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Specht property for some varieties of Jordan algebras of almost polynomial growth

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ABSTRACT

Let F be a field of characteristic zero. In [25] it was proved that UJ_2, the Jordan algebra of 2×2 upper triangular matrices, can be endowed up to isomorphism with either the trivial grading or three distinct non-trivial $\mathbb{Z}_2$-gradings or by a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. In this paper we prove that the variety of Jordan algebras generated by UJ_2 endowed with any G-grading has the Specht property, i.e., every T_G-ideal containing the graded identities of UJ_2 is finitely based. Moreover, we prove an analogue result about the ordinary identities of A_1, a suitable infinitely generated metabelian Jordan algebra defined in [27].

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1. Introduction

Given a set X of indeterminates and a field F, it is possible to construct an F-algebra $F\{X\}$ which is free in a variety $\mathcal{C}$ of algebras. The objects of that free algebra are usually called "$\mathcal{C}$-polynomials". The ideals of $F\{X\}$ invariant under $\mathcal{C}$-homomorphisms are called T-ideals. The study of T-ideals leads to the so-called *Specht problem*, whether every T-ideal is finitely generated as a T-ideal. In the case $\mathcal{C}$ is the class of associative algebras and F is a field of characteristic 0, the Specht problem has a positive answer given by Kemer [22] whereas it fails in positive characteristic as showed by several authors such as Kanel-Belov in [3] and Belov in [21], Grishin in [17] (ground field of characteristic 2) and Shchigolev in [30]. It is a remarkable fact that the three papers cited above have been published in the same issue of the same journal and it seems their authors presented such results in 1998 at the Seminar of Latyshev at the Moscow State University in the order cited above. We also want to cite the paper by Gupta and Krasilnikov [18] which presented a simple counterexample in characteristic 2 and a paper by Shchigolev (see [29]) about non-finitely generated T-spaces. If the class $\mathcal{C}$ coincides with the class of Lie algebras in characteristic 0 we have a result by Iltyakov [20] in which he proved that the Specht problem has a positive solution for finite dimensional Lie algebras. Moreover in [4] the authors proved that a Lie algebra has the Specht property if its codimension sequence is polynomially bounded. In the general case we have no definite answer in characteristic 0 although we have counterexamples in the case of positive characteristic. For the purpose, see the works by Vaughan-Lee [33] (characteristic 2) and Drensky [11] (characteristic $p > 0$). Recently in [14] the authors were able to construct a variety of non-associative algebras which does not satisfy the Specht property via a sophisticated construction of varieties of algebras with slow growth of their codimension sequence. The latter examples have the additional exotic property that the codimension grows as $n^{3+\alpha}$, where α is any positive real number strictly less than 1. Carrying on with examples of non-Spechtian varieties, a very interesting examples was obtained by Drenski in [10]. In particular the author gave an example of an anticommutative algebra whose variety generated by is of quadratic growth although non-finitely based.

Even if $\mathcal{C}$ is the class of Jordan algebras, one can get only partial answers to the Specht problem. Indeed, in [34] Vais and Zelmanov proved that any finitely generated Jordan algebra in characteristic 0 has the Specht property by showing that it has the same identities of a finite dimensional generalized Jordan pair. Unfortunately, we do not know yet whether the answer is positive nor negative in case of infinitely generated Jordan algebras.

We can also generalize the Specht problem for classes of algebras graded by a group G. In particular, in case of associative G-graded algebras in characteristic 0, where G is any finite group, a positive answer to the problem was found in [1] and [32], whereas in case of G-graded Lie or Jordan algebras we have experimental results, such as in [13] in which the authors proved the Specht property of $sl_2(F)^G$, the Lie algebra of 2×2 traceless matrices over a field F of characteristic 0 graded by any finite abelian group G, or in [31] in which a similar result was achieved for B_n, the finite dimensional Jordan algebra of a non-degenerate symmetric bilinear form graded by $\mathbb{Z}_2$, the cyclic group of order 2, always in characteristic 0.

The goal of this paper is twofold. On one hand, we get a positive solution to the Specht problem in case of $UJ_2(F)$, the Jordan algebra of 2×2 upper triangular matrices over a field F of characteristic zero, graded by any finite abelian group. In particular, we shall consider the classification of the G-gradings on $UJ_2(F)$ given in [25], that is a particular case of a latter result by Koshlukov and Yukihide in which the authors gave such a classification for $UJ_n(F)$, $n \geq 2$ (see [24]).

On the other hand, we shall prove the Specht property for varieties of Jordan algebras with trivial grading of almost polynomial growth. Recall that a variety $\mathcal{V}$ has almost polynomial growth if its codimension sequence grows exponentially and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, its codimension sequence $c_n(\mathcal{U})$ grows polynomially. In a forthcoming paper by Martino (see [26]), it was proved that up to equivalence, the only variety of finite dimensional special Jordan algebras of almost polynomial growth is generated by $UJ_2(F)$. Moreover, in [27] the authors introduced an infinitely generated metabelian Jordan algebra, denoted by A_1 that generates another variety of almost polynomial growth. Thus, in the last sections we shall prove that $\mathrm{var}(UJ_2)$ and $\mathrm{var}(A_1)$ have the Specth property. We highlight that the first part of this statement, i.e., any T-ideal containing that of $UJ_2(F)$ is finitely generated, is a particular case of [34]. We chose to include it here since its proof involves some interesting non-trivial techniques. In particular, all results are stated in the language of well-quasi-ordered sets used for the first time by the author in [7] to establish positive results on the Specht problem for groups. Later this method, also known as the Higman–Cohen method, was used for groups, Lie and associative algebras.

2. Preliminaries

All fields we refer to are of characteristic 0 unless explicitly written.

Let X be a countable set of indeterminates and let $\mathcal{J}(X)$ be the free Jordan algebra generated by the set X over F. We say that a polynomial $f(x_1, \ldots, x_n) \in \mathcal{J}(X)$ is a polynomial identity for the Jordan algebra J if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in J$. In this case we write $f \equiv 0$. The identities of J form a T-ideal of $\mathcal{J}(X)$, i.e., an ideal closed under all endomorphisms of the free Jordan algebra. Let us denote by $\mathrm{Id}(J) = \{f \in \mathcal{J}(X) |\ f \equiv 0 \text{ on } J\}$ the T-ideal of polynomial identities of J. It is well-known (see for example [15, Theorem 1.3.7]) that, in characteristic 0, $\mathrm{Id}(J)$ is determined by the multilinear polynomials it contains. Recall that a multilinear polynomial is an element of the vector subspace

$$P_n = span_F\langle \{x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} |\ \sigma \in S_n\} \rangle,$$

where S_n is the symmetric group and $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ stands for a monomial with all possible brackets arrangement. Thus, the relatively free algebra $\frac{\mathcal{J}(X)}{\mathrm{Id}(J)}$ is determined by the sequence of vector subspaces

$$P_n(J) = \frac{P_n}{P_n \cap \mathrm{Id}(J)},\ n \geq 1.$$

In this way, we can attach to the Jordan algebra J a numerical sequence $c_n(J)$ called the codimension sequence, by defining

$$c_n(J) = \dim_F P_n(J).$$

Remark that in general the codimensions are bounded only by an over-exponential function

$$c_n(J) \leq \frac{1}{n}\binom{2n-2}{n-1} n!,$$

where $\frac{1}{n}\binom{2n-2}{n-1}$ is the Catalan number. Nevertheless, one can improve this bound in some special settings. For instance, in [28] a celebrated theorem of Regev states that any associative algebra satisfying a non-trivial polynomial identity (PI-algebra) has the sequence of codimensions exponentially bounded. A similar result was obtained in the setting of finite dimensional Jordan algebras (see [12] and [16]). We shall refer to the growth of the Jordan algebra J as the asymptotic behaviour of its codimension sequence.

Given a non-empty set $S \subseteq \mathcal{J}(X)$, the class of all Jordan algebras J such that $f \equiv 0$ on J for all $f \in S$, is called variety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Similarly, given a Jordan algebra J, the variety of Jordan algebras generated by J, $\mathrm{var}(J)$, is the class of all Jordan algebras satisfying the identities of J. Hence we say that $A \in \mathrm{var}(J)$ if and only if $\mathrm{Id}(J) \subseteq \mathrm{Id}(A)$. It is clear that there exists a one-to-one correspondence between T-ideals and varieties, thus given a variety $\mathcal{V}$, we can naturally define $\mathrm{Id}(\mathcal{V})$, $P_n(\mathcal{V})$ and $c_n(\mathcal{V})$. The growth of $\mathcal{V}$ will be the asymptotic behaviour of $c_n(\mathcal{V})$. Moreover, we say that $\mathcal{V}$ has almost polynomial growth if its codimension sequence is exponentially (but not polynomially) bounded and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n(\mathcal{U})$ grows polynomially.

Let now define an action of the symmetric group S_n on P_n: if $\sigma \in S_n$ and $f(x_1,\dots,x_n) \in P_n$, then $\sigma \cdot f(x_1,\dots,x_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Such an action induces a natural S_n-action on $P_n(J)$ that becomes an S_n-module. Hence an S_n-character arises denoted by $\chi_n(J)$ and called n-th cocharacter of J. For all $n \geq 1$ the sequence $\{\chi_n(J)\}_{n\geq 1}$ is called cocharacter sequence of J. Since $\mathrm{char}F = 0$, by complete reducibility $\chi_n(J)$ can be written as

$$\chi_n(J) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the character associated to the partition λ and m_λ is the corresponding multiplicity.

Finally, from now on let us denote by G any finite abelian group and by J a G-graded Jordan algebra over F. Recall that J is a G-graded Jordan algebra if $J = \bigoplus_{g\in G} J_g$ is a direct sum of subspaces such that $J_g J_h \subseteq J_{gh}$, for all $g, h \in G$.

The free G-graded Jordan algebra $\mathcal{J}^G(X)$ is the G-graded Jordan algebra freely generated by the set $X = \bigcup_{g\in G} X^g$, where for any $g \in G$ the sets $X^g = \{x_i^g \mid i \geq 1\}$ of variables of homogeneous degree g are countable and pairwise disjoint. A polynomial f of $\mathcal{J}(X)$ is a G-graded polynomial identity of J if it vanishes under all graded substitutions, i.e., for any $g \in G$, we evaluate the variables x_i^g by elements of the homogeneous component J_g. We denote by $\mathrm{Id}^G(J)$ the ideal of $\mathcal{J}^G(X)$ of G-graded polynomial identities of J. It is easily checked that $\mathrm{Id}^G(J)$ is a T_G-ideal, i.e., an ideal invariant under all G-graded endomorphisms of $\mathcal{J}^G(X)$. We say that J is a graded PI-algebra if $\mathrm{Id}^G(J) \neq 0$. As in the ordinary case, one can define $\mathcal{V}^G = \mathcal{V}^G(S)$ the variety of G-graded Jordan algebras defined by the set $S \subseteq \mathcal{J}^G(X)$ as the set of all G-graded Jordan algebras such that $f \equiv 0$ for all $f \in S$.

