-isomorphic simple $\varepsilon A \varepsilon$-modules. Thus, the labelling set Λ' of the isomorphism classes of simple $\varepsilon A \varepsilon$-modules can be considered as a subset of Λ in a natural way. Moreover, by Proposition 2.6, it suffices to show that the idempotent ε satisfies the following property: If $(i,r) \trianglelefteq (j,s)$ are elements in Λ and if $\varepsilon \cdot D_{(i,r)} \neq \{0\}$ then also $\varepsilon \cdot D_{(j,s)} \neq \{0\}$. This will be done in Theorem 7.7. The main reason for introducing the partial order $\trianglelefteq$ in Section 4 is that this property is not satisfied for the partial order $\leq$, as we shall see in Example 7.9 below.

The following proposition establishes quickly the set Λ for the finite split category algebra $A = k_\kappa\mathsf{C}$ and the subset $\Lambda' \subseteq \Lambda$.

7.3. Proposition. (a) *For $L, M \in S$, one has $\mathscr{J}(L) = \mathscr{J}(M)$ if and only if $q(L) = q(M)$. In particular, the elements $e_i := \Delta(G_i) \in \mathsf{C}_{G_i,G_i} \subseteq S$, with $i = 1, \ldots, n$, form a set of representatives of the $\mathscr{J}$-classes of C. For $i = 1, \ldots, n$, we have $e_i^\circ = e_i$, where $-^\circ : \mathsf{C} \to \mathsf{C}$ is the functor in 7.1(b).*

(b) *For $i \in \{1, \ldots, n\}$, the element e_i is an idempotent and $\Gamma_{e_i} = \{\Delta_\alpha(G_i) \mid \alpha \in \mathrm{Aut}(G_i)\}$; in particular, $\mathrm{Aut}(G_i) \cong \Gamma_{e_i}$ via the map $\alpha \mapsto \Delta_\alpha(G_i)$. Moreover, the*

2-cocycle $\kappa \in Z^2(\mathsf{C}, k^\times)$ restricts to a constant 2-cocycle on Γ_{e_i} with value $|G_i|^{-1}$, and the k-linear map

$$k_\kappa\Gamma_{e_i} \to k\mathrm{Aut}(G_i),\ \Delta_\alpha(G_i) \mapsto |G_i|^{-1} \cdot \alpha\,, \quad (\alpha \in \mathrm{Aut}(G_i)) \tag{20}$$

defines a k-algebra isomorphism. If also $j \in \{1, \ldots, n\}$ and $L \in \mathsf{C}_{G_i,G_j}$ then $\kappa(e_i, L) = |G_i|^{-1}$.

(c) *For $(i, r) \in \Lambda$, one has $\varepsilon \cdot D_{(i,r)} \neq \{0\}$ if and only if $\mathrm{Inn}(G_i)$ acts trivially on $T_{(i,r)}$, when viewed as $k\mathrm{Aut}(G_i)$-module via the isomorphism in (b).*

(d) *For $i = 1, \ldots, n$, we set $S_i := \mathscr{J}(e_i)$. Then, for $i, j \in \{1, \ldots, n\}$, one has $S_i \leq_{\mathscr{J}} S_j$ if and only if there is a section (P, K) of G_j with $G_i \cong P/K$. In particular, the ordering $G_1, \ldots, G_n$ has the property that if $\mathscr{J}(e_i) \leq_{\mathscr{J}} \mathscr{J}(e_j)$ then $i \leqslant j$, as required in 3.1(b).*

Proof. Assertions (a) and (b) follow immediately from [1, Proposition 6.3, Proposition 6.4], [17, Lemma 2.1], and the definition of κ.

Part (c) follows immediately from [1, Corollary 7.4] and its proof.

To prove part (d), note that, in the notation of 7.1, we have $e_j = (G_j, 1, \mathrm{id}, G_j, 1)$. Suppose first that $S_i \leq_{\mathscr{J}} S_j$, that is, $S*e_i*S_i \subseteq S*e_j*S$, by 3.1(b). Thus $e_i = L*e_j*M$, for some $L, M \in S$. But this implies that G_i is isomorphic to a subquotient of G_j, see [1, Lemma 2.7].

Conversely, suppose that there is a section (P, K) of G_j such that $P/K \cong G_i$. Then $e := (P, K, \mathrm{id}, P, K)$ is an idempotent in C with $\mathscr{J}(e) = \mathscr{J}(e_i) = S_i$, by part (a). Moreover, $e = e * e_j * e$, thus $S * e * S \subseteq S * e_j * S$, implying $S_i = \mathscr{J}(e_i) = \mathscr{J}(e) \leq_{\mathscr{J}} \mathscr{J}(e_j) = S_j$. $\square$

7.4. Notation. (a) For $i \in \{1, \ldots, n\}$, let $e_i := \Delta(G_i)$ and set $S_i := \mathscr{J}(e_i)$, as in Proposition 7.3. In consequence of Proposition 7.3, $S_1, \ldots, S_n$ are then precisely the distinct $\mathscr{J}$-classes of C, and e_i is an idempotent endomorphism in S_i, for $i \in \{1, \ldots, n\}$.

Also, for $i \in \{1, \ldots, n\}$, we shall from now on identify the group Γ_{e_i} with the automorphism group $\mathrm{Aut}(G_i)$, and the twisted group algebra $k_\kappa\Gamma_{e_i}$ with the untwisted group algebra $k\mathrm{Aut}(G_i)$, via the isomorphisms in Proposition 7.3(b). In particular, every $k_\kappa\Gamma_{e_i}$-module can and will from now on be viewed as a $k\mathrm{Aut}(G_i)$-module.

(b) Suppose that $i, j \in \{1, \ldots, n\}$ are such that $S_j <_{\mathscr{J}} S_i$, so that we have the $\Gamma_{e_i} \times \Gamma_{e_j}$-action on $S_j \cap e_i * S * e_j$ introduced in 5.4. Thus, via the isomorphisms $\Gamma_{e_i} \cong \mathrm{Aut}(G_i)$ and $\Gamma_{e_j} \cong \mathrm{Aut}(G_j)$, we also have a left $\mathrm{Aut}(G_i) \times \mathrm{Aut}(G_j)$-action on $S_j \cap e_i * S * e_j$ via

$${}^{(\alpha,\beta)}L := \Delta_\alpha(G_i) * L * \Delta_{\beta^{-1}}(G_j) \quad (\alpha \in \mathrm{Aut}(G_i), \beta \in \mathrm{Aut}(G_j), L \in S_j \cap e_i * S * e_j)\,. \tag{21}$$

As before, we shall denote the stabilizer of $L \in S$ in $\mathrm{Aut}(G_i) \times \mathrm{Aut}(G_j)$ simply by $\mathrm{stab}(L)$, whenever i and j are apparent from the context.

7.5. Remark. (a) By Proposition 7.3(b), we are able to apply Corollary 6.7. So, suppose that $(i,r),(j,s) \in \Lambda$ are such that $S_j <_{\mathscr{J}} S_i$. Then, by Corollary 6.7, Proposition 5.1 and Proposition 5.2, the following are equivalent:

(i) $f_{(i,r)} \cdot J_j \cdot f_{(j,s)} \nsubseteq J_{j-1}$;
(ii) $f_{(j,s)} \cdot J_j \cdot f_{(i,r)} \nsubseteq J_{j-1}$;
(iii) there is some $L \in S_j \cap e_i * S * e_j$ with $f_{(i,r)} \cdot L \cdot f_{(j,s)} \neq 0$;
(iv) there is some $M \in S_j \cap e_j * S * e_i$ with $f_{(j,s)} \cdot M \cdot f_{(i,r)} \neq 0$.

Note also that the set $S_j \cap e_i * S * e_j$ consists precisely of those subgroups $(P, K, \eta, G_j, 1)$ of $G_i \times G_j$, where (P, K) is a section of G_i, and $\eta : G_j \to P/K$ is an isomorphism.

(b) In the proof of Lemma 7.6 below we shall often only be interested to see whether certain products of elements in $A = k_\kappa \mathsf{C}$ are non-zero, without determining the coefficients at the standard basis elements explicitly. Therefore, given $a, b \in A$, we shall write $a \sim b$ if there is some $\lambda \in k^\times$ such that $a = \lambda b$.

With this convention we, in particular, deduce the following description of the block idempotent $f_{(i,r)}$ of $k_\kappa \Gamma_{e_i}$: view $T_{(i,r)}$ as a simple $k\mathrm{Aut}(G_i)$-module via the isomorphism (20), and let $\chi_{(i,r)}$ be the character of $\mathrm{Aut}(G_i)$ afforded by $T_{(i,r)}$. Then the corresponding block idempotent of $k\mathrm{Aut}(G_i)$ is

$$f'_{(i,r)} := \frac{\chi_{(i,r)}(1)}{u^2 v |\mathrm{Aut}(G_i)|} \sum_{\alpha \in \mathrm{Aut}(G_i)} \chi_{(i,r)}(\alpha^{-1})\alpha\,,$$

where $\chi_{(i,r)} = u(\psi_1 + \cdots + \psi_v)$ is a decomposition into absolutely irreducible characters over a suitable extension field of k. Thus, applying (20) again, we get

$$\begin{aligned} f_{(i,r)} &= \frac{|G_i| \cdot \chi_{(i,r)}(1)}{u^2 v |\mathrm{Aut}(G_i)|} \sum_{\alpha \in \mathrm{Aut}(G_i)} \chi_{(i,r)}(\alpha^{-1}) \Delta_\alpha(G_i) \\ &\sim \sum_{\alpha \in \mathrm{Aut}(G_i)} \chi_{(i,r)}(\alpha^{-1}) \Delta_\alpha(G_i)\,. \end{aligned} \tag{22}$$

The next lemma will be the key step towards establishing Theorem 7.7, our main result of this section.