Furthermore, one can define P_n^G as the vector space spanned by all multilinear monomials $x_{\sigma(1)}^{g_{\sigma(1)}} \cdots x_{\sigma(n)}^{g_{\sigma(n)}}$, $\sigma \in S_n$, $g_1, \dots, g_n \in G$, in the graded variables of the set X and by $P_n^G(J)$ the quotient vector space $\frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(J)}$.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_s$ as a sum of non-negative integers. Define $P_{n_1,\dots,n_s} \subseteq P_n^G$ as the space of multilinear graded polynomials in which the first n_1 variables $x_1^{g_1}, \dots, x_{n_1}^{g_1}$ are of homogeneous degree $g_1, \dots$, the last n_s variables $x_{n-n_s+1}^{g_s}, \dots, x_n^{g_s}$ are of homogeneous degree g_s. Notice that given such $n_1, \dots, n_s$, there are $\binom{n}{n_1,\dots,n_s}$ subspaces isomorphic to $P_{n_1,\dots,n_s}$ where $\binom{n}{n_1,\dots,n_s}$ denotes the multinomial coefficient. It is clear that P_n^G is the direct sum of such subspaces with $n_1 + \cdots + n_s = n$. Moreover such decomposition is inherited by $P_n^G \cap \mathrm{Id}^G(J)$ and we consider the spaces $P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(J)$. In light of these remarks, one defines

$$P_{n_1,\dots,n_s}(J) = \frac{P_{n_1,\dots,n_s}}{P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(J)}.$$

The space $P_{n_1,\dots,n_s}(J)$ is naturally endowed with a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module in the following way: the group $S_{n_1} \times \cdots \times S_{n_s}$ acts on the left on $P_{n_1,\dots,n_s}$ by permuting the variables of the same homogeneous degree; hence S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2 and so on. Since $\mathrm{Id}^G(J)$ is invariant under this action, $P_{n_1,\dots,n_s}(J)$ has a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module and we denote by $\chi_n^G(J)$ its character.

If $\lambda(1) \vdash n_1, \dots, \lambda(s) \vdash n_s$, are partitions, then we write $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$.

Since $\mathrm{char} F = 0$, by complete reducibility, $\chi_n^G(J)$ can be written as a sum of irreducible characters in the following way:

$$\chi_n^G(J) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)},$$

where $m_{\langle\lambda\rangle}$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ in $\chi_n^G(J)$. We call $\chi_n^G(J)$ the n-th graded cocharacter of J.

Recall that the multiplicities in the cocharacter sequence are equal to the maximal number of linearly independent highest weight vectors, according to the representation theory of GL_n. We also recall that a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row of the corresponding Young tableaux (see [8, Chapter 12] for more details).

3. Finite basis property for sets

The finite basis property for sets was first studied in [19] by G. Higman and in an unpublished manuscript by P. Erdos and R. Rado. Authors like B. H. Neumann and J. B. Kruskal also studied the finite basis property for sets which is also known as theory of well-quasi-ordering.

A binary relation $\leq$ on a set A is a quasi-order if $\leq$ is reflexive and transitive, i.e., (i) $a \leq a$ for all $a \in A$, and (ii) $a \leq b$ and $b \leq c$ imply $a \leq c$. Every partial order is a quasi-order but not worth the contrary. For example, if f, $g \in \mathcal{J}^G(X)$ then

$$f \leq g \Leftrightarrow g \in \langle f \rangle_{T_G}, \tag{1}$$

where $\langle f \rangle_{T_G}$ denotes the T_G-ideal generated by f, is a quasi-order in $\mathcal{J}^G(X)$ but it is not in general a partial order. If B is a subset of a quasi-ordered set A, the closure of B is defined as

$$\overline{B} = \{a \in A \mid \text{exists } \ b \in B \text{ such that } b \leq a\}.$$

A closed subset is a set that coincides with its own closure, i.e., $B = \overline{B}$. We say that the quasi-ordered set A has the **finite basis property (f.b.p.)** if every closed subset of A is the closure of a finite set. Every well-ordered set has f.b.p. (because every non-empty subset is the closure of a single element). In particular $\mathbb{N}$ the set of natural numbers with standard ordering has f.b.p.. However, $\mathbb{Z}$ the set of integers has not the f.b.p.. In general a totally ordered set A has f.b.p. if and only if A is a well-ordered set. Below we present some equivalent definitions for f.b.p..

Theorem 1. *[19, Theorem 2.1] The following conditions on a quasi-ordered set A are equivalent.*

(1) *Every closed subset of A is the closure of a finite subset;*
(2) *If B is any subset of A, there is a finite B_0 such that $B_0 \subset B \subset \overline{B_0}$;*

(3) *Every infinite sequence of elements* $\{a_i\}_{i\geq 0}$ *of* A *has an infinite ascending subsequence*

$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_k} \leq \cdots ;$$

(4) *There exists neither an infinite strictly descending sequence in* A *nor an infinite one of mutually incomparable elements of* A*.*

It is a consequence of the above theorem that every subset B of a quasi-ordered set A that satisfies f.b.p. has finite minimal elements (from which the name well-quasi-ordering).

The next proposition will be very often used in this work.

Proposition 1. *Let* $(A_1, \leq_{A_1})$, $(A_2, \leq_{A_2})$, …, $(A_k, \leq_{A_k})$ *be quasi-ordered sets satisfying f.b.p.*

(1) *The disjoint union of* A_1, A_2, …, A_k *endowed with the quasi-order where* $a \leq b$ *if and only if* a, $b \in A_i$ *and* $a \leq_{A_i} b$ *for some* $i \in \{1, 2, \ldots, k\}$ *satisfies f.b.p.*
(2) *The cartesian product* $A_1 \times A_2 \times \cdots \times A_k$ *endowed with the quasi-order where* $(a_1, a_2, \ldots, a_k) \leq (b_1, b_2, \ldots, b_k)$ *if and only if* $a_i \leq_{A_i} b_i$ *for any* $i \in \{1, 2, \ldots, k\}$ *satisfies f.b.p.*

Let $S(A)$ be the set of finite subsets of A where A is a quasi-ordered set. We define for P, $Q \in S(A)$, $P \leq Q$ if and only if there is one-to-one increasing map of P into Q. For instance if $A = \mathbb{N}$ the set of non-negative integers we define a quasi-order on the set $S(\mathbb{N})$ of finite sequences of non-negative integers in the following way: $a = (a_1, \ldots, a_n) \leq (a'_1, \ldots, a'_{n'}) = a'$ if and only if there is a subsequence $a'' = (a'_{i_1}, \ldots, a'_{i_n})$ of a' $(i_1 < \cdots < i_{n'})$ such that $a_j \leq a'_{i_j}$ for all $j \in \{1, \ldots, n\}$. Then Erdos and Rado proved in an unpublished manuscript the following result which can be found in [19] too:

Theorem 2. *If* A *has the f.b.p., so has* $S(A)$*.*

As seen above the free graded Jordan algebra $\mathcal{J}^G(X)$ is a quasi-ordered set if we define for $f, g \in \mathcal{J}^G(X)$, $f \leq g$ if and only if $g \in \langle f \rangle_{T_G}$. If I is a T_G-ideal of $\mathcal{J}^G(X)$, the quasi-order on $\mathcal{J}^G(X)$ is inherited by $\dfrac{\mathcal{J}^G(X)}{I}$. Hence, if $f, g \in \mathcal{J}^G(X)$, we set

$$f \leq g \text{ if and only if } g \in \langle \{f\} \cup I \rangle_{T_G} . \tag{2}$$

In this case we say that g is a consequence of f modulo I or simply that g is a consequence of f. When $f \leq g$ and $g \leq f$ we say that f is equivalent to g and we write $f \equiv g$. We observe that if $\mathcal{B} \subseteq \mathcal{J}^G(X)$ then $\overline{\mathcal{B}} \subseteq \langle \mathcal{B} \rangle_{T_G}$ modulo I and consequently $\langle\, \overline{\mathcal{B}}\, \rangle_{T_G} = \langle \mathcal{B} \rangle_{T_G}$ modulo I where $\overline{\mathcal{B}}$ is a closure of $\mathcal{B}$.

Definition 1. Let A be a G-graded Jordan algebra. We say that $\mathrm{Id}^G(A)$ has the **Specht property** if any T_G-ideal I such that $I \supseteq \mathrm{Id}^G(A)$, has a finite basis, that is, I is finitely generated as a T_G-ideal. Moreover, we say that a variety $\mathcal{V}$ has the Specht property if the corresponding T_G-ideal has the Specht property.

The following remark draws up the technique that we will apply in order to prove the Specht property for a variety of G-graded Jordan algebras.

Remark 1. Fix a variety $\mathcal{V}$ of graded Jordan algebras such that $\mathrm{Id}^G(\mathcal{V})$ is finitely generated and let $\mathcal{L} = \frac{\mathcal{J}^G(X)}{\mathrm{Id}^G(\mathcal{V})}$ be the corresponding relatively free algebra. A strategy to give a positive answer to the Specht problem for $\mathcal{V}$ is:

(1) Find a set of polynomials $\mathcal{B} \subseteq \mathcal{L}$ (not necessarily finite) such that for every T_G-ideal I of $\mathcal{L}$,

$$I = \langle \mathcal{B}' \rangle_{T_G} \text{ for some } \mathcal{B}' \subseteq \mathcal{B}.$$

(2) Show that $(\mathcal{B}, \leq)$ satisfies **f.b.p.** where $\leq$ is the quasi-order given by the consequence, i.e., $f \leq g$ if and only if g is a consequence of f in $\mathcal{L}$.

In fact, suppose that there is a set $\mathcal{B}$ satisfying (1) and (2) and let I be a T_G-ideal of $\mathcal{L}$. There exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $I = \langle \mathcal{B}' \rangle_{T_G}$. Since $(\mathcal{B}, \leq)$ satisfies f.b.p. by Theorem 1, there exists a finite set $\mathcal{B}_0 \subseteq \mathcal{B}'$ such that $\mathcal{B}_0 \subseteq \mathcal{B}' \subseteq \overline{\mathcal{B}_0}$. Therefore

$$I = \langle \mathcal{B}' \rangle_{T_G} = \langle \, \overline{\mathcal{B}_0} \, \rangle_{T_G} = \langle \mathcal{B}_0 \rangle_{T_G}.$$

If F is a field of characteristic 0 and $\mathrm{Id}^G(\mathcal{V})$ is finitely generated, a natural set that satisfies step (1) is a set of highest weight vectors generating irreducible modules whose characters appear with non-zero multiplicity in the decomposition of the cocharacter of the variety $\mathcal{V}$. If a concrete list of highest weight vectors for $\mathcal{V}$ is known, the next step is to show that this list satisfies f.b.p. with the quasi-order inherited by $\mathcal{L}$.

In the proof of the main results of this paper our strategy is showing that for any set $\mathcal{S}$ of highest weight vectors there is a finite subset $\mathcal{S}_0$ such that all elements in $\mathcal{S}$ follow from those in $\mathcal{S}_0$. When handling highest weight vectors corresponding to multipartitions appearing with multiplicity 1 in the cocharacter sequence, in order to prove the Specht property it suffices using Remark 1. Otherwise the usual way to handle the problem is the following. We argue for the ungraded case only being the generalization to the graded case a simple restatement.

Step 1. If the highest weight vectors of degree n are linear combination of $f_1^n, \dots, f_k^n$, $k = k(n)$, we order linearly the f_i's in order to select a leading term namely $f_{i_0}^n$.

Step 2. We define a partial quasi-order on the set of all f_i^n's (for any n and any $i = 1, \ldots, k(n)$) and we prove the set of these highest weight vectors satisfies f.b.p. with respect to the partial quasi-order above.

Step 3. We prove that if

$$f^{n_1} = \sum_{i=1}^{k_1} \alpha_i f_i^{n_1},\ f^{n_2} = \sum_{j=1}^{k_2} \beta_j f_j^{n_2},\ \alpha_i, \beta_j \in F,$$

are two highest weight vectors with leading terms $f_{i_0}^{n_1}$, $f_{j_0}^{n_2}$ respectively, and $f_{i_0}^{n_1} \leq f_{j_0}^{n_2}$, then there exists a highest weight vector

$$v^{n_2} = \sum_{j=1}^{k_2} \gamma_j f_j^{n_2},\ \gamma_j \in F,$$

which is a consequence of f^{n_1} and its leading term is exactly $f_{j_0}^{n_2}$.

Step 4. We consider the set $\mathcal{L}$ of all leading terms of the set $\mathcal{S}$ of all highest weight vectors in the T-ideal. By **Step 2**, $\mathcal{L}$ has a finite subset $\mathcal{L}_0$ such that every element in $\mathcal{L}$ is bigger than some element of $\mathcal{L}_0$. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be the finite subset with leading terms in $\mathcal{L}_0$. Let $f^{n_1} \in \mathcal{S}_0$ and $f^{n_2} \in \mathcal{S}$ such that their leading terms $f_{i_0}^{n_1}$, $f_{j_0}^{n_2}$ respectively, are such that $f_{i_0}^{n_1} \leq f_{j_0}^{n_2}$. By **Step 3** the leading term of

$$f_{j_0}^{n_2} - \frac{\beta_{j_0}}{\gamma_{j_0}} v$$

is smaller than the leading term of $f_{j_0}^{n_2}$ and by inductive arguments is a consequence of $\mathcal{S}_0$.

4. Specht property for G-graded identities of UJ_2

In this section we prove the Specht property for G-graded identities of UJ_2, where G is any finite abelian group.

Throughout the paper we let $UJ_2(F) = UJ_2$ be the Jordan algebra of upper triangular matrices of order 2 over the field F with the product $x \circ y = \dfrac{xy + yx}{2}$ for all x, $y \in UJ_2$. We fix the basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

of $J = UJ_2$. Hence $a \circ a = a^2 = 1$, $b^2 = b \circ b = 0$ and $a \circ b = 0$. Here we simply write F to denote the scalar matrices in J. Up to isomorphism, J can be endowed with the following non isomorphic gradings (see [25, Theorem 1]):

1. $G = \{0\}$ The trivial grading: $J_0 = J$;
2. $G = \mathbb{Z}_2$:
   (a) The scalar grading: $J_0 = F$, $J_1 = Fa \oplus Fb$;
   (b) The associative grading: $J_0 = F \oplus Fb$, $J_1 = Fa$;
   (c) The classical grading: $J_0 = F \oplus Fa$, $J_1 = Fb$.
3. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (The Klein grading): $J_{(0,0)} = F$, $J_{(0,1)} = Fa$, $J_{(1,0)} = Fb$, $J_{(1,1)} = \{0\}$.

We denote by S, A, C and K the scalar, associative, classical and Klein grading, respectively. Thus, for instance, $\mathrm{Id}^C(UJ_2)$ and $\mathrm{var}^C(UJ_2)$ will denote the T_2-ideal of graded identities and the variety of $\mathbb{Z}_2$-graded Jordan algebras generated by UJ_2 endowed with the classical grading, respectively.

4.1. The S-grading

In this section we consider the scalar grading on $J = UJ_2$, i.e., $G = \mathbb{Z}_2$ and $J = J_0 \oplus J_1$ where $J_0 = F$ and $J_1 = Fa \oplus Fb$. As above we write $X = Y \cup Z$ where the variables $y_i \in Y$ have homogeneous degree 0, the variables $z_i \in Z$ have homogeneous degree 1.

A basis of the graded identities of UJ_2 and its cocharacter sequence with the S-grading is described below. On this purpose we recall the definition of *associator* between three elements u_1, u_2, u_3 of a given algebra:

$$(u_1, u_2, u_3) := (u_1u_2)u_3 - u_1(u_2u_3).$$

Of course we have an analogous definition for associators of length n, where n is an odd number greater than 3.

Theorem 3. *[23, Proposition 8] The following polynomials are a basis of $Id^S(UJ_2)$*

$$(y_1, y_2, y_3), \quad (y, z_1, z_2), \quad (z_1, y, z_2), \quad (z_1, z_2, z_3)z_4.$$

The next theorem describes the graded cocharacter sequence.

Theorem 4. *[6, Theorem 2] Let $n \geq 0$ and let*

$$\chi_n^S(UJ_2) = \sum_{\lambda,\mu} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be the S-graded cocharacter of UJ_2. Then $m_{\lambda,\mu} = 1$ if and only if $\lambda = (r)$ and $\mu = (s)$, where $r + s = n$ and r, s not simultaneously equal to 0. In all other cases $m_{\lambda,\mu} = 0$.

As a consequence of the previous result, we have the description of the highest weight vectors whose characters appear with non-zero multiplicity in the decomposition of $\chi_n^S(UJ_2)$. In particular, the highest weight vectors are of the form:

$$y^r z^s$$

for all r, s not simultaneously equal to 0. Let $\mathcal{B}$ be a set of all highest weight vectors and let $B \subseteq \mathbb{N}^2$ be the set of pars (r,s) such that $y^r z^s \in \mathcal{B}$. By Proposition 1, B has f.b.p. if we define a natural quasi-order by saying that $(r,s) \leq (r', s')$ if and only if $r \leq r'$ and $s \leq s'$.

We shall show that the quasi-order $\leq$ in B induces the quasi-order in $\mathcal{B}$. More precisely we have the following result.

Lemma 1. *For all r, s not simultaneously equal to 0, if $(r,s) \leq (r',s')$ then $y^r z^s \leq y^{r'} z^{s'}$.*

Proof. The statement readily follows from the fact that $y^{r'} z^{s'} \equiv [y^{r'-r}(y^r z^s)] z^{s'-s}$, since the even variables y lie in the associative and commutative centre of the relatively free graded algebra $\frac{\mathcal{J}\langle X \rangle}{\mathrm{Id}^S(UJ_2)}$. □

Now we can prove the Specht property for $\mathrm{Id}^S(UJ_2)$.

Theorem 5. *Let F be a field of characteristic 0. Then var$^S(UJ_2)$ has the Specht property.*

Proof. Let $\mathcal{U}$ be a subvariety of $\mathrm{var}^S(UJ_2)$ and let I be the corresponding T_2-ideal of graded identities. If $\mathcal{U} = \mathrm{var}^S(UJ_2)$, i.e., $I = \mathrm{Id}^S(UJ_2)$, then the result follows from Theorem 3.

So let us suppose $\mathcal{U} \subsetneq \mathrm{var}^S(UJ_2)$ and $I \subsetneq \mathrm{Id}^S(UJ_2)$. By Remark 1 it suffices to show that $(\mathcal{B}, \leq)$ satisfies f.b.p. where $\mathcal{B}$ is the set all highest weight vectors described above and $\leq$ is the quasi-order given by the consequence. Let $\mathcal{B}'$ be a subset of $\mathcal{B}$ and B' a subset of $B = \{(r,s) : y^r z^s \in \mathcal{B}\}$ corresponding to $\mathcal{B}'$, i.e., $B' = \{(r,s) : y^r z^s \in \mathcal{B}'\}$. Since $B' \subseteq B \subseteq \mathbb{N}^2$ and by Proposition 1 $\mathbb{N}^2$ has f.b.p., there is a finite set $B_0 \subseteq B'$ such that $B_0 \subseteq B' \subseteq \overline{B_0}$. Consider $\mathcal{B}_0 = \{y^r z^s : (r,s) \in B_0\} \subseteq \mathcal{B}'$ and $y^r z^s \in \mathcal{B}'$. This implies that $(r,s) \in B' \subseteq \overline{B_0}$ and therefore there is $(r_0, s_0) \in B_0$ where $(r_0, s_0) \leq (r,s)$. By the previous lemma $y^{r_0} z^{s_0} \leq y^r z^s$ where $y^{r_0} z^{s_0} \in \mathcal{B}_0$. Thus $y^r z^s \in \overline{\mathcal{B}_0}$ and consequently $\mathcal{B}_0 \subseteq \mathcal{B}' \subseteq \overline{\mathcal{B}_0}$ where $\mathcal{B}_0$ is a finite set. □

4.2. The A-grading

Let us consider now the associative grading on $J = UJ_2$, i.e., $G = \mathbb{Z}_2$ and $J = J_0 \oplus J_1$ where $J_0 = F \oplus Fb$ and $J_1 = Fa$.

As in the previous cases, let us recall the basis of the T_2-ideal of UJ_2 and its graded cocharacter sequence.

Theorem 6. *[23, Proposition 6] The following polynomials are a basis of $Id^A(UJ_2)$*

$$(y_1, y_2, y_3),\ (z_1, y, z_2),\ (z_1, z_2, z_3),\ (y_1, z, y_2),\ (z, y_1, y_2),\ (z_1 z_2, x_1, x_2),\ (x_1, z_1 z_2, x_2).$$

Theorem 7. *[6, Theorem 3] Let $n \geq 0$ and let*

$$\chi_n^A = \sum_{\lambda,\mu} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be the A-graded cocharacter of UJ_2. Then $m_{\lambda,\mu} = 1$ if either

1) $\lambda = (r)$, $\mu = (s)$, *for all* $r \geq 0$, $r + s = n$ *and* s *odd.*
2) $\lambda = (n)$, $\mu = \emptyset$.
3) $\lambda = (r)$, $\mu = (s)$, *for all* $r \leq 1$, $r + s = n$ *and* s *even.*
4) $\lambda = (1, 1)$, $\mu = (s)$, s *even.*
5) $\lambda = (p + q, p)$, $\mu = (s)$, *for all* $2p + q > 2$, $2p + q + s = n$ *and* s *even.*

Moreover, $m_{\lambda,\mu} = 2$ if $\lambda = (2)$ and $\mu = (s)$, where s is even, and $m_{\lambda,\mu} = r - 2$ if $\lambda = (r)$ and $\mu = (s)$, where $r > 2$ and s even. In all other cases $m_{\lambda,\mu} = 0$.