7.6. Lemma. *Let $(i,r),(j,s) \in \Lambda$ be such that $S_j <_{\mathscr{J}} S_i$, and suppose that $L \in S_j \cap e_i * S * e_j$ is such that $f_{(i,r)} \cdot L \cdot f_{(j,s)} \neq 0$. Then one has*

(a) $\mathrm{stab}_{\mathrm{Aut}(G_i)\times\mathrm{Aut}(G_j)}(L) \cdot (1 \times \mathrm{Inn}(G_j)) \leq (\mathrm{Inn}(G_i) \times 1) \cdot \mathrm{stab}_{\mathrm{Aut}(G_i)\times\mathrm{Aut}(G_j)}(L)$*;*

(b) *if* $\mathrm{Inn}(G_i)$ *acts trivially on the simple* $k\mathrm{Aut}(G_i)$*-module* $T_{(i,r)}$ *then* $\mathrm{Inn}(G_j)$ *acts trivially on the simple* $k\mathrm{Aut}(G_j)$*-module* $T_{(j,s)}$*.*

Proof. For ease of notation, set $A_i := \mathrm{Aut}(G_i)$, $A_j := \mathrm{Aut}(G_j)$, $I_i := \mathrm{Inn}(G_i)$, and $I_j := \mathrm{Inn}(G_j)$.

Since $L \in S_j \cap e_i * S * e_j$, we deduce from Proposition 7.3 that $L = (P, K, \eta, G_j, 1)$, for some $1 \leq K \trianglelefteq P \leq G_i$.

To prove (a), note first that $\mathrm{stab}(L)(1 \times I_j)$ and $(I_i \times 1)\mathrm{stab}(L)$ are indeed subgroups of $A_i \times A_j$, since $I_i \trianglelefteq A_i$ and $I_j \trianglelefteq A_j$. Note further that it suffices to show that $1 \times I_j \leq (I_i \times 1)\mathrm{stab}(L)$.

For $(\alpha, \beta) \in A_i \times A_j$, we have

$$\begin{aligned}(\alpha, \beta) \in \mathrm{stab}(L) &\Leftrightarrow \Delta_\alpha(G_i) * L * \Delta_{\beta^{-1}}(G_j) = L \\ &\Leftrightarrow (\alpha(P), \alpha(K), \bar{\alpha} \circ \eta \circ \beta^{-1}, G_j, 1) = (P, K, \eta, G_j, 1)\,,\end{aligned}$$

where $\bar{\alpha}$ is the isomorphism $P/K \to \alpha(P)/\alpha(K)$ induced by α.

Now, given $\beta \in \mathrm{Inn}(G_j)$, there is some $g \in G_j$ with $\beta = c_g$. Let $h \in P \leq G_i$ be such that $\eta(g) = hK$, and set $\alpha := c_h \in \mathrm{Inn}(G_i)$. Since $h \in P$ and $K \trianglelefteq P$, we get $\alpha(P) = P$, $\alpha(K) = K$ as well as

$$\begin{aligned}(\bar{\alpha} \circ \eta \circ \beta^{-1})(x) &= \bar{\alpha}(\eta(g^{-1}xg)) = \bar{\alpha}(h^{-1}K \cdot \eta(x) \cdot hK) \\ &= hK \cdot h^{-1}K \cdot \eta(x) \cdot hK \cdot h^{-1}K = \eta(x)\,,\end{aligned}$$

for all $x \in G_j$. Thus $(\alpha, \beta) \in \mathrm{stab}(L)$, and

$$(1, \beta) = (\alpha, \beta) \cdot (\alpha^{-1}, 1) \in \mathrm{stab}(L)(I_i \times 1)\,,$$

implying $1 \times I_j \leq \mathrm{stab}(L)(I_i \times 1)$. This proves assertion (a).

To prove assertion (b), recall from (22) that

$$f_{(i,r)} \sim \sum_{\alpha \in A_i} \chi_{(i,r)}(\alpha^{-1})\Delta_\alpha(G_i) \quad \text{and} \quad f_{(j,s)} \sim \sum_{\beta \in A_j} \chi_{(j,s)}(\beta^{-1})\Delta_\beta(G_j)\,,$$

and recall from Remark 7.2 that

$$\varepsilon_i = |Z(G_i)| \cdot \sum_{\alpha \in I_i} \Delta_\alpha(G_i) \tag{23}$$

is an idempotent in $k_\kappa \Gamma_{e_i}$ that, up to a non-zero scalar, corresponds under the isomorphism in (20) to the principal block idempotent of kI_i. Since $I_i \trianglelefteq A_i$, the element ε_i, viewed in kA_i, is stable under A_i-conjugation. Thus, ε_i is a central idempotent of $k_\kappa \Gamma_{e_i}$. Similarly, ε_j is an idempotent in $Z(k_\kappa \Gamma_{e_j})$.

Now, assume that I_i acts trivially on $T_{(i,r)}$, but I_j does not act trivially on $T_{(j,s)}$. Then we get $0 \neq \varepsilon_i \cdot T_{(i,r)} = \varepsilon_i f_{(i,r)} \cdot T_{(i,r)}$, thus $\varepsilon_i f_{(i,r)} \neq 0$ and

$$\varepsilon_i f_{(i,r)} \sim f_{(i,r)} \sim f_{(i,r)} \varepsilon_i\,.$$

On the other hand, we have

$$\varepsilon_j f_{(j,s)} = 0 = f_{(j,s)} \varepsilon_j\,;$$

for otherwise we would have $\varepsilon_j f_{(j,s)} \sim f_{(j,s)} \sim f_{(j,s)}\varepsilon_j$, and so I_j would act trivially on $T_{(j,s)} = f_{(j,s)} \cdot T_{(j,s)} = \varepsilon_j f_{(j,s)} \cdot T_{(j,s)}$, contradicting our assumption.

Therefore, we also have $0 = f_{(i,r)} \cdot L \cdot f_{(j,s)}\varepsilon_j$ and $0 \neq f_{(i,r)} \cdot L \cdot f_{(j,s)} \sim \varepsilon_i f_{(i,r)} \cdot L \cdot f_{(j,s)}$.

Our final step will be to show that

$$(\chi^*_{(i,r)} \times \chi_{(j,s)}, 1)_{\mathrm{stab}(L)(1\times I_j)} = 0 \neq (\chi^*_{(i,r)} \times \chi_{(j,s)}, 1)_{(I_i\times 1)\mathrm{stab}(L)}\,, \tag{24}$$

which will then, by (a), lead to a contradiction completing the proof of (b). Here 1 simply denotes the trivial character of $\mathrm{stab}(L)(1 \times I_j)$ and $(I_i \times 1)\mathrm{stab}(L)$, respectively.

By (22) and (23), we have

$$\begin{aligned}
0 = f_{(i,r)} \cdot L \cdot f_{(j,s)}\varepsilon_j &\sim \sum_{\alpha\in A_i}\sum_{\beta\in A_j}\sum_{g\in G_j} \chi_{(i,r)}(\alpha^{-1})\chi_{(j,s)}(\beta^{-1}) \cdot {}^{(\alpha,\beta^{-1})}L * \Delta_g(G_j)\\
&= \sum_{\alpha\in A_i}\sum_{\beta\in A_j}\sum_{g\in G_j} \chi_{(i,r)}(\alpha^{-1})\chi_{(j,s)}(\beta^{-1}) \cdot {}^{(\alpha,c_{g^{-1}}\circ\beta^{-1})}L\\
&= \sum_{\alpha\in A_i}\sum_{\beta\in A_j}\sum_{g\in G_j} \chi_{(i,r)}(\alpha^{-1})\chi_{(j,s)}(\beta) \cdot {}^{(\alpha,c_{g^{-1}}\circ\beta)}L\,.
\end{aligned}\tag{25}$$

Fixing $(\alpha_0, \beta_0) \in A_i \times A_j$, the coefficient at ${}^{(\alpha_0,\beta_0)}L$ in (25) equals

$$\begin{aligned}
&\sum_{\substack{\alpha\in A_i\\ \beta\in A_j\\ g\in G_j\\ {}^{(\alpha,c_g^{-1}\circ\beta)}L={}^{(\alpha_0,\beta_0)}L}} \chi_{(i,r)}(\alpha^{-1})\chi_{(j,s)}(\beta)\\
&= \sum_{g\in G_j}\sum_{(\sigma,\tau)\in\mathrm{stab}(L)} \chi_{(i,r)}(\sigma^{-1}\circ\alpha_0^{-1})\chi_{(j,s)}(c_g\circ\beta_0\circ\tau)\\
&= \sum_{g\in G_j}\sum_{(\sigma,\tau)\in\mathrm{stab}(L)} \chi_{(i,r)}(\sigma^{-1}\circ\alpha_0^{-1})\chi_{(j,s)}(\beta_0\circ\tau\circ c_g)\\
&\sim \sum_{(\sigma,\tau)\in\mathrm{stab}(L)(1\times I_j)} \chi_{(i,r)}(\sigma^{-1}\circ\alpha_0^{-1})\chi_{(j,s)}(\beta_0\circ\tau)\\
&= (\chi^*_{(i,r)} \times \chi_{(j,s)})\left((\alpha_0,\beta_0)\cdot(\mathrm{stab}(L)(1\times I_j))^+\right)\,.
\end{aligned}$$

Here $(\mathrm{stab}(L)(1 \times I_j))^+ := \sum_{(\sigma,\tau)\in\mathrm{stab}(L)(1\times I_j)}(\sigma, \tau) \in k[A_i \times A_j]$.