In the same paper, it was also proved that the highest weight vectors whose characters appear with non-zero multiplicity in the decomposition of $\chi_n^A(UJ_2)$ are of the form (here we use the standard notation $\overline{y_1}\cdots\overline{y_2} := y_1\cdots y_2 - y_2\cdots y_1$):

(1a) $(y^r z)z^{s-1}$ for s odd;
(1b) y^r for all $r \geq 1$;
(1c) $y^r z^s$ for $r \in \{0, 1\}$ and s even;
(1d) $(((y_1 z)z)y_2)z^{s-2}$ for s even;
(1e) $((((y_1^q \underbrace{\bar{y_1}\cdots\tilde{y_1}}_{p})z)z)\underbrace{\bar{y_2}\cdots\tilde{y_2}}_{p})z^{s-2}$ for s even, or a linear combination of the following ones:
(2a) (i) $y^2 z^s$ for s even;
(ii) $((y^2 z)z)z^{s-2}$ for s even ≥ 2;
(3a) $(((y^i z)z)y^{r-i})z^{s-2}$ for $r > 2$, s even, $i \in \{1, \dots, r-2\}$.

Let us denote by

$$\mathcal{B}_{1a}, \mathcal{B}_{1b}, \mathcal{B}_{1c}, \mathcal{B}_{1d}, \mathcal{B}_{1e}, \mathcal{B}_{2a(i)}, \mathcal{B}_{2a(ii)}, \mathcal{B}_{3a}$$

the sets of highest weight vectors associated to $(1a)$, $(1b)$, $(1c)$, $(1d)$, $(1e)$, $(2a(i))$, $(2a(ii))$ and $(3a)$, respectively. We consider the following sets which are clearly in one-to-one correspondence with the highest weight vectors described above. These sets are the following:

$$B_{1a} = \{(r, s) : s \text{ is odd}\};$$
$$B_{1b} = \{r : r \geq 1\};$$

$$\begin{aligned}
B_{1c} &= \{(r,s) : 0 \le r \le 1, s \text{ is even}\};\\
B_{1d} &= \{s : s \text{ is even}\};\\
B_{1e} &= \{(q,p,s) : s \text{ is even}\};\\
B_{2a(i)} &= \{s : s \text{ is even}\};\\
B_{2a(ii)} &= \{s : s \text{ is even}\};\\
B_{3a} &= \{(i, r-i, s) : 0 \le i \le r-1, s \text{ is even}, r > 2\}\\
&= \{(i,j,s) : s \text{ is even and } j \ge 1\}.
\end{aligned}$$

As in the previous section, we shall show that the natural quasi-order $\le$ in $B_{1a} \cup B_{1b} \cup B_{1c} \cup B_{1d} \cup B_{1e} \cup B_{2a(i)} \cup B_{2a(ii)} \cup B_{3a}$ induces the quasi-order $\le$ in $\mathcal{B}_{1a} \cup \mathcal{B}_{1b} \cup \mathcal{B}_{1c} \cup \mathcal{B}_{1d} \cup \mathcal{B}_{1e} \cup \mathcal{B}_{2a(i)} \cup \mathcal{B}_{2a(ii)} \cup \mathcal{B}_{3a}$ where $f \le g$ if and only if f, $g \in \mathcal{B}_i$ for some i and g is a consequence of f. In order to reach the goal, first we prove the following technical lemma.

Lemma 2. *The following statements hold modulo $Id^A(UJ_2)$.*

(1) $(z^t(yz))z \equiv z^{t+1}(yz)$ *for all* $t \ge 0$.
(2) $(Y_1z)(yz) \equiv ((Y_1y)z)z$ *where* Y_1, Y_2 *are products of even variables.*
(3) $(((Y_1z)z)Y_2)z^r \le (((Y_1z)z)Y_2y)z^r$ *where r is even* ≥ 2 *and* Y_1, Y_2 *are products of even variables.*
(4) $(((Y_1z)z)Y_2)z^r \le ((((Y_1y)z)z)Y_2)z^r$ *where r is even* ≥ 2 *and* Y_1, Y_2 *are products of even variables.*

Proof. First let us prove statement (1). If $t = 0$ then (1) trivially follows, so let us suppose $t > 0$. If t is odd, then $(z^t(yz))z \equiv ((yz)z)z^t \equiv (yz)z^{t+1} \pmod{\mathrm{Id}^A(UJ_2)}$, since $(z_1, z_2, z_3) \equiv 0$. Conversely, if t is even, then due to $(z_1, y, z_2) \equiv 0$, $z^t = zz^{t-1}$ with $t-1$ odd and $(z_1z_2, x_1, x_2) \equiv 0$, we get

$$(z^t(yz))z \equiv ((yz)z)z^t \equiv z^{t+1}(yz) \pmod{\mathrm{Id}^A(UJ_2)}.$$

Furthermore, since (z_1, y, z_2), (y_1, z, y_2), (y_1, y_2, z) are graded identities of UJ_2^A it follows that

$$(Y_1z)(yz) \equiv [(Y_1z)y]z \equiv [Y_1(yz)]z \equiv ((Y_1y)z)z \pmod{\mathrm{Id}^A(UJ_2)},$$

so we get statement (2).

Statement (3) is easily proved by remarking that because of r is even and $(y_1, y_2, y_3) \equiv 0$, then

$$[(((Y_1z)z)Y_2)z^r]y \le [((Y_1z)z)Y_2](yz^r) \le [(((Y_1z)z)Y_2)y]z^r \le [((Y_1z)z)Y_2y]z^r$$

Finally, let us partially linearize the variable z of $(((Y_1z)z)Y_2)z^r$. We get the following consequence:

$$\sum_{\sigma\in S_{r+2}}(((((Y_1z_{\sigma(1)})z_{\sigma(2)})Y_2)(z_{\sigma(3)}\cdots z_{\sigma(r+2)}).$$

If we replace in the previous polynomial z_1 by yz and z_i by z for all $i\neq 1$, we obtain

$$(((Y_1(yz))z)Y_2)z^r+(((Y_1z)(yz))Y_2)z^r+\sum_{t=0}^{r}(((Y_1z)z)Y_2)((\cdots((z^t(yz))\underbrace{z)\cdots z)z}_{r-t-1}).$$

By taking into account the previous statements, we have that $(((Y_1(yz))z)Y_2)z^r$ and $(((Y_1z)(yz))Y_2)z^r$ are equivalent to $((((Y_1y)z)z)Y_2)z^r$ modulo $\mathrm{Id}^A(UJ_2)$. Moreover, by (1) for all $0\leq t\leq r$,

$$(((Y_1z)z)Y_2)((\cdots((z^t(yz))\underbrace{z)\cdots z)z}_{r-t-1})\equiv(((Y_1z)z)Y_2)((yz)z^{r-1})\ (\mathrm{mod}\ \mathrm{Id}^A(UJ_2)).$$

Thus, in order to prove statement (4), it suffices to show that $(((Y_1z)z)Y_2)((yz)z^{r-1})$ is equivalent to $((((Y_1y)z)z)Y_2)z^r$ modulo $\mathrm{Id}(UJ_2)$. Since r is even and (y_1,y_2,y_3) is an identity, we have

$$(((Y_1z)z)Y_2)((yz)z^{r-1})\equiv[(((Y_1z)z))(z^{r-1}(yz))]Y_2\ (\mathrm{mod}\ \mathrm{Id}^A(UJ_2)).$$

Now remark that

$$\begin{aligned}[(((Y_1z)z))(z^{r-1}(yz))]Y_2&\equiv[((Y_1z)z)z^{r-1}](yz)]Y_2\equiv[[(Y_1z)z^r](yz)]Y_2\\&\equiv[(z^r(Y_1z))(yz)]Y_2\ (\mathrm{mod}\ \mathrm{Id}^A(UJ_2)),\end{aligned}$$

since (z_1z_2,x_1,x_2) and (z_1,z_2,z_3) are identities and $r-1$ is odd. Moreover, due to the fact that r is even and greater than 2, $z^r=zz^{r-1}$ and $(z_1z_2,x_1,x_2)\equiv(x_1,z_1z_2,x_2)\equiv 0$, it turns out that

$$[(z^r(Y_1z))(yz)]Y_2\equiv[((Y_1z)(yz))Y_2]z^r\ (\mathrm{mod}\ \mathrm{Id}^A(UJ_2)).$$

Therefore by statement (2)

$$[((Y_1z)(yz))Y_2]z^r\equiv[[((Y_1y)z)z]Y_2]z^r\ (\mathrm{mod}\ \mathrm{Id}^A(UJ_2))$$

and we are done. □

Lemma 3. *We have*

(1) $(y^r z)z^{s-1} \leq (y^{r'} z)z^{s'-1}$ *where* $(r, s) \leq (r', s')$ *and* s, s' *odd;*
(2) $y^r \leq y^{r'}$ *where* $r \leq r'$ *for all* r, $r' \geq 1$*;*
(3) $yz^s \leq yz^{s'}$ *and* $z^s \leq z^{s'}$ *where* $s \leq s'$ *and* s, s' *even;*
(4) $(((y_1 z)z)y_2)z^{s-2} \leq (((y_1 z)z)y_2)z^{s'-2}$ *where* $s \leq s'$ *and* s, s' *even;*
(5) $((((y_1^q \underbrace{\bar{y_1} \cdots \tilde{y_1}}_{p})z)z) \underbrace{\bar{y_2} \cdots \tilde{y_2}}_{p})z^{s-2} \leq ((((y_1^{q'} \underbrace{\bar{y_1} \cdots \tilde{y_1}}_{p'})z)z) \underbrace{\bar{y_2} \cdots \tilde{y_2}}_{p'})z^{s'-2}$ *where* $(q, p, s) \leq (q', p', s')$ *and* s, s' *even* ≥ 2*;*
(6) $y^2 z^s \leq y^2 z^{s'}$ *and* $((y^2 z)z)z^{s-2} \leq ((y^2 z)z)z^{s'-2}$ *where* $s \leq s'$ *for* s *even;*
(7) $(((y^i z)z)y^j)z^{s-2} \leq (((y^{i'} z)z)y^{j'})z^{s'-2}$ *where* $(i, j, s) \leq (i', j', s')$ *and* j, $j' \geq 1$, s, s' *even.*

Proof. The statement (1) follows from

$$[y^{r'-r}[(y^r z)z^{s-1}]]z^{s'-s} \equiv [[y^{r'-r}(y^r z)]z^{s-1}]z^{s'-s} \equiv [(y^{r'} z)z^{s-1}]z^{s'-s}$$
$$\equiv (y^{r'} z)z^{s'-1} \pmod{\mathrm{Id}^A(UJ_2)},$$

since (y_1, z, y_2) and (z, y_1, y_2) are graded identities of UJ_2. The statement (2) is trivial.