Hence, altogether this yields

$$\begin{aligned}
0 &= f_{(i,r)} \cdot L \cdot f_{(j,s)}\varepsilon_j\\
&\sim \sum_{\substack{(\alpha,\beta)\in\\ [A_i\times A_j/\mathrm{stab}(L)]}} (\chi^*_{(i,r)} \times \chi_{(j,s)})\left((\alpha,\beta)\cdot(\mathrm{stab}(L)(1\times I_j))^+\right)\cdot {}^{(\alpha,\beta)}L\,,
\end{aligned}$$

where $[A_i \times A_j/\mathrm{stab}(L)]$ denotes a set of representatives of the left cosets $A_i \times A_j/\mathrm{stab}(L)$. Thus, we have $0 = f_{(i,r)} \cdot L \cdot f_{(j,s)}\varepsilon_j$ if and only if $(\chi^*_{(i,r)} \times \chi_{(j,s)})((\alpha, \beta) \cdot (\mathrm{stab}(L)(1 \times I_j))^+) = 0$, for all $(\alpha, \beta) \in A_i \times A_j$. By [1, Lemma 7.3], the latter condition is in turn satisfied if and only if $(\chi^*_{(i,r)} \times \chi_{(j,s)})\,((\mathrm{stab}(L)(1 \times I_j))^+) = 0$, that is, if and only if $(\chi^*_{(i,r)} \times \chi_{(j,s)}, 1)_{\mathrm{stab}(L)(1\times I_j)} = 0$.

A completely analogous calculation gives

$$0 \neq \varepsilon_i f_{(i,r)} \cdot L \cdot f_{(j,s)} \sim \sum_{\substack{(\alpha,\beta)\in \\ [A_i \times A_j/\mathrm{stab}(L)]}} (\chi^*_{(i,r)} \times \chi_{(j,s)})\left((\alpha,\beta)((I_i \times 1)\mathrm{stab}(L))^+\right) \cdot {}^{(\alpha,\beta)}L\,,$$

so that $0 \neq \varepsilon_i f_{(i,r)} \cdot L \cdot f_{(j,s)}$ holds if and only if there is some $(\alpha,\beta) \in A_i \times A_j$ with $(\chi^*_{(i,r)} \times \chi_{(j,s)})\,((\alpha,\beta)((I_i \times 1)\mathrm{stab}(L))^+) \neq 0$. By [1, Lemma 7,3] again, this is equivalent to $(\chi^*_{(i,r)} \times \chi_{(j,s)})\,(((I_i \times 1)\mathrm{stab}(L))^+) \neq 0$, which is equivalent to $(\chi^*_{(i,r)} \times \chi_{(j,s)}, 1)_{(I_i\times 1)\mathrm{stab}(L)} \neq 0$.

To summarize, we have now established (24), which completes the proof of assertion (b). $\square$

7.7. Theorem. *For $i \in \{1,\ldots,n\}$, let $\varepsilon_i := |Z(G_i)| \cdot \sum_{\alpha\in \mathrm{Inn}(G_i)} \Delta_\alpha(G_i)$, and let further $\varepsilon := \varepsilon_{\mathsf{C}} := \sum_{i=1}^n \varepsilon_i$ (see Remark 7.2). Then the following hold:*

(a) *The twisted category algebra $A = k_\kappa\mathsf{C}$ is quasi-hereditary, both with respect to $(\Lambda,\leq)$ and to $(\Lambda,\trianglelefteq)$. For $(i,r) \in \Lambda$, the corresponding standard and costandard A-modules $\Delta_{(i,r)}$ and $\nabla_{(i,r)}$ with respect to both $\leq$ and $\trianglelefteq$ satisfy*

$$\Delta_{(i,r)} \cong Ae'_i \otimes_{e'_iAe'_i} \tilde{T}_{(i,r)} \quad \text{and} \quad \nabla_{(i,r)} \cong \mathrm{Hom}_{e'_iAe'_i}(e'_iA, \tilde{T}_{(i,r)}) \cong (Ae'_i \otimes_{e'_iAe'_i} \tilde{T}^\circ_{(i,r)})^\circ$$

with respect to the anti-involution $-^\circ\colon A \to A$ from 7.1(b).

(b) *Suppose that $(i,r),(j,s) \in \Lambda$ are such that $(i,r) \lhd (j,s)$. If $\varepsilon \cdot D_{(i,r)} \neq \{0\}$ then also $\varepsilon \cdot D_{(j,s)} \neq \{0\}$.*

(c) *The element ε is an idempotent in A. Moreover, the condensed algebra $\varepsilon A\varepsilon$ is quasi-hereditary with respect to the partial order induced by $\trianglelefteq$ on $\Lambda' := \{(i,r) \in \Lambda \mid \varepsilon \cdot D_{(i,r)} \neq \{0\}\}$. The corresponding standard and costandard modules are precisely the modules $\Delta'_{(i,r)} := \varepsilon \cdot \Delta_{(i,r)}$ and $\nabla'_{(i,r)} := \varepsilon \cdot \nabla_{(i,r)}$, respectively, for $(i,r) \in \Lambda'$. For every $(i,r) \in \Lambda'$, one also has an isomorphism $(\nabla'_{(i,r)})^\circ \cong \Delta'_{(i,r)^\circ} = \Delta'_{(i,r^\circ)}$ of $\varepsilon A\varepsilon$-modules.*

Proof. Assertion (a) is immediate from Theorem 4.3, Corollary 4.5, Proposition 6.6, and the properties of the duality functor $-^\circ : \mathsf{C} \to \mathsf{C}$, see the last paragraph of 7.1(b). To prove (b), let $(i,r),(j,s) \in \Lambda$ be such that $(i,r) \lhd (j,s)$, that is, there exist $m \in \mathbb{N}$ and suitable $(i_0,r_0) = (i,r),(i_1,r_1),\ldots,(i_{m-1},r_{m-1}),(i_m,r_m) = (j,s) \in \Lambda$ such that

$$S_{i_{q+1}} <_{\mathscr{J}} S_{i_q} \quad \text{and} \quad f_{(i_q,r_q)} \cdot J_{i_{q+1}} \cdot f_{(i_{q+1},r_{q+1})} \not\subseteq J_{i_{q+1}-1}\,,$$

for all $q = 0,\ldots,m-1$. Since $\varepsilon \cdot D_{(i,r)} \neq 0$, we deduce from [1, Corollary 7.6] that $\mathrm{Inn}(G_i)$ acts trivially on $T_{(i,r)}$. By Proposition 5.1(b) and Lemma 7.6(b), $\mathrm{Inn}(G_{i_1})$ acts trivially on $T_{(i_1,r_1)}$. Thus $\varepsilon \cdot D_{(i_1,r_1)} \neq 0$, by [1, Corollary 7.6] again. Iteration of this argument implies $\varepsilon \cdot D_{(j,s)} \neq 0$, as claimed.

As already mentioned in Remark 7.2 and shown in [1], ε is an idempotent in A. Moreover, for every $i = 1, \ldots, n$, we have $\varepsilon_i^\circ = \sum_{g\in G_i} \Delta_g(G_i)^\circ = \sum_{g\in G_i} \Delta_{g^{-1}}(G_i) = \varepsilon_i$, and thus also $\varepsilon^\circ = \varepsilon$. Assertion (c) now follows immediately from (a), (b), Proposition 2.6, (16), and Proposition 6.6(a). $\square$

7.8. Remark. (a) Theorem 7.7(a) remains true if one only requires that the morphisms of C satisfy the slightly technical condition (10) in [1]. But the assumption of C being section-closed was needed in [1, Corollary 7.6], and thus in the proofs of Lemma 7.6 and Theorem 7.7(b)–(c). So if C satisfies condition (10) in [1] and is section-closed then parts (b) and (c) of Theorem 7.7 still remain true.

(b) Part (c) of Theorem 7.7 gives a different proof of one of the main results of Webb in [21, Section 7] in the case that C is finite. Lifting the finite case to the infinite case is possible using standard techniques. In fact, if $\mathsf{C}' \subseteq \mathsf{C}$ is a full subcategory whose objects are again closed under taking subsections then all the constructions for C' arise from those of C by multiplying with the idempotent $\varepsilon_{\mathsf{C}'}$. However, also with this approach we do not see how to prove the finiteness condition on injective objects for an infinite category C (see [21, Theorem 7.2]).

In order to compare our approach with the one in [21], we first claim that if $\varepsilon D_{(i,r)} \neq \{0\}$ then this module corresponds to the simple biset functor $S_{G_i,T_{(i,r)}}$ (in the notation in [21]). In fact, by [4, Theorem 4.3.10 and Lemma 4.3.9], $S_{G_i,T_{(i,r)}}$ is characterized by the following two properties:

(i) G_i has minimal order among all group $G_j \in \mathrm{Ob}(\mathsf{C})$ with the property that $S_{G_i,T_{(i,r)}}(G_j) \neq \{0\}$, and

(ii) $S_{G_i,T_{(i,r)}}(G_i) \cong T_{(i,r)}$ as $k\mathrm{Out}(G_i)$-modules.

To prove the claim, recall from [21] that evaluation of a biset functor at a group G_j translates into multiplying the corresponding $\varepsilon A\varepsilon$-module with ε_j. Suppose that $\varepsilon \cdot D_{(i,r)} \neq \{0\}$. Then, by [1, Corollary 7.6], $\mathrm{Inn}(G_i)$ acts trivially on $T_{(i,r)}$, so that $T_{(i,r)}$ can be viewed as a simple $k\mathrm{Out}(G_i)$-module. Now, $D_{(i,r)}$ satisfies property (ii), since

$$\varepsilon_i\varepsilon D_{(i,r)} = \varepsilon_i D_{(i,r)} = \varepsilon_i e_i D_{(i,r)} \cong \varepsilon_i \tilde{T}_{(i,r)} \cong T_{(i,r)}$$

as $k\mathrm{Out}(G_i)$-modules, by (6). In order to prove (i), suppose that $|G_j| < |G_i|$, so that $S_i \not\trianglelefteq_{\mathscr{J}} S_j$. Thus $S_j \cdot D_{(i,r)} = \{0\}$, by [17, Theorem 1.2, Proposition 5.1]. From this we get

$$\varepsilon_j\varepsilon D_{(i,r)} = \varepsilon_j D_{(i,r)} = \varepsilon_j e_j D_{(i,r)} = \{0\}\,.$$

Therefore, also the standard modules and costandard modules constructed in [21] must coincide with ours, since the ones constructed in [21] are the standard modules with respect to the partial order we call $\leq$, and since $\leq$ is a refinement of $\trianglelefteq$. However, knowing that they are also the standard modules with respect to $\trianglelefteq$ is an improvement, since it imposes restrictions on possible composition factors occurring in the standard modules (see Example 7.9).

(c) The standard modules $\varepsilon \cdot \Delta_{(i,r)}$ appear also in [5] as the functors $\overline{L}_{H,V}$, see the paragraph preceding [5, Lemma 4.3] for a definition. There they play an important role in the determination of simple biset functors, but without any investigation of quasi-hereditary structures. That our standard modules coincide with these functors can also be seen from their definition using (5).

7.9. Example. To conclude this section, we shall illustrate some of our previous results by an explicit example. We shall, in particular, see that the relation $\sqsubseteq$ in Definition 4.1 is in general not transitive. Furthermore, we shall show that the partial order $\leq$ on Λ is in general a proper refinement of $\trianglelefteq$. Throughout this example, let $k := \mathbb{C}$.

(a) With the notation as in 7.1, we consider the category C whose objects are the following finite groups:

$$G_1 := \{1\},\ G_2 := C_2,\ G_3 := C_3,\ G_4 := C_4,\ G_5 := C_2 \times C_2\,,$$
$$G_6 := \mathfrak{S}_3,\ G_7 := D_8,\ G_8 := \mathfrak{A}_4,\ G_9 := \mathfrak{S}_4\,.$$

Here, D_8 denotes the dihedral group of order 8. In particular, $\mathrm{Ob}(\mathsf{C})$ is a section-closed set. We shall determine the relation $\sqsubseteq$ via Corollary 5.5(a), and encode it in Table 1. To this end, we first list, for each $G \in \mathrm{Ob}(\mathsf{C})$, the isomorphism type of $\mathrm{Aut}(G)$ as well as the ordinary irreducible $\mathrm{Aut}(G)$-characters. The $\mathrm{Aut}(G)$-characters that restrict trivially to $\mathrm{Inn}(G)$ are set in boldface, as these are precisely the characters leading to simple A-modules not annihilated by ε.

i	G_i	$\mathrm{Aut}(G_i)$	$\mathrm{Irr}(\mathrm{Aut}(G_i))$
1	$\{1\}$	$\{1\}$	$\boldsymbol{\chi_{(1,1)}} := 1$
2	C_2	$\{1\}$	$\boldsymbol{\chi_{(2,1)}} := 1$
3	C_3	C_2	$\boldsymbol{\chi_{(3,1)}} := 1, \boldsymbol{\chi_{(3,2)}} := \mathrm{sgn}$
4	C_4	C_2	$\boldsymbol{\chi_{(4,1)}} := 1, \boldsymbol{\chi_{(4,2)}} := \mathrm{sgn}$
5	$C_2 \times C_2$	$\mathfrak{S}_3$	$\boldsymbol{\chi_{(5,1)}} := 1, \boldsymbol{\chi_{(5,2)}} := \mathrm{sgn}, \boldsymbol{\chi_{(5,3)}} := \nu_2$
6	$\mathfrak{S}_3$	$\mathfrak{S}_3$	$\boldsymbol{\chi_{(6,1)}} := 1, \chi_{(6,2)} := \mathrm{sgn}, \chi_{(6,3)} := \nu_2$
7	D_8	D_8	$\boldsymbol{\chi_{(7,1)}} := 1, \chi_{(7,2)} := \tau, \boldsymbol{\chi_{(7,3)}} := \mu, \chi_{(7,4)} := \mu', \chi_{(7,5)} := \chi$
8	$\mathfrak{A}_4$	$\mathfrak{S}_4$	$\boldsymbol{\chi_{(8,1)}} := 1, \boldsymbol{\chi_{(8,2)}} := \mathrm{sgn}, \chi_{(8,3)} := \chi_2, \chi_{(8,4)} := \chi_3, \chi_{(8,5)} := \chi_3'$
9	$\mathfrak{S}_4$	$\mathfrak{S}_4$	$\boldsymbol{\chi_{(9,1)}} := 1, \chi_{(9,2)} := \mathrm{sgn}, \chi_{(9,3)} := \chi_2, \chi_{(9,4)} := \chi_3, \chi_{(9,5)} := \chi_3'$

Here, by abuse of notation, 1 always denotes the trivial character, and sgn denotes the sign character, for each of the relevant groups. Moreover, ν_2 is the natural character of $\mathfrak{S}_3$ of degree 2, χ_3 is the natural character of $\mathfrak{S}_4$ of degree 3, $\chi_3' = \chi_3 \cdot \mathrm{sgn}$, and χ_2 is the unique irreducible $\mathfrak{S}_4$-character of degree 2.

As for the characters of $\mathrm{Aut}(D_8) \cong D_8$, we have $\deg(\tau) = \deg(\mu) = \deg(\mu') = 1$ and $\deg(\chi) = 2$. Moreover, taking D_8 to be the Sylow 2-subgroup of $\mathfrak{S}_4$ generated by $(1,2)$ and $(1,3)(2,4)$, an explicit isomorphism $D_8 \xrightarrow{\sim} \mathrm{Aut}(D_8)$ is given by the map

$$\varphi : D_8 \to \mathrm{Aut}(D_8)\,; \begin{cases} (1,2) \mapsto \begin{cases} (1,2) \mapsto (1,2) \\ (1,3)(2,4) \mapsto (1,4)(2,3) \end{cases} \\ (1,3)(2,4) \mapsto \begin{cases} (1,2) \mapsto (1,3)(2,4) \\ (1,3)(2,4) \mapsto (1,2)\,. \end{cases} \end{cases}$$

Table 1
The relation $\sqsubseteq$ in Example 7.9.