Statements (3), (4) and (6) readily follow from the fact that s and $s' - s$ are even and $(y_1, y_2, y_3) \equiv 0$.

In order to prove statement (5) we use the transitivity of the quasi-order, i.e., we prove that $(q, p, s) \leq (q', p, s)$ implies $f_{q,p,s} \leq f_{q',p,s}$, $(q, p, s) \leq (q, p', s)$ implies $f_{q,p,s} \leq f_{q,p',s}$ and $(q, p, s) \leq (q, p, s')$ implies $f_{q,p,s} \leq f_{q,p,s'}$, for all q, q', p, p', s, s' integers, where

$$f_{q,p,s} = f_{q,p,s}(y_1, y_2, z) = ((((y_1^q \underbrace{\bar{y_1} \cdots \tilde{y_1}}_{p})z)z) \underbrace{\bar{y_2} \cdots \tilde{y_2}}_{p})z^{s-2}.$$

Clearly, $f_{q,p,s} \leq f_{q,p,s'}$ when $s \geq s'$ and s, s' are even, since $f_{q,p,s'} \equiv f_{q,p,s} z^{s'-s}$.

Now suppose $q \leq q'$, then $f_{q,p,s} \leq f_{q',p,s}$ by (4) of Lemma 2.

Finally, without loss of generality we may suppose $p' = p + 1$. The general statement will follow by a standard induction argument. By [6, Lemma 5] we have

$$f_{q,p,s} = \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} ((((y_1^{q+j} y_2^{p-j})z)z)y_1^{p-j} y_2^j)z^{s-2}.$$

On the other hand, by statements (3) and (4) of Lemma 2, for all $0 \leq j \leq p$

$$((((y_1^{q+j} y_2^{p-j})z)z)y_1^{p-j} y_2^j)z^{s-2} \leq ((((y_1^{q+j} y_2^{(p+1)-j})z)z)y_1^{(p+1)-j} y_2^j)z^{s-2},$$

therefore $f_{q,p,s} \leq f_{q,p+1,s}$.

The proof of (7) is analogous to that of (5). $\square$

Our next goal is proving the Specht property for $\mathrm{var}^A(UJ_2)$.

Theorem 8. *Let F be field of characteristic 0, then $var^A(UJ_2)$ has the Specht property.*

Proof. Let $\mathcal{U}$ be a subvariety of $\mathrm{var}^A(UJ_2)$. If $\mathcal{U} = \mathrm{var}^A(UJ_2)$, then the result follows from Theorem 6.

So let us suppose $\mathcal{U} \subsetneq \mathrm{var}^A(UJ_2)$ and let us denote by I the corresponding T-ideal of graded identities. As noted above, not all the highest weight vectors do correspond to multipartitions appearing with multiplicity 1 in the cocharacter sequence of $\mathrm{var}^A(UJ_2)$. Hence we divide the proof into two parts being the first one the one involving the cases from (1a) to (1e). We start handling the cases (1a)–(1e). By Remark 1 it suffices to show that $(\mathcal{B}_{1a} \cup \mathcal{B}_{1b} \cup \mathcal{B}_{1c} \cup \mathcal{B}_{1d} \cup \mathcal{B}_{1e}, \leq)$ satisfies f.b.p. where $f \leq g$ if and only if $f,\, g \in \mathcal{B}_i$ for some i and g is a consequence of f. If we set $\mathcal{B} = \mathcal{B}_{1a} \cup \mathcal{B}_{1b} \cup \mathcal{B}_{1c} \cup \mathcal{B}_{1d} \cup \mathcal{B}_{1e}$ and $B = B_{1a} \cup B_{1b} \cup B_{1c} \cup B_{1d} \cup B_{1e}$, then by Lemma 3 we get the claim as outlined in the proof of Theorem 5.

It remains to prove the cases (2a) and (3a). We have to remark the two cases have to be manipulated separately because they correspond to different modules. Hence we start with the case (2a) and we notice any highest weight vector can be written as $h = \alpha f + \beta g$, $\alpha, \beta \in F$, f of type (2a)(i) and g of type (2a)(ii). We have to follow the four steps as outlined in Section 3, then we have to order the highest weight vectors. We always consider the elements (2a)(i) greater than the elements (2a)(ii) and among them we give an order depending on the exponent of the variable z and this completes **Step 1**. By Lemma 3 the set $(\mathcal{B}_{2a(i)} \cup \mathcal{B}_{2a(ii)}, \leq)$ satisfies f.b.p. where $f \leq g$ if and only if $f,\, g \in \mathcal{B}_i$ for some i and g is a consequence of f and this is enough to complete **Step 2**. In order to get **Step 3** we have simply to notice that if

$$h_1 = \alpha_1 f_1 + \beta_1 g_1 \text{ and } h_2 = \alpha_2 f_2 + \beta_2 g_2$$

are highest weight vectors such that $f_1 \leq f_2$, then $f_2 = f_1 z^t$ for some $t \geq 0$, then we set $g = g_1 z^t$ and we consider

$$h = \alpha f_2 + \beta g$$

which is the required highest weight vector and we are done because the other cases are completely analogous.

The case (3a) may be treated similarly to the first case. Actually we order the elements $(3a)$ depending on the exponent $s-2$ of z and **Step 1** is over. **Step 2** follows again from Lemma 3. If

$$h_1 = \sum_{i=1}^{r-2} \alpha_i f_i^{(i,r-i,s)} \text{ and } h_2 = \sum_{j=1}^{r'-2} \alpha_j f_j^{(j,r'-j,s')}$$

are two highest weight vectors such that their leading terms are $f_1 := f_{i_0}^{(i_0,r-i_0,s)}$, $f_2 := f_{j_0}^{(j_0,r'-j_0,s')}$ respectively and $f_1 \leq f_2$, then $f_2 = \phi(f_1 z^{s'-s} y^{r'-j_0-r+i_0})$, where ϕ is a partial linearization of the polynomial in the odd variables as outlined in the proof of Statement (3) of Lemma 2. In light of this we set

$$h := \phi(\sum_{i=1}^{r-2} \alpha_i((f_i^{(i,r-i,s)} z^{s'-s}) y^{r'-j-r+i}))$$

that is the required highest weight vector and **Step 3** is completed. □

4.3. The C-grading

In this section we fix the classical grading on $J = UJ_2$, i.e., $G = \mathbb{Z}_2$ and $J = J_0 \oplus J_1$ where $J_0 = F \oplus Fa$ and $J_1 = Fb$. In the free $\mathbb{Z}_2$-graded Jordan algebra $\mathcal{J}\langle X\rangle$ we write $X = Y \cup Z$, the disjoint union of two countable sets and we require that the variables $y_i \in Y$ have homogeneous degree 0, the variables $z_i \in Z$ have homogeneous degree 1. Moreover, we denote by x_i any kind of variable.

The next theorems give a basis of the graded identities of UJ_2 with the C-grading and describe the corresponding cocharacter sequence.

Theorem 9. *[23, Proposition 12] The following polynomials are a basis of* $Id^C(UJ_2)$

$$(x_1x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4), \quad (y_1, y_2, y_3), \quad z_1z_2, \quad (y_1, z, y_2).$$

Theorem 10. *[6, Theorem 1] Let* $n \geq 0$ *and let*

$$\chi_n^C(UJ_2) = \sum_{\lambda,\mu} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be the C-graded cocharacter of UJ_2. *Then* $m_{\lambda,\mu} = 1$ *if and only if* $\lambda = (r)$ *and* $\mu = (s)$, *where* $r + s = n$, $s \leq 1$ *and* r, s *not simultaneously equal to zero. In all other cases* $m_{\lambda,\mu} = 0$.

In the case of the C-grading, the Specht property for UJ_2 is proved assuming only that the base field is infinite of characteristic different from 2.

Let I be a T_2-ideal such that $\mathrm{Id}^C(UJ_2) \subsetneq I$ and $f \in I$. Since F is an infinite field we can assume that f is a multihomogeneous polynomial. By Proposition 12 of [23]

$$f \equiv f(y_1, ..., y_s, z) \equiv \underbrace{y_1(\cdots(y_1}_{a_1}(y_2(\cdots(\underbrace{y_s\cdots(y_s}_{a_s} z))\cdots)))\cdots) \pmod{\mathrm{Id}^C(UJ_2)},$$

where $a_i \geq 0$ for all $i \in \{1, \ldots, s\}$

or

$$f \equiv f(y_1, ..., y_s) \equiv y_1^{b_1} \dots y_s^{b_s} \pmod{\mathrm{Id}^C(UJ_2)},\ b_i \geq 0 \text{ for all } i \in \{1, \dots, s\}.$$

Then I is defined by a subset of set of polynomial in $\mathcal{J}(X)$ of the form:

(1) $\underbrace{y_1(\cdots(y_1}_{a_1}(y_2(\cdots(\underbrace{y_s\cdots(y_s}_{a_s} z))\cdots)))\cdots) \pmod{\mathrm{Id}^C(UJ_2))}$, where $a_i \geq 0$ for all $i \in \{1, \dots, s\}$

(2) $y_1^{b_1} \dots y_s^{b_s} \pmod{\mathrm{Id}^C(UJ_2))}$, $b_j \geq 0$ for all $j \in \{1, \dots, s\}$

Let us denote by $\mathcal{B}_1$ and $\mathcal{B}_2$ the set of polynomials associated to (1) and (2) respectively. We consider the following sets which are clearly in one-to-one correspondence with polynomials of the type (1) and (2) respectively.

$$B_1 = \{(a_1, \dots, a_r) : r > 0, b_i > 0 \text{ for all } i\};$$

$$B_2 = \{(b_1, \dots, b_s) : s > 0, a_j > 0 \text{ for all } j\}.$$

Since B_1 and B_2 are subsets of $S(\mathbb{N})$, B_1 and B_2 satisfy f.b.p. with the quasi-order induced by the quasi-order of $S(\mathbb{N})$.

To simplify the notation, we will denote $\underbrace{y_1(\cdots(y_1}_{a_1}(y_2(\cdots(\underbrace{y_s\cdots(y_s}_{a_s} z))\cdots)))\cdots)$ by $Y^a z$ where $a = (a_1, \dots, a_s)$.

Lemma 4. *Let $a = (a_1, \dots, a_r)$, $a' = (a'_1, \dots, a'_{r'}) \in B_1$ and $b = (b_1, \dots, b_s)$, $b' = (b'_1, \dots, b'_{s'}) \in B_2$.*

(1) *If $a \leq a'$ then $Y^a z \leq Y^{a'} z$.*
(2) *If $b \leq b'$ then $y_1^{b_1} \dots y_s^{b_s} \leq y_1^{b'_1} \dots y_s^{b' s'}$.*

Proof. Let us prove statement (1). Since $a = (a_1, \dots, a_r) \leq (a'_1, \dots, a'_{r'}) = a'$ there is a subsequence $a'' = (a'_{i_1}, \dots, a'_{i_r})$ of a' ($i_1 \leq \cdots \leq i_r$) such that $a_j \leq a'_{i_j}$ for all $j \in \{1, \dots, r\}$. Define $f(y_1, \dots, y_r, z) = Y^a z$ and let $f(y_{i_1}, \dots, y_{i_r}, z)$ be the polynomial obtained replacing the variable y_j by y_{i_j} for each $j \in \{1, \dots, r\}$ in $f(y_1, \dots, y_r, z)$. Then by Proposition 12 of [23]

$$\tilde{Y} y_{i_1}^{a'_{i_1} - a_1} \cdots y_{i_r}^{a'_{i_r} - a_r} f(y_{i_1}, \dots, y_{i_r}, z) \equiv Y^{a'} z$$

where

$$\tilde{Y} = \prod_{k \in \{1, \cdots, n\} - \{i_1, \dots, i_r\}} y_k^{a'_k}$$

and the last equivalence is obtained replacing the variable y_{i_j} by y_j for each $j \in \{1, \ldots, r\}$. Therefore $Y^a z = f(y_{i_1}, \ldots, y_{i_r}, z) \leq Y^{a'} z$. Similar arguments can be applied in the proof of statement (2). $\square$

Theorem 11. *Let F be an infinite field of characteristic different from 2. Then $var^C(UJ_2)$ has the Specht property.*

Proof. The result follows from Theorem 9 if $\mathcal{U} = \mathrm{var}^C(UJ_2)$.

Let I a T_2-ideal such that $\mathrm{Id}^C(UJ_2) \subsetneq I$. By Remark 1 it suffices to show that $(\mathcal{B}_1 \cup \mathcal{B}_2, \leq)$ satisfies f.b.p. where $f \leq g$ if f, $g \in \mathcal{B}_i$ for some $i \in \{1, 2\}$ and g is a consequence of f. Let us set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $B = B_1 \cup B_2$, then by taking into account Lemma 4, the proof follows verbatim the one of Theorem 5. $\square$

It is interesting to mention the following. The fact that in the case of characteristic 0 the multiplicities of the irreducible components in the cocharacter sequence of $\mathrm{var}^C(UJ_2)$ are 0 and 1 only, implies that the lattice of the subvarieties of $\mathrm{var}^C(UJ_2)$ is distributive (see the paper [2] for more details). With some additional work this leads to the description of this lattice.

4.4. The K-grading

Now we deal with the Klein grading on $J = UJ_2$, i.e., $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $J = J_{(0,0)} \oplus J_{(1,0)} \oplus J_{(0,1)} \oplus J_{(1,1)}$ where $J_{(0,0)} = F$, $J_{(0,1)} = Fa$, $J_{(1,0)} = Fb$, $J_{(1,1)} = \{0\}$. Here we write $X = Y \cup Z \cup T \cup W$, the disjoint union of four countable sets and we require that the variables $y_i \in Y$ have homogeneous degree $(0, 0)$, the variables $z_i \in Z$ have homogeneous degree $(1, 0)$, the variables $t_i \in T$ have homogeneous degree $(0, 1)$, the variables $w_i \in W$ have homogeneous degree $(1, 1)$.

In [5] the authors gave a description of the generators of the T-ideal of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded polynomial identities of UJ_2 and the corresponding graded cocharacter sequence.

Theorem 12. *[5, Lemma 3.5] The following polynomials are a basis of $Id^K(UJ_2)$.*

$$(y_1, y_2, x), \quad (y_1, x, y_2), \quad (z_1, z_2, y), \quad (z_1, y, z_2), \quad (z_1, z_2, z_3), \quad zt, \quad t_1 t_2, \quad w.$$

Theorem 13. *[5, Lemma 3.6] Let $n \geq 0$ and let*

$$\chi_n^K(UJ_2) = \sum_{\lambda,\mu,\nu,\eta} m_{\lambda,\mu,\nu,\eta} \chi_{\lambda,\mu,\nu,\eta}$$

be the K-graded cocharacter of UJ_2. Then $m_{\lambda,\mu,\nu,\eta} = 1$ if either

1) $\lambda = (r)$, $\mu = (s)$, $\nu = \emptyset$, $\eta = \emptyset$ *for all* $r, s \geq 0$ *and* $r + s = n$;
2) $\lambda = (r)$, $\mu = (s)$, $\nu = (1)$, $\eta = \emptyset$ *for all* $r \geq 0$, $r + s + 1 = n$ *and* s *even;*
3) $\lambda = (n - 1)$, $\mu = \emptyset$, $\nu = (1)$, $\nu = \emptyset$.

In all other cases $m_{\lambda,\mu,\nu,\eta} = 0$.

The highest weight vectors whose characters appear with non-zero multiplicity in the decomposition of $\chi_n^K(UJ_2)$ are of the form:

(1) $y^r z^s$ for all r, s not simultaneously equal to 0;
(2) $y^r z^s w$ for all $r \geq 0$ and s even;
(3) $y^r w$ for $r \geq 0$.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be the sets of all highest weight vectors corresponding to (1), (2) and (3) respectively. Let us consider B_1, B_2 being the set of pairs (r, s) such that $y^r z^s \in \mathcal{B}_1$, $y^r z^s w \in \mathcal{B}_2$ respectively and B_3 the set of positive integers r where $y^r w \in \mathcal{B}_3$. By Proposition 1, B_1, B_2 and B_3 have f.b.p. and are pairwise disjoint.

As in the case of the C-grading we shall show that the natural quasi-order $\leq$ in $B_1 \cup B_2 \cup B_3$ induces the quasi-order $\leq$ in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where $f \leq g$ if $f, g \in \mathcal{B}_i$ for some $i \in \{1, 2, 3\}$ and g is a consequence of f. Notice that, although the quasi-order $\leq$ is the same for $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ (given by the consequence), it will be necessary to compare only polynomials $f, g \in \mathcal{B}_i$ for some $i \in \{1, 2, 3\}$.

Lemma 5. *Let* $(r, s), (r', s') \in B_1$, $(t, u), (t', u') \in B_2$ *and* $v, v' \in B_3$.

(1) *If* $(r, s) \leq (r', s')$ *then* $y^r z^s \leq y^{r'} z^{s'}$;
(2) *If* $(t, u) \leq (t', u')$ *then* $y^t z^u w \leq y^{t'} z^{u'} w$;
(3) *If* $r \leq r'$ *then* $y^r w \leq y^{r'} w$.

Proof. The statement (1) follows from the fact that $y^{r'} z^{s'} \equiv [y^{r'-r}(y^r z^s)]z^{s'-s}$ modulo $\mathrm{Id}^K(UJ_2)$, since the even variables y lie in the associative and commutative centre of the relatively free graded algebra $\frac{\mathcal{J}\langle X\rangle}{\mathrm{Id}^S(UJ_2)}$ and (y, z_1, z_2) is an identity of UJ_2.

Moreover, since $t' - t$ is even, then $z^{t'-t}$ is an even variable that lies in the associative and commutative centre. Thus $z^{s'-s}[y^{r'-r}(y^r z^s w)] \equiv y^{r'} z^{s'} w$ modulo $\mathrm{Id}^K(UJ_2)$ and the statement (2) follows.

Similar arguments prove (3). $\square$

We are now in a position to prove the Specht property for $\mathrm{var}^K(UJ_2)$.

Theorem 14. *Let F be a field of characteristic 0. Then $var^K(UJ_2)$ has the Specht property.*

Proof. Let $\mathcal{U}$ be a subvariety of $\mathrm{var}^K(UJ_2)$. If $\mathcal{U} = \mathrm{var}^K(UJ_2)$, then the result follows from Theorem 12.

So let us suppose $\mathcal{U} \subsetneq \mathrm{var}^K(UJ_2)$ and let us denote by I the corresponding T-ideal of graded identities. By Remark 1 it suffices to show that $(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3, \leq)$ satisfies f.b.p. where $f \leq g$ if $f, g \in \mathcal{B}_i$ for some $i \in \{1, 2, 3\}$ and g is a consequence of f. Let us set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and $B = B_1 \cup B_2 \cup B_3$, then by taking into account Lemma 5, the proof follows verbatim the one of Theorem 5. $\square$

4.5. The trivial grading

We finally deal with the trivial grading, i.e., the ordinary polynomial identities of UJ_2. In [26] it was proved that the algebra of 2×2 upper triangular matrices is the only finite dimensional special Jordan algebra that generates a variety of almost polynomial growth. Furthermore, in [6] and [23] the authors computed a basis of the T-ideal of identities of UJ_2 and the corresponding cocharacter sequence.

Lemma 6. *[23, Theorem 19] The following polynomials are a basis of $Id(UJ_2)$*

$$(x_1x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4), \qquad (x_1, (x_2, x_3, x_4), x_5).$$

Remark 2. If B is an associator then $x_1(x_2B) \equiv x_2(x_1B)$ modulo $\mathrm{Id}(UJ_2)$.

Proof. Since $(x_1, (x_2, x_3, x_4), x_5) \in \mathrm{Id}(UJ_2)$, then we have that

$$x_1(x_2B) = (x_1B)x_2 - (x_1, B, x_2) \equiv (x_1B)x_2 = x_2(x_1B) \pmod{\mathrm{Id}(UJ_2)}. \quad \square$$

The proof of the next Lemma uses the relation between proper and ordinary cocharacter sequences. We refer to [8, Chapters 4.3 and 12.5] for an exhaustive survey about proper polynomials, proper cocharacters and Littlewood–Richardson rule. The book [8] considers associative algebras only but the same results hold for any unitary algebra, see Proposition 1.5 in [9].

Lemma 7. *Let $n \geq 1$ and let*

$$\chi_n(UJ_2) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

be the n-cocharacter of the Jordan algebra UJ_2. If $\lambda = (n)$ then $m_\lambda = 1$, if either $\lambda = (p + q, p)$ or $\lambda = (p + q, p, 1)$, $p > 0$, then $m_\lambda = \lceil \frac{q+1}{2} \rceil$ if p is odd and $m_\lambda = \lceil \frac{q+2}{2} \rceil$ if p is even.

In all other cases $m_\lambda = 0$.