		i	1	2	3		4		5			6			7					8					9				
		G_i	$\{1\}$	C_2	C_3		C_4		C_2^2			$\mathfrak{S}_3$			D_8					$\mathfrak{A}_4$					$\mathfrak{S}_4$				
		$\mathrm{Aut}(G_i)$	$\{1\}$	$\{1\}$	C_2		C_2		$\mathfrak{S}_3$			$\mathfrak{S}_3$			D_8					$\mathfrak{S}_4$					$\mathfrak{S}_4$				
G_i	$\mathrm{Aut}(G_i)$	r	**1**	**1**	**1**	**2**	**1**	**2**	**1**	**2**	**3**	**1**	2	3	**1**	2	**3**	4	5	**1**	**2**	3	4	5	**1**	2	3	4	5
$\{1\}$	$\{1\}$	**1**	**1**	**1**	**1**	·	**1**	·	**1**	·	**1**	**1**	·	1	**1**	·	**1**	·	1	**1**	·	1	1	·	**1**	·	1	1	·
C_2	$\{1\}$	**1**	·	**1**	**1**	·	**1**	·	**1**	·	**1**	**1**	·	1	**1**	·	**1**	·	1	**1**	·	1	·	·	**1**	·	1	1	·
C_3	C_2	**1**	·	·	**1**	·	·	·	·	·	·	**1**	·	·	·	·	·	·	·	**1**	·	·	1	·	**1**	·	·	1	·
		2	·	·	·	**1**	·	·	·	·	·	·	1	·	·	·	·	·	·	·	**1**	·	·	1	·	1	·	·	1
C_4	C_2	**1**	·	·	·	·	**1**	·	·	·	·	·	·	·	**1**	·	·	·	·	·	·	·	·	·	**1**	·	1	·	·
		2	·	·	·	·	·	**1**	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	1
C_2^2	$\mathfrak{S}_3$	**1**	·	·	·	·	·	·	**1**	·	·	·	·	·	**1**	·	**1**	·	·	**1**	·	·	·	·	**1**	·	1	·	·
		2	·	·	·	·	·	·	·	**1**	·	·	·	·	·	·	**1**	·	1	·	**1**	·	·	·	**1**	1	1	1	·
		3	·	·	·	·	·	·	·	·	**1**	·	·	·	**1**	·	**1**	·	1	·	·	1	·	·	**1**	·	1	1	·
$\mathfrak{S}_3$	$\mathfrak{S}_3$	**1**	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	1	·
		2	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	1
		3	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	1	1	1
D_8	D_8	**1**	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·	·	·	·	·	·	**1**	·	1	·	·
		2	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	1
		3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·	·	·	·	**1**	·	1	·	·
		4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	1
		5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	1	1	1	·
$\mathfrak{A}_4$	$\mathfrak{S}_4$	**1**	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·	**1**	·	·	·	·
		2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·	1	·	·	·
		3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1	·	·
		4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1	·
		5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1
$\mathfrak{S}_4$	$\mathfrak{S}_4$	**1**	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	**1**	·	·	·	·
		2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
		3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·
		4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·
		5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1

With this convention, we get $\ker(\mu) = \langle\varphi((1,2)), \varphi((3,4))\rangle$, $\ker(\mu') = \langle\varphi((1,2)(3,4)), \varphi((1,3)(2,4))\rangle$, and $\ker(\tau)$ is the unique cyclic subgroup of $\mathrm{Aut}(D_8)$ of order 4.

Now, whenever $i, j \in \{1, \ldots, 9\}$ are such that $S_j <_{\mathscr{J}} S_i$, that is, G_j is isomorphic to a subquotient of G_i, we proceed as follows, according to Corollary 5.5: we determine a set of representatives of the $\mathrm{Aut}(G_i) \times \mathrm{Aut}(G_j)$-orbits on

$$S_j \cap e_i * S * e_j = \{(P, K, \eta, G_j, 1) \mid K \trianglelefteq P \leq G_i : P/K \cong G_j\}\,.$$

For every such representative L, we decompose the permutation character $\mathrm{Ind}_{\mathrm{stab}(L)}^{\mathrm{Aut}(G_i)\times\mathrm{Aut}(G_j)}(1)$ into a sum of irreducible characters. This determines the relation $\sqsubset$: if $\chi_{(i,r)} \times \chi^*_{(j,s)}$ is a constituent of the above permutation character then $(i,r) \sqsubset (j,s)$. We indicate this with an entry 1 in Table 1, and with $\cdot$ otherwise.

As above, if $G_i \in \mathrm{Ob}(\mathsf{C})$ and if $\chi_{(i,r)} \in \mathrm{Irr}(\mathrm{Aut}(G_i))$ restricts trivially to $\mathrm{Inn}(G_i)$ then we set the entries involving $\chi_{(i,r)}$ in boldface, since these characters lead precisely to the simple A-modules not annihilated by ε. Note that all characters in the above table are self-dual. So, in our particular example, we have $\chi^*_{(j,s)} = \chi_{(j,s)}$, for all $(j,s) \in \Lambda$.

For instance, letting $i := 9$ and $j := 6$, we have $G_i = \mathfrak{S}_4$ and $G_j = \mathfrak{S}_3$, thus $S_j <_{\mathscr{J}} S_i$. Representatives of the $\mathrm{Aut}(G_i) \times \mathrm{Aut}(G_j)$-orbits of $S_j \cap e_i * S * e_j$ are given by $L := \Delta(\mathfrak{S}_3)$ and $M := (\mathfrak{S}_4, V_4, \eta, \mathfrak{S}_3, 1)$, where V_4 is the normal Klein four-group in $\mathfrak{S}_4$ and $\eta : \mathfrak{S}_3 \to \mathfrak{S}_4/V_4$ is any fixed isomorphism. We have $\mathrm{Aut}(G_i) = \mathrm{Inn}(G_i) \cong G_i$ and $\mathrm{Aut}(G_j) = \mathrm{Inn}(G_j) \cong G_j$. Identifying G_i with $\mathrm{Aut}(G_i)$ and G_j with $\mathrm{Aut}(G_j)$, we get $\mathrm{stab}(L) = \Delta(\mathfrak{S}_3)$ and $\mathrm{stab}(M) = \Delta(\mathfrak{S}_3)(1 \times V_4)$. Furthermore,

$$\mathrm{Ind}_{\mathrm{stab}(L)}^{\mathfrak{S}_4\times\mathfrak{S}_3}(1) = 1 \times 1 + \mathrm{sgn} \times \mathrm{sgn} + \chi_3 \times 1 + \chi_3' \times \mathrm{sgn} + \chi_2 \times \nu_2 + \chi_3 \times \nu_2 + \chi_3' \times \nu_2\,,$$

and

$$\mathrm{Ind}_{\mathrm{stab}(M)}^{\mathfrak{S}_4\times\mathfrak{S}_3}(1) = \chi_2 \times \nu_2 + 1 \times 1 + \mathrm{sgn} \times \mathrm{sgn}\,.$$

(b) Now Table 1 shows that the relation $\sqsubseteq$ on Λ is not transitive, since $(9,2) \sqsubset (7,5)$ and $(7,5) \sqsubset (1,1)$, whereas $(9,2) \not\sqsubset (1,1)$.

(c) From Table 1 we, moreover, see that the partial order $\trianglelefteq$ on Λ is indeed coarser than $\leq$: consider $i = 8$ and $j = 3$, so that $G_i = \mathfrak{A}_4$ and $G_j = C_3$. The group G_j can either be realized as a maximal subgroup or as a minimal quotient group of $\mathfrak{A}_4$. So we infer that $(i,r) \sqsubset (j,s)$ if and only if $(i,r) \vartriangleleft (j,s)$, for $i = 8$, $j = 3$, $1 \leq r \leq 5$, $1 \leq s \leq 2$. By Table 1, we have

$$(i,r) \sqsubset (j,s) \Leftrightarrow (r,s) \in \{(1,1), (1,4), (2,2), (2,5)\}\,,$$

whereas $(i,r) < (j,s)$, for all $1 \leq r \leq 5$, $1 \leq s \leq 2$.

After condensation with ε, we obtain $(i,r) \vartriangleleft (j,s)$ in $(\Lambda', \trianglelefteq)$ if and only if $(r,s) \in \{(1,1), (2,2)\}$, but $(i,r) < (j,s)$ for all combinations of $r, s \in \{1, 2\}$.

(d) As indicated in Remark 7.2, Theorem 7.7(b) does, in general, not hold with $\leq$ instead of $\trianglelefteq$. To see this in our current example, take $(i,r) := (9,1)$ and $(j,s) := (8,3)$. Then $(i,r) < (j,s)$, since $G_j = \mathfrak{A}_4$ is isomorphic to a subquotient of $G_i = \mathfrak{S}_4$. From Proposition 7.3(c) we deduce that $\varepsilon \cdot D_{(i,r)} \neq \{0\}$, but $\varepsilon \cdot D_{(j,s)} = \{0\}$.

8. Application II: Brauer algebras

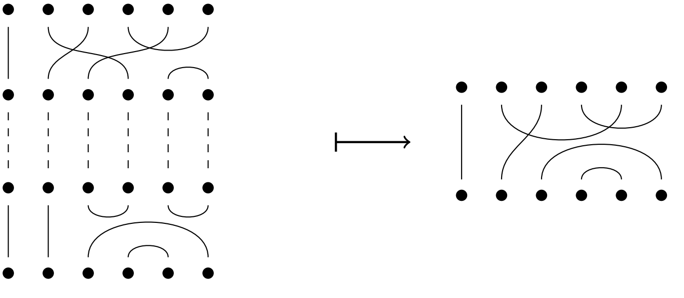
As mentioned in the Introduction, several classes of diagram algebras arise naturally as twisted category algebras. In fact, in all these examples one deals with monoid algebras, that is, the underlying category has only one object.

Throughout this section, let $n \in \mathbb{N}$, and let $\mathfrak{S}_n$ be the symmetric group of degree n. Permutations in $\mathfrak{S}_n$ will always be composed from right to left, so that, for instance, we have $(1,2)(2,3) = (1,2,3) \in \mathfrak{S}_n$, whenever $n \geq 3$. Moreover, let k be a field such that $n!$ is invertible in k, and let $\delta \in k^\times$.

8.1. Brauer algebras. Consider the set S of (n,n)-*Brauer diagrams*; each of these consists of n *northern nodes*, labelled by $1,\dots,n$, and n *southern* nodes, labelled by $\bar{1},\dots,\bar{n}$, and each node is connected by an edge to precisely one other node. Egdes connecting a pair of northern or southern nodes are called *arcs*, and edges connecting a northern with a southern node are called *propagating lines*. In other words, the elements of S can be viewed as equivalence relations of the set $\{1,\dots,n\}\cup\{\bar{1},\dots,\bar{n}\}$ whose equivalence classes contain precisely two elements.

Given (n,n)-Brauer diagrams t and t', their composition $t \circ t'$ is defined by first taking the *concatenation of* t *above* t', and then deleting all cycles from the resulting diagram.

For instance, suppose that $n = 6$, $t = \{\{1,\bar{1}\},\{2,\bar{4}\},\{3,\bar{2}\},\{4,6\},\{5,\bar{3}\},\{\bar{5},\bar{6}\}\}$, and $t' = \{\{1,\bar{1}\},\{2,\bar{2}\},\{3,4\},\{5,6\},\{\bar{3},\bar{6}\},\{\bar{4},\bar{5}\}\}$. Then we get:



Hence $t \circ t' = \{\{1,\bar{1}\},\{2,5\},\{3,\bar{2}\},\{4,6\},\{\bar{3},\bar{6}\},\{\bar{4},\bar{5}\}\}$.

In this way, S becomes a finite monoid whose identity element is the diagram that connects each pair of opposite nodes by a propagating line.

The map

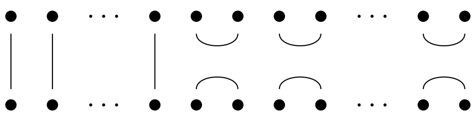
$$\alpha : S \times S \to k^\times,\ (t,t') \mapsto \delta^{m(t,t')}, \tag{26}$$

where $m(t, t')$ is the number of cycles in the concatenation of t above t', defines a 2-cocycle of the monoid S with values in $k^\times$. The resulting twisted monoid algebra

$$B_n(\delta) := k_\alpha S \tag{27}$$

is called the *Brauer algebra* of degree n over k with parameter δ.

The $\mathscr{J}$-classes of the monoid S have been determined by Mazorchuk in [19]; we shall recall the result in Proposition 8.2 below. In order to do so, it will be convenient to use the following notation: let $d := \lfloor \frac{n}{2} \rfloor$, and for $i = 1, \ldots, d+1$, let e_i be the diagram each of whose $n - 2(d-i+1)$ leftmost northern nodes is joined to its opposite southern node by a propagating line; the remaining $2(d-i+1)$ edges of e_i are arcs, each connecting a pair of neighbouring northern or southern nodes. That is, e_i has shape



8.2. Proposition. *(See [19, Theorem 7], [22, Section 8].) Keep the notation as in 8.1. Then, one has the following:*

(a) *Brauer diagrams $t, t' \in S$ belong to the same $\mathscr{J}$-class if and only if they have the same number of propagating lines. Moreover, the diagrams $e_1, \ldots, e_{d+1}$ are idempotents in S, and they form a set of representatives of the distinct $\mathscr{J}$-classes of S. Furthermore,*

$$\mathscr{J}(e_1) <_{\mathscr{J}} \mathscr{J}(e_2) <_{\mathscr{J}} \cdots <_{\mathscr{J}} \mathscr{J}(e_{d+1})\,.$$

(b) *For $i = 1, \ldots, d+1$, the group Γ_{e_i} consists of those diagrams $t \in S$ with arcs*

$$\{n-2j+1, n-2j+2\},\ \{\overline{n-2j+1}, \overline{n-2j+1}\} \quad (j = 1, \ldots, d-i+1)\,,$$

and whose remaining northern and southern nodes are connected by propagating lines. In particular, there is a group isomorphism

$$\mathfrak{S}_{n_i} \to \Gamma_{e_i},\ \sigma \mapsto t_\sigma\,, \tag{28}$$

where $n_i := n - 2(d-i+1)$, and t_σ is the Brauer diagram in Γ_{e_i} with propagating lines $\{\sigma(1), \bar{1}\}, \ldots, \{\sigma(n_i), \overline{n_i}\}$.

(c) *For $i \in \{1, \ldots, d+1\}$ and $x, x' \in \Gamma_{e_i}$, one has $\alpha(x, x') = \alpha(e_i, e_i) = \delta^{d-i+1}$; in particular, α restricts to a constant 2-cocycle on Γ_{e_i}, and the map*

$$k_\alpha \Gamma_{e_i} \to k\Gamma_{e_i},\ t \mapsto \delta^{d-i+1} \cdot t \tag{29}$$

is a k-algebra isomorphism.

(d) *For each $t \in S$, let t° be the diagram that is obtained by reflecting t about the horizontal axis. Then the resulting map $-^\circ : S \to S$, $t \mapsto t^\circ$, satisfies Hypotheses 6.1(ii)–(v), with respect to the 2-cocycle α in (26). In particular, S is a regular monoid.*

8.3. Remark. In accordance with our notation in Section 3, we again set $S_i := \mathscr{J}(e_i)$, for $i = 1, \ldots, d+1$, so that, by Proposition 8.2(a), $S_1, \ldots, S_{d+1}$ are the distinct $\mathscr{J}$-classes of S.

Furthermore, for $i = 1, \ldots, d+1$, we may identify Γ_{e_i} with the symmetric group $\mathfrak{S}_{n_i}$, via the isomorphism (28), and the twisted group algebra $k_\alpha\Gamma_{e_i}$ with the untwisted group algebra $k\mathfrak{S}_{n_i}$ via the isomorphism in (29). Note that, since we are assuming $n! \in k^\times$, we also ensure that the group orders $|\mathfrak{S}_{n_1}|, \ldots, |\mathfrak{S}_{n_{d+1}}|$ are invertible in k. Hence, the isomorphism classes of simple $k\mathfrak{S}_{n_i}$-modules are parametrized by the partitions of n_i. More precisely, suppose that $\{\lambda_{(i,1)}, \ldots, \lambda_{(i,l_i)}\}$ is the set of partitions of n_i. Then, for $r = 1, \ldots, l_i$, the simple $k\mathfrak{S}_{n_i}$-module $T_{(i,r)}$ can be chosen to be the *Specht $k\mathfrak{S}_{n_i}$-module* $S^{\lambda_{(i,r)}}$. For details concerning the representation theory of symmetric groups, we refer to [14] and [15].

Now, Theorem 4.3 and Corollary 4.5 as well as the results from Sections 5 and 6 apply, and we obtain the following result. Here we again set $d := \lfloor \frac{n}{2} \rfloor$, so that $\Lambda = \{(i,r) \mid 1 \leq i \leq d+1,\ 1 \leq r \leq l_i\}$.

8.4. Theorem. *The Brauer algebra $B_\delta(n)$ is quasi-hereditary with respect to $(\Lambda, \trianglelefteq)$. The corresponding standard $B_\delta(n)$-module labelled by $(i,r) \in \Lambda$ is isomorphic to $\Delta_{(i,r)} := B_\delta(n)e'_i \otimes_{e'_i B_\delta(n) e'_i} \tilde{T}_{(i,r)}$, and the costandard $B_\delta(n)$-module labelled by (i,r) is isomorphic to $\nabla_{(i,r)} \cong \mathrm{Hom}_{e'_i B_\delta(n) e'_i}(e'_i B_\delta(n), \tilde{T}_{(i,r)}) \cong \Delta^\circ_{(i,r)}$. Moreover, every simple $B_\delta(n)$-module is self-dual with respect to the map $-^\circ : S \to S$ in Proposition 8.2 and the resulting duality introduced in 2.3; in particular, $B_\delta(n)$ is a BGG-algebra.*

Proof. By Theorem 4.3 and Proposition 8.2, $B_\delta(n)$ is quasi-hereditary with respect to $(\Lambda, \trianglelefteq)$, and the standard modules are as claimed. Moreover, the costandard modules are as claimed, by Corollary 4.5 and Proposition 6.6, taking into account that $k_\alpha\Gamma_{e_i} \cong k\mathfrak{S}_{n_i}$ and that every simple $k\mathfrak{S}_{n_i}$-module is self-dual; see [14, Theorem 11.5]. Finally, since, by Proposition 6.6 again, $D_{(i,r)}$ is the head of $\Delta_{(i,r)}$, and $D^\circ_{(i,r)}$ is isomorphic to the head of $\Delta_{(i,r)^\circ} \cong \Delta_{(i,r^\circ)} \cong \Delta_{(i,r)}$, we get $D_{(i,r)} \cong D^\circ_{(i,r)}$. $\square$

8.5. Remark. It has been known that, under our assumptions on k and δ, the k-algebra $B_\delta(n)$ is quasi-hereditary, see [16, Theorem 1.3]. The underlying partial order on the set Λ that is usually considered is the one in which (i,r) is strictly smaller than (j,s) if and only if $j < i$. As shown in Proposition 8.6 (a), this partial order coincides with the partial order $\leq$ from (7). Part (c) of Proposition 8.6 determines the partial order $\trianglelefteq$ from Definition 4.1 explicitly. Here, for $i = 1, \ldots, d+1$, we again identify the subgroup Γ_{e_i} of S with the symmetric group $\mathfrak{S}_{n_i}$ via the isomorphism (28). Furthermore, if $j \in \{1, \ldots, i-1\}$ then we denote by $\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}$ the standard Young subgroup

$$\mathfrak{S}_{n_j} \times \langle (n_j+1, n_j+2) \rangle \times \langle (n_j+3, n_j+4) \rangle \times \cdots \times \langle (n_i-1, n_i) \rangle$$

of $\mathfrak{S}_{n_i}$.