Proof. By [6], the proper cocharacter of UJ_2 is

$$\psi_n(UJ_2) = \begin{cases} \chi_{(n-1,1)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\chi_n(UJ_2) = \sum_{k=0}^{n} \chi_k \otimes \psi_{n-k}(UJ_2) = \chi_{(n)} + \sum_{p>0} \left(m_{(p+q,p)}\chi_{(p+q,p)} + m_{(p+q,p,1)}\chi_{(p+q,p,1)} \right).$$

Here $\otimes$ stands for the multiplication in the Littlewood–Richardson rule.

This proves immediately the cases $\lambda = (n)$ and $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_3 > 1$ or $\lambda_4 > 0$.

The case $\lambda = (p+q, p)$ (with similar arguments for $\lambda = (p+q, p, 1)$) is handled as follows. Clearly, $\chi_{(p+q,p)}$ is obtained from $\chi_{(k)} \otimes \psi_{n-k}(UJ_2)$ when $n-k$ is odd, i.e., from $\chi_{(k)} \otimes \chi_{(2s,1)}$, $p \leq 2s \leq p+q$. For p even, we have the possibilities $s = p, p+2, \dots, p+q-\varepsilon$, $\varepsilon = 0, 1$, depending on the parity of q, i.e., $\lceil \frac{q+2}{2} \rceil$ possibilities. For p odd the possibilities for s are $s = p+1, p+3, \dots, p+q-\varepsilon$, $\varepsilon = 0, 1$, i.e., $\lceil \frac{q+1}{2} \rceil$ possibilities. $\square$

By using the Littlewood–Richardson rule, it was also proved in [6, Theorem 4] that the highest weight vectors associated to the partition $\lambda = (p+q, p)$ whose characters appear with a non-zero multiplicity in the decomposition of $\chi_n(UJ_2)$ are of the form:

$$f_{t,u,v} = \underbrace{x_1 \cdots x_1}_{t} \underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u} (\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \underbrace{x_1, \dots, x_1}_{v}), \tag{3}$$

where $p = u+1$, $q = t+v$ and $u+v$ odd. Thus it is clear that each highest weight vector is uniquely determined by a triple of positive integers (t, u, v).

Next we consider the following set which is in a one-to-one correspondence with the set of highest weight vectors:

$$B = \{(t, u, v) : \ u \text{ is even and } v \text{ is odd or } u \text{ is odd and } v \text{ is even}\}.$$

We shall consider a quasi-order on B and we shall show that it induces the quasi-order (2) in the corresponding subset of highest weight vectors $\mathcal{B}$.

We start by defining a natural quasi-order $\leq$ on B as follows:

$$(t, u, v) \leq (t', u', v') \text{ if } t \leq t', u \leq u', v \leq v'.$$

Clearly, B has the f.b.p. Thus, in order to reach our goal, as in the previous section, by transitivity it suffices to prove that for all t, t', u, u', v, v' we have: $(t, u, v) \leq (t', u, v)$ implies $f_{t,u,v} \leq f_{t',u,v}$, $(t, u, v) \leq (t, u', v)$ implies $f_{t,u,v} \leq f_{t,u',v}$ and $(t, u, v) \leq (t, u, v')$ implies $f_{t,u,v} \leq f_{t,u,v'}$. The next lemmas go in this direction.

Lemma 8. *Let $(t, u, v), (t', u, v) \in B$, then $(t, u, v) \leq (t', u, v)$ implies $f_{t,u,v} \leq f_{t',u,v}$.*

Proof. Since the brackets in (3) are right-normed, it is clear that

$$f_{t',u,v} = \underbrace{x_1 \cdots x_1}_{t'-t} f_{t,u,v},$$

thus $f_{t,u,v} \leq f_{t',u,v}$. $\square$

Lemma 9. *Let* $(t,u,v),(t,u,v') \in B$*, then* $(t,u,v) \leq (t,u,v')$ *implies* $f_{t,u,v} \leq f_{t,u,v'}$*.*

Proof. With abuse of notation, let

$$f_{t,u,v} = AC,$$

where $A = \underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}$ and $C = (\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \underbrace{x_1, \dots, x_1}_{v})$.

Remark that, since $u+v$ and $u'+v'$ are odd, we have $v \equiv v' \pmod 2$, thus we can consider $f_{t,u,v'} = A(B, x_1, x_1)$. The general statement will follow by a standard induction argument. By applying Remark 2 we get

$$\begin{aligned} f_{t,u,v'} &= A(C, x_1, x_1) = A((Cx_1)x_1) - A(C(x_1x_1)) \equiv x_1(x_1(AC)) - (x_1x_1)(AC) \\ &= x_1(x_1 f_{t,u,v}) - (x_1x_1) f_{t,u,v} \pmod{\mathrm{Id}(UJ_2)}. \end{aligned}$$

Thus $f_{t,u,v} \leq f_{t,u,v'}$. $\square$

Lemma 10. *Let* $(t,u,v),(t,u',v) \in B$*, then* $(t,u,v) \leq (t,u',v)$ *implies* $f_{t,u,v} \leq f_{t,u',v}$*.*

Proof. As in the previous lemma, since $u \equiv u' \pmod 2$, we can consider

$$\begin{aligned} f_{t,u,v} &= \underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \underbrace{x_1, \dots, x_1}_{v}) \text{ and} \\ f_{t,u',v} &= \underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}\tilde{\bar{\bar{x}}}_2\widehat{x}_2(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \tilde{\bar{\bar{x}}}_1, \widehat{x}_1, \underbrace{x_1, \dots, x_1}_{v}). \end{aligned}$$

We remark that in the proof of [23, Theorem 19], the authors showed that one can always reorder the variables that lie inside the associator except the one in the second position, therefore using this fact and Remark 2, we get

$$\begin{aligned} f_{t,u',v} &= \underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}\tilde{\bar{\bar{x}}}_2\widehat{x}_2(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \tilde{\bar{\bar{x}}}_1, \widehat{x}_1, \underbrace{x_1, \dots, x_1}_{v}) \\ &\equiv \underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}\tilde{\bar{\bar{x}}}_2\widehat{x}_2(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \underbrace{x_1, \dots, x_1}_{v}, \tilde{\bar{\bar{x}}}_1, \widehat{x}_1) \\ &\equiv \tilde{\bar{\bar{x}}}_2\widehat{x}_2\underbrace{x_1 \cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u}(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \dots, \tilde{x}_1}_{u}, \underbrace{x_1, \dots, x_1}_{v}, \tilde{\bar{\bar{x}}}_1, \widehat{x}_1) \pmod{\mathrm{Id}(UJ_2)}. \end{aligned}$$

By expanding the last associator and by using once again Remark 2, we have

$$\begin{aligned}
f_{t,u',v} &= \Big[\big(\tilde{\bar{x}}_2\widehat{x}_2\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v})\big)\tilde{\bar{x}}_1\Big]\widehat{x}_1 \\
&- \Big[\ \tilde{\bar{x}}_2\widehat{x}_2\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v})\Big]\big(\tilde{\bar{x}}_1\widehat{x}_1\big) \\
&= \tilde{\bar{x}}_1\widehat{x}_1\tilde{\bar{x}}_2\widehat{x}_2\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v}) \\
&- \big(\tilde{\bar{x}}_1\widehat{x}_1\big)\Big[\ \tilde{\bar{x}}_2\widehat{x}_2\big(\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v})\big)\Big] \\
&\equiv -\big(\tilde{\bar{x}}_1\widehat{x}_1\big)\Big[\ \tilde{\bar{x}}_2\widehat{x}_2 f_{t,u,v}\Big] \pmod{\mathrm{Id}(UJ_2)}.
\end{aligned}$$

Thus $f_{t,u,v} \le f_{t,u',v}$ and we are done. □

Lemma 11. *Let $(t,u,v),(t,u',v') \in B$, such that either u,v' are odd and v,u' are even or u,v' are even and v,u' are odd, then $(t,u,v) \le (t,u',v')$ implies $f_{t,u,v} \le f_{t,u',v'}$.*

Proof. Let us suppose $u' = u+1$ and $v' = v+1$. The general statement will follow by a standard induction argument.

We write

$$\begin{aligned}
f_{t,u,v} &= \underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v}) \text{ and} \\
f_{t,u',v'} &= \underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}\widehat{x}_2(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\widehat{x}_1,\underbrace{x_1,\ldots,x_1}_{v},x_1).
\end{aligned}$$

Let expand the last alternation of $f_{t,u',v'}$:

$$\begin{aligned}
f_{t,u',v'} &= \underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}x_2(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},x_1,\underbrace{x_1,\ldots,x_1}_{v},x_1) \\
&- \underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}x_1(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},x_2,\underbrace{x_1,\ldots,x_1}_{v},x_1).
\end{aligned}$$

Using Remark 2 and reordering opportunely the variables inside the associator, we get

$$\begin{aligned}
f_{t,u',v'} &\equiv x_2\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v},x_1,x_1) \\
&- x_1\underbrace{x_1\cdots x_1}_{t}\underbrace{\bar{\bar{x}}_2\cdots\tilde{x}_2}_{u}(\bar{x}_1,\bar{x}_2,\underbrace{\bar{\bar{x}}_1,\ldots,\tilde{x}_1}_{u},\underbrace{x_1,\ldots,x_1}_{v},x_2,x_1) \pmod{\mathrm{Id}(UJ_2)}.
\end{aligned}$$

Finally, applying the same arguments as in Lemma 9, we have

$$f_{t,u',v'} \equiv x_2(x_1(x_1 f_{t,u,v}) - (x_1x_1)f_{t,u,v}) - x_1(x_2(x_1 f_{t,u,v}) - (x_1x_2)f_{t,u,v} \ (\mathrm{mod}\ \mathrm{Id}(UJ_2)).)\ (\mathrm{mod}\ \mathrm{Id}(UJ_2)).$$

Hence $f_{t,u,v} \leq f_{t,u',v'}$ and we are done. $\square$

If $\lambda = (p+q, p, 1)$, then the highest weight vector associated to λ is of the form

$$g_{t,u,v} = \underbrace{x_1 \cdots x_1}_{t} \underbrace{\bar{\bar{x}}_2 \cdots \tilde{x}_2}_{u} \bar{x}_3(\bar{x}_1, \bar{x}_2, \underbrace{\bar{\bar{x}}_1, \ldots, \tilde{x}_1}_{u}, \underbrace{x_1, \ldots, x_1}_{v}). \tag{4}$$

With similar arguments as in the previous case, we can find a set B' which is in a one-to-one correspondence with the set of highest weight vectors $\mathcal{B}'$ of the $g_{t,u,v}$'s. It turns out that analogous statements of the ones of Lemmas 8, 9, 10 and 11 hold.