8.6. Proposition. *Let $(i,r),(j,s) \in \Lambda$. Then one has the following:*

(a) $(i,r) < (j,s)$ *if and only if* $j < i$.

(b) *For all* $x \in \Gamma_{e_i}$, $t \in e_i \circ S$ *and* $u \in S \circ e_i$, *one has* $\alpha(x,t) = \delta^{d-i+1} = \alpha(u,x)$. *If* $j < i$, *then* $S_j \cap e_i \circ S \circ e_j$ *is a transitive* $\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}$*-set via the action defined in 5.4 and the isomorphism (28). Moreover, in this case one also has* $e_j \in S_j \cap e_i \circ S \circ e_j$.

(c) *One has*

$$(i,r) \lhd (j,s) \Leftrightarrow j < i \text{ and } T_{(i,r)} \mid M^{(i)}_{(j,s)},$$

where $M^{(i)}_{(j,s)}$ *denotes the* $k\mathfrak{S}_{n_i}$*-module*

$$M^{(i)}_{(j,s)} := \mathrm{Ind}^{\mathfrak{S}_{n_i}}_{\mathfrak{S}_{n_j} \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2}(T_{(j,s)} \times k \times \cdots \times k).$$

Before proving the proposition, we mention (without proof) the following well-known lemma that will be used repeatedly in the proof below. As usual, given finite groups G and H, we identify left $k[G \times H]$-modules with (kG, kH)-bimodules, and vice versa.

8.7. Lemma. *Let G and H be finite groups such that $|G|$ and $|H|$ are invertible in k and assume that k is a splitting field for $G \times H$. Let M be a left $k[G \times H]$-module and assume that X is an irreducible left kG-module and Y is an irreducible left kH-module. Then, $X \otimes_k Y^*$ is a constituent of M if and only if X is a constituent of $M \otimes_{kH} Y$.*

We are now prepared to prove Proposition 8.6:

Proof of Proposition 8.6. Assertion (a) is clear, by the description of the $\mathscr{J}$-classes of S in Proposition 8.2(a).

To prove the first assertion in (b), let $x \in \Gamma_{e_i}$ and $t \in e_i \circ S$. Note that cycles in the construction of $x \circ t$ can only result from southern arcs of x and northern arcs of t. Since $x \in \Gamma_{e_i}$, and $t \in e_i \circ S$, the diagram of x has precisely $d-i+1$ southern arcs connecting consecutive nodes starting from the right, and t has the $d-i+1$ matching northern arcs (and possibly more, which are irrelevant). Thus there are precisely $d-i+1$ cycles, so that $\alpha(x,t) = \delta^{d-i+1}$. Similarly, we obtain $\alpha(u,x) = \delta^{d-i+1}$, for $u \in S \circ e_i$.

Now suppose that $j < i$, so that also $n_j = n - 2(d-j+1) < n - 2(d-i+1) = n_i$; in particular, $\mathfrak{S}_{n_j} < \mathfrak{S}_{n_i}$. Since $e_j = e_i \circ e_j$, we have $e_j \in S_j \cap e_i \circ S \circ e_j$. Now let $t \in S_j \cap e_i \circ S \circ e_j$ be arbitrary. Then t has precisely n_j propagating lines, each connecting one of the n_j leftmost southern nodes with one of the n_i leftmost northern nodes. In other words, there is an injection $\iota : \{1, \ldots, n_j\} \to \{1, \ldots, n_i\}$ such that t has the following propagating lines:

$$\{\iota(1), \bar{1}\}, \ldots, \{\iota(n_j), \bar{n}_j\}.$$

Multiplying t from the left by a suitable permutation in $\mathfrak{S}_{n_i}$, we may suppose that $\iota(q) \leq n_j$, for all $q = 1, \ldots, n_j$. Then, for $\sigma \in \mathfrak{S}_{n_i}$ with $\sigma(\iota(m)) = m$ for $m = 1, \ldots, n_j$ and $\sigma(m) = m$ for $m = n_j + 1, \ldots, n_i$, we get $e_j = \sigma \cdot t$. Therefore, $S_j \cap e_i \circ S \circ e_j$ is actually a transitive left $\mathfrak{S}_{n_i}$-set, thus also a transitive left $\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}$-set.

It remains to verify assertion (c). To this end, we first determine when $(i, r) \sqsubset (j, s)$ holds. Note that the hypothesis of Corollaries 5.5 and 6.7 are satisfied by Proposition 8.2(c) and by part (b). Therefore, we obtain

$$(i, r) \sqsubset (j, s) \Leftrightarrow j < i \text{ and } T_{(i,r)} \otimes T_{(j,s)} \mid \operatorname{Ind}_{L_{i,j}}^{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(k)\,, \tag{30}$$

where $L_{i,j} := \operatorname{stab}_{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(e_j)$. Note that here we again used the fact that the simple $k\mathfrak{S}_{n_j}$-module $T_{(j,s)}$ is self-dual. So, by Lemma 8.7, we infer that

$$(i, r) \sqsubset (j, s) \Leftrightarrow j < i \text{ and } T_{(i,r)} \mid k[(\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j})/L_{i,j}] \otimes_{k\mathfrak{S}_{n_j}} T_{(j,s)}\,. \tag{31}$$

Now suppose that $j < i$, so that $n_i - n_j = 2(i - j)$. In order to describe $L_{i,j}$, let first $W_{i,j}$ denote the subgroup of $\mathfrak{S}_{n_i}$ defined by

$$\begin{aligned} W_{i,j} :=\ &\langle (n_j + 1, n_j + 2),\ (n_j + 1, n_j + 3)(n_j + 2, n_j + 4), \\ &(n_j + 1, n_j + 3, \ldots, n_j + 2(i - j) - 1)(n_j + 2, n_j + 4, \ldots, n_j + 2(i - j))\rangle\,. \end{aligned} \tag{32}$$

Then $W_{i,j}$ is isomorphic to the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_{i-j}$, and we have $L_{i,j} = \Delta(\mathfrak{S}_{n_j}) \cdot (W_{i,j} \times 1) \leq \mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}$. Thus, writing $L_{i,j}$ as a quintuple as in 7.1(b), this gives

$$L_{i,j} = (\mathfrak{S}_{n_j} \times W_{i,j}, W_{i,j}, \eta_{i,j}, \mathfrak{S}_{n_j}, 1)\,, \tag{33}$$

where $\eta_{i,j} : \mathfrak{S}_{n_j} \xrightarrow{\sim} (\mathfrak{S}_{n_j} \times W_{i,j})/W_{i,j}$, $\sigma \mapsto (\sigma, 1)W_{i,j}$.

Consequently, we have shown the following:

$$\begin{aligned} (i, r) \lhd (j, s) \Leftrightarrow\ &\exists\, q \in \mathbb{N},\ (i_0, r_0), \ldots, (i_q, r_q) \in \Lambda : \\ &\qquad (i, r) = (i_0, r_0) \sqsubset (i_1, r_1) \sqsubset \cdots \sqsubset (i_q, r_q) = (j, s) \\ \Leftrightarrow\ &\exists\, q \in \mathbb{N},\ (i_0, r_0), \ldots, (i_q, r_q) \in \Lambda : \\ &\qquad j = i_q < \ldots < i_1 < i_0 = i \text{ and} \\ &T_{(i_p,r_p)} \mid k[(\mathfrak{S}_{n_{i_p}} \times \mathfrak{S}_{n_{i_{p+1}}})/L_{i_p,i_{p+1}}] \otimes_{k\mathfrak{S}_{n_{i_{p+1}}}} T_{(i_{p+1},r_{p+1})} \ \ (0 \leq p \leq q - 1) \\ \Leftrightarrow\ &\exists\, q \in \mathbb{N},\ (i_0, r_0), \ldots, (i_q, r_q) \in \Lambda : \\ &\qquad j = i_q < \ldots < i_1 < i_0 = i \text{ and} \\ &T_{(i_0,r_0)} \mid k[(\mathfrak{S}_{n_{i_0}} \times \mathfrak{S}_{n_{i_1}})/L_{i_0,i_1}] \otimes_{k\mathfrak{S}_{n_{i_1}}} \cdots \otimes_{k\mathfrak{S}_{n_{i_{q-1}}}} k[(\mathfrak{S}_{n_{i_{q-1}}} \times \mathfrak{S}_{n_{i_q}})/L_{i_{q-1},i_q}] \\ &\qquad \otimes_{k\mathfrak{S}_{n_{i_q}}} T_{(i_q,r_q)}\,. \end{aligned} \tag{34}$$

Suppose that $(i_0, r_0), \ldots, (i_q, r_q) \in \Lambda$ are such that $j = i_q < \ldots < i_1 < i_0 = i$. Then the $(k\mathfrak{S}_{n_i}, k\mathfrak{S}_{n_j})$-bimodule $k[\mathfrak{S}_{n_{i_0}} \times \mathfrak{S}_{n_{i_1}}/L_{i_0,i_1}] \otimes_{k\mathfrak{S}_{n_{i_1}}} \cdots \otimes_{k\mathfrak{S}_{n_{i_{q-1}}}} k[\mathfrak{S}_{n_{i_{q-1}}} \times \mathfrak{S}_{n_{i_q}}/L_{i_{q-1},i_q}]$ is isomorphic to kX, where X is the $(\mathfrak{S}_{n_i}, \mathfrak{S}_{n_j})$-biset

$$X := (\mathfrak{S}_{n_{i_0}} \times \mathfrak{S}_{n_{i_1}}/L_{i_0,i_1}) \times_{\mathfrak{S}_{n_{i_1}}} \cdots \times_{\mathfrak{S}_{n_{i_{q-1}}}} (\mathfrak{S}_{n_{i_{q-1}}} \times \mathfrak{S}_{n_{i_q}}/L_{i_{q-1},i_q})\,.$$

For a precise definition of the tensor product of bisets, see [4, Definition 2.3.11]. Since $p_2(L_{i_p,i_{p+1}}) = \mathfrak{S}_{n_{i_{p+1}}}$, for $p = 0, \ldots, q-1$, the Mackey formula for tensor products of bisets in [4, Lemma 2.3.24] gives $X \cong \mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}/L$, where

$$\begin{aligned} L := (\mathfrak{S}_{n_j} \times W_{i_{q-1},i_q} \times W_{i_{q-2},i_{q-1}} \times \cdots \times W_{i_0,i_1}, W_{i_{q-1},i_q} \times W_{i_{q-2},i_{q-1}} \times \cdots \\ \times W_{i_0,i_1}, \eta, \mathfrak{S}_{n_j}, 1) \end{aligned} \tag{35}$$

and $\eta(\sigma) := (\sigma, 1, \ldots, 1)(W_{i_{q-1},i_q} \times \cdots \times W_{i_0,i_1})$, for $\sigma \in \mathfrak{S}_{n_j}$. Set $W := W_{i_{q-1},i_q} \times \cdots \times W_{i_0,i_1}$. Then, by [4, Lemma 2.3.26], we also know that, as a functor from $k\mathfrak{S}_{n_j}$**-mod** to $k\mathfrak{S}_{n_i}$**-mod**, tensoring with kX over $k\mathfrak{S}_{n_j}$ is equivalent to $\mathrm{Ind}_{\mathfrak{S}_{n_j} \times W}^{\mathfrak{S}_{n_i}} \circ \mathrm{Inf}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j} \times W}$. Therefore, this implies

$$\begin{aligned} &(i,r) \lhd (j,s) \\ &\quad \Leftrightarrow \exists\, q \in \mathbb{N},\ j = j_q < \ldots < i_1 < i_0 = i :\ T_{(i,r)} \mid \mathrm{Ind}_{\mathfrak{S}_{n_j} \times W}^{\mathfrak{S}_{n_i}}(\mathrm{Inf}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j} \times W}(T_{(j,s)}))\,, \\ &\qquad \text{for } W := W_{i_{q-1},i_q} \times \cdots \times W_{i_0,i_1}\,. \end{aligned} \tag{36}$$

Recall again that, by Lemma 8.7, the condition on the right-hand side of (36) holds if and only if

$$T_{(i,r)} \otimes T_{(j,s)} \mid \mathrm{Ind}_L^{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(k)\,;$$

here L is the group in (35), which contains the subgroup

$$M := (\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}, (\mathfrak{S}_2)^{i-j}, \eta', \mathfrak{S}_{n_j}, 1)\,, \tag{37}$$

where $\eta'(\sigma) = (\sigma, 1, \ldots, 1)\mathfrak{S}_2^{i-j}$, for $\sigma \in \mathfrak{S}_{n_j}$. Hence, if $T_{(i,r)} \otimes T_{(j,s)} \mid \mathrm{Ind}_L^{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(k)$ then we have $T_{(i,r)} \otimes T_{(j,s)} \mid \mathrm{Ind}_M^{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(k)$ as well. Conversely, if $T_{(i,r)} \otimes T_{(j,s)} \mid \mathrm{Ind}_M^{\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_j}}(k)$ then $T_{(i,r)} \mid \mathrm{Ind}_{\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}}^{\mathfrak{S}_{n_i}}(\mathrm{Inf}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}}(T_{(j,s)}))$, again by Lemma 8.7 and [4, Lemma 2.3.26]. But then we may consider the chain $j < j+1 < j+2 < \ldots < i-1 < i$ and the group $W := W_{j+1,j} \times \cdots \times W_{i,i-1} = (\mathfrak{S}_2)^{i-j}$. So our above considerations imply $(i,r) \lhd (j,s)$.

To summarize, we have now shown that

$$(i,r) \lhd (j,s) \Leftrightarrow j < i \quad \text{and} \quad T_{(i,r)} \mid \mathrm{Ind}_{\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}}^{\mathfrak{S}_{n_i}}(\mathrm{Inf}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j} \times (\mathfrak{S}_2)^{i-j}}(T_{(j,s)}))\,. \tag{38}$$

Since $\mathrm{Inf}_{\mathfrak{S}_{n_j}}^{\mathfrak{S}_{n_j}\times(\mathfrak{S}_2)^{i-j}}(T_{(j,s)}) \cong T_{(j,s)} \otimes k \otimes \cdots \otimes k$, for $j < i$, this completes the proof of (c). $\square$

The next example will show that, also in the case where the twisted category algebra is a Brauer algebra, the partial order $\leq$ on Λ is a proper refinement of $\trianglelefteq$, and that the relation $\sqsubseteq$ in Definition 4.1 is not transitive.

8.8. Example. Let $n := 6$, so that $d = \frac{6}{2} = 3$; in particular, there are four $\mathscr{J}$-classes of S. Moreover, for simplicity let $k := \mathbb{C}$.

(a) If $i = 3$ and $j = 2$ then $n_i = 4$ and $n_j = 2$. The isomorphism classes of simple $k\mathfrak{S}_4$-modules are labelled by the partitions of 4, and the isomorphism classes of simple $k\mathfrak{S}_2$-modules are labelled by the partitions of 2. We thus choose our notation in such a way that the simple $k\mathfrak{S}_{n_i}$-module $T_{(i,r)}$ corresponds to partition λ_r, and the simple $k\mathfrak{S}_{n_j}$-module $T_{(j,s)}$ corresponds to the partition λ_s:

r	1	2	3	4	5	s	1	2
λ_r	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)	λ_s	(2)	(1^2)

By the Littlewood–Richardson Rule [14, Theorem 16.4], we obtain

$$\mathrm{Ind}_{\mathfrak{S}_2\times\mathfrak{S}_2}^{\mathfrak{S}_4}(T_{(j,1)}\otimes k) \cong T_{(i,1)} \oplus T_{(i,2)} \oplus T_{(i,3)} \quad \text{and} \quad \mathrm{Ind}_{\mathfrak{S}_2\times\mathfrak{S}_2}^{\mathfrak{S}_4}(T_{(j,2)}\otimes k) \cong T_{(i,2)} \oplus T_{(i,4)}\,,$$

so that $(i,r) \lhd (j,s)$ if and only if $(r,s) \in \{(1,1),(2,1),(3,1),(2,2),(4,2)\}$, by Proposition 8.6(c). On the other hand, $(i,r) < (j,s)$, for all $1 \leq r \leq 5$ and $1 \leq s \leq 2$.

(b) Now let $l := 4$, so that $n_l = 6$. Again, we choose our labelling such that the simple $k\mathfrak{S}_6$-module $T_{(l,t)}$ corresponds to the partition λ_t of 6:

t	1	2	3	4	5	6	7	8	9	10	11
λ_t	(6)	$(5,1)$	$(4,2)$	$(4,1^2)$	(3^2)	$(3,2,1)$	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$	(1^4)

Using the notation introduced in the proof of Proposition 8.6, we have

$$L_{4,2} = \mathrm{stab}_{\mathfrak{S}_6\times\mathfrak{S}_2}(e_2) = \Delta(\mathfrak{S}_2)(W_{4,2}\times 1) \text{ with } W_{4,2} \cong \mathfrak{S}_2 \wr \mathfrak{S}_2\,,$$
$$L_{4,3} = \mathrm{stab}_{\mathfrak{S}_6\times\mathfrak{S}_4}(e_3) = \Delta(\mathfrak{S}_4)(W_{4,3}\times 1) \text{ with } W_{4,3} \cong \mathfrak{S}_2\,,$$
$$L_{3,2} = \mathrm{stab}_{\mathfrak{S}_4\times\mathfrak{S}_2}(e_2) = \Delta(\mathfrak{S}_2)(W_{3,2}\times 1) \text{ with } W_{3,2} \cong \mathfrak{S}_2\,.$$

Recall that, by (30), we have $(i,r) \sqsubset (j,s)$ if and only if $T_{(i,r)}\otimes T_{(j,s)} \mid \mathrm{Ind}_{L_{i,j}}^{\mathfrak{S}_{n_i}\times\mathfrak{S}_{n_j}}(k)$. By a character computation with MAGMA [3], we infer that $(3,2) \sqsubset (2,1)$ and $(l,5) \sqsubset (3,2)$, whereas $(l,5) \not\sqsubset (2,1)$.

8.9. Remark. Via the partial order $\trianglelefteq$ we, in particular, obtain information on the decomposition numbers of the Brauer algebra $B_\delta(n)$, that is, on the composition factors of standard modules. It should be pointed out that further information on decomposition numbers of Brauer algebras over fields of characteristic 0 can, for instance, be found in [8].

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