Finally, if $\lambda = (n)$, then the corresponding highest weight vector is of the form $h_n = x^n$. Thus one can collect them in a set B'' that is in a one-to-one correspondence with $\mathcal{B}'' = \{n : n \geq 1\}$. It is clear that if $n \leq n'$ then $h_n \leq h'_n$ and so B'' has the f.b.p.

We are now in a position to prove the main theorem of this section.

Theorem 15. *Let F be a field of characteristic zero and let I be a T-ideal containing $Id(UJ_2)$. Then I is finitely generated as T-ideal.*

Proof. If $I = \mathrm{Id}(UJ_2)$, then we have nothing to prove since Lemma 6 ensures us that I is finitely generated. So let us suppose that $I \supsetneq \mathrm{Id}(UJ_2)$.

Let now focus our attention to the highest weight vectors of the type (3), since the statement for the ones of type (4) will follow analogously.

Since the multiplicities of the characters corresponding to such highest weight vectors are greater than 1, we have to follow the four steps described in section 3. For any fixed $n \geq 1$, we choose to order the polynomials of degree n in $\mathcal{B}$ in the following way: $f_{t,u,v} \prec f_{t',u',v'}$ if and only if either $t < t'$ or $t = t'$ and $u < u'$ or $t = t'$ and $u = u'$ and $v < v'$. Recall that $n = t + 2u + v + 2$. Hence we completed **Step 1**. By Lemmas 8–11, the quasi-order defined on B induces a quasi-order on $\mathcal{B}$, thus $\mathcal{B}$ satisfies the f.b.p. and **Step 2** is complete. In order to get **Step 3**, let

$$f^{n_1} = \sum_{i=1}^{k_1} \alpha_i f_{t_i,u_i,v_i} \text{ and } f^{n_2} = \sum_{j=1}^{k_2} \beta_j f_{t_j,u_j,v_j}$$

be two highest weight vectors of degree n_1 and n_2 with leading terms $f_{t_{i_0},u_{i_0},v_{i_0}}$ and $f_{t_{j_0},u_{j_0},v_{j_0}}$ according to $\prec$, respectively. Recall that $t_i + 2u_i + v_i + 2 = n_1$ for all $1 \leq i \leq k_1$ and $t_j + 2u_j + v_j + 2 = n_2$ for all $1 \leq j \leq k_2$. Moreover, let $f_{t_{i_0},u_{i_0},v_{i_0}} \leq f_{t_{j_0},u_{j_0},v_{j_0}}$.

By the proofs of Lemmas 8–11, one gets that it is possible to obtain $f_{t_{j_0},u_{j_0},v_{j_0}}$ properly multiplying $f_{t_{i_0},u_{i_0},v_{i_0}}$ by some variables. In order to simplify the notation, let $f_{t_{j_0},u_{j_0},v_{j_0}} = \varphi(f_{t_{i_0},u_{i_0},v_{i_0}})$ be such multiplication. Thus it is clear that

$$v^{n_2} = \frac{\beta_{j_0}}{\alpha_{i_0}} \varphi(f^{n_1})$$

is a consequence of f^{n_1} and has the same leading term of f^{n_2}, hence it is the required highest weight vector and **Step 3** is done.

Finally, since the multiplicity corresponding to the partition $\lambda = (n)$, $n \geq 1$ is equal to 1, we have nothing more to prove. The proof of the theorem is now complete. □

5. Specht property for the metabelian algebra A_1

In this section we shall prove the Specht property for the variety generated by the metabelian Jordan algebra introduced in [27].

Let A_1 be the Jordan algebra generated by the elements t, a_i, b_i, $i \geq 1$, such that

$$\begin{aligned} &a_i a_j = b_i b_j = a_i b_j = t b_i = t w t = 0, \\ &t w a_i a_j = 0, \quad t w b_i b_j = 0 \\ &t w a_i b_j a_k = -t w a_k b_j a_i, \\ &t w b_i a_j b_k = -t w b_k a_j b_i, \end{aligned}$$

for all i, j, k, where w is any word in the alphabet of the generators. Here we are considering monomials with left-normed brackets.

Theorem 16. *[27, Theorem 1] Let A_1 be the Jordan algebra defined above, then $Id(A_1) = \langle (x_1x_2)(x_3x_4) \rangle^T$. Moreover, if*

$$\chi_n(A_1) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

is the n-th cocharacter of A_1, then $m_\lambda = 1$ if either $\lambda = (3, 2^{k-1}, 1^{n-2k-1})$ or $\lambda = (2^k, 1^{n-2k}.)$ In all other cases $m_\lambda = 0$.

Theorem 17. *[27, Theorem 2] Every proper subvariety of $var(A_1)$ has polynomial growth.*

Notice that in light of the previous theorems, A_1 is an infinitely generated Jordan algebra such that any product of its elements has left-normed brackets and furthermore it generates a variety of almost polynomial growth.

The highest weight vector associated to the partition $\lambda = (3, 2^{k-1}, 1^{n-2k-1})$ is of the form

$$f_{n,k} = x_1 \bar{x}_1 \tilde{x}_1 \bar{x}_2 \tilde{x}_2 \cdots \bar{x}_k \tilde{x}_k \bar{x}_{k+1} \bar{x}_{k+2} \cdots \bar{x}_{n-k-1}, \tag{5}$$

whereas the highest weight vector associated to the partition $\lambda = (2^k, 1^{n-2k}.)$ is of the form

$$g_{n,k} = \bar{x}_1\tilde{x}_1\bar{x}_2\tilde{x}_2\cdots\bar{x}_k\tilde{x}_k\bar{x}_{k+1}\bar{x}_{k+2}\cdots\bar{x}_{n-k}. \tag{6}$$

In order to prove the Specht property for $\mathrm{var}(A_1)$, by following the lines of [27], we first prove some technical lemmas.

Lemma 12. *The identities*

$$x_1x_2yx_3 + x_1x_3yx_2 + x_2x_3yx_1 \equiv 0 \text{ and} \tag{7}$$

$$z_1z_2\cdots z_sxyx \equiv 0,\ \ s \geq 2 \tag{8}$$

hold in var(A_1).

Proof. Recall that the identity $y(xx)x \equiv yx(xx)$ holds in every Jordan algebra. Then, by taking into account the identity $(x_1x_2)(x_3x_4)$, we get that the previous becomes

$$xxyx \equiv 0. \tag{9}$$

Now, the complete linearization of the latter identity gives us (7).

Moreover, if one partially linearizes (9) replacing x by $x+z$ and considers the multihomogeneous component of degree 1 in the z and degree 2 in the x, we get $2zxyx+x^2yz \equiv 0$. Finally, by replacing z by the product $z_1z_2\cdots z_s$, $s \geq 2$, we obtain identity (8). $\square$

Lemma 13. *Let $\mathcal{U}$ be a proper subvariety of var(A_1). If either $f_{n,k} \equiv 0$ or $g_{n,k} \equiv 0$ on $\mathcal{U}$ for some n and k, then $f_{n',k'} \equiv 0$ for all $n' \geq n$ and $k' \geq n-k-1$ and $g_{n',k'} \equiv 0$ for all $n' \geq n$ and $k' > n-k-1$.*

Proof. First, let us suppose that

$$f_{n,k} = x_1\bar{x}_1\tilde{x}_1\bar{x}_2\tilde{x}_2\cdots\bar{x}_k\tilde{x}_k\bar{x}_{k+1}\bar{x}_{k+2}\cdots\bar{x}_{n-k-1} \equiv 0$$

for some n and k and let us replace x_1 by $z_1z_2+x_1$. If we consider the linear component in z_1 and z_2 and if we multiply by $x_{k+1}, x_{k+2}, \ldots, x_{n-k-1}$, then we can apply identities (7) and (8) in order to get the following consequence

$$z_1z_2x_1x_1x_2x_2\cdots x_{n-k-1}x_{n-k-1}.$$

Thus, it is clear that $f_{n',k'} \equiv 0$ for all $n' \geq n$ and $k' \geq n-k-1$.

Now let us suppose

$$g_{n,k} = \bar{x}_1\tilde{x}_1\bar{x}_2\tilde{x}_2\cdots\bar{x}_k\tilde{x}_k\bar{x}_{k+1}\bar{x}_{k+2}\cdots\bar{x}_{n-k} \equiv 0$$

for some n and k. By multiplying $g_{n,k}$ by x_1 and by applying identity (7), we get as consequence

$$-x_1\bar{x}_1\tilde{x}_1\bar{x}_2\tilde{x}_2\cdots\bar{x}_k\tilde{x}_k\bar{x}_{k+1}\bar{x}_{k+2}\cdots\bar{x}_{n-k}-x_1\tilde{x}_1\bar{x}_1\bar{x}_2\tilde{x}_2\cdots\bar{x}_k\tilde{x}_k\bar{x}_{k+1}\bar{x}_{k+2}\cdots\bar{x}_{n-k}\equiv 0$$

Hence, with arguments similar to those used in the previous case, we get the claim. □

We now can prove the Specht property for $\mathrm{var}(A_1)$.

Theorem 18. *Let F be a field of characteristic zero and let $\mathcal{U}\subseteq var(A_1)$. Then $Id(\mathcal{U})$ is finitely generated as T-ideal.*

Proof. If $\mathcal{U}=\mathrm{var}(A_1)$, then by Theorem 16 we have nothing to prove, so let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}(A_1)$ and let $\mathrm{Id}(\mathcal{U})$ be the corresponding T-ideal.

Let us consider the sets

$$\mathcal{B}_1=\{f_{n,k}:\ n,k\in\mathbb{N}\}\text{ and}$$
$$\mathcal{B}_2=\{g_{n,k}:\ n,k\in\mathbb{N}\}.$$

We define a total order on $\mathcal{B}_1$ and $\mathcal{B}_2$ by stating that $f_{n,k}\le f_{n',k'}$ (resp. $g_{n,k}\le g_{n',k'}$) if $n\le n'$ or $n=n'$ and $n-k-1\le n'-k'-1$.

Among the generators of $\mathrm{Id}(\mathcal{U})$ let now consider $f_{N,K}$ and $g_{M,L}$ as the minimal highest weight vectors of $\mathcal{B}_1$ and $\mathcal{B}_2$ with respect to the above order. Then, by taking into account Lemma 13, it is clear that $f_{n,k}$ and $g_{m,l}$ are consequences of $f_{N,K}$ and $g_{M,L}$, respectively, if $n\ge N$ or $n-k-1\ge N-K-1$ and $m\ge M$ or $m-l-1\ge M-L-1$. It readily follows that a basis of $\mathrm{Id}(\mathcal{U})$ contains $f_{N,K}$, $g_{M,L}$ and a finite list of highest weight vectors $f_{n,k}$ and $g_{m,l}$ such that $n<N$ or $n-k-1<N-K-1$ and $m<M$ or $m-l-1<M-L-1$. Hence $\mathrm{Id}(\mathcal{U})$ is finitely generated and $\mathrm{var}(A_1)$ has the Specht property. □

